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CONVERGENCE IN MEASURE OF LOGARITHMIC MEANS OF DOUBLE FOURIER SERIES

Baramidze L., Goginava U.

Abstract. We establish condition which guarantees convergence in measure of logarithmic means of the two-dimensional Fourier series.

Keywords and phrases: Two-dimensional Fourier series, convergence in measure, summability

AMS subject classification (2010): 43A50.

Let $\mathbb{T}^2 := [-\pi, \pi)^2$ denote a cube in the 2-dimensional Euclidean space $\mathbb{R}^2$. The elements of $\mathbb{R}^2$ are denoted by (x, y).

The notation $a \lesssim b$ in the paper stands for $a \leq cb$, where c is an absolute constant.

We denote by $L_0(\mathbb{T}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{T}^2$. $\operatorname{mes}(A)$ is the Lebesgue measure of the set $A \subset \mathbb{T}^2$.

We denote by $L_p(\mathbb{T}^2)$ the class of all measurable functions f that are 2π-periodic with respect to all variables and satisfy

$$\|f\|_p := \left(\int_{\mathbb{T}^2} |f|^p \right)^{1/p} < \infty.$$

The $weak - L_1(\mathbb{T}^2)$ space consists of all measurable, 2π-periodic relative to each variable functions f for which

$$\|f\|_{weak-L_1(\mathbb{T}^2)} := \sup_{\lambda} \lambda \operatorname{mes}\left\{ (x, y) \in \mathbb{T}^2 : |f(x, y)| > \lambda \right\} < \infty.$$

Let $f \in L_1(\mathbb{T}^2)$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{n,m=-\infty}^{+\infty} \widehat{f}(n, m) e^{i(nx+my)},$$

where

$$\widehat{f}(n, m) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x, y) e^{-i(nx+my)} dx dy$$

are the Fourier coefficients of the function f. The rectangular partial sums are defined as follows:

$$S_{NM}(f; x, y) := \sum_{n=-N}^{N} \sum_{m=-M}^{M} \widehat{f}(n, m) e^{i(nx+my)}.$$

In the literature the notion of the Riesz's logarithmic means of a Fourier series is known. The n-th Riesz logarithmic mean of the Fourier series of the integrable function

f is defined by

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_k(f)}{k+1}, l_n:=\sum_{k=0}^{n}\frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of its Fourier series. This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [13, 15]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [12, 2].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k}\sum_{k=0}^{n} q_k S_{n-k}(f).$$

If $q_k = \frac{1}{k+1}$, then we get the (Nörlund) logarithmic means:

$$L_n(f;x) := \frac{1}{l_n}\sum_{k=0}^{n}\frac{S_{n-k}(f)}{k+1}. \tag{1}$$

Although, it is a kind of "reverse" Riesz's logarithmic means. In [5] some convergence and divergence properties of the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions, and in the Lebesgue space L are proved.

In one of his last papers [14] Tkebuchava constructed a set of logarithmic summation methods which contains both of the above mentioned logarithmic summation methods as limit cases. Namely, for any integers n, n_0 such that $0 \leq n_0 \leq n$ let Tkebuchava's means T_{n,n_0} be defined by

$$\begin{aligned} & T_{n,n_0}(f;x) \\ :& = \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{S_k(f;x)}{n_0-k+1} + S_{n_0}(f;x) + \sum_{k=n_0+1}^{n}\frac{S_k(f;x)}{k-n_0+1}\right), \end{aligned}$$

where

$$l(n,n_0) := \sum_{k=0}^{n_0-1}\frac{1}{n_0-k+1} + 1 + \sum_{k=n_0+1}^{n}\frac{1}{k-n_0+1}.$$

It is clear that $l(n,n_0) \asymp \log n$. This summation method includes the Riesz (for $n_0 = 0$) and Nörlund (for $n_0 = n$) logarithmic methods, too.

Define the kernels F_{n,n_0} of Tkebuchava's means by

$$F_{n,n_0} := \frac{1}{l(n,n_0)}\left(\sum_{k=0}^{n_0-1}\frac{D_k}{n_0-k+1} + D_{n_0} + \sum_{k=n_0+1}^{n}\frac{D_k}{k-n_0+1}\right).$$

Tkebuchava [14] gave estimates of kernels. Namely, the following theorem holds.

Theorem T (Tkebuchava). *Let* $0 \leq n_0 \leq n$. *Then*

$$1 + \frac{\log^2(n_0+2)}{\log(n+2)} \lesssim \|F_{n,n_0}\|_{L_1(\mathbb{T})} \lesssim 1 + \frac{\log^2(n_0+2)}{\log(n+2)}.$$

The mixed logarithmic means of double Fourier series are defined by

$$\left(L_n\circ R_m\right)(f;x,y):=\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{S_{n-i,j}\left(f;x,y\right)}{\left(i+1\right)\left(j+1\right)}.$$

The Nörlund logarithmic means and Riesz logarithmic means of double Fourier series are defined by

$$\left(L_n\circ L_m\right)(f;x,y):=\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{S_{n-i,m-j}\left(f;x,y\right)}{\left(i+1\right)\left(j+1\right)},$$

$$\left(R_n\circ R_m\right)(f;x,y):=\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{S_{i,j}\left(f;x,y\right)}{\left(i+1\right)\left(j+1\right)},$$

respectively.

It is evident that

$$\left(L_n\circ L_m\right)(f;x,y)=\frac{1}{\pi^2}\int\limits_{\mathbb{T}^2}f\left(s,t\right)F_n\left(x-s\right)F_m\left(y-t\right)dsdt,$$

$$\left(R_n\circ R_m\right)(f;x,y)=\frac{1}{\pi^2}\int\limits_{\mathbb{T}^2}f\left(s,t\right)G_n\left(x-s\right)G_m\left(y-t\right)dsdt$$

and

$$\left(L_n\circ R_m\right)(f;x,y)=\frac{1}{\pi^2}\int\limits_{\mathbb{T}^2}f\left(s,t\right)F_n\left(x-s\right)G_m\left(y-t\right)dsdt,$$

where

$$F_n\left(u\right):=\frac{1}{l_n}\sum_{i=0}^{n}\frac{D_{n-i}\left(u\right)}{i+1},G_n\left(u\right):=\frac{1}{l_n}\sum_{i=0}^{n}\frac{D_{i}\left(u\right)}{i+1}.$$

Let $L_Q=L_Q(\mathbb{T}^2)$ be the Orlicz space ([10], Ch 2) generated by Young function Q, i.e. Q is a convex continuous even function such that $Q(0)=0$ and

$$\lim_{u\to+\infty}\frac{Q\left(u\right)}{u}=+\infty,\quad\lim_{u\to0}\frac{Q\left(u\right)}{u}=0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{T}^2)}=\inf\{k>0:\int\limits_{\mathbb{T}^2}Q(\left|f\right|/k)\leq1\}.$$

In particular, if $Q(u)=u\log^{\beta}(1+u)$ $(u,\beta>0)$, then the corresponding space will be denoted by $L\log^{\beta}L(\mathbb{T}^2)$.

The rectangular partial sums of double Fourier series $S_{n,m}\left(f;x,y\right)$ of the function $f\in L_p\left(\mathbb{T}^2\right),1<p<\infty$ converge in L_p norm to the function f, as $n\to\infty$ [16]. In the case $L_1(\mathbb{T}^2)$ this result does not hold . But for one dimensional case and for $f \in L_1(\mathbb{T})$, the operator $S_n(f)$ is of weak type (1,1) [17]. This estimate implies convergence of $S_n(f;x)$ in measure on $\mathbb{T}$ to the function $f \in L_1(\mathbb{T})$. However, for double Fourier series this result does not hold [9, 11]. Moreover, it is proved that quadratical partial sums $S_{n,n}(f;x,y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbb{T}^2)$. On the other hand, it is well-known that if the function $f \in L\log L(\mathbb{T}^2)$, then rectangular partial sums $S_{n,m}(f;x,y)$ converge in measure on $\mathbb{T}^2$.

Classical regular summation methods often improve the convergence of Fourier seeries. For instance, the Fejér means of the double Fourier series of the function $f \in L_1(\mathbb{T}^2)$ converge in $L_1(\mathbb{T}^2)$ norm to the function f [16]. These means present the particular case of the Nörlund means.

It is well known that the method of Nörlund logarithmic means of double Fourier series is weaker than the Cesáro method of any positive order. In [7] it is proved, that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than Orlicz space $L\log L(\mathbb{T}^2)$. Thus, not all classic regular summation methods can improve the convergence in measure of double Fourier series.

The results for summability of logarithmic means of Walsh-Fourier series can be found in [3, 4, 6, 5, 13, 15].

In [7] the mixed logarithmic means $(L_n \circ R_m)$ of rectangular partial sums multiple Fourier series are considered and it is proved that these means are acting from space $L(\mathbb{T}^2)$ into space $weak - L_1(\mathbb{T}^2)$. This fact implies that mixed logarithmic means of rectangular partial sums of double Fourier series converge in measure. In particular, the following is true.

Theorem GG1(Goginava, Gogoladze). *Let $f \in L_1(\mathbb{T}^2)$. Then*

$$(R_n \circ L_m)(f;x,y) \to f \text{ in measure on } \mathbb{T}^2, \text{ as } n,m \to \infty.$$

Theorem GG2 (Goginava, Gogoladze) *Let $f \in L\log L(\mathbb{T}^2)$. Then*

$$(L_n \circ L_m)(f;x,y) \to f \text{ in measure on } \mathbb{T}^2, \text{ as } n,m \to \infty.$$

Theorem GG3 (Goginava, Gogoladze). *Let $L_Q(\mathbb{T}^2)$ be an Orlicz space, such that*

$$L_Q\left(\mathbb{T}^d\right) \nsubseteq L\log L\left(\mathbb{T}^2\right).$$

Then the set of the functions from the Orlicz space $L_Q(\mathbb{T}^2)$ with logarithmic means $(L_n \circ L_m)(f)$ of rectangular partial sums of double Fourier series convergent in measure on $\mathbb{T}^2$ is of first Baire category in $L_Q(\mathbb{T}^2)$.

For any integers n, n_0, m such that $0 \leq n_0 \leq n$ we put

$$(T_{n,n_0} \circ L_m)(f;x,y) = f * (F_{n,n_0} \times F_m).$$

It is easy to show that

$$\left(T_{n,n_0}\circ L_m\right)(f;x,y)=\frac{1}{\pi^2}\int\limits_{\mathbb{T}^2} f(s,t)\,F_{n,n_0}(x-s)\,F_m(y-t)\,dsdt.$$

This summation method includes the $(R_n\circ L_m)$ (for $n_0=0$) and $(L_n\circ L_m)$ (for $n_0=n$) methods, too.

On the basis of the above facts we can formulate the following problem:

Let $f\in L_1(\mathbb{T}^2)$. What condition on the $n_0=n_0(n)$ ensure the convergence in measure on $\mathbb{T}^2$ of the $(T_{n,n_0}\circ L_m)$ means of the two-dimensional trigonometric Fourier series?

A solution of this problem is given in

Theorem 1. *a)Let $f\in L_1(\mathbb{T}^2)$ and*

$$\log n_0(n)=O\left(\sqrt{\log n}\right).$$

Then

$$\left(T_{n,n_0}\circ L_m\right)(f;x,y)\to f \text{ in measure on } \mathbb{T}^2,\quad \text{as}\quad n,m\to\infty.$$

b) Let

$$\overline{\lim_{n\to\infty}}\frac{\log n_0(n)}{\sqrt{\log n}}=\infty.$$

Then the set of the functions from the space $L_1(\mathbb{T}^2)$ with logarithmic means $(T_{n,n_0}\circ L_m)(f)$ of rectangular partial sums of double Fourier series convergent in measure on $\mathbb{T}^2$ is of first Baire category in $L_1(\mathbb{T}^2)$.

In order to prove Theorem we apply the reasoning of ([1], Ch. 1) formulated as the following proposition in a particular case.

Theorem G. *Let $H:L_1(\mathbb{T}^2)\to L_0(\mathbb{T}^2)$ be a linear continuous operator, which commutes with family of translations $\mathcal{E}$, i. e. $\forall E\in\mathcal{E}\quad \forall f\in L_1(\mathbb{T}^2)\quad HEf=EHf$. Let $\|f\|_{L_1(\mathbb{T}^2)}=1$ and $\lambda>1$. Then for any $1\le r\in\mathbb{N}$ under condition $mes\{(x,y)\in\mathbb{T}^2:|Hf|>\lambda\}\ge\frac{1}{r}$ there exist $E_1,...,E_r,E_1',...,E_r'\in\mathcal{E}$ and $\varepsilon_i=\pm1,\quad i=1,...,r$ such that*

$$\mathrm{mes}\{(\mathrm{x},\mathrm{y})\in\mathbb{T}^2:\left|\mathrm{H}\left(\sum_{\mathrm{i}=1}^{\mathrm{r}}\varepsilon_{\mathrm{i}}\mathrm{f}(\mathrm{E_i x},\mathrm{E_i' y})\right)\right|>\lambda\}\ge\frac{1}{8}.$$

Theorem GGT (Gát, Goginava, Tkebuchava). *Let $\{H_m\}_{m=1}^{\infty}$ be a sequence of linear continuous operators, acting from the space $L_1(\mathbb{T}^2)$ into the space $L_0(\mathbb{T}^2)$. Suppose that there exists the sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from the unit ball $S(0,1)$ of space $L_1(\mathbb{T}^2)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\nu_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0=\inf_k \mathrm{mes}\{(x,y)\in\mathbb{T}^2:|H_{m_k}\xi_k(x,y)|>\nu_k\}>0.$$

Then K - the set of functions f from the space $L_1(\mathbb{T}^2)$, for which the sequence $\{H_m f\}$ converges in measure to an a. e. finite function is of first Baire category in the space $L_1(\mathbb{T}^2)$.

The proof of Lemma GGT can be found in [3].

Set

$$\alpha_{km}:=\frac{\pi\left(12k+1\right)}{6\left(m+1/2\right)},\beta_{km}:=\frac{\pi\left(12k+5\right)}{6\left(m+1/2\right)},\gamma_{m}:=\frac{\pi}{6\left(m+1/2\right)},$$

$$J_{m}:=\bigcup_{k=1}^{\left[\frac{\sqrt{m+1}-5}{12}\right]}\left[\alpha_{km}+\gamma_{m},\beta_{km}-\gamma_{m}\right].$$

Lemma T (Tkebuchava). *Let $0\leq z\leq\gamma_{n}$ and $x\in J_{n}$. Then*

$$F_{n,n_{0}}\left(x-z\right)\gtrsim\frac{\log\left(n_{0}+2\right)}{x\log\left(n+2\right)}.$$

The proof of Lemma T can be found in [6].

Proof of Theorem 1. a) In [8] it is proved that the one dimensional operator $L_{m}\left(f\right):=f\ast F_{m}$ (see (1)) is of weak type $(1,1)$, i. e. for $f\in L_{1}\left(\mathbb{T}^{1}\right)$ we have

$$\left\Vert L_{m}\left(f\right)\right\Vert_{weak-L_{1}\left(\mathbb{T}^{1}\right)}\lesssim\left\Vert f\right\Vert_{L_{1}\left(\mathbb{T}^{1}\right)}. \tag{2}$$

On the other hand, Tkebuchava in [14] proved that

$$\sup_{n}\left\Vert F_{n,n_{0}}\right\Vert_{L_{1}\left(\mathbb{T}\right)}<\infty$$

when

$$\log n_{0}=O\left(\sqrt{\log n}\right). \tag{3}$$

Set

$$\Omega:=\left\{(x,y)\in\mathbb{T}^{\mathbf{2}}:\left|\left(\mathbf{T_{n,n_{0}}}\circ\mathbf{L_{m}}\right)\left(\mathbf{f},\mathbf{x},\mathbf{y}\right)\right|>\lambda\right\}.$$

Then from (2) and (3) we have

$$\begin{aligned}&\lambda\operatorname{mes}\left(\Omega\right) \\ =\ &\lambda\int\limits_{\mathbb{T}^{2}}\mathbb{I}_{\Omega}\left(x,y\right)dxdy=\lambda\int\limits_{\mathbb{T}}\left(\int\limits_{\mathbb{T}}\mathbb{I}_{\Omega}\left(x,y\right)dy\right)dx \\ \lesssim\ &\left\Vert\left(f\ast F_{n,n_{0}}\right)\left(f\right)\right\Vert_{L_{1}\left(\mathbb{T}^{2}\right)}\lesssim\left\Vert f\right\Vert_{L_{1}\left(\mathbb{T}^{2}\right)},\end{aligned} \tag{4}$$

where $\mathbb{I}_{\mathbb{E}}$ is a characteristic function of the set E.

By virtue of standart argument (see [17]) we can prove the validity of part a) from the estimation (4).

Now, we prove part b). Let

$$\overline{\lim_{n\to\infty}}\frac{\log n_{0}\left(n\right)}{\sqrt{\log n}}=\lim_{k\to\infty}\frac{\log n_{0}\left(n_{k}\right)}{\sqrt{\log n_{k}}}=\infty.$$

By Lemma GGT the proof of Theorem will be complete if we show that there exists for the sequences of integers $\{n_{k}:k\geq1\}$ and $\{\nu_{k}:k\geq1\}$ increasing to infinity, and a sequence of functions $\{\xi_k : n \geq 1\}$ from the unit bull $S(0,1)$ of space $L_1(\mathbb{T}^2)$, such that for all n

$$\operatorname{mes}\{(x,y)\in\mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)}\circ L_{n_k}\right)(\xi_k;x,y)\right| > \nu_k\} \geq \frac{1}{8}. \tag{5}$$

First, we prove that

$$\operatorname{mes}\left\{(x,y)\in\mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)}\circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};x,y\right)\right| \gtrsim n_k^{3/2}\right\} \tag{6}$$

$$\gtrsim \frac{\log^2 n_0(n_k)}{n_k^{3/2}\log n_k}.$$

From Lemma T we have

$$\begin{aligned}
&\left(T_{n_k,n_0(n_k)}\circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};x,y\right)\\
=\ & \frac{1}{\gamma_{n_k}^2}\frac{1}{\pi^2}\int_{[0,\gamma_{n_k}]^2} F_{n_k,n_0(n_k)}(x-u)\,F_{n_k}(y-v)\,dudv\\
\gtrsim\ & \frac{\log n_0(n_k)}{\log n_k}\frac{1}{xy},\ (x,y)\in J_{n_k}\times J_{n_k}.
\end{aligned}$$

Set

$$s_{i,n_k} := \frac{\sqrt{n_k}\log n_0(n_k)}{i\log n_k}.$$

Then we can write

$$\begin{aligned}
&\operatorname{mes}\left\{(x,y)\in\mathbb{T}^2 : \left|\left(T_{n_k,n_0(n_k)}\circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};x,y\right)\right| \gtrsim n_k^{3/2}\right\}\\
\geq\ & \operatorname{mes}\left\{(x,y)\in J_{n_k}\times J_{n_k} : \left|\left(T_{n_k,n_0(n_k)}\circ L_{n_k}\right)\left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};x,y\right)\right| \gtrsim n_k^{3/2}\right\}\\
\geq\ & \operatorname{mes}\left\{(x,y)\in J_{n_k}\times J_{n_k} : \frac{\log n_0(n_k)}{\log n_k}\frac{1}{xy} \gtrsim n_k^{3/2}\right\}\\
=\ & \operatorname{mes}\left\{(x,y)\in J_{n_k}\times J_{n_k} : y \lesssim \frac{\log n_0(n_k)}{x n_k^{3/2}\log n_k}\right\}\\
\gtrsim\ & \frac{1}{n_k^2}\sum_{i=1}^{\left[\frac{\sqrt{n_0(n_k)+1}-5}{12}\right]}\sum_{l=1}^{s_{i,n_k}}\\
=\ & c\sum_{i=1}^{\left[\frac{\sqrt{n_0(n_k)+1}-5}{12}\right]}\frac{\sqrt{n_k}\log n_0(n_k)}{i n_k^2\log n_k}\\
\gtrsim\ & \frac{\log^2 n_0(n_k)}{n_k^{3/2}\log n_k},
\end{aligned}$$

Hence (6) is proved.

