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EFFECTIVE SOLUTION OF THE DIRICHLET BVP OF THERMOELASTICITY WITH MICROTEMPERATURES FOR AN ELASTIC SPACE WITH A SPHERICAL CAVITY

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Abstract. In the present paper the linear theory of thermoelasticity with microtemperatures is considered. The representation of regular solution for the equations of steady vibration of the 3D theory of thermoelasticity with microtemperatures is obtained. We use it for explicitly solving Dirichlet boundary value problem (BVP) for an elastic space with a spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords and phrases: Thermoelasticity with microtemperatures, absolutely and uniformly convergent series, spherical harmonic.

AMS subject classification (2010): 74F05, 74G05.

1. Introduction

A thermodynamic theory for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was established by Grot [1]. The linear theory of thermoelasticity with microtemperatures was presented in [2], where the existence theorems were proved and the continuous dependence of solutions of the initial data and body loads were established. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze [3]. The representations of the Galerkin type and general solutions of the system in this theory were obtained by Scalia, Svanadze and Tracinà [4]. The 3D linear theory of thermoelasticity for microstretch elastic materials with microtemperatures was constructed by Iesan [5], where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved. A wide class of external BVPs of steady vibrations is investigated by Svanadze [6]. Effective solution of the Dirichlet and the Neumann BVPs of the linear theory of thermoelasticity with microtemperatures for a spherical ring are obtained in [7-8].

The two-dimensional model of thermoelasticity with microtemperatures is considered by Basheleishvili, Bitsadze and Jaiani in [9,10,11,12]. In particular, fundamental and singular solutions of the system of equations of the equilibrium of the 2D thermoelastisity theory with microtemperatures were constructed. Uniqueness and existence theorems of some basic boundary value problems of the 2D thermoelasticity with microtemperatures are proved and the explicit solutions of boundary value problems for the half-plane are constructed.

In the present paper the linear theory of thermoelasticity with microtemperatures is considered. The representation of regular solution for the equations of steady vibrations of the 3D theory of thermoelasticity with microtemperatures is obtained. We use it for explicitly solving Dirichlet boundary value problem (BVP) of steady vibrations for an elastic space with spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

2. Basic equations

We consider an isotropic elastic material with microtemperatures. Let us assume that D^+ is a ball, of radius R_1, centered at point $O(0,0,0)$ in space E_3 and S is a spherical surface of radius R_1. Denote by D^--whole space with a spherical cavity. $\overline{D^+} := D^+ \bigcup S$, $D^- := E_3 \backslash \overline{D^+}$. Let $\mathbf{x} := (x_1, x_2, x_3) \in E_3$, $\partial\mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$.

The basic homogeneous system of equations of steady vibrations in the linear theory of thermoelasticity with microtemperatures has the following form [2]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - \beta grad\theta + \varrho\omega^2\mathbf{u} = 0 \tag{1}$$

$$k_6\Delta\mathbf{w} + (k_4+k_5) graddiv\mathbf{w} - k_3 grad\theta + k_8\mathbf{w} = 0 \tag{2}$$

$$(k\Delta + a_0)\theta + \beta_0 div\mathbf{u} + k_1 div\mathbf{w} = 0 \tag{3}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector, $\mathbf{w} = (w_1, w_2)^T$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads), $a_0 = i\omega a T_0$, $\beta_0 = i\omega\beta T_0$, $k_8 = ib\omega - k_2$, $b > 0$, a, λ, μ, β, k, k_j, $j = 1, ..., 6$, are constitutive coefficients, Δ is the 3D Laplace operator and ω is the oscillation frequency $(\omega > 0)$. The superscript "T" denotes transposition.