Then by the virtue of Theorem G there exists $E_1, ..., E_{r_k}, E'_1, ..., E'_{r_k} \in \mathcal{E}$ and $\varepsilon_1, ..., \varepsilon_{r_k} = \pm 1$ such that

$$\begin{aligned} \operatorname{mes}\{(x,y) \quad &\in \quad \mathbb{T}^2 : \left| \sum_{i=1}^{r_k} \varepsilon_i \left(T_{n_k, n_0(n_k)} \circ L_{n_k} \right) \left(\frac{\mathbb{I}_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2}; E_i x, E'_i y \right) \right| \\ &\gtrsim \quad n_k^{3/2} \} > \frac{1}{8}, \end{aligned} \tag{7}$$

where

$$r_k \sim \frac{n_k^{3/2} \log n_k}{\log^2 n_0 (n_k)}.$$

Denote

$$\nu_k = \frac{\log^2 n_0 (n_k)}{\log n_k}$$

and

$$\xi_k (x, y) = \frac{1}{r_k} \sum_{i=1}^{r_k} \varepsilon_i \frac{\mathbb{I}_{[0,\gamma_{n_k}]^2} (E_i x, E'_i y)}{\gamma_{n_k}^2}.$$

Thus, from (7) we obtain (5).

Finally, we prove that $\xi_k \in S(0,1)$. Indeed,

$$\|\xi_k\|_{L_1(\mathbb{T}^2)} \leq \frac{1}{r_k} \sum_{i=1}^{r_k} \frac{\left\| \mathbb{I}_{[0,\gamma_{n_k}]^2} \right\|_{L_1(\mathbb{T}^2)}}{\gamma_{n_k}^2} \leq 1.$$

Hence, $\xi_k \in S(0,1)$, and Theorem is proved.

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BOUNDARY VALUE PROBLEMS OF THE FULLY COUPLED THEORY OF ELASTICITY FOR SOLIDS WITH DOUBLE POROSITY FOR HALF-PLANE

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Abstract. In the paper the two-dimensional version of steady vibration in the fully coupled linear theory of elasticity for solids with double porosity is considered. Using the Fourier integrals, some basic boundary value problems are solved explicitly (in quadratures) for the half-plane.

Keywords and phrases: Porous media, double porosity, fully coupled theory of elasticity.

AMS subject classification (2010): 74F10, 74G05.

Introduction

Porous media theories play an important role in many branches of engineering, including material science, the petroleum industry, chemical engineering, and soil mechanics, as well as biomechanics.

In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis [1]. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1],[2], [3], where analytical solutions of the relevant equations are also given.This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. The basic results and the historical information on the theory of porous media were summarized by R.de Boer [4]. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [5-8].

In the last years many authors have investigated different types of problems of the 2-dimensional and 3-dimensional theories of elasticity for materials with double porosity, publishing a large number of papers (some of these results can be seen in [9-20] and references therein). There the explicit solutions on some BVPs in the form of series and in quadratures are given in a form useful for engineering practice.

The purpose of this paper is to consider the two-dimensional version of steady vibration in the fully coupled linear theory of elasticity for solids with double porosity. Using the Fourier integrals, some basic boundary value problems in the fully coupled linear theory of elasticity are solved explicitly (in quadratures) for the half-plane.

2. Basic equations. Boundary value problems

Let R^2_+ denote the upper half-plane $x_2 > 0$. The boundary of R^2_+ which is x_1-axis we denoted by S : Let $\mathbf{x} := (x_1, x_2) \in R^2_+$, $\partial\mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. We assume the domain R^2_+ to be filled with an isotropic elastic material with double porosity.

The governing homogeneous system of the theory of steady vibration in the fully coupled linear theory of elasticity for materials with double porosity has the form [9]

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) + \rho_1\omega^2\mathbf{u} = 0,\\
&i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 = 0, \qquad (1)\\
&i\omega\beta_2 div\mathbf{u} + (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 = 0,
\end{aligned}$$

where $\mathbf{u}(\mathbf{x}) = \mathbf{u}(u_1, u_2)$ is the displacement vector in a solid, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ are the pore and fissure fluid pressures respectively. $a_j = i\omega\alpha_j - \gamma$, $\omega > 0$ is the oscillation frequency, $\rho_1 > 0$ is the reference mass density, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, are constitutive coefficients, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively. $k_j = \frac{\kappa_j}{\mu'}$, $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the Laplace operator. Throughout this article it is assumed that $\beta_1^2 + \beta_2^2 > 0$. Vectors, if needed, we consider as column matrices.

Here we state the following BVPs.

Find a solution $\boldsymbol{U}(\boldsymbol{u}, p_1, p_2) \in C^2(R^2_+)$ to the Eqs. (1) in R^2_+ , satisfying one of the following boundary conditions (BCs) on S :

Problem 1.

$$\mathbf{u}^+ = \mathbf{f}(x_1), \quad p_1^+ = f_3(x_1) \quad p_2^+ = f_4(x_1), \quad x_1 \in S, \qquad (2)$$

Problem 2.

$$\begin{aligned}
&u_1^+ = f_1(x_1),\\
&(\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{u})_2^+ = f_2(x_1), \quad p_1 = f_3(x_1), \quad p_2 = f_4(x_1), \qquad (3)
\end{aligned}$$

Problem 3.

$$(\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{u})_1^+ = f_1(x_1), \quad u_2^+ = f_2(x_1), \quad \frac{\partial p_1}{\partial x_2} = f_3(x_1), \quad \frac{\partial p_2}{\partial x_2} = f_4(x_1). \qquad (4)$$

The symbol $(.)^+$ denotes the limit on S from R^2_+,

$$\lim_{R^2_+\ni\mathbf{x}\to x_1\in S}\mathbf{u}=\mathbf{f}(x_1),\qquad \lim_{R^2_+\ni\mathbf{x}\to x_1\in S}p_1=\mathbf{f}_3(x_1),\qquad \lim_{R^2_+\ni\mathbf{x}\to x_1\in S}p_2=\mathbf{f}_4(x_1),$$
$$\lim_{R^2_+\ni\mathbf{x}\to x_1\in S}\left[\mathbf{P}\left(\frac{\partial}{\partial\mathbf{x}},\mathbf{n}\right)\mathbf{U}\right]_\alpha=f_\alpha(x_1),\quad \alpha=1,2,$$

the functions f_j, $j=1,2,3,4$, are prescribed, $\mathbf{n}:=(0,1)$ is a unit normal vector,

$$\mathbf{P}(\partial\mathbf{x},\mathbf{n})\mathbf{U}=\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}-\mathbf{n}(\beta_1p_1+\beta_2p_2),\tag{5}$$

$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}$ is the following vector

$$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}:=\begin{pmatrix}\mu\frac{\partial}{\partial x_2} & \mu\frac{\partial}{\partial x_1}\\ \lambda\frac{\partial}{\partial x_1} & \mu_0\frac{\partial}{\partial x_2}\end{pmatrix}\mathbf{u},\quad \mu_0:=\lambda+2\mu.$$

In the domain of regularity the regular solution $\mathbf{U}=(\mathbf{u},p_1,p_2)\in C^2(D)$ of system (1) is represented as the sum (see appendix 1)

$$\begin{aligned}&\mathbf{u}(\mathbf{x})=-grad\sum_{m=1}^{3}\frac{\varphi_m(\mathbf{x})}{\lambda_m^2}+\mathbf{u}^{(4)}(\mathbf{x}),\quad div\mathbf{u}^{(4)}(\mathbf{x})=0,\\ &p_1(\mathbf{x})=\sum_{m=1}^{3}B_m\varphi_m(\mathbf{x}),\quad p_2(\mathbf{x})=\sum_{m=1}^{3}C_m\varphi_m(\mathbf{x}),\end{aligned}\tag{6}$$

where

$$\begin{aligned}&(\Delta+\lambda_m^2)\varphi_m(\mathbf{x})=0,\quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}(\mathbf{x})=0,\quad div\mathbf{u}^{(4)}(\mathbf{x})=0,\\ &B_m=-\frac{i\omega}{\delta_m}\left[\beta_1(a_2-k_2\lambda_m^2)-\beta_2(a_{12}-k_{12}\lambda_m^2)\right],\\ &C_m=-\frac{i\omega}{\delta_m}\left[\beta_2(a_1-k_1\lambda_m^2)-\beta_1(a_{21}-k_{21}\lambda_m^2)\right],\\ &\delta_m=(k_1k_2-k_{12}k_{21})\lambda_m^4-k_0\lambda_m^2+a_1a_2-a_{12}a_{21},\\ &\beta_1B_m+\beta_2C_m=-\frac{i\omega}{\delta_m}(\alpha_{12}-\alpha_{11}\lambda_m^2).\end{aligned}$$

λ_j^2, $j=1,2,3$, are roots of cubic algebraic equation

$$\begin{aligned}&\mu_0\alpha_0\xi^3-[\mu_0k_0+i\omega\alpha_{11}+\rho_1\omega^2\alpha_0]\xi^2\\ &+[\mu_0(a_1a_2-a_{12}a_{21})+i\omega\alpha_{12}+\rho_1\omega^2k_0]\xi-\rho_1\omega^2(a_1a_2-a_{12}a_{21})=0,\\ &\alpha_{11}=k_2\beta_1^2+k_1\beta_2^2-\beta_1\beta_2(k_{12}+k_{21}),\alpha_{12}=a_2\beta_1^2+a_1\beta_2^2-\beta_1\beta_2(a_{12}+a_{21}),\\ &\alpha_0=k_1k_2-k_{12}k_{21},k_0=a_1k_2+a_2k_1-k_{12}a_{21}-k_{21}a_{12},\lambda_4^2=\frac{\rho_1\omega^2}{\mu}.\end{aligned}\tag{7}$$

Let us assume that

$$\widehat{\mathbf{F}}(x_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{F}(\xi)\exp(-ix_1\xi)d\xi$$

and the inversion formula

$$\mathbf{F}(\xi)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\widehat{\mathbf{F}}(x_1)\exp(ix_1\xi)dx_1$$

is valid.

The Fourier integral theorem holds if both $\mathbf{F}$ and its Fourier transform are absolutely integrable and $\mathbf{F}$ is bounded and continuous at the point x_1. [24]

In what follows we assume, that the vector $\mathbf{f}$, and the functions f_3, f_4 are absolutely integrable, bounded, and continuous on S, moreover $\widehat{\mathbf{f}}$, $\widehat{f}_3$, and $\widehat{f}_4$ are absolutely integrable on S.

Theorem 1. *Problem 1 has at most one regular solution in the domain D.*

Theorem 1 can be proved similarly to the corresponding theorem in the classical theory of elasticity (for details see [25]).

Solution of Problem 1 for a half-plane

The solution of Problem 1 is sought in the form (6). Let us assume that the functions $\varphi_m(\mathbf{x})$, $m=1,2,3$, and $\mathbf{u}^{(4)}(\mathbf{x})$ are sought in the form [23]

$$\begin{aligned}
&\varphi_m=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\alpha_m(\xi)exp(-x_2r_m)exp[ix_1\xi]d\xi,\quad k=1,2,3,\\
&\mathbf{u}^{(4)}(\mathbf{x})=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\boldsymbol{\alpha}^{\mathbf{(4)}}(\xi)exp(-x_2r_4)exp[ix_1\xi]d\xi,\\
&r_m^2=\xi^2-\lambda_m^2,\quad \boldsymbol{\alpha}^{\mathbf{(4)}}=(\alpha_1^{(4)},\alpha_2^{(4)}),
\end{aligned}\tag{8}$$

where $\boldsymbol{\alpha}^{(4)}$ and α_m are absolutely integrable on S unknown values.

It is not difficult to prove that (8) satisfy equations $(\Delta+\lambda_m^2)\varphi_m=0$, $m=1,2,3$, $(\Delta+\lambda_4^2)\mathbf{u}^{(4)}=0$ for arbitrary α_m and $\boldsymbol{\alpha}^{\mathbf{(4)}}$, respectively.

By substituting in (6) the expressions of $\varphi_m(\mathbf{x})$ and $\mathbf{u}^{(4)}$ from (8), passing to the limit as $x_2\to 0$, and taking into account boundary conditions, for determining the unknown values α_m, $k=1,2,3$ and $\boldsymbol{\alpha}^{\mathbf{(4)}}$, we obtain the following system of algebraic equations

$$\begin{aligned}
&\xi^2\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2}+r_4\alpha_2^{(4)}=i\xi\widehat{f}_1,\qquad \sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}+\alpha_2^{(4)}=\widehat{f}_2,\\
&\sum_{m=1}^{3}B_m\alpha_m=\widehat{f}_3,\qquad \sum_{m=1}^{3}C_m\alpha_m=\widehat{f}_4,\qquad i\xi\alpha_1^{(4)}-r_4\alpha_2^{(4)}=0.
\end{aligned}\tag{9}$$

It easy to show that the determinant of system (9) has the form

$$\Delta_1=$$

$$=-\omega^2 d\left\{\frac{(\lambda_2^2-\lambda_3^2)(r_4\lambda_1^2+r_1\lambda_4^2)}{\lambda_1^2\delta_2\delta_3(r_1+r_4)}-\frac{(\lambda_1^2-\lambda_3^2)(r_4\lambda_2^2+r_2\lambda_4^2)}{\lambda_2^2\delta_1\delta_3(r_2+r_4)}+\frac{(\lambda_1^2-\lambda_2^2)(r_4\lambda_3^2+r_3\lambda_4^2)}{\lambda_3^2\delta_1\delta_2(r_3+r_4)}\right\}$$

$$d=(\beta_1a_2-\beta_2a_{12})(\beta_2k_1-\beta_1k_{21})-(\beta_2a_1-\beta_1a_{21})(\beta_1k_2-\beta_2k_{12}).$$

Due to Theorem 1 we conclude that the determinant of system (9) different from zero and system (9) is uniquely solvable.

From (9) we find

$$
\begin{aligned}
&\Delta_1\alpha_1 = -\left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_1 + \left[\frac{C_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{C_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_3\\
&-\left[\frac{B_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{B_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,\\
&\Delta_1\alpha_2 = \left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_2 - \left[\frac{C_3}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{C_1}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_3\\
&+\left[\frac{B_3}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{B_1}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,\\
&\Delta_1\alpha_3 = -\left[i\xi\widehat{f}_1 - r_4\widehat{f}_2\right]\eta_3 + \left[\frac{C_2}{\lambda_1^2}(r_1r_4-\xi^2) - \frac{C_1}{\lambda_2^2}(r_2r_4-\xi^2)\right]\widehat{f}_3\\
&-\left[\frac{B_3}{\lambda_2^2}(r_2r_4-\xi^2) - \frac{B_2}{\lambda_3^2}(r_3r_4-\xi^2)\right]\widehat{f}_4,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_1\alpha_2^{(4)} = \left[\frac{r_1-r_3}{\lambda_1^2\lambda_3^2}B_2 + \frac{r_2-r_1}{\lambda_1^2\lambda_2^2}B_3 + \frac{r_3-r_2}{\lambda_2^2\lambda_3^2}B_1\right]\xi^2\widehat{f}_4\\
&-\left[\frac{r_1-r_3}{\lambda_1^2\lambda_3^2}C_2 + \frac{r_2-r_1}{\lambda_1^2\lambda_2^2}C_3 + \frac{r_3-r_2}{\lambda_2^2\lambda_3^2}C_1\right]\xi^2\widehat{f}_3,\\
&-\frac{\omega^2 d(a_1a_2-a_{12}a_{21})}{\lambda_1^2\lambda_2^2\lambda_3^2\delta_1\delta_2\delta_3}(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\xi^2\widehat{f}_2,\\
&+\left[\frac{r_1\eta_1}{\lambda_1^2} - \frac{r_2\eta_2}{\lambda_2^2} + \frac{r_3\eta_3}{\lambda_3^2}\right]i\xi\widehat{f}_1, \quad i\xi\alpha_1^{(4)} = r_4\alpha_2^{(4)},\\
&\eta_1 = \frac{\omega^2 d}{\delta_2\delta_3}(\lambda_2^2-\lambda_3^2), \quad \eta_2 = \frac{\omega^2 d}{\delta_1\delta_3}(\lambda_1^2-\lambda_3^2), \quad \eta_3 = \frac{\omega^2 d}{\delta_1\delta_2}(\lambda_1^2-\lambda_2^2).
\end{aligned}
$$

Substituting the obtained values in (6), we obtain the desired solution of the BVP in quadratures.