We will suppose that the following assumptions on the constitutive coefficients hold [2]

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad a > 0, \quad b > 0, \quad k > 0,$$

$$3k_4 + k_5 + k_6 > 0, \quad k_6 \pm k_5 > 0, \quad (k_1 + k_3 T_0)^2 < 4T_0 k k_2.$$

Definition 1. A vector-function $\mathbf{U}(U_1, U_2, U_3, U_4, U_5, U_6, U_7)$ defined in the domain D^- is called regular if [6]

1.
$$\mathbf{U} \in C^2(D^-) \cap C^1(\overline{D^-}),$$

2.
$$\mathbf{U} = \sum_{j=1}^{5} \mathbf{U}^{(j)}(\mathbf{x}), \quad U^{(j)} = (U_1^{(j)}, U_2^{(j)}, U_3^{(j)}, U_4^{(j)}, U_5^{(j)}, U_6^{(j)}, U_7^{(j)}),$$
$$U^{(j)} \in C^2(D^-) \cap C^1(\overline{D^-}), \tag{4}$$

3.
$$(\Delta + \lambda_j^2) U_l^{(j)} = 0, \tag{5}$$

and

$$\left(\frac{\partial}{\partial |\mathbf{x}|} - i\lambda_j \right) U_l^{(j)} = e^{i\lambda_j |\mathbf{x}|} o(|\mathbf{x}|^{-1}), \quad for \quad |\mathbf{x}| \geq 1, \tag{6}$$

$$U_m^{(5)} = U_7^{(4)} = U_7^{(5)} = 0, \quad m = 1, 2, 3, \quad j = 1, 2, .., 5, \quad l = 1, 2, ..., 7,$$

where $\lambda_j^2, \quad j = 1, 2, 3$ are roots of equation $D(-\xi) = 0$, where

$$D(\Delta) = (\mu_0\Delta + \rho\omega^2)k_1k_3\Delta + (k_7\Delta + k_8)[\beta\beta_0\Delta + (\mu_0\Delta + \rho\omega^2)(k\Delta + a_0)],$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{1}{\mu_0 k k_7}\left[\mu_0(a_0k_7 + kk_8 + k_1k_3) + \rho\omega^2 kk_7 + \beta\beta_0 k_7\right],$$

$$\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = \frac{1}{\mu_0 k k_7}\left[k_8(\mu_0 a_0 + \beta\beta_0) + \rho\omega^2(a_0k_7 + kk_8 + k_1k_3)\right],$$

$$\lambda_1^2\lambda_2^2\lambda_3^2 = \frac{a_0k_8\rho\omega^2}{\mu_0 k k_7} = \frac{a_0\mu k_6\lambda_4^2\lambda_5^2}{\mu_0 k k_7}, \quad \mu_0 = \lambda + 2\mu, \quad k_7 = k_4 + k_5 + k_6,$$

the constants λ_4^2 and λ_5^2 are determined by the formulas

$$\lambda_4^2 = \frac{\rho\omega^2}{\mu} > 0, \quad \lambda_5^2 = \frac{k_8}{k_6}.$$

The quantities $\lambda_j^2, \quad j = 1, 2, 3, 5$ are complex numbers and are chosen so as to ensure positivity of their imaginary part, i.e. it is assumed that $Im\lambda_j^2 > 0$.

Equations in (6) are Sommerfeld-Kupradze type radiation conditions in the linear theory of thermoelastisity with microtemperatures.

The external Dirichlet BVP is formulated as follows:

Find in the unbounded domain D^- a regular solution $\mathbf{U}(\mathbf{u}, \mathbf{w}, \theta)$ of the equations (1),(2),(3) by the boundary conditions

$$\mathbf{u}^- = \mathbf{F}^-(\mathbf{y}), \quad \mathbf{w}^- = \mathbf{f}^-(\mathbf{y}), \quad \theta^- = f_7^-(\mathbf{y}), \quad \mathbf{y} \in S,$$

where $\mathbf{F}^-(f_1, f_2, f_3)$, $\mathbf{f}^-(f_4, f_5, f_6)$, f_7^- are prescribed functions on S.

The following theorem is valid [6].

Theorem 1. *The external Dirichlet BVP admit at most one regular solution.*

3. Expansion of regular solutions

The following theorem is valid [6].