Solution of Problem 2 for a half-plane

A solution is sought in the form (6),(8). Keeping in mind BCs and

$$
[Pu]_2 = -\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{(r_4^2+\xi^2)\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2}exp(-x_2r_m) + 2r_4\alpha_2^{(4)}exp(-x_2r_4)\right\}exp(ix_1\xi)d\xi,
$$

after passing to the limit, as $x_2 \to 0$, we get the following system of algebraic equations

$$
\begin{aligned}
&\xi^2\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2} + r_4\alpha_2^{(4)} = i\xi\widehat{f}_1, \quad (r_4^2+\xi^2)\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2} + 2r_4\alpha_2^{(4)} = -\frac{\widehat{f}_2}{\mu},\\
&\sum_{m=1}^{3}B_m\alpha_m = \widehat{f}_3, \quad \sum_{m=1}^{3}C_m\alpha_m = \widehat{f}_4, \quad i\xi\alpha_1^{(4)} - r_4\alpha_2^{(4)} = 0.
\end{aligned}
$$

From here we obtain

$$
\begin{aligned}
&\sum_{m=1}^{3}\frac{\alpha_m}{\lambda_m^2}=\left[\frac{\widehat{f}_2}{\mu}+2i\xi\widehat{f}_1\right]\frac{1}{\lambda_4^2},\\
&\sum_{m=1}^{3}B_m\alpha_m=\widehat{f}_3,\quad \sum_{m=1}^{3}C_m\alpha_m=\widehat{f}_4,
\end{aligned}
\tag{10}
$$

The determinant of system (10) has the form

$$
D_2=-\frac{\omega^2 d}{\lambda_1^2\lambda_2^2\lambda_3^2\delta_1\delta_2\delta_3}[a_1a_2-a_{12}a_{21}](\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\neq 0
$$

By elementary calculation, from (10) we obtain

$$
\begin{aligned}
&\alpha_m D_2=(-1)^m\left\{\left[\frac{\widehat{f}_2}{\mu}+2i\xi\widehat{f}_1\right]\eta_m+c_m\widehat{f}_3-b_m\widehat{f}_4\right\},\quad m=1,2,3,\\
&\alpha_2^{(4)}=-\frac{1}{\lambda_4^2 r_4}\left[i\xi(\xi^2+r_4^2)\widehat{f}_1+\xi^2\frac{\widehat{f}_2}{\mu}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&\eta_1=\frac{\omega^2 d}{\lambda_4^2\delta_2\delta_3}(\lambda_2^2-\lambda_3^2),\quad \eta_2=\frac{\omega^2 d}{\lambda_4^2\delta_1\delta_3}(\lambda_1^2-\lambda_3^2),\quad \eta_3=\frac{\omega^2 d}{\lambda_4^2\delta_1\delta_2}(\lambda_1^2-\lambda_2^2),\\
&c_1=\frac{C_3}{\lambda_2^2}-\frac{C_2}{\lambda_3^2},\quad c_2=\frac{C_3}{\lambda_1^2}-\frac{C_1}{\lambda_3^2},\quad c_3=\frac{C_2}{\lambda_1^2}-\frac{C_1}{\lambda_2^2},\\
&b_1=\frac{B_3}{\lambda_2^2}-\frac{B_2}{\lambda_3^2},\quad b_2=\frac{B_3}{\lambda_1^2}-\frac{B_1}{\lambda_3^2},\quad b_3=\frac{B_2}{\lambda_1^2}-\frac{B_1}{\lambda_2^2}.
\end{aligned}
$$

Substituting the obtained values in (6) and taking into account the following formula [24]

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-x_2r_m)\exp[i\xi(x_1-y_1)]\frac{1}{r_m}d\xi=i\sqrt{\frac{\pi}{2}}H_0^{(1)}(i\lambda_m r),
$$

where $H_0^{(1)}(i\lambda_m r)$ is the first kind Hankel function of zero order,

$$
r^2=(x_1-y_1)^2+x_2^2,\quad r_m^2=\xi^2-\lambda_m^2,\quad m=1,2,3
$$

we obtain

$$
\begin{aligned}
\varphi_m=&\frac{i(-1)^{m+1}}{2D_2}\int_{-\infty}^{+\infty}\left[\frac{\eta_m}{\mu}f_2(y_1)+c_mf_3(y_1)-b_mf_4(y_1)\right]\frac{\partial}{\partial x_2}H_0^{(1)}(i\lambda_m r)dy_1\\
&+\frac{i(-1)^{m+1}}{2D_2}\eta_m\int_{-\infty}^{+\infty}f_1(y_1)\frac{\partial^2}{\partial x_1\partial x_2}H_0^{(1)}(i\lambda_m r)dy_1,
\end{aligned}
$$

$$u_1^{(4)}=-\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[2\frac{\partial^3}{\partial x_1^2\partial x_2}H_0^{(1)}(\lambda_4 r)+\lambda_4^2\frac{\partial}{\partial x_2}H_0^{(1)}(i\lambda_4 r)\right]f_1(y_1)dy_1$$
$$-\frac{i}{2\lambda_4^2\mu}\int\limits_{-\infty}^{+\infty}f_2(y_1)\frac{\partial^2}{\partial x_1\partial x_2}H_0^{(1)}(i\lambda_4 r)dy_1$$

$$u_2^{(4)}=\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[-\frac{\partial^3}{\partial x_1^3}H_0^{(1)}(i\lambda_4 r)+\lambda_4^2\frac{\partial^3}{\partial x_1\partial x_2^2}H_0^{(1)}(i\lambda_4 r)\right]f_1(y_1)dy_1$$
$$-\frac{i}{2\lambda_4^2\mu}\int\limits_{-\infty}^{+\infty}f_2(y_1)\frac{\partial^2}{\partial x_1^2}H_0^{(1)}(i\lambda_4 r)dy_1,$$

Solution of Problem 3 for a half-plane

A solution is sought in the form (6),(8). Keeping in mind BCs, after passing to the limit, as $x_2\to 0$, we get the following system of algebraic equations

$$-2\xi^2\sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}-(r_4^2+\xi^2)\alpha_2^{(4)}=\frac{i\xi\widehat{f}_1}{\mu},\qquad \sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}+\alpha_2^{(4)}=\widehat{f}_2,$$
$$\sum_{m=1}^{3}B_m r_m\alpha_m=-\widehat{f}_3,\qquad \sum_{m=1}^{3}C_m r_m\alpha_m=-\widehat{f}_4,\qquad i\xi\alpha_1^{(4)}-r_4\alpha_2^{(4)}=0.$$

From here we get

$$\begin{gathered}\sum_{m=1}^{3}\frac{r_m\alpha_m}{\lambda_m^2}=-\frac{1}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+(\xi^2+r_4^2)\widehat{f}_2\right]\\ \sum_{m=1}^{3}B_m r_m\alpha_m=-\widehat{f}_3,\quad \sum_{m=1}^{3}C_m r_m\alpha_m=-\widehat{f}_4,\end{gathered}\tag{11}$$

It is easily seen that the determinant of system (11) has the form

$$D_3=-\frac{\omega^2 d r_1 r_2 r_3}{\lambda_1^2\lambda_2^2\lambda_3^2\delta_1\delta_2\delta_3}[a_1a_2-a_{12}a_{21}](\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)=r_1r_2r_3D_2\neq 0$$

By elementary calculation, from (11) we obtain

$$\alpha_m=\frac{(-1)^m}{r_m D_2}\left\{\frac{\eta_m}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+(\xi^2+r_4^2)\widehat{f}_2\right]+c_m\widehat{f}_3-b_m\widehat{f}_4\right\},$$
$$\alpha_1^{(4)}=\frac{r_4}{\lambda_4^2}\left[\frac{\widehat{f}_1}{\mu}-2i\xi\widehat{f}_2\right],\quad \alpha_2^{(4)}=\frac{1}{\lambda_4^2}\left[\frac{i\xi\widehat{f}_1}{\mu}+2\xi^2\widehat{f}_2\right].$$

Finally we have

$$\varphi_m=\frac{i}{2D_2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\eta_m\frac{\partial}{\partial x_1}H_0^{(1)}(i\lambda_m r)\frac{f_1(y_1)}{\mu}dy_1$$
$$-\frac{i}{2D_2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\eta_m\left(2\frac{\partial^2}{\partial x_1^2}H_0^{(1)}(i\lambda_m r)+\lambda_4^2H_0^{(1)}(i\lambda_m r)\right)f_2(y_1)dy_1$$
$$+\frac{i}{2D_2}\int\limits_{-\infty}^{+\infty}[C_mf_3(y_1)-b_mf_4(y_1)]H_0^{(1)}(i\lambda_m r)dy_1,$$

$$u_1^{(4)}=\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[\frac{\partial^2}{\partial x_2^2}H_0^{(1)}(i\lambda_4 r)\frac{f_1(y_1)}{\mu}-2\frac{\partial^3}{\partial x_1\partial x_2^2}H_0^{(1)}(i\lambda_4 r)f_2(y_1)\right]dy_1,$$
$$u_2^{(4)}=\frac{i}{2\lambda_4^2}\int\limits_{-\infty}^{+\infty}\left[-\frac{\partial^2}{\partial x_1\partial x_2}H_0^{(1)}(i\lambda_4 r)\frac{f_1(y_1)}{\mu}+2\frac{\partial^3}{\partial x_2\partial x_1^2}H_0^{(1)}(i\lambda_4 r)f_2(y_1)\right]dy_1,$$

Appendix 1. A Representation of Regular Solutions

Theorem 2. *If $\boldsymbol{U}:=(\boldsymbol{u},p_1,p_2)$ is a regular solution of the homogeneous system (1), then $\boldsymbol{u}$, $div\boldsymbol{u}$, p_1 and p_2 satisfy the equations*

$$\begin{aligned}&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)\mathbf{u}=0,\\&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)div\mathbf{u}=0\\&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)p_j=0,\quad j=1,2.\end{aligned}\tag{12}$$

where λ_j^2, $j=1,2,3$, are roots of equation (7).

Proof. Let $\mathbf{U}=(\mathbf{u},p_1,p_2)$ be a regular solution of the equations (1). Upon taking the divergence operation, from (1) we get

$$\begin{aligned}&(\mu_0\Delta+\rho\omega^2)div\mathbf{u}-\beta_1\Delta p_1-\beta_2\Delta p_2=0,\quad \mu_0=\lambda+2\mu,\\&i\omega\beta_1div\mathbf{u}+(k_1\Delta+a_1)p_1+(k_{12}\Delta+a_{12})p_2=0,\\&i\omega\beta_2div\mathbf{u}+(k_{21}\Delta+a_{21})p_1+(k_2\Delta+a_2)p_2=0,\end{aligned}$$

Rewrite the latter system as follows

$$D(\Delta)\Psi:=\begin{pmatrix}\mu_0\Delta+\rho\omega^2 & -\beta_1\Delta & -\beta_2\Delta\\ i\omega\beta_1 & k_1\Delta+a_1 & k_{12}\Delta+a_{12}\\ i\omega\beta_2 & k_{21}\Delta+a_{21} & k_2\Delta+a_2\end{pmatrix}\Psi=0,\tag{13}$$

where $\Psi=(div\mathbf{u},p_1,p_2)$.

By the direct calculation, we get

$$det\mathbf{D} = \mu_0\alpha_0(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2),$$

Clearly, from system (13) it follows that

$$\begin{aligned}&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)div\mathbf{u}=0,\\&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)p_j=0,\quad j=1,2.\end{aligned}\tag{14}$$

Further, applying the operator $(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)$ to equation (1), and using the last relations we obtain

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)\mathbf{u}=0\tag{15}$$

The last formulas (14),(15) prove the theorem.

Theorem 3. T*he regular solution* $\boldsymbol{U}=(\boldsymbol{u},p_1,p_2)$ *of system (1) admits in the domain of regularity a representation*

$$\mathbf{U}=(\overset{\mathbf{1}}{\mathbf{u}}+\overset{\mathbf{2}}{\mathbf{u}},p_1,p_2),\tag{16}$$

where $\overset{\mathbf{1}}{\mathbf{u}}$, *and* $\overset{\mathbf{2}}{\mathbf{u}}$ *are the regular vectors, satisfying the conditions*

$$\begin{aligned}&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)\overset{\mathbf{1}}{\mathbf{u}}=0,\quad rot\overset{\mathbf{1}}{\mathbf{u}}=0,\\&(\Delta+\lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}}=0,\quad div\overset{\mathbf{2}}{\mathbf{u}}=0.\end{aligned}$$

Proof. Let $\mathbf{U}=(\mathbf{u},p_1,p_2)$ be a regular solution of system (1). Using the identity

$$\Delta\mathbf{w}=graddiv\mathbf{w}-rotrot\mathbf{w},\tag{17}$$

from Eq.(1) we obtain

$$\mathbf{u}=-\frac{\mu_0}{\rho\omega^2}graddiv\mathbf{u}+\frac{\mu}{\rho\omega^2}rotrot\mathbf{u}+\frac{1}{\rho\omega^2}grad(\beta_1p_1+\beta_2p_2),$$

Let

$$\overset{\mathbf{1}}{\mathbf{u}}:=-\frac{\mu_0}{\rho\omega^2}graddiv\mathbf{u}+\frac{1}{\rho\omega^2}grad(\beta_1p_1+\beta_2p_2),\tag{18}$$

$$\overset{\mathbf{2}}{\mathbf{u}}:=\frac{\mu}{\rho\omega^2}rotrot\mathbf{u}.\tag{19}$$

Clearly

$$\mathbf{u}=\overset{\mathbf{1}}{\mathbf{u}}+\overset{\mathbf{2}}{\mathbf{u}},\quad rot\overset{\mathbf{1}}{\mathbf{u}}=0,\quad div\overset{\mathbf{2}}{\mathbf{u}}=0.\tag{20}$$

Using the identity $\Delta\overset{\mathbf{2}}{\mathbf{u}}=-rotrot\overset{\mathbf{2}}{\mathbf{u}}$, from (19) we obtain

$$(\Delta+\lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}}=0.\tag{21}$$

Taking into account the relations (14),(15),(18) and (19) we can easily prove the following

Theorem 4. *In the domain of regularity the regular solution* $\boldsymbol{U} = (\boldsymbol{u}, p_1, p_2) \in C^2(D)$ *of system (1) is represented as the sum*

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{x}) = -grad \sum_{m=1}^{3} \frac{\varphi_m(\boldsymbol{x})}{\lambda_m^2} + \boldsymbol{u}^{(2)}(\boldsymbol{x}), \\ &p_1(\boldsymbol{x}) = \sum_{m=1}^{3} B_m \varphi_m(\boldsymbol{x}), \quad p_2(\boldsymbol{x}) = \sum_{m=1}^{3} C_m \varphi_m(\boldsymbol{x}), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &(\Delta + \lambda_m^2)\varphi_m(\boldsymbol{x}) = 0, \quad (\Delta + \lambda_4^2)\boldsymbol{u}^{(2)}(\boldsymbol{x}) = 0, \quad div\boldsymbol{u}^{(2)}(\boldsymbol{x}) = 0, \\ &B_m = -\frac{i\omega}{\delta_m}\left[\beta_1(a_2 - k_2\lambda_m^2) - \beta_2(a_{12} - k_{12}\lambda_m^2)\right], \\ &C_m = -\frac{i\omega}{\delta_m}\left[\beta_2(a_1 - k_1\lambda_m^2) - \beta_1(a_{21} - k_{21}\lambda_m^2)\right], \\ &\delta_m = (k_1k_2 - k_{12}k_{21})\lambda_m^4 - k_0\lambda_m^2 + a_1a_2 - a_{12}a_{21}. \end{aligned}$$

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FUNDAMENTAL SOLUTION IN THE FULLY COUPLED THEORY OF ELASTICITY FOR SOLIDS WITH DOUBLE POROSITY

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Abstract. In this paper the 2D fully coupled quasi-static theory of poroelasticity for materials with double porosity is considered. For these equations the fundamental and some other matrixes of singular solutions are constructed in terms of elementary functions. The properties of single and double layer potentials are studied.

Keywords and phrases: Double porosity, fundamental solution.

AMS subject classification (2010): 74F10, 35E05.

Introduction

The theory of consolidation for elastic materials with double porosity was presented in [1-3]. The theory of Aifantis unifies the models of Barenblatt for porous media with double porosity [4] and Biot's model for porous media with single porosity [5]. However, Aifantis' quasi-static theory ignored the cross-coupling effects between the volume change of the pores and fissures in the system. This deficiency was eliminated and cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solid with double porosity in [6]. In [6,7] the cross-coupled terms were included in Darcy's law for solid with double porosity.

The double porosity concept was extended for multiple porosity media in [8, 9]. The basic equations of the thermo-hydro-mechanical coupling theory for elastic materials with double porosity were presented in [10-12]. The theory of multiporous media, as originally developed for the mechanics of naturally fractured reservoirs, has found applications in blood perfusion. The double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity. An extensive review of the results in the theory of bone poroelasticity can be found in the survey papers [13-15]. For a history of developments and a review of main results in the theory of porous media see [16].

The fundamental solutions have occupied a special place in the theory of PDEs. They are encountered in many mathematical, mechanical, physical and engineering applications. Indeed, the application of fundamental solutions to a recently developed area of boundary element methods has provided a distinct advantage in the fact that an integral representation of solution of a boundary value problem by fundamental solution is often more easily solved by numerical methods than a differential equation with specified boundary and initial conditions. Recent advances in the area of boundary element methods, where the theory of fundamental solutions plays a pivotal role, has provided a prominent place in research of problems in the theories of PDEs, applied mathematics, continuum mechanics and quantum physics. The fundamental solutions in the linear theories of elasticity and thermoelasticity for materials with microstructures are constructed by means of elementary functions by several authors [17-20].

In this paper the 2D fully coupled quasi-static theory of poroelasticity for materials with double porosity is considered. For these equations the fundamental and some other matrixes of singular solutions are constructed in terms of elementary functions. The properties of single and double layer potentials are studied.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D space E^2. Let D^+ be a bounded 2D domain surrounded by the curve S and let D^- be the complement of $D^+ \cup S$. $D_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. Let us assume that the domain D is filled with an isotropic material with double porosity.

The system of homogeneous equations in the 2D fully coupled quasi-static linear theory of elasticity for solids with double porosity can be written as follows

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0,$$

$$\begin{aligned} &i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 = 0,\\ &i\omega\beta_2 div\mathbf{u} + (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 = 0, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. $a_j = i\omega\alpha_j - \gamma, a_{ij} = i\omega\alpha_{ij} + \gamma$, $\omega > 0$ is the oscillation frequency, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively; α_{12} and α_{21} are the cross-coupling compressibility for fluid flow at the interface between the two-pore systems at a microscopic level. λ, μ, are constitutive coefficients, $k_j = \frac{\kappa_j}{\mu'}$, $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$, μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface beetween the matrix and fissure phases, Δ is the Laplacian. The superscript "T" denotes transposition.

We assume that the inertial energy density of solid with double porosity is a positive definite quadratic form. Thus, the constitutive coefficients satisfy the conditions

$$\mu > 0, \quad k_1 > 0, \quad a_1 a_2 > a_{12}a_{21}, \quad k_1 k_2 > k_{12}k_{21}, \quad \gamma > 0.$$

We introduce the matrix differential operator with constant coefficients:

$$\mathbf{A}(D_x, \omega) = (A_{ij})_{4\times 4},$$

where

$$
\begin{aligned}
&A_{lj} := \delta_{lj}\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l,j=1,2,\\
&A_{j3} := -\beta_1\frac{\partial}{\partial x_j}, \quad A_{j4} := -\beta_2\frac{\partial}{\partial x_j} \quad j=1,2\\
&A_{3j} := i\omega\beta_1\frac{\partial}{\partial x_j}, \quad A_{4j} := i\omega\beta_2\frac{\partial}{\partial x_j} \quad j=1,2 \quad A_{33} := k_1\Delta + a_1,\\
&A_{34} := k_{12}\Delta + a_{12}, \quad A_{43} := k_{21}\Delta + a_{21}, \quad A_{44} := k_2\Delta + a_2,
\end{aligned}
$$

δ_{lj} is the Kronecker delta. Then the system (1) can be rewritten as

$$
\mathbf{A}(D_x,\omega)\mathbf{U} = 0, \tag{2}
$$

where

$$
\mathbf{U} := (\mathbf{u}, p_1, p_2).
$$

The conjugate system of the equation (1) is

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - i\omega grad(\beta_1 p_1 + \beta_2 p_2) = 0,\\
&\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{21}\Delta + a_{21})p_2 = 0,\\
&\beta_2 div\mathbf{u} + (k_{12}\Delta + a_{12})p_1 + (k_2\Delta + a_2)p_2 = 0,
\end{aligned}
\tag{3}
$$

$$
\widetilde{\mathbf{A}}(D_x,\omega)\mathbf{U} = \mathbf{A}^T(-D_x,\omega)\mathbf{U} = 0.
$$

We assume that $\mu\mu_0(k_1k_2 - k_{12}k_{21}) \neq 0$, where $\mu_0 := \lambda + 2\mu$. Obviously, if the last condition is satisfied, then $\mathbf{A}(D_x,\omega)$ is the elliptic differential operator.

3. The basic fundamental matrix

In this section, we will construct the basic fundamental matrix of system (2). We introduce the matrix differential operator $\mathbf{B}(\partial\mathbf{x})$ consisting of cofactors of elements of the matrix $\mathbf{A}^T$ divided on $\mu\mu_0(k_1k_2 - k_{12}k_{21})$:

$$
\mathbf{B}(D_x) = (B_{ij})_{4\times 4},
$$

where

$$
\begin{aligned}
B_{lj} = &\frac{\delta_{lj}}{\mu}\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2) - \xi_l\xi_j\frac{i\omega}{\alpha_0}(\alpha_{12} + \alpha_{11}\Delta)\\
&-\xi_l\xi_j\frac{\lambda+\mu}{\alpha_0}[(k_1k_2 - k_{12}k_{21})\Delta\Delta + k_0\Delta + a_1a_2 - a_{12}a_{21}],
\end{aligned}
$$

$$B_{3j}=-\frac{i\omega\mu}{\alpha_0}\xi_j\Delta[(\beta_1k_2-\beta_2k_{12})\Delta+\beta_1a_2-\beta_2a_{12}],$$
$$B_{4j}=-\frac{i\omega\mu}{\alpha_0}\xi_j\Delta[(\beta_1k_{21}-\beta_2k_1)\Delta+\beta_1a_{21}-\beta_2a_1],$$
$$B_{j4}=-\frac{\mu}{\alpha_0}\xi_j\Delta[(\beta_1k_{12}-\beta_2k_1)\Delta+\beta_1a_{12}-\beta_2a_1],$$
$$B_{j3}=\frac{\mu}{\alpha_0}\xi_j\Delta[(\beta_1k_2-\beta_2k_{21})\Delta+\beta_1a_2-\beta_2a_{21}],\quad \xi_j=\frac{\partial}{\partial x_j},\quad l,j=1,2,$$
$$B_{33}=\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0k_2\Delta+\mu_0a_2+i\omega\beta_2^2],\quad B_{44}=\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0k_1\Delta+\mu_0a_1+i\omega\beta_1^2],$$
$$B_{43}=-\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0k_{21}\Delta+\mu_0a_{21}+i\omega\beta_1\beta_2],\quad B_{34}=-\frac{\mu}{\alpha_0}\Delta\Delta[\mu_0k_{12}\Delta+\mu_0a_{12}+i\omega\beta_1\beta_2],$$
$$k_0=a_1k_2+a_1k_1-k_{12}a_{21}-k_{21}a_{12},\quad \mu_0=\lambda+2\mu,\quad \alpha_0=\mu\mu_0(k_1k_2-k_{12}k_{21}),$$

δ_{lj} is the Kronecker delta.

Substituting the vector $\mathbf{U}(\mathbf{x})=\mathbf{B}(\partial\mathbf{x})\mathbf{\Psi}$ into (1), where $\mathbf{\Psi}$ is a four-component vector function, we get

$$\Delta\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\mathbf{\Psi}=0,$$

λ_j^2 are roots of equation

$$\mu_0(k_1k_2-k_{12}k_{21})\xi^2-(\mu_0k_0+i\omega\alpha_{11})\xi+\mu_0(a_1a_2-a_{12}a_{21})+i\omega\alpha_{12}=0, \tag{4}$$

$$\alpha_{11}=k_2\beta_1^2+k_1\beta_2^2-\beta_1\beta_2(k_{12}+k_{21}),$$
$$\alpha_{12}=a_2\beta_1^2+a_1\beta_2^2-\beta_1\beta_2(a_{12}+a_{21}).$$

Whence, after some calculations, the function Ψ can be represented as

$$\Psi=\frac{r^2(\ln r-1)}{4\lambda_1^2\lambda_2^2}-\frac{1}{\lambda_1^2-\lambda_2^2}\left[\frac{\varphi_1-\ln r}{\lambda_1^4}-\frac{\varphi_2-\ln r}{\lambda_2^4}\right] \tag{5}$$

where

$$\varphi_m=\frac{\pi}{2i}H_0^{(1)}(\lambda_m r),$$

$H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

$$H_0^{(1)}(\lambda_m r)=\frac{2i}{\pi}J_0(\lambda_m r)\ln r+\frac{2i}{\pi}\left(\ln\frac{\lambda_m}{2}+C-\frac{i\pi}{2}\right)J_0(\lambda_m r)$$
$$-\frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}\left(\frac{1}{k}+\frac{1}{k-1}+...+1\right)$$
$$J_0(\lambda_m r)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k},\quad r^2=(x_1-y_1)^2+(x_2-y_2)^2,\ \ m=1,2.$$

Substituting (5) into $\mathbf{U}=\mathbf{B\Psi}$, we obtain the matrix of fundamental solutions for the equation (1) which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})=\|\ \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y})\ \|_{4\times4}$$

where

$$\Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\ln r}{\mu}\delta_{kj} - \frac{\partial^2 \Psi_{11}}{\partial x_k \partial x_j}, \quad \Gamma_{j3}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\partial \Psi_{13}}{\partial x_j}, \quad k,j = 1,2,$$

$$\Gamma_{j4}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{\partial \Psi_{14},}{\partial x_j} \quad \Gamma_{3j}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{\partial \Psi_{31}}{\partial x_j} \quad \Gamma_{4j}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\partial \Psi_{41}}{\partial x_j},$$

$$\Gamma_{33}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[m_{12}\varphi_2 - m_{11}\varphi_1\right], \quad m_{1j} = -\mu_0 k_2 \lambda_j^2 + \mu_0 a_2 + i\omega\beta_2^2$$

$$\Gamma_{44}(\mathbf{x}\text{-}\mathbf{y}) = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[m_{22}\varphi_2 - m_{21}\varphi_1\right], \quad m_{2j} = -\mu_0 k_1 \lambda_j^2 + \mu_0 a_1 + i\omega\beta_1^2$$

$$\Gamma_{34}(\mathbf{x}\text{-}\mathbf{y}) = \frac{-\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[n_{12}\varphi_2 - n_{11}\varphi_1\right], \quad n_{1j} = -\mu_0 k_{12} \lambda_j^2 + \mu_0 a_{12} + i\omega\beta_1\beta_2$$

$$\Gamma_{43}(\mathbf{x}\text{-}\mathbf{y}) = \frac{-\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\left[n_{22}\varphi_2 - n_{21}\varphi_1\right], \quad n_{2j} = -\mu_0 k_{21} \lambda_j^2 + \mu_0 a_{21} + i\omega\beta_1\beta_2,$$

$$j = 1,2, \quad \Psi_{11} = \left[(\lambda+\mu)(a_1 a_2 - a_{12}a_{21}) + i\omega\alpha_{12}\right]\frac{r^2(\ln r - 1)}{4\alpha_0\lambda_1^2\lambda_2^2}$$

$$+\frac{i\omega\mu}{\mu_0\alpha_0(\lambda_1^2 - \lambda_2^2)}\sum_1^2 (-1)^j \left(\alpha_{11} - \frac{\alpha_{12}}{\lambda_j^2}\right)\frac{\varphi_j - \ln r}{\lambda_j^2},$$

$$\Psi_{13} = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\sum_1^2 (-1)^j m_{j3}(\varphi_j - \ln r),$$

$$m_{j3} = \beta_1 k_2 - \beta_2 k_{21} - \frac{\beta_1 a_2 - \beta_2 a_{21}}{\lambda_j^2}, \quad j = 1,2,$$

$$\Psi_{31} = \frac{i\omega\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\sum_1^2 (-1)^j m_{3j}(\varphi_j - \ln r),$$

$$m_{3j} = \beta_1 k_2 - \beta_2 k_{12} - \frac{\beta_1 a_2 - \beta_2 a_{12}}{\lambda_j^2}, \quad j = 1,2,$$

$$\Psi_{14} = \frac{\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\sum_1^2 (-1)^j m_{j4}(\varphi_j - \ln r),$$

$$m_{j4} = \beta_1 k_{12} - \beta_2 k_1 - \frac{\beta_1 a_{12} - \beta_2 a_1}{\lambda_j^2}, \quad j = 1,2$$

$$\Psi_{41} = \frac{i\omega\mu}{\alpha_0(\lambda_1^2 - \lambda_2^2)}\sum_1^2 (-1)^j m_{4j}(\varphi_j - \ln r)$$

$$m_{4j} = \beta_1 k_{21} - \beta_2 k_1 - \frac{\beta_1 a_{21} - \beta_2 a_1}{\lambda_j^2}, \quad j = 1,2$$

Clearly

$$\frac{\pi}{2i}H_0^{(1)}(\lambda r) = \ln|\mathbf{x}-\mathbf{y}| - \frac{\lambda^2}{4}|\mathbf{x}-\mathbf{y}|^2 \ln|\mathbf{x}-\mathbf{y}| + const + O(|\mathbf{x}-\mathbf{y}|^2).$$

It is evident that all elements of $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ are single-valued functions on the whole plane and they have a logarithmic singularity at most. It can be shown that columns of the matrix $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ are solutions to the system (3) with respect to $\mathbf{x}$ for any $\mathbf{x} \neq \mathbf{y}$. By applying the methods, as in the classical theory of elasticity, we can directly prove the following;

Theorem 3. *The elements of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$ have a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$ and each column of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$, considered as a vector, is a solution of the system (4) at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$.*

Let us consider the matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x}) := \mathbf{\Gamma}^T(-\mathbf{x})$. The following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x})$ may be easily verified:

Theorem 4. *Each column of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$, considered as a vector, satisfies the associated system $\widetilde{\boldsymbol{A}}(\partial \boldsymbol{x})\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y}) = 0$, at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$ and the elements of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$ have a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$.*

4. Singular matrix of solutions

Using the basic fundamental matrix, we will construct the so-called singular matrix of solutions and study their properties.

Write now the expressions for the components of the stress vector, which acts on an elements of the arc with the normal $\mathbf{n} = (n_1, n_2)$. Denoting the stress vector by $\boldsymbol{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{u}$, we have

$$\boldsymbol{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{6}$$

where

$$\boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \begin{pmatrix} \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_1\dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1\dfrac{\partial}{\partial x_2} + \mu\dfrac{\partial}{\partial s} \\ (\lambda+\mu)n_2\dfrac{\partial}{\partial x_1} - \mu\dfrac{\partial}{\partial s} & \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_2\dfrac{\partial}{\partial x_2} \end{pmatrix}\mathbf{u},$$

$$\frac{\partial}{\partial s} = n_2\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_2}.$$

We introduce the following notation $\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n})$, $\widetilde{\boldsymbol{R}}(\partial\mathbf{x}, \mathbf{n})$

$$\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n}) = \begin{pmatrix} T_{11}(\partial x, n) & T_{12}(\partial x, n) & -\beta_1 n_1 & -\beta_2 n_1 \\ T_{21}(\partial x, n) & T_{22}(\partial x, n) & -\beta_1 n_2 & -\beta_2 n_2 \\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & k_{12}\dfrac{\partial}{\partial n} \\ 0 & 0 & k_{21}\dfrac{\partial}{\partial n} & k_2\dfrac{\partial}{\partial n} \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial\mathbf{x}, \mathbf{n}) = \begin{pmatrix} T_{11}(\partial x, n) & T_{12}(\partial x, n) & -i\omega n_1\beta_1 & -i\omega n_1\beta_2 \\ T_{21}(\partial x, n) & T_{22}(\partial x, n) & -i\omega n_2\beta_1 & -i\omega n_2\beta_2 \\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & k_{21}\dfrac{\partial}{\partial n} \\ 0 & 0 & k_{12}\dfrac{\partial}{\partial n} & k_2\dfrac{\partial}{\partial n} \end{pmatrix},$$

By Applying the operator $\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y}$ and the operator $\widetilde{\boldsymbol{R}}(\partial\mathbf{x},\mathbf{n})$ to the matrix $\widetilde{\boldsymbol{\Gamma}}(\mathbf{x}\text{-}\mathbf{y})$, we shall construct the so-called singular matrix of solutions respectively

$$\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})=\|R_{pq}\|_{4\times 4},\quad \widetilde{R}(\partial x,n)\widetilde{\Gamma}(y-x)=\|\widetilde{R}_{pq}\|_{4\times 4},$$

The elements R_{pq} are following:

$$\begin{aligned}
&R_{pp}=\frac{\partial\ln r}{\partial n}+(-1)^p\,2\mu\frac{\partial^2}{\partial x_1\partial x_2}\frac{\partial\Psi_{11}}{\partial s},\quad p=1,2\\
&R_{12}=\frac{\partial\ln r}{\partial s}-2\mu\frac{\partial^2}{\partial x_2^2}\frac{\partial\Psi_{11}}{\partial s},\quad R_{21}=-\frac{\partial\ln r}{\partial s}+2\mu\frac{\partial^2}{\partial x_1^2}\frac{\partial\Psi_{11}}{\partial s},\\
&R_{13}=2\mu\frac{\partial}{\partial x_2}\frac{\partial\Psi_{13}}{\partial s},\quad R_{23}=-2\mu\frac{\partial}{\partial x_1}\frac{\partial\Psi_{13}}{\partial s},\quad R_{14}=-2\mu\frac{\partial}{\partial x_2}\frac{\partial\Psi_{14}}{\partial s},\\
&R_{24}=2\mu\frac{\partial}{\partial x_1}\frac{\partial\Psi_{14}}{\partial s},\qquad R_{3j}=\frac{\partial}{\partial x_j}\frac{\partial}{\partial n}(k_{12}\Psi_{41}-k_1\Psi_{31}),\\
&R_{4j}=\frac{\partial}{\partial x_j}\frac{\partial}{\partial n}(k_2\Psi_{41}-k_{21}\Psi_{31}).\ \ j=1,2\\
&R_{33}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_1m_{12}-k_{12}n_{22})\varphi_2-(k_1m_{11}-k_{12}n_{11})\varphi_1\right\},\\
&R_{44}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_2m_{22}-k_{21}n_{12})\varphi_2-(k_2m_{21}-k_{21}n_{11})\varphi_1\right\},\\
&R_{34}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(m_{22}k_{12}-k_1n_{12})\frac{\partial(\varphi_2-\varphi_1)}{\partial n},\\
&R_{43}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(m_{12}k_{21}-k_2n_{22})\frac{\partial(\varphi_2-\varphi_1)}{\partial n},
\end{aligned}$$

Similarly we obtain the matrix

$$\widetilde{R}(\partial x,n)\widetilde{\Gamma}(y-x)=\|\widetilde{R}_{pq}\|_{4\times 4},$$

where

$$\begin{aligned}
&\widetilde{R}_{pq}=R_{pq},\quad p,q=1,2,\quad \widetilde{R}_{13}=2\mu\frac{\partial}{\partial x_2}\frac{\partial\psi_{31}}{\partial s}\quad \widetilde{R}_{14}=-2\mu\frac{\partial}{\partial x_2}\frac{\partial\psi_{41}}{\partial s},\\
&\widetilde{R}_{23}=-2\mu\frac{\partial}{\partial x_1}\frac{\partial\psi_{31}}{\partial s},\quad \widetilde{R}_{24}=2\mu\frac{\partial}{\partial x_1}\frac{\partial\psi_{41}}{\partial s},\quad \widetilde{R}_{3j}=\frac{\partial}{\partial n}\frac{\partial(k_{21}\psi_{14}-k_1\psi_{13})}{\partial x_j},\\
&\widetilde{R}_{4j}=\frac{\partial}{\partial n}\frac{\partial(-k_{12}\psi_{13}+k_2\psi_{14})}{\partial x_j},\quad j=1,2,\\
&\widetilde{R}_{34}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(k_{21}m_{22}-k_1n_{22})\frac{\partial}{\partial n}(\varphi_2-\varphi_1),\\
&\widetilde{R}_{43}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}(k_{12}m_{12}-k_2n_{12})\frac{\partial}{\partial n}(\varphi_2-\varphi_1),\\
&\widetilde{R}_{33}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_1m_{12}-k_{21}n_{12})\varphi_2-(k_1m_{11}-k_{21}n_{11})\varphi_1\right\},\\
&\widetilde{R}_{44}=\frac{\mu}{\alpha_0(\lambda_1^2-\lambda_2^2)}\frac{\partial}{\partial n}\left\{(k_2m_{22}-k_{21}n_{22})\varphi_2-(k_2m_{21}-k_{21}n_{21})\varphi_1\right\},
\end{aligned}$$

Let us consider the matrix $[\boldsymbol{R}(\partial\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})]^{*}$, which is obtained from $\boldsymbol{R}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})=(R_{pq})_{4\times4}$ by transposition of the columns and rows and the variables x and y (analogously $\left[\tilde{\boldsymbol{R}}(\partial\mathbf{y},\mathbf{n})\tilde{\boldsymbol{\Gamma}}(\mathbf{y}\text{-}\mathbf{x})\right]^{T}$.) We can state the following:.

Theorem 5. *Every column of the matrix* $[\boldsymbol{R}(\partial\boldsymbol{y},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x})]^{T}$, *considered as a vector, is a solution of the system* $\widetilde{\boldsymbol{A}}(\partial\boldsymbol{x})=0$ *at any point* $\boldsymbol{x}$ *if* $\boldsymbol{x}\neq\boldsymbol{y}$ *and the elements of the matrix*$[\mathbf{R}(\partial\boldsymbol{y},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x})]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 6. *Every column of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial\boldsymbol{y},\boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x})\right]^{T}$, *considered as a vector, is a solution of the system* $\boldsymbol{A}(\partial\boldsymbol{x})\boldsymbol{U}=0$ *at any point* $\boldsymbol{x}$ *if* $\boldsymbol{x}\neq\boldsymbol{y}$ *and the elements of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial\boldsymbol{y},\boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x})\right]^{T}$, *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Let us introduce the following single and double layer potentials : The vector-functions defined by the equalities

$$\mathbf{V}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_{S}\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

$$\widetilde{\boldsymbol{V}}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_{S}\boldsymbol{\Gamma}^{T}(\mathbf{y}-\mathbf{x})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

will be called single layer potentials, while the vector-functions defined by the equalities

$$\mathbf{W}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_{S}\left[\boldsymbol{P}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})\right]^{T}\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S$$

$$\widetilde{\boldsymbol{W}}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_{S}\left[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}^{T}(\mathbf{x}-\mathbf{y})\right]^{T}\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S$$

will be called double layer potentials. Here $\mathbf{g}$ and $\mathbf{h}$ are the continuous (or Hölder continuous) vectors and S is a closed Lyapunov curve.

We can state the following:

Theorem 7. *The vector* $\boldsymbol{W}(\boldsymbol{x};\mathbf{h})$ *is a solution of the system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\boldsymbol{U}=\mathbf{0}$ *at any point* $\boldsymbol{x}$ *and* $\boldsymbol{x}\neq\boldsymbol{y}$. *The elements of the matrix* $[\boldsymbol{P}(\partial_{\boldsymbol{y}},\boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}-\mathbf{x})]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 8. *The vector* $\widetilde{\boldsymbol{W}}(\boldsymbol{x};\mathbf{h})$ *is a solution of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}})\boldsymbol{U}=\mathbf{0}$ *at any point* $\boldsymbol{x}$ *and* $\boldsymbol{x}\neq\boldsymbol{y}$. *The elements of the matrix* $\left[\widetilde{\boldsymbol{P}}(\partial_{\boldsymbol{y}},\boldsymbol{n})\boldsymbol{\Gamma}^{T}(\boldsymbol{x}-\mathbf{y})\right]^{T}$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 9. *If* $S\in C^{1,\eta}(S)$, $\boldsymbol{g},\mathbf{h}\in C^{0,\delta}(S)$, $\quad 0<\delta<\eta\leq1$, *then the vectors* $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{h})$, $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{g})$, $\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{h})$ *and* $\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{g})$ *are the regular vector-functions in* $D^{+}(D^{-})$, *and when the point* $\boldsymbol{x}$ *tends to any point* $\boldsymbol{z}$ *of the boundary* S *from inside or from* *outside we have the following formulas:*

$$[\mathbf{W}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \left[\boldsymbol{P}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{z})\right]^T \mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\widetilde{\boldsymbol{W}}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{z}-\mathbf{y})\right]^T \mathbf{h}(\mathbf{y})d_{\mathbf{y}}S$$

$$[\boldsymbol{P}(\partial_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \boldsymbol{P}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{z}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

$$[\widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\widetilde{\boldsymbol{V}}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \widetilde{\boldsymbol{P}}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{z})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

Here the integrals are singular and understood as the principal value.

Theorem 10. *The potentials* $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{g})$ *and* $\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{h})$ *are solutions of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}})\boldsymbol{U} = \mathbf{0}$ *and the potentials* $\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{g})$ *and* $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{h})$ *are solutions of the system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\boldsymbol{U} = \mathbf{0}$ *in both domains* D^+ *and* D^-*.*

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ON HIGHER ORDER "ALMOST LINEAR" FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PROPERTY A AND B

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Abstract. An operator differential equation is considered. A particular case of this equations is the ordinary differential equation

$$u^{(n)}(t) + p(t)\big|u(t)\big|^{\mu(t)} \operatorname{sign} u(t) = 0,$$

where $p \in L_{\mathrm{loc}}(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$. This equation is "almost linear" if the condition $\liminf\limits_{t\to+\infty} \mu(t) = 1$ holds, while if $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$ or $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$, then the equation is an essentially nonlinear differential equation. "Almost linear" differential equations are considered and sufficient condition are established for oscillation of solutions.

Keywords and phrases: Property A, property B, oscillation.

AMS subject classification (2010): 34K11.

Introduction

This work deals with study of oscillatory properties of solutions of a functional-differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $F : C(R_+; R) \to L_{\mathrm{loc}}(R_+; R)$ is a continuous mapping. Let $\tau \in C(R_+; R_+)$, $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. Denote by $V(\tau)$ the set of continuous mappings F satisfying the condition: $F(x)(t) = F(y)(t)$ holds for any $t \in R_+$ and $x, y \in C(R_+; R)$ provided that $x(s) = y(s)$ for $s \geq \tau(t)$. For any $t_0 \in R_+$, we denote by $H_{t_0,\tau}$ the set of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geq t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$. Throughout the work whenever the notation $V(\tau)$ and $H_{t_0,\tau}$ occurs, it will be understood, unless specified otherwise that the function τ satisfies the conditions stated above.

It will always be assumed that either the condition

$$F(u)(t)\, u(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}, \tag{1.2}$$

or the condition

$$F(u)(t)\, u(t) \leq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of equation (1.1), if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$\overline{u}^{(n)}(t) + F(\overline{u})(t) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1 We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big| \downarrow 0 \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2 We say that equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies either (1.4) or

$$\big|u^{(i)}(t)\big| \uparrow +\infty, \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even and either is oscillatory or satisfies (1.5) when n is odd.

The ordinary differential equation with deviating argument

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0 \tag{1.6}$$

is a particular case of equation (1.1), where $p \in L_{\text{loc}}(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$. In the case $\lim\limits_{t \to +\infty} \mu(t) = 1$, we call differential equation (1.6) "almost linear", while if $\liminf\limits_{t \to +\infty} \mu(t) \neq 1$ or $\limsup\limits_{t \to +\infty} \mu(t) \neq 1$, then we call equation (1.6) essentially nonlinear generalized Emden-Fowler type differential equation.

Everywhere below we assume that the inequality

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \big|u(s)\big|^{\mu(s)} d_s\, r_i(s,t) \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}, \tag{1.7}$$

holds, where

$$\begin{gathered} \mu \in C(R_+; (0, +\infty)), \quad \tau_i, \sigma_i \in C(R_+; R_+), \quad \tau_i(t) \leq \sigma_i(t) \\ \text{for} \quad t \in R_+, \quad \lim_{t \to +\infty} \tau_i(t) = +\infty \quad (i = 1, \dots, m), \end{gathered} \tag{1.8}$$

$r_i : R_+ \times R_+ \to R_+$ are nondecreasing in the first argument and Lebesgue integrable in the second argument on any finite subsegment of $[0, +\infty)$.

Study of oscillatory properties of differential equation of type (1.1) begin in 1990. Namely, in [1,2] for the first time a new approach was used for establishing oscillatory properties. Investigation of "almost linear" (essentially nonlinear) differential equations, in our opinion for the first time, was carried out [3,4] ([5–7]).

In the present paper the both cases of Properties **A** and **B** will be studied for "almost linear" differential equations.

2. Necessary conditions of the existence of monotone solutions

Let $t_0 \in R_+$, $\ell \in \{1, \dots, n-1\}$. By U_{ℓ,t_0} we denote the set of proper solutions of equation (1.1) satisfying the conditions

$$\begin{gathered} u^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_0 \quad (i = 0, \dots, \ell-1), \\ (-1)^{i+\ell} u^{(i)}(t) \geq 0 \quad \text{for} \quad t \geq t_0 \quad (i = \ell, \dots, n-1). \end{gathered} \tag{2.1$_\ell$}$$

Theorem 2.1 *Let* $F \in V(\tau)$, *conditions* (1.2) ((1.3)), (1.6), (1.7) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$, $\ell+n$ *be odd* ($\ell+n$ *be even*),

$$\int_0^{+\infty} t^{n-\ell} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(\ell-1)\mu(s)} d_s r_i(s,t) = +\infty, \tag{2.2$_\ell$}$$

$$\int_0^{+\infty} t^{n-\ell-1} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{\ell\mu(s)} d_s r_i(s,t) = +\infty, \tag{2.3$_\ell$}$$

and

$$\liminf_{t\to+\infty} \mu(t) > 0. \tag{2.4}$$

Moreover, let $U_{\ell,t_0} \neq \varnothing$ *for some* $t_0 \in R_+$. *Then there exist* $\lambda \in [\ell-1,\ell]$ *such that*

$$\limsup_{\varepsilon\to 0+} \Big(\liminf_{t\to+\infty} g_\ell(t,\lambda,\varepsilon)\Big) \le (\ell-1)!\,(n-\ell-1)!,$$

where

$$g_\ell(t,\lambda,\varepsilon) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\big(\overline{\sigma}(s)\big)^{-h_\varepsilon(\lambda)}$$

$$\times \int_{t_0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\lambda+h_{1\varepsilon}(\lambda)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds, \tag{2.4}$$

$$\overline{\sigma}(t) = \max\big\{ \max(s,\sigma_1(s),\dots,\sigma_m(s)) : 0 \le s \le t\big\},$$

$$h_{1\varepsilon}(\lambda) = \begin{cases} 0 & \text{for } \lambda = \ell, \\ \varepsilon & \text{for } \lambda \in [\ell-1,\ell), \end{cases}$$

$$h_{2\varepsilon}(\lambda) = \begin{cases} 0 & \text{for } \lambda = \ell-1, \\ \varepsilon & \text{for } \lambda \in (\ell-1,\ell], \end{cases} \qquad h_\varepsilon(\lambda) = h_{1\varepsilon}(\lambda) + h_{2\varepsilon}(\lambda). \tag{2.5}$$

Theorem 2.2 *Let the conditions of Theorem* 2.1 *be fulfilled and*

$$\liminf_{t\to+\infty} \frac{t}{\sigma_i(t)} > 0 \quad (i=1,\dots,m). \tag{2.7}$$

Then there exist $\lambda \in [\ell-1,\ell]$ *such that*

$$\limsup_{\varepsilon\to 0+} \Big(\liminf_{t\to+\infty} g_{\ell,1}(t,\lambda,\varepsilon)\Big) \le (\ell-1)!\,(n-\ell-1)!,$$

where

$$g_{\ell,1}(t,\lambda,\varepsilon) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)}(s-t)^{n-\ell-1} \int_{t_0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell}$$

$$\times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda+h_{1\varepsilon}(\lambda))\mu(\xi_1)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds, \tag{2.8}$$

$h_{1\varepsilon}$, $h_{2\varepsilon}$ and h_ε *are given by* (2.6).

3. Sufficient conditions of nonexistence of monotone solutions

Theorem 3.1 *Let* $F \in V(\tau)$, *conditions* (1.2) ((1.3)), (1.6), (1.7), (2.2$_\ell$)–(2.4) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$, *with* $\ell + n$ *odd* ($\ell + n$ *even*), *and for any* $\lambda \in [\ell - 1, \ell]$

$$\limsup_{\varepsilon \to 0+} \Big(\liminf_{t \to +\infty} g_\ell(t, \lambda, \varepsilon) \Big) > (\ell - 1)!\, (n - \ell - 1)!. \tag{3.1$_\ell$}$$

Then for any $t_0 \in R_+$, $U_{\ell, t_0} = \varnothing$, *where* g_ℓ, $h_{1\varepsilon}$, $h_{2\varepsilon}$ *and* h_ε *are defined by* (2.5) and (2.6).

Theorem 3.2 *Let* $F \in V(\tau)$, *conditions* (1.2) ((1.3)), (1.6), (1.7), (2.2$_\ell$)–(2.4) *and* (2.7) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$, *with* $\ell + n$ *odd* ($\ell + n$ *even*) *and for any* $\lambda \in [\ell - 1, \ell]$

$$\limsup_{\varepsilon \to 0+} \Big(\liminf_{t \to +\infty} g_{\ell 1}(t, \lambda, \varepsilon) \Big) > (\ell - 1)!\, (n - \ell - 1)!. \tag{3.2$_\ell$}$$

Then for any $t_0 \in R_+$, $U_{\ell, t_0} = \varnothing$, *where* $g_{\ell 1}$, $h_{1\varepsilon}$, $h_{2\varepsilon}$ *and* h_ε *are defined by* (2.6) *and* (2.8).

4. Functional differential equation with property A

Relying on the results obtained in Section 3, in Sections 4 and 5 we establish sufficient conditions for equation (1.1) to have Properties **A** and **B**.

Theorem 4.1 *Let* $F \in V(\tau)$, *conditions* (1.2), (1.6), (1.7) *and* (2.4) *be fulfilled and for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and* $\lambda \in [\ell - 1, \ell]$ *conditions* (2.2$_\ell$), (2.3$_\ell$) *and* (3.1$_\ell$) *hold. If moreover,* (2.3$_0$) *holds when* n *is odd, then equation* (1.1) *has Property* **A**.

Theorem 4.2 *Let* $F \in V(\tau)$, *conditions* (1.2), (1.6), (1.7), (2.4), (2.7) *be fulfilled and for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and* $\lambda \in [\ell - 1, \ell]$ *conditions* (2.2$_\ell$), (2.3$_\ell$) *and* (3.2$_\ell$) *hold. If moreover,* (2.3$_0$) *holds when* n *is odd, then equation* (1.1) *has Property* **A**.

Theorem 4.3 *Suppose* $F \in V(\tau)$, *condition* (1.2) *be fulfilled and for large* $t_0 \in R_+$

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} p_i(t) \int_{\alpha_i t}^{\beta_i t} |u(s)|^{1 - \frac{d}{\ln s}} ds \;\; \text{for } t \geq t_0, \;\; u \in H_{t_0, \tau} \tag{4.1}$$

and

$$\liminf_{t \to +\infty} \frac{1}{t} \int_0^t s^{n+1} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds > \frac{1}{m} \max \Bigg(\prod_{i=1}^m \Big(\beta_i^{1+\lambda} - \alpha_i^{1+\lambda} \Big)^{-\frac{1}{m}} \times$$
$$\times e^{\lambda d} (1 + \lambda) \lambda (\lambda - 1) \cdots (\lambda - n + 1) : \lambda \in [0, n-1] \Bigg).$$

Then equation (1.1) *has Property* **A**, *where*

$$p_i \in L_{\mathrm{loc}}(R_+; R_+), \;\; 0 < \alpha_i < \beta_i < +\infty \;\; (i = 1, \dots, m), \;\; d \in [0, +\infty). \tag{4.2}$$

Theorem 4.4 *Suppose $F \in V(\tau)$, condition* (1.2) *be fulfilled and for large $t_0 \in R_+$*

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} p_i(t)\big|u(\alpha_i t)\big|^{1-\frac{d}{\ln t}} \ \text{ for } \ t \geq t_0, \ \ u \in H_{t_0,\tau} \tag{4.3}$$

and

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^n\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds >$$
$$> \frac{1}{m}\ \max\bigg(\Big(\prod_{i=1}^{m}\alpha_i^{-\frac{\lambda}{m}}\Big)e^{\lambda d}\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\bigg).$$

Then equation (1.1) *has Property* **A**, *where*

$$p_i \in L_{\mathrm{loc}}(R_+; R_+), \ \ \alpha_i \in (0, +\infty) \ \ (i = 1, \dots, m), \quad d \in [0, +\infty). \tag{4.4}$$

5. Functional differential equation with property B

Theorem 5.1 *Let $F \in V(\tau)$, conditions* (1.3), (1.6), (1.7), (2.4) *be fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ even and $\lambda \in [\ell-1, \ell]$ conditions* (2.2_ℓ), (2.3_ℓ) *and* (3.1_ℓ) *hold. If moreover,* (2.3_0) *when n is even, and* (2.2_n) *hold then equation* (1.1) *has Property* **B**.

Theorem 5.2 *Let $F \in V(\tau)$, conditions* (1.3), (1.6), (1.7), (2.4), (2.7) *be fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell+n$ even and $\lambda \in [\ell-1, \ell]$ conditions* (2.2_ℓ), (2.3_ℓ) *and* (3.2_ℓ) *hold. If moreover,* (2.3_0) *when n is even, and* (2.2_n) *hold then equation* (1.1) *has Property* **B**.

Theorem 5.3 *Suppose $F \in V(\tau)$, conditions* (1.3), (4.1), (4.2) *be fulfilled and*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^{n+1}\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\ \max\Big(-\prod_{i=1}^{m}\big(\beta_i^{1+\lambda} - \alpha_i^{1+\lambda}\big)^{-\frac{1}{m}} \times$$
$$\times e^{\lambda d}(1+\lambda)\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\Big).$$

Then equation (1.1) *has Property* **B**.

Theorem 5.4 *Suppose $F \in V(\tau)$, conditions* (1.3), (4.3), (4.4) *be fulfilled and*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^n\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds >$$
$$> \frac{1}{m}\ \max\Big(-\prod_{i=1}^{m}\alpha_i^{-\frac{\lambda}{m}} \cdot e^{\lambda d} \cdot \lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\Big).$$

Then equation (1.1) *has Property* **B**.

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IN THE EUROPEAN UNION WITH THE GEORGIAN LANGUAGE - THE AIMS AND BASEMENTS OF THE PROJECT "ONE MORE STEP TOWARDS GEORGIAN TALKING SELF-DEVELOPING INTELLECTUAL CORPUS"

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Abstract. The paper shortly overviews the aims and fundamentals of the two years project "A One More Step Towards Georgian Talking Self-Developing Intellectual Corpus" and the paper "Strategic Research Agenda for Multilingual Europe 2020" by the META-NET technological board. Also, taking into account the national aim of defending the Georgian language from the danger of digital extinction, as well as, the national aim of joining with the Georgian language the European Union, which according to the strategic research agenda of the Meta-Net is planned to become completely free from language barriers, the current paper underlines that the prioritization of the task of the complete technological foundation of the Georgian language, i.e. the task of creation of the Georgian thinking, speaking and translating system is the question of vital necessity for the Georgian society.

Keywords and phrases: Georgian self-developing intellectual corpus, technological alphabet of the Georgian language, logical grammar of the Georgian language

AMS subject classification (2010): 03B65, 68T50, 68Q55, 91F20.

Introduction

In 2010-2012 with the financial support of the European commission, there was carried out a research "Europe's Languages in the Digital Age" [1]. As a result, in 2012, Meta-Net published a press-release "At Least 21 European Languages in Danger of Digital Extinction - Good News and Bad News on the European Day of Languages" [2], and also Strategic Research Agenda for Multilingual Europe 2020" [3]. These publications, which are very important for us, are overviewed in the paper "Open Letter To The Georgian National Academy Of Sciences Id Est The Fact That European Languages Are At The Danger, Makes It Clear That The Georgian Language Is At Especially High Quality Danger! Id Est, Once Again For Defending The Rights Of The Georgian Language!! Id Est, It's Time To Take Care Of The Georgian Language!!! Short Version" [4]. - Here the main thing is that for today, in the European Union, processes are going on in concordance with the Strategic Research Agenda for Multilingual Europe 2020 with the aims of building such new Europe whose every citizen will be able to have access to any kind of service, knowledge, media, and technologies with their own mother language and, according to this agenda, in this new Europe, there will be no language barriers in communication, and there will be freely accessible high quality translations of domain independent as well as domain specific contents.

The coordinator of Meta-Net, Prof. Hans Uszkoreit, scientific director at German Research Center for Artificial Intelligence (DFKI) says the following: "The results of our study are most alarming. The majority of European languages are severely under-resourced and some are almost completely neglected. In this sense, many of our languages are not yet future-proof." [2]

This all in sum once again make clear the urgent necessity of declaring as one of the main state priorities of Georgia the researches aimed at defending the Georgian language from the danger of digital extinction. There is also a clear necessity of formation a united Georgian group of researchers, which via collaboration with Meta-Net, will work on the tasks of complete mathematical and technological foundation of the Georgian language, in other words, on the task of creation of the high quality Georgian thinker, talker and translator system. - Without this type of system it will be impossible to join the European Union with the Georgian language, as well as, to defend the Georgian language from the danger of the digital extinction. For us it is clear that if we do not act in this way, and if we again do not manage properly the local processes with the aim of creation Georgian thinker, talker and translator system, i.e. if we continue chaotic, uncoordinated activities, like it is the case today, then the Georgian language will have the future about which Dr. Georg Rehm said in [2]: "There are dramatic differences in language technology support between the various European languages and technology areas. The gap between 'big' and 'small' languages still keeps widening. We have to make sure that we equip all smaller and under-resourced languages with the needed base technologies, otherwise these languages are doomed to digital extinction." - We say the same: We should be certain that we will be capable to defend the Georgian language from the very high danger of digital extinction in the digital age [5-8], and therefore, we should not act chaotically, but in an ordered manner, so that we could minimize today the existing gap instead of making it even bigger.

The aims and basement of the two year project "A one more step towards Georgian talking self-developing intellectual corpus". In 2012, in the Center for Georgian Language Technology at the Georgian Technical University, there was started a long-term project "The Technological Alphabet of the Georgian Language" [9 - 11] with K.Pkhakadze's leadership;[1] in the confines of this project, now center works on the №31/70 project "Foundation of the logical grammar of the Georgian language and its applications in the information technologies" financed by Shota Rustaveli National Science foundation. In addition to it, within this long-term project, the center in March 2014 accomplished a project №048 "Internet Versions of a Number of Developable (Learnable) Systems Necessary for Creating The Technological Alphabet of the Georgian Language "[2] financed by Georgian Technical University. Also, in 2012, there were started the two doctoral theses in the doctoral program "Informatics" at the Georgian Technical University, namely: Giorgi Chichua's doctoral thesis - "Georgian Speech Synthesis and Recognition", and Merab Chikvinidze's doctoral thesis -

[1]This long-term project was elaborated via the further development and completion of a state priority program of the Iv. Javakhishvili Tbilisi State University "Free and Complete Programming Inclusion of a Computer in the Georgian Natural Language System" [12 - 13], which was going on in previous years with K.Pkhakadze's leadership.

[2]The results of this project were successfully presented on the seminar "The Technological Alphabet Of The Georgian Language - One Of The Main Georgian Challenges Of The 21st Century" held on 14 April 2014 that was dedicated to the day of the Georgian language.

"Georgian grammar checker (analyzer)" [14].

In 2014, on the basis of the results achieved within these above mentioned projects and doctoral theses, the center worked out a two year project "One More Step Towards Georgian Talking Self-Developing Intellectual Corpus", which is one more subproject of the long-term project "The Technological Alphabet of the Georgian Language" of the Center for Georgian Language Technology. This project, with which the Center applied for financing to Shota Rustaveli National Science Foundation, aims at building up a complete version of the Georgian self-developing intellectual corpus via further developing the trial version of the Georgian self-developing intellectual corpus, which is already created by us [15-23]. Thus, to build up the Georgian talking self-developing intellectual corpus means to create an automatically developing complete Georgian web-corpus which will be equipped with: the logic of the Georgian natural language systems; with the intellectual procedures constructed on the basis of this logic; and, also, with the Georgian technological alphabet, which is constructed on the basis of this logic and these intellectual procedures, in other words, with the Georgian talking Intellectual System, i.e., with the Georgian written and spoken texts analyzer and generator systems, which are necessary to realize full scale human computer intellectual interaction by means of the Georgian language. Besides it, to build up the Georgian talking self-developing intellectual corpus means to equip it with the two-way translator systems from Georgian to foreign languages, which, in turn, will be constructed on the basis of the above-mentioned Georgian talking intellectual system.

Obviously, it is impossible to build the above-described Georgian Talking Self-Developing Intellectual Corpus in the confines of one two-year project. Therefore, this two year project aims at building above-described Georgian corpus as complete as it is possible, and, also, the project aims to provide the Georgian language with all the necessary resources that are needed in order to be able to participate in those processes that are already going on in concordance with the strategic research agenda for multilingual Europe 2020. - In our opinion, this is the only way to defend the Georgian language from digital extinction in the digital age.

Below, we will very briefly present those results on which the project is based on; they are as follows:

1. A trial version of the Georgian self-developing multilingual and multimodal intellectual web-corpus [15], which despite that it is still only trial one contains already over 144 126 000 words, among which 2 267 700 words are mutually different, and it is already equipped with trial versions of the Georgian intellectual procedures and technological systems, which are listed below and some of which even are unique (see: http://geoanbani.com/Corpus/):

–Taggers, descriptors and generators of the words of the types of V, N and A [16];

–Self-developing syntactic/orthographic spellcheckers and Georgian orthographic corrector [17];

–Georgian-Mathematical/Georgian-English-German translators [18];

–Speech recognizers based on teaching and studying principles [19, 20];

–Georgian e-text and web-page reader [21];

–Georgian multilingual speech assistant and Georgian Spoken Support for Persons with Speech Disorder [22];

–Georgian Multi-lingual Spoken Lexicon and Georgian Extension of Google Translator [23].

2. The foundations of the logical grammar of the Georgian language [24-28], which is elaborated within the project №31/70 "Foundations of the Logical Grammar of Georgian Language and its Applications in the Information Technologies", and which, on the one hand, is the first logical grammar of the natural Georgian language system. On the other hand, the above-listed intellectual procedures and technological systems are created on the basis of this logical grammar of the Georgian language.

The importance and benefits of the two year project "A one more step towards Georgian talking self-developing intellectual corpus".

For today, the Georgian language in the sense of language resources (resources, data and knowledge basis) and technologies (tools, technologies, applications) is very poorly supported. Even more, the Georgian language is alarmingly lagging compared to almost any of those 21 European languages, which according to the research "Europe's Languages in the Digital Age" [1- 3] done by META-NET, are under the danger of digital extinction in the digital age. All these together clearly indicate the urgent necessity of reducing this lagging as much as it is possible and as soon as it is possible. The aim of two year project "One More Step Towards Georgian Self-Developing Intellectual Corpus" is to reduce this lagging in the shortest possible period, and consequently, to radically change the current state of affairs.

Indeed, in the case of successful completion of the project, which is truly realistic taking into account our existing results that serve as the foundation for the project, in the summer 2017, there will be already built the Georgian self-developing intellectual corpus, i.e. the Self-developing Georgian-net, which will be equipped with the continuously developing Georgian text analyzer *(such as: automatic descriptor of tokens and descriptive databases (that define knowledge and logic of the corpus), automatic extender of intellectual procedures; morphological and syntactic structure generators for words and composed linguistic expressions; the hybrid morphological, syntactic and semantic checker; the Information/knowledge extractor, question-answerer, and logical problem solver-checker), speech processor (such as: the Georgian e-texts semantic reader equipped with possibility to built in it users own voice; the recognizer of synthesized and natural speeches; the various kinds of segmentators of voice and subtitled voice data), automatic translator (such as: the rule based Georgian-English-German and Georgian-Mathematical translators; the hybrid Georgian-English-German translator; the Georgian extension of Google translator; the Georgian spoken lexicon)* and the corpus voice manager systems. In addition, the Georgian-net, i.e. the Georgian self-developing intellectual corpus, from the day of its launch, will extend automatically itself with Georgian and Georgian-foreign texts freely available in the web in a such a way that it will be able to record the source and date of entrance of any newly added Georgian words in it and, accordingly, in the Georgian web space. - It is absolutely obvious that here very shortly but almost completely described the Georgian self-developing intellectual corpus or, shortly, the Georgian-net, from the point of view of technological support, will essentially reduce the existing alarming lagging with technologically advanced languages.

Besides, if we take into account that within the project it is planned to build Georgian_Thinker&Talker&Translator_1 web-system and mobile apps some of its modules (they are: Georgian multilingual spoken lexicon, Georgian extension of Google translate, Georgian multilingual speech assistant, Georgian e-text and web-page reader), and also to publish monographic work "The Georgian Web-Corpus: Aims, Methods, and Recommendations", it gets even clearer that the project has very high or even groundbreaking importance for the scientific community that is concerned with building Georgian information technology systems.

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ON ESTIMATION OF THE INCREMENT OF SOLUTION FOR A CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATION CONSIDERING DELAY PARAMETER PERTURBATION

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Abstract. The estimation of the increment of solution is obtained with respect to small parameter for nonlinear delay functional differential equation with the continuous initial condition. Moreover, value of the increment is calculated at the initial moment. This estimation plays an important role in proving the variation formulas of solution.

Keywords and phrases: Controlled delay functional-differential equation, variation formula of solution, effect of delay perturbation, continuous initial condition.

AMS subject classification (2010): 34K99.

Let R_x^n be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T is the sign of transposition; suppose that $O \subset R_x^n$ and $V \subset R_u^r$ are open sets. Let the n-dimensional function $f(t, x, y, u)$ satisfy the following conditions: for almost all $t \in I = [a, b]$, the function $f(t, \cdot) : O^2 \times V \to R_x^n$ is continuously differentiable; for any $(x, y, u) \in O^2 \times V$, the functions

$$f(t, x, y, u), f_x(t, x, y, u), f_y(t, x, y, u), f_u(t, x, y, u)$$

are measurable on I; for arbitrary compacts $K \subset O, U \subset V$ there exists a function $m_{K,U}(t) \in L(I, [0, \infty))$, such that for any $(x, y, u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t, x, y, u) | + | f_x(t, x, y, u) | + | f_y(t, x, y, u) | + | f_u(t, x, y, u) | \leq m_{K,U}(t).$$

Furthermore, let $0 < \tau_1 < \tau_2$ be given numbers and let E_φ be the space of continuous functions $\varphi : I_1 \to R_x^n$, where $I_1 = [\hat{\tau}, b], \hat{\tau} = a - \tau_2; \Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ is a set of initial functions; let E_u be the space of bounded measurable functions $u : I \to R_u^r$ and let $\Omega = \{u \in E_u : clu(I) \subset V\}$ be a set of control functions, where $u(I) = \{u(t) : t \in I\}$ and $clu(I)$ is closer of the set $u(I)$.

To each element $\mu = (t_0, \tau, \varphi, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times \Phi \times \Omega$ we assign the controlled delay functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau), u(t)) \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2}$$

Condition (2) is said to be a continuous initial condition since always $x(t_0) = \varphi(t_0)$.

Definition 1. Let $\mu = (t_0, \tau, \varphi, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times E_\varphi \times E_u$ we introduce the set of variations:

$$V = \{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta u) \in E_\mu - \mu_0 : |\ \delta t_0\ | \leq \alpha, |\ \delta\tau\ | \leq \alpha,$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, |\ \lambda_i\ | \leq \alpha, i = \overline{1,k}\},$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta u_i \in E_u - u_0, i = \overline{1,k}$ are fixed functions ; $\alpha > 0$ is a fixed number.

Theorem 1([1]). *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$ and let $K_0 \subset O$ and $U_0 \subset V$ be compact sets containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$ and $clu_0(I)$, respectively. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover,*

$$\begin{cases} x(t; \mu_0 + \varepsilon\delta\mu) \in K_0, t \in [\hat{\tau}, t_{10} + \delta_1], \\ u_0(t) + \varepsilon\delta u(t) \in U_0, t \in I. \end{cases} \tag{3}$$

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Theorem 1 allows one to define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\begin{cases} \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \\ (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V. \end{cases}$$

Theorem 2. *Let the following conditions hold:*

1. the function $\varphi_0(t), t \in I_1$ is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;

2. there exist compact sets $K_0 \subset O$ and $U_0 \subset V$ containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$ and $clu_0(I)$, respectively, such that the function $f(t, x, y, u)$ is bounded on the set $I \times K_0^2 \times U_0$;

3. there exist the limits

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \lim_{w \to w_0} f(w, u_0(t)) = f^-,$$

where $w = (t, x, y) \in (a, t_{00}] \times O^2, w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_0))$. *Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that*

$$\max_{t \in [\hat{\tau}, t_{10} + \delta_2]} |\ \Delta x(t; \varepsilon\delta\mu)\ | \leq O(\varepsilon\delta\mu) \tag{4}$$

for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V^-$, *where* $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$. *Moreover,*

$$\Delta x(t_{00}; \varepsilon\delta\mu) = \varepsilon\Big[\delta\varphi(t_{00}) + \{\dot{\varphi}_0^- - f^-\}\delta t_0\Big] + o(\varepsilon\delta\mu).$$

Here the symbols $O(t;\varepsilon\delta\mu), o(t;\varepsilon\delta\mu)$ stand for quantities that have the corresponding order of smallness with respect to ε uniformly with respect to t and $\delta\mu$.

Theorem 3. *Let the conditions 1 and 2 of Theorem 2 hold and there exist the limits*

$$\lim_{t\to t_{00}+} \dot{\varphi}_0(t) = \dot{\varphi}_0^+, \lim_{w\to w_0} f(w, u_0(t)) = f^+, w \in [t_{00}, b) \times O^2.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that inequality (4) is valid for arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V^+$, *where* $V^+ = \{\delta\mu \in V : \delta t_0 \ge 0\}$. *Moreover,*

$$\Delta x(t_{00} + \varepsilon\delta t_0; \varepsilon\delta\mu) = \varepsilon\Big[\delta\varphi(t_{00}) + \{\dot{\varphi}_0^+ - f^+\}\delta t_0\Big] + o(\varepsilon\delta\mu).$$

Theorems 2 and 3 are proved by the scheme given in [2,3].

Theorem 4. *Let the conditions of Theorems 2 and 3 hold. Moreover,*

$$\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ := \hat{f}.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that inequality (4) is valid for arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$ *and*

$$\Delta x(t_{00} + \varepsilon\delta t_0; \varepsilon\delta\mu) = \varepsilon\Big[\delta\varphi(t_{00}) + \hat{f}\delta t_0\Big] + \gamma(\varepsilon\delta\mu), \tag{5}$$

where

$$\gamma(\varepsilon\delta\mu) = \begin{cases} o(\varepsilon\delta\mu) + \hat{O}(\varepsilon\delta\mu) & for \ \ \delta t_0 \le 0, \\ o(\varepsilon\delta\mu) & for \ \ \delta t_0 \ge 0. \end{cases}$$

Here $\hat{O}(\varepsilon\delta\mu) = 0$ *for* $\delta t_0 = 0$.

Proof. It is clear that inequality (4) holds for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_2] \times V$ and formula (5) is valid for $\delta t_0 \ge 0$.

Let $\delta t_0 \le 0$ then

$$\Delta x(t_{00} + \varepsilon\delta t_0; \varepsilon\delta\mu) - \Delta x(t_{00}; \varepsilon\delta\mu) = \int_{t_{00}}^{t_{00}+\varepsilon\delta t_0} \dot{\Delta} x(t; \varepsilon\delta\mu)dt$$

$$= \int_{t_{00}}^{t_{00}+\varepsilon\delta t_0} [f(t, x(t; \mu_0 + \varepsilon\delta\mu), x(t-\tau; \mu_0 + \varepsilon\delta\mu), u(t)) - \dot{\varphi}_0(t)]dt = \hat{O}(\varepsilon\delta\mu),$$

(see (3) and the conditions *1* and *2*), i.e.

$$\begin{aligned} \Delta x(t_{00} + \varepsilon\delta t_0; \varepsilon\delta\mu) &= \Delta x(t_{00}; \varepsilon\delta\mu) + \hat{O}(\varepsilon\delta\mu) \\ &= \varepsilon\Big[\delta\varphi(t_{00}) + \hat{f}\delta t_0\Big] + o(\varepsilon\delta\mu) + \hat{O}(\varepsilon\delta\mu). \end{aligned}$$

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THE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE OF FINDING FULL-STRENGTH CONTOUR INSIDE THE POLYGON

Svanadze K.

Abstract. In the present work we consider the problem of statics of the linear theory of elastic mixture of finding a full-strength contour for a finite doubly-connected domain whose outer boundary is a convex polygon, while the inner boundary is a smooth closed contour. It is assumed that absolutely smooth rigid punches are applied to every link of the polygon. The punches are under the action of external normal contractive forces. The goal of the problem is to find an unknown contour under the condition that tangential normal stress vector on it takes constant value.

Keywords and phrases: Elastic mixture, conformal mapping, Riemann-Hilbert problem, Kolosov-Muskhelishvili type formulas.

AMS subject classification (2010): 74B05.

1. Introduction

The problems of the plane theory of elasticity for infinite domains weakened by equally strong holes have been studied by many authors, particularly in [1], [9] the same problem for simple and doubly-connected domains with partially unknown boundaries are investigated in [2], [10] etc. The mixed boundary value problems of the plane theory of elasticity for domain with partially unknown boundaries have been studied by R. Bantsuri [3]. Analogous problem in the case of the plane theory of elastic mixtures is considered in [15].

In [14] using the method suggested by R. Bantsury in [4], the author gives a solution of the mixed problem of the plane theory of elasticity for a finite multiply connected domain with a partially unknown boundary having the axis of symmetry. Analogous problem in the case of the plane theory of elastic mixtures has been studied in [16]. The problem of statics of the plane theory of elasticity of finding an equally strong contour for square which is weakened by a hole and by cuttings at vertices have been investigated in [5] by R. Bantsuri and G. Kapanadze. The analogous problem in the case of the plane theory of elastic mixtures has been studied in [17].

In the work of R. Bantsuri and G. Kapanadze [6] the problem of statics of the plane theory of elasticity of finding a full-strength contour inside the polygon are considered.

In the present paper in the case of the plane theory of elastic mixtures we study the problem analogous to that solved in [6]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili's formula [17] and the method developed in [6].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [8]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0 \tag{2.1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

$$U = (u_1 + iu_2, u_3 + iu_4)^T, \quad u' = (u_1, u_2)^T \quad and \quad u'' = (u_3, u_4)^T \quad ,$$

are partial displacements,

$$K = -\frac{1}{2}em^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\triangle_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix}, \quad \triangle_0 = m_1 m_3 - m_2^2,$$

$$m_k = e_k + \frac{1}{2}e_{3+k}, \quad e_1 = a_2/d_2 \quad e_2 = -c/d_2, \quad e_3 = a_1/d_2, \quad d_2 = a_1 a_2 - c^2,$$

$$a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1,$$

$$e_3 + e_6 = a/d_1, \quad d_1 = ab - c_0^2, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho,$$

$$b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4, \quad \rho = \rho_1 + \rho_2, \quad a = a_1 + b_1, \quad b = a_2 + b_2$$

$$c_0 = c + d, \quad d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$ are elasticity modules characterizing mechanical properties of a mixture, ρ_1 and ρ_2 are its particular densities. The elastic constants $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of the inequality [13].

In [7] M. Basheleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2}ze\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.2}$$

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right], \tag{2.3}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions;

$$A = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix} \quad B = \mu e, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\frac{\partial}{\partial(x)} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial n(x)} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2},$$

$n = (n_1, n_2)^T$ is the unit vector of the outer normal, $(TU)_p, p = \overline{1,4}$, the stress components [7]

$$(TU)_1 = r'_{11}n_1 + r'_{21}n_2, \quad (TU)_2 = r'_{12}n_1 + r'_{22}n_2,$$
$$(TU)_3 = r''_{11}n_1 + r''_{21}n_2, \quad (TU)_4 = r''_{12}n_1 + r''_{22}n_2,$$
$$r'_{11} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_2}(\mu_1 u_2 + \mu_3 u_4), \quad r'_{21} = -a_1\omega' - c\omega'' + 2\frac{\partial}{\partial x_1}(\mu_1 u_2 + \mu_3 u_4),$$
$$r'_{12} = a_1\omega' + c\omega'' + 2\frac{\partial}{\partial x_2}(\mu_1 u_1 + \mu_3 u_3), \quad r'_{22} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_1}(\mu_1 u_1 + \mu_3 u_3),$$
$$r''_{11} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_2}(\mu_3 u_2 + \mu_2 u_4), \quad r''_{21} = -c\omega' - a_2\omega'' + 2\frac{\partial}{\partial x_1}(\mu_3 u_2 + \mu_2 u_4),$$
$$r''_{12} = c\omega' + a_2\omega'' + 2\frac{\partial}{\partial x_2}(\mu_3 u_1 + \mu_2 \mu_3), \quad r''_{22} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_1}(\mu_3 u_1 + \mu_2 u_3),$$
$$\theta'' = duv\nu', \quad \theta'' = duv\nu'', \quad \omega' = rotu', \omega'' = rotu''.$$

Introduce the vectors:

$$\tau^{(1)} = (r'_{11}, r''_{11})^T, \tau^{(2)} = (r'_{22}, r''_{22})^T, \tau = \tau^{(1)} + \tau^{(2)}, \tag{2.4}$$

$$\eta^{(1)} = (r'_{21}, r''_{21})^T, \eta^{(2)} = (r'_{12}, r''_{12})^T, \eta = \eta^{(1)} + \eta^{(2)}, \quad \varepsilon^* = \eta^{(1)} - \eta^{(2)}. \tag{2.5}$$

Let (n, S) be the right rectangular system, where S and n are respectively, the tangent and the normal of the curve L at the point $t = t_1 + it_2$. Assume that $n = (n_1, n_2)^T = (cos\alpha, sin\alpha)^T$ and $S^0 = (-n_2, n_1)^T = (-sin\alpha, cos\alpha)^T$, where α is the angle of inclination of the normal n to the ox_1 axis.

Introduce the vectors

$$U_n = (u_1 n_1 + u_2 n_2, u_3 n_1 + u_4 n_2)^T, \qquad U_S = (u_2 n_1 - u_1 n_2, u_4 n_1 - u_3 n_2)^T, \tag{2.6}$$

$$\sigma_n = \begin{pmatrix} (TU)_1 n_1 + (TU)_2 n_2 \\ (TU)_3 n_1 + (TU)_4 n_2 \end{pmatrix}, \qquad \sigma_S = \begin{pmatrix} (TU)_2 n_1 - (TU)_1 n_2 \\ (TU)_4 n_1 - (TU)_3 n_2 \end{pmatrix}, \tag{2.7}$$

$$\sigma_t = \begin{pmatrix} \left[r'_{21}n_1 - r'_{11}n_2, r'_{22}n_1 - r'_{12}n_2\right]^T S^0 \\ \left[r''_{21}n_1 - r''_{11}n_2, r''_{22}n_1 - r''_{12}n_2\right]^T S^0 \end{pmatrix} \tag{2.8}$$

Let us call the vector (2.8) the tangential normal stress vector in the linear theory of elastic mixture.

After elementary calculations we obtain

$$\sigma_n = \tau^{(1)}cos^2\alpha + \tau^{(2)}sin^2\alpha + \eta sin\alpha cos\alpha,$$
$$\sigma_t = \tau^{(1)}sin^2\alpha + \tau^{(2)}cos^2\alpha - \eta sin\alpha cos\alpha,$$
$$\sigma_s = \frac{1}{2}\left[(\tau^{(2)} - \tau^{(1)})sin2\alpha + \eta cos2\alpha - \varepsilon^*\right].$$

Direct calculations allow us to check that on L [15]

$$\sigma_n + \sigma_t = \tau = 2(2E - A - B)Re\varphi'(t), \tag{2.9}$$

$$\sigma_n + 2\mu\left(\frac{\partial U_s}{\partial S} + \frac{U_n}{\varrho_0}\right) + i\left[\sigma_S - 2\mu\left(\frac{\partial U_n}{\partial S} - \frac{U_s}{\varrho_0}\right)\right] = 2\varphi'(t), \tag{2.10}$$

$$[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_L = -i\int_L e^{i\alpha}(\sigma_n + i\sigma_s)ds, \tag{2.11}$$

where $\det(2E-A-B) > 0$, $\frac{1}{\varrho_0}$ is the curvature of L at the point $t = t_1 + it_2$. Everywhere in the sequel it will be assumed that the components U_n and U_s are bounded [8].

Formulas (2.2), (2.3), (2.9) and (2.10) are analogous in the linear theory of elastic mixtures to those of Kolosov-Muskhelishvili [12].

3. Statement of the problem and the method of its solving

In the present work we consider the problem of statics of the linear theory of elastic mixture of finding a full-strength contour for a finite doubly-connected domain whose outer boundary is a convex polygon, while the inner boundary is a smooth closed unknown contour. It is assumed that the unknown contour is free from external stresses and absolutely smooth rigid punches are applied to the polygon boundary; the punches are under action of normal contractive forces.

Our problem is to find strained state of the polygon (with a hole) and analytic form of the unknown contour under the condition that the tangential normal stress vector (2.8) on it takes constant value (the condition of the unknown contour full-strength).

Statement of the problem. Let smooth rigid punches be applied to the boundary of a convex polygon which is weakened by an internal hole, and let the punches be under the action of external normal contractive forces; the hole boundary is free from external forces.

We consider the problem: Find elastic equilibrium of the polygon and analytic form of an unknown contour under the condition that the tangential normal stress vector on it takes constant value $\sigma_t = K^0$, $K^0 = (K_1^0, K_2^0)^T = const$.

By D we denote a doubly-connected domain whose internal boundary is a smooth closed curve L_1 (an unknown part of the boundary), and the external boundary is a polygon L_0. By A_j^0 $(j = \overline{1,n})$ we denote vertices (and their affixes) or the polygon (G_0) and assume that the point $z = 0$ lies inside the contour L_1. The positive direction on $L = L_0 \bigcup L_1$ is taken that which leaves the domain D on the left.

It is not difficult to note that in the case under consideration the $\sigma_S = 0$ (see (2.7)) on the entire boundary of D, and the $U_n(t)$ (see (2.6)) is a piecewise constant (unknown) vector on L_0.

Relying on the analogous Kolosov-Muskhelishvilis formulas (2.9) - (2.11) the above posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in domain D, by the following boundary conditions on $L = L_0 \bigcup L_1$:

$$Re\varphi'(t) = H, \quad t \in L_1, \quad H = \frac{1}{2}(2E-A-B)^{-1}K^0, \tag{3.1}$$

$$Im\varphi'(t) = 0, \quad t \in L_0, \tag{3.2}$$

$$Ree^{-i\alpha(t)}\left[(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}\right]=C(t), \qquad t\in L_0, \tag{3.3}$$

$$(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}=0, \quad t\in L_1, \tag{3.4}$$

where $\alpha(t)$ is the angle lying between the ox_1-axis and external normal to the boundary at the point $t\in L_0$,

$$C(t)=Re\{-i\int_{A^0}^{t}\sigma(t_0)expi[\alpha(t_0)-\alpha(t)]dS_0+(\delta^{(1)}+i\delta^{(2)})exp(-i\alpha(t))\}, t\in L_0,$$

$\delta^{(j)}=(\delta_1^{(j)},\delta_2^{(j)})^T, \quad (j=1,2)$, are arbitrary real constant vectors.

Moreover if $t\in L_0$ then we can write

$$Ret\; e^{-i\alpha(t)}=Ree^{-i\alpha(t)}A^0(t),$$

where $A^0(t)=A_k^0$ for $t\in A_k^0A_{k+1}^0$.

Since $\alpha(t)$ is the piecewise constant function, we obtain for $C(t)$ the representation

$$C(t)=\sum_{j=1}^{k}P^{(j)}sin(\alpha_k-\alpha_j)+\delta^{(1)}cos\alpha_k+\delta^{(2)}sin\alpha_k=C_k,$$

for $t\in A_k^0A_{k+1}^0, \quad k=\overline{1,n}, \quad (A_{k+1}^0\equiv A_1^0)$ where α_k is the value of the function $\alpha(t)$ on $A_k^0A_{k+1}^0$,

$$P^{(j)}=-\int_{S_j}^{S_{j+1}}\sigma_n(S)ds, \quad j=\overline{1,n}, \quad \sum_{k=1}^{n}P^{(k)}\cos\alpha_k=\sum_{k=1}^{n}P^{(k)}sin\alpha_k=0,$$

$$P^{(j)}=(P_1^{(j)},P_2^{(j)})^T,$$

(the equilibrium conditions), Thus, $C(t)$ is the piecewise constant vector-function containing n arbitrary real constants to be defined in the sequel.

Now note that, the conditions (3.1) and (3.2) is the Keldysh-Sedov problem having a solution [11]

$$\varphi(z)=Hz=\frac{1}{2}(2E-A-B)^{-1}K^0z, \qquad z\in D \tag{3.5}$$

(an arbitrary constant is assumed to be equal to zero).

Let the function $z=\omega(\zeta)$ map conformally a circular ring $G(1<|\zeta|<R)$ onto the domain D. We assume that the contour $l_0(|\zeta|=R)$ turns into L_0 and the contaur $l_1(|\zeta|=1)$ into L_1.

By virtue of (3.3), (3.4) and (3.5) for the vector-functions $\psi_0(\zeta)=\psi[\omega(\zeta)]$ holomorphic in the ring G, we obtain the following boundary value problem:

$$Ree^{-i\alpha(\xi)}[\frac{1}{2}K^0\omega(\xi)-2\mu\psi_0(\xi)]=-C(\xi), \quad |\xi|=R, \tag{3,6}$$

$$\frac{1}{2}K^0\omega(\sigma)-2\mu\overline{\psi_0(\sigma)}=0 \quad |\sigma|=1. \tag{3.7}$$

Note that on l_0 there takes place the equality

$$\frac{1}{2}Ree^{-i\alpha(\sigma)}K^0\omega(\sigma)=\frac{1}{2}K^0f_0(\sigma)=F_0(\sigma), \tag{3.8}$$

where $f_0(\sigma) = Re[e^{-i\alpha(\sigma)}A^0(\sigma)]$, $\quad A^0(\sigma) = A^0_k$, $\quad \sigma \in l_0^{(k)}$ (l_0^k are the arcs of the circumference l_0 corresponding to the sides L_0^k) $k=\overline{1,n}$.

Let us consider a new unknown vector-function $W(\zeta) = (W_1, W_2)^T$ defined by the formula

$$W(\zeta)=\begin{cases}\frac{1}{2}\overline{K^0\omega(\zeta)}, & 1<|\zeta|<R,\\ 2\mu\psi_0\left(\frac{1}{\bar\zeta}\right), & \frac{1}{R}<|\zeta|<1.\end{cases} \tag{3.9}$$

By the conditions (3.7) and (3.8) we can conclude that $W(\zeta)$ is the vector-function, holomorphic in the ring $G^*(\frac{1}{R}<|\zeta|<R)$ and satisfying the boundary conditions

$$\begin{aligned}&Ree^{-\alpha(\xi)}W(\xi)=F_0(\xi), \quad \xi\in l_0,\\&Ree^{-\alpha(\sigma)}W(\sigma)=F_0^*(\sigma), \quad \sigma\in l_0^*,\end{aligned} \tag{3.10}$$

where l_0^* the circumference $|\zeta| = \frac{1}{R}$, $\quad F_0^*(\sigma) = C(\sigma) + F_0(\sigma)$.

Since $F_0(\xi)$ and $F_0^*(\sigma)$ are the piecewise constant vector-functions, from (3.10) by means of multiplication by the abscissa s, with respect to the vector-function $W'(\zeta)$ we obtain the boundary value problem

$$Re[i\sigma e^{-i\alpha(\sigma)}W'(\sigma)]=0, \qquad \sigma\in l_0Ul_0^* \tag{3.11}$$

Consider now the polygon (G_1) lying completely inside the contour L_1 and similar to the polygon (G_0); the corresponding vertices lie on one and the same ray emanating from the point $z = 0$ (the similarity coefficient q remains unfixed yet).

We denote by A_j^* (that is, $A_j^* = q^{-1}A_j^0$,), vertices of the polygon (G_1) and by L_0^* the boundary.

By D^* we denote the doubly-connected domain which is bounded by the polygons (G_1) and (G_0), and as the positive direction on the boundary of D^* $(L_0 \bigcup L_0^*)$ we choose that which leaves the domain D^* on the left.

Let the function $z = \omega_0(\zeta)$ map conformally the circular ring $G^*(R^{-1} < |\zeta| < R)$ onto the domain D^* (this can be achieved by the choice of q). Assume that $(|\zeta| = R)$ corresponds to L_0 and l_0^* $(|\zeta| = R^{-1})$ corresponds to L_0^*.

Taking into account that on l_0 and l_0^* the equalities:

$$Re[e^{-i\alpha(\xi)}\frac{1}{2}K^0\omega_0(\xi)]=F_0(\xi), \quad \xi\in l_0,$$

$$Re[e^{-i\alpha(\sigma)}\frac{1}{2}K^0\omega_0(\sigma)]=\frac{1}{q}F_0(\sigma), \quad \sigma\in l_0^*, \tag{3.12}$$

take place, we obtain with respect to the vector-function $\frac{1}{2}K^0\omega_0'(\zeta)$ the boundary value problem (3.11). Thus the vector-functions $W'(\zeta)$ and $\frac{1}{2}K^0\omega_0'(\zeta)$ satisfy one and the same boundary conditions on $l_0Ul_0^*$

Taking into account the results cited in [6], we can conclude that the necessary and sufficient condition for solving the problem (3.11) is of the form

$$\prod_{k=1}^{n}\left(\frac{a_k}{R^2}\right)^{\gamma_k-1}\left(\frac{a_k}{q}\right)^{1-\gamma_k}=1, \tag{3.13}$$

and the solution itself is given by the formula

$$W'(z)=\nu\prod_{k=1}^{n}\left(\frac{a_k}{R}\right)^{\frac{1}{2}(\gamma_k-1)}\left(1-\frac{\zeta}{a_k}\right)^{\gamma_k-1}\left(1-\frac{a_k}{\zeta R^2}\right)^{\gamma_k-1}T(\zeta)[\zeta^2T(R^2\zeta)]^{-1}, \tag{3.14}$$

where by a_k we denote the preimages of the points A_k^0 $(a_k\in l_0)$, $k=\overline{1,n}$, $\nu=(\nu_1,\nu_2)^T$ is an arbitrary real constant vector,$\pi\gamma_k$ is the innear angle at the vertex A_k, $k=\overline{1,n}$ and

$$T(\zeta)=\prod_{j=1}^{\infty}\prod_{k=1}^{n}\left(1-\frac{a_k}{R^{4j}\zeta}\right)^{\gamma_k-1}\left(1-\frac{\zeta}{R^{4j}a_k}\right)^{\gamma_k-1}.$$

Since $\sum_{k=1}^{n}(\gamma_k-1)=-2$ form (3.13) we get the relation $q=R^2$.

On the basis of the above results we can conclude that the problem of finding a full-strength contour inside the polygon is closely connected with the problem of conformal mapping of a doubly-connected domain, bounded by polygons, onto the circular ring. In order that the above-mentioned problems (3.10) and (3.12) be identical, it is necessary that the equality (see [6])

$$\left(1-\frac{1}{R^2}\right)F_0(\sigma)=C(\sigma),\quad \sigma\in l_0^*, \tag{3.15}$$

hold, or what is the same thing,

$$\begin{aligned}\frac{1}{2}\left(1-\frac{1}{R^2}\right)K^0(A_m^{(1)}cos\alpha_m+A_m^{(2)}sin\alpha_m)=\\ =\sum_{j=1}^{m}P^{(j)}sin(\alpha_m-\alpha_j)+\delta^{(1)}cos\alpha_m+\delta^{(2)}sin\alpha_m,\end{aligned} \tag{3.16}$$

where $A_m^0=A_m^{(1)}+iA_m^{(2)}$. $m=\overline{1,n}$.

If we choose the constants $P^{(j)}=(P_1^{(j)},P_2^{(j)})^T$, $j=\overline{1,n}$ and $\delta^{(1)},\delta^{(2)}$ (two of $P^{(j)}$ are expressed through the rest ones) in such a way that the equality (3.16) holds, we obtain $W(\zeta)=\frac{1}{2}K^0\omega_0(\zeta)$, and hence the equation of the unknown contour L_1 will be

$$t=\omega_0(\sigma)=\frac{2}{K_1^0}W_1(\sigma)=\frac{2}{K_2^0}W_2(\sigma),\quad \sigma\in l_1$$

and the vector-function $2\mu\psi_0(\zeta)$ will be represented in the form $2\mu\psi_0(\zeta)=\frac{1}{2}K^0\overline{\omega_0\left(\frac{1}{\zeta}\right)}$, $\zeta\in G$.

As an example, we consider the case with the rectilinear polygon (G_0). Assume that to every polygon side are applied punches whose middle is under the action of normal concentrated force $-P$, $(P=(P_1,P_2)^T)$.

The coordinate origin is at the center of the polygon (G_0) and the $ox_1 - axis$ is perpendicular to the side A_1^0, A_2^0. Owing to the symmetry in the case we may assume that

$$A_k^0 = exp\left[-\frac{\pi i}{n} + \frac{2\pi i}{n}(k-1)\right]; \qquad \alpha_k = \frac{2\pi}{n}(k-1) \quad a_k = Rexp\left[\frac{2\pi i}{n}(k-1)\right].$$

It can be shown that the function $f_0(\sigma) = Re\left[e^{-i\alpha(\sigma)}A^0(\sigma)\right]$ is constant: $f_0(\sigma) = rcos\frac{\pi}{n}$, and the vector-function $C(t)$ in this case has the form

$$C(t) = \frac{P}{2\sin\frac{\pi}{n}}\left[\cos\frac{\pi}{n} - cos\frac{\pi}{n}(2k-1)\right] + \nu^{(1)}\cos\frac{2\pi}{n}(k-1)+$$

$$\nu^{(2)}\sin\frac{2\pi}{n}(k-1) = \frac{1}{2}P\left[ctg\frac{\pi}{n} - \cos\frac{2\pi}{n}(k-1)ctg\frac{\pi}{n} + \sin\frac{2\pi}{n}(k-1)\right]+$$

$$+\nu^{(1)}\cos\frac{2\pi}{n}(k-1) + \nu^{(2)}\sin\frac{2\pi}{n}(k-1).$$

Taking $\nu^{(1)} = \frac{1}{2}Pctg\frac{\pi}{n}$; $\quad \nu^{(2)} = -\frac{1}{2}P$, we get $C(t) = -\frac{1}{2}Pctg\frac{\pi}{n}$ and hence (3.15) results in the relation

$$K^0 = \frac{PR^2}{r(R^2-1)\sin\frac{\pi}{n}}. \tag{3.17}$$

In particular, if we assume that the polygon side is equal to unity, i.e. $a_n = 2r\sin\frac{\pi}{n} = 1$, then from (3.17) we obtain

$$K^0 = \frac{2PR^2}{R^2-1},$$

whence we conclude that $K_j^0 > 2P_j$; $\quad (j=1,2)$ and also, when R increases (i.e. when the hole shrinks to the point) $K^0 \to 2P$, while as $R \to 1$ i.e., when K^0 increases and does not exceed critical value, the hole contour approaches to that of the polygon.

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INVERSE PROBLEM ABOUT TRANSITION IN A FIXED POINT FOR LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In the paper the following inverse problem is considered: find such initial functions that the value of corresponding solution at given moment is equal to a fixed vector. On the basis of necessary conditions an algorithm is provided for the approximate solution of the inverse problem.

Keywords and phrases: Inverse problem, neutral functional differential equation, necessary optimality conditions.

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Let $\mathbb{R}^n$ be an n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$ with

$$| x |^2 = \sum_{i=1}^{n} (x^i)^2.$$

Let $K_1 \subset \mathbb{R}^n, K_2 \subset \mathbb{R}^n$ be convex compact sets, let $\tau(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$ be continuously differentiable scalar functions (delay functions) satisfying the conditions

$$\tau(t) < t,\ \eta(t) < t,\ \dot{\tau}(t) > 0,\ \dot{\eta}(t) > 0.$$

Let $t_0 < t_1$ be given numbers with $\tau(t_1) > t_0$ and $\eta(t_1) > t_0$. By Δ_1 and Δ_2 we denote, respectively, the sets of measurable initial functions $\varphi : [\hat{\tau}, t_0] \to K_1$ and $g : [\hat{\tau}, t_0] \to K_2$, where $\hat{\tau} = t_0 - \max\{\tau(t_0), \eta(t_0)\}$.

To each element (initial data) $w = (\varphi(t), g(t)) \in W = \Delta_1 \times \Delta_2$ we assign the linear neutral functional differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(\tau(t)) + C(t)\dot{x}(\eta(t)) \tag{1}$$

with the initial condition

$$\begin{cases} x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \quad (\varphi(t_0) = \varphi(t_0-)), \\ \dot{x}(t) = g(t), t \in [\hat{\tau}, t_0), \end{cases} \tag{2}$$

where $A(t), B(t), C(t), t \in [t_0, t_1]$, are given continuous matrix functions with appropriate dimensions.

Definition. Let $w = (\varphi(t), g(t)) \in W$, a function $x(t) = x(t; w) \in \mathbb{R}^n, t \in [\hat{\tau}, t_1]$ is called a solution of differential equation (1) with the initial condition (2) or a solution corresponding to the element w if $x(t)$ satisfies the initial condition (2) is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere.

For every element $w \in W$ there exists a unique solution $x(t; w)$ defined on the interval $[\hat{\tau}, t_1]$.

Introduce the set

$$Y=\Big\{y\in\mathbb{R}^n:\exists w\in W,\ x(t_1;w)=y\Big\}.$$

The inverse problem. Let $y\in Y$ be a given vector. Find element $w\in W$ such that the following condition holds

$$x(t_1;w)=y.$$

The vector y, as a rule, by distinct error is beforehand given. Thus instead of the vector y we have $\hat{y}$ (so called observed vector) which is an approximation to the y and in general, $\hat{y}\notin Y$. Therefore it is natural to change the posed inverse problem by the following approximate problem.
The approximate inverse problem. Find an element $w\in W$ such that the deviation

$$\frac{1}{2}|x(t_1;w)-\hat{y}|^2$$

takes the minimal value.

It is clear that the approximate inverse problem is equivalent to the following optimization problem:

$$\dot{x}(t)=A(t)x(t)+B(t)x(\tau(t))+C(t)\dot{x}(\eta(t)) \tag{3}$$

$$x(t)=\varphi(t),t\in[\hat{\tau},t_0],\dot{x}(t)=g(t),t\in[\hat{\tau},t_0), \tag{4}$$

$$J(w)=\frac{1}{2}|x(t_1;w)-\hat{y}|^2\to\min,w\in W. \tag{5}$$

Problem (3)-(5) is called an optimal control problem corresponding to the inverse problem.

Theorem 1.([1]) *There exists an optimal element $w_0=(\varphi_0(t),g_0(t))$ for problem (3)-(5).*

Theorem 2.([1]) *Let $w_0=(\varphi_0(t),g_0(t))\in W$ be an optimal element. Then the following conditions hold:*
1) the condition for the initial function $\varphi_0(t)$

$$\psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t)\varphi_0(t)=\max_{\varphi\in K_1}\psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t)\varphi,$$

$$t\in[\tau(t_0),t_0],$$

where $\gamma(t)$ is the inverse function of $\tau(t)$;
2) the condition for the initial function $g_0(t)$

$$\psi(\rho(t))C(\rho(t))\dot{\rho}(t)g_0(t)=\max_{g\in K_2}\psi(\rho(t))C(\rho(t))\dot{\rho}(t)g,$$

$$t\in[\eta(t_0),t_0].$$

where $\rho(t)$ is the inverse function of $\eta(t)$.
Here $(\psi(t),\chi(t))$ is solution of the system

$$\begin{cases}\dot{\chi}(t)=-\psi(t)A(t)-\psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t),\\ \psi(t)=\chi(t)+\psi(\rho(t))C(\rho(t))\dot{\rho}(t)\end{cases} \tag{6}$$

with the initial condition

$$\psi(t_1) = \chi(t_1) = -(x_0(t_1) - \hat{y})^T, \psi(t) = \chi(t) = 0, t > t_1.$$

Let the optimal element $w_0 = (\varphi_0(t), g_0(t))$ be unique and conditions *1)* and *2)* give the unique initial functions $\varphi(t)$ and $g(t)$, respectively.

The algorithm. Let $\varphi_1(t) \in \Delta_1$ and $g_1(t) \in \Delta_2$ be starting approximation of the initial functions. We construct the sequences

$$\{\varphi_k(t)\}, \{g_k(t)\}, \{x_k(t)\}, \{\psi_k(t)\}, \{\chi_k(t)\}$$

by the following process:

3) for given $\varphi_1(t)$ and $g_1(t)$ find $x_1(t)$: the solution of the differential equation (3) with the initial condition

$$x(t) = \varphi_1(t), t \in [\tau(t_0), t_0], \dot{x}(t) = g_1(t), t \in [\eta(t_0), t_0);$$

4) find $\psi_1(t)$ and $\chi_1(t)$: the solution of the differential equation (6) with the initial condition

$$\psi(t_1) = \chi(t_1) = -(x_1(t_1) - \hat{y}), \psi(t) = \chi(t) = 0, t > t_1;$$

5) find the next iterations $\varphi_2(t)$ and $g_2(t)$ from *1)* and *2)*, respectively.

6) if

$$| J(w_1) - J(w_2) | \leq \varepsilon$$

stop, where $w_1 = (\varphi_1(t), g_1(t)), w_2 = (\varphi_2(t), g_2(t))$ and ε is a given number.

If

$$| J(w_1) - J(w_2) | > \varepsilon$$

go to 3).

Theorem 3. *The following relations are valid:*

$$\lim_{k\to\infty} \varphi_k(t) = \varphi_0(t) \text{ weakly in the space } L[\tau(t_0), t_0];$$

$$\lim_{k\to\infty} g_k(t) = g_0(t) \text{ weakly in the space } L[\sigma(t_0), t_0];$$

$$\lim_{k\to\infty} x_k(t) = x_0(t) \text{ uniformly for } t \in [t_0, t_1];$$

$$\lim_{k\to\infty} \sup_{[t_0,t_1]} | \psi_k(t) - \psi(t) | = 0;$$

$$\lim_{k\to\infty} \chi_k(t) = \chi(t) \text{ uniformly for } t \in [t_0, t_1].$$

Moreover, $w_0 = (\varphi_0(t), g_0(t))$ *is an optimal element,* $x_0(t) = x(t; w_0)$ *is an optimal trajectory,* $(\psi(t), \chi(t))$ *is the solution of equation (6) corresponding to* w_0.

Theorem 3 is proved by the scheme given in [2].

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მოხსენებები, ტომი 41, 2015

ფურიეს მწკრივების ლოგარითმული საშუალოების ზომით კრებადობის შესახებ

ლ. ბარამიძე, უ. გოგინავა

ცნობილია, რომ თუ f არის $L\log L$ კლასიდან, მაშინ ორმაგი ფურიეს მწკრივის ლოგარითმული საშუალოები ზომით კრებადია. ამასთან, ცნობილია რომ, თუ განვიხილავთ ნებისმიერ ორლიჩის სივრცეს, რომელიც არ არის ქვესივრცე $L\log L$-ის, მაშინ, იმ ფუნქციების სიმრავლე, რომელთა ორმაგი ფურიეს მწკრივის ლოგარითმული საშუალოები არის ზომით კრებადი, არის ბერის პირველი კატეგორიის სიმრავლე. გ. ტყებუჩავამ შემოიღო ლოგარით-მული საშუალოები, რომელთა კერძო შემთხვევაა ნორლუნდის ლოგარითმული საშუალოები. წარმოდგენილ ნაშრომში მიღებულია აუცილებელი და საკმარისი პირობა, რომელიც უზრუნველყოფს ამ საშუალოების ზომით კრებადობას.

დრეკადობის ბმული თეორიის სასაზღვრო ამოცანები ორგვარი ფოროვნობის მქონე ნახევარსიბრტყისათვის

ლ. ბიწაძე

ნაშრომში განხილულია დრეკადობის ბმული თეორიის მდგრადი რხევის სასაზღვრო ამოცანები ორგვარი ფოროვნობის მქონე ნახევარსიბრტყისათვის. მიღებულია სასაზღვრო ამოცანების ამონახსნები კვადრატურებში.

დრეკადობის ბმულ თეორიაში ფუნდამენტური ამონახსნის აგება ორგვარი ფოროვნობის მქონე სხეულებისათვის

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თითქმის წრფივი ფუნქციონალურ დიფერენციალური განტოლებები A და B თვისებებით

რ. კოპლატაძე

შრომაში განხილულია ოპერატორულ დიფერენციალური განტოლება. შესწავლილია მოცემული განტოლების ამონახსნების ასიმპტოტური ყოფაქცევა უსასრულობის მიდამოში, როდესაც განტოლების მარჯვენა მხარეს გააჩნია თითქმის წრფივი მინორანტი. კერძოდ, მოყვანილია საკმრისი პირობები იმისა, რომ მოცემულ განტოლებას გააჩნდეს A ან B თვისება.

ევროკავშირში ქართული ენით - პროექტის „კიდევ ერთი ნაბიჯი მოსაუბრე ქართული თვითგანვითარებადი ინტელექტუალური კორპუსისაკენ“ მიზნები და საფუძვლები

კ. ფხაკაძე, მ. ჩიქვინიძე, გ. ჩიჩუა, ი. ბერიაშვილი, ნ. ფხაკაძე, დ. კურცხალია

ნაშრომში მოკლედაა მიმოხილული მეტა-ქსელის ტექნოლოგიური საბჭოს ჩვენთვის მეტად მნიშვნელოვანი გამოქვეყნება სათაურით „სტრატეგიული კვლევითი გეგმა 2020 წლის მრავალენოვანი ევროპისათვის“ და, ასევე, საქართველოს ტექნიკური უნივერსიტეტის ქართული ენის ტექნოლოგიზების სასწავლო-სამეცნიერო ცენტრში შოთა რუსთაველის ეროვნული სამეცნიერო ფონდის საგრანტო მხარდაჭერით მოქმედი პროექტის „კიდევ ერთი ნაბიჯი მოსაუბრე ქართული თვითგანვითარებადი ინტელექტუალური კორპუსისაკენ“ მიზნები და საფუძვლები. ამასთან, დასკვნის სახით ხაზი ესმება იმას, რომ ქართული ენის სრული და ამომწურავი ტექნოლოგიური დამუშავება ანუ ქართული მოაზროვნე, მოსაუბრე და მთარგნელი სისტემის შემუშავება სასიცოცხლოდ მნიშვნელოვანია ქართული ენის ციფრული კვდომის საფრთხისგან დაცვისა და, შესაბამისად, მეტა-ქსელის სტრატეგიული კვლევითი გეგმის მიხედვით ენობრივი ბარიერებისაგან სრულიად თავისუფალ სივრცედ გარდაქმნის პროცესში მყოფ ევროკავშირში ქართველი ერის ღირსეული ანუ ქართული ენით შესვლის მიზნების გათვალისწინებით.

ამონახსნის ნაზრდის შეფასების შესახებ სამართი ფუნქციონალურ-დიფერენციალური განტოლებისათვის დაგვიანების პარამეტრის შეშფოთების გათვალისწიებით

თ. შავაძე

მიღებულია ამონახსნის ნაზრდის შეფასება მცირე პარამეტრის მიმართ არაწრფივი დაგვიანებულ არგუმენტიანი სამართი ფუნქციონალურ-დიფერენციალური განტოლებისათვის უწყვეტი საწყისი პირობით. გარდა ამისა, ამონახსნის ნაზრდი გამოთვლილია საწყის მომენტში.

დრეკად ნარევთა თეორიის სტატიკის ამოცანა მრავალკუთხედში თანაბრადმტკიცე კონტურის მოძებნის შესახებ

კ. სვანაძე

ნაშრომში განხილულია დრეკად ნარევთა წრფივი თეორიის სტატიკის ამოცანა, რომელიც ეხება თანაბრადმტკიცე კონტურის მოძებნას სასრული ორადბმული არისათვის, რომლის გარე საზღვარი წარმოადგენს ამოზნექილ მრავალკუთხედს, ხოლო შიგა საზღვარი–გლუვ შეკრულ კონტურს. იგულისხმე-ბა, რომ მრავალკუთხედის გვერდებზე მოდებულია აბსოლუტურად გლუვი ხისტი შტამპები, რომელზედაც მოქმედებენ ნორმალური კუმშავი ძალები, ხოლო შიგა კონტური (საზღვრის უცნობი ნაწილი) თავისუფალია გარეგანი დატვირთვისაგან. ამოცანა მდგომარეობს განისაზღვროს საძიებელი კონტურის ანალიზური სახე იმ პირობით, რომ მასზე ე.წ. ტანგენციალური ნორმალური ძაბვის ვექტორი ღებულობდეს მუდმივ მნიშვნელობას.

შებრუნებული ამოცანა ფიქსირებულ წერტილში გადასვლის შესახებ წრფივი ნეიტრალური ფუნქციონალურ-დიფერენციალური განტოლებებისთვის

თ. თადუმაძე

ნაშრომში განხილულია შემდეგი ამოცანა: იპოვეთ ისეთი საწყისი ფუნქციები, რომლის შესაბამისი ამონახსნის მნიშვნელობა მოცემულ მომენტში ემთხვევა მოცემულ ვექტორს. მოყვანილია ამოცანის მიახლოებითი ამოხსნის ალგორითმი.

C O N T E N T S

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