Theorem 2. *The regular solution $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta) \in C^2(D^-)$ of system (1-3) for $\boldsymbol{x} \in D^-$, is represented as the sum*

$$\mathbf{u} = \sum_{j=1}^{4} \mathbf{u}^{(j)}(\mathbf{x}), \quad \mathbf{w} = \sum_{j=1,2,3,5} \mathbf{w}^{(j)}(\mathbf{x}), \quad \theta = \sum_{j=1}^{3} \theta^{(j)}, \tag{7}$$

where

$$\mathbf{u}^{(j)} = \left[\prod_{l=1;l\neq j}^{4} \frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_j^2}\right]\mathbf{u}, \quad j=1,2,3,4,$$

$$\mathbf{w}^{(p)} = \left[\prod_{l=1,2,3,5} \frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_p^2}\right]\mathbf{w}, \quad l\neq p, \quad p=1,2,3,5, \tag{8}$$

$$\theta^{(q)} = \left[\prod_{l=1}^{3} \frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_q^2}\right]\theta, \quad l\neq q, \quad q=1,2,3.$$

$\mathbf{u}^{(j)}, \mathbf{w}^{(j)}$*and* $\theta^{(j)}$ *are regular functions satisfying the following conditions*

$$(\Delta+\lambda_j^2)\mathbf{u}^{(j)}=0, \quad (\Delta+\lambda_l^2)\mathbf{w}^{(l)}=0, \quad (\Delta+\lambda_m^2)\theta^{(m)}=0,$$

$$j=1,2,3,4, \quad l=1,2,3,5, \quad m=1,2,3.$$

Thus, the regular in D^- solution of system (1-3) is represented as a sum of functions $\mathbf{u}^{(j)}, \quad \mathbf{w}^{(j)}, \quad \theta^{(j)}$, which satisfy Helmholtz' equations in D^-.

Lemma 1. *In the domain of regularity the regular solution of system (1),(3) can be represented in the form*

$$\mathbf{u} = a_1 grad\varphi_1 + a_2 grad\varphi_2 + a_3 grad\varphi_3 + \mathbf{u}^{(4)},$$

$$\mathbf{w} = b_1 grad\varphi_1 + b_2 grad\varphi_2 + b_3 grad\varphi_3 + \mathbf{w}^{(5)}, \tag{9}$$

$$\theta = \varphi_1 + \varphi_2 + \varphi_3,$$

where

$$(\Delta+\lambda_j^2)\varphi_j=0, \quad j=1,2,3, \quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}=0,$$
$$div\mathbf{u}^{(4)}=0, \quad (\Delta+\lambda_5^2)\mathbf{w}^{(5)}=0, \quad div\mathbf{w}^{(5)}=0, \tag{10}$$

a_j *and* b_j, $\quad j=1,2,3,$ *are constants.*

Proof. Replacing $\mathbf{u}$, $\mathbf{w}$ and θ by their values from (8), and substituting $\mathbf{u}, \quad \mathbf{w}, \quad \theta$ into (1),(3), after some calculations we obtain

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)(\mathbf{u}^{(1)}+\mathbf{u}^{(2)}+\mathbf{u}^{(3)}) =$$

$$grad\left[-\frac{(\lambda+\mu)k_1k_3}{\beta_0}(\lambda_1^2\varphi_1+\lambda_2^2\varphi_2+\lambda_3^2\varphi_3)+\beta(k_7\Delta+k_8)(\varphi_1+\varphi_2+\varphi_3)\right. \tag{11}$$
$$\left.+\frac{(\lambda+\mu)}{\beta_0}(k\Delta+a_0)(k_7\Delta+k_8)(\varphi_1+\varphi_2+\varphi_3)\right].$$

Equation (11) is satisfied by

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(1)} =$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_1^2)(k_8-k_7\lambda_1^2)-k_1k_3\lambda_1^2]+\beta(k_8-k_7\lambda_1^2)\right\}grad\varphi_1,$$

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(2)}=$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_2^2)(k_8-k_7\lambda_2^2)-k_1k_3\lambda_2^2]+\beta(k_8-k_7\lambda_2^2)\right\}grad\varphi_2,$$

$$(\mu\Delta+\rho\omega^2)(k_7\Delta+k_8)\mathbf{u}^{(3)}=$$

$$\left\{\frac{(\lambda+\mu)}{\beta_0}[(a_0-k\lambda_3^2)(k_8-k_7\lambda_3^2)-k_1k_3\lambda_3^2]+\beta(k_8-k_7\lambda_3^2)\right\}grad\varphi_3.$$

last identity gives

$$\mathbf{u}^{(1)}=a_1grad\varphi_1,\quad \mathbf{u}^{(2)}=a_2grad\varphi_2\quad \mathbf{u}^{(3)}=a_3grad\varphi_3 \tag{12}$$

where

$$a_1=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_1^2},\quad a_2=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_2^2},\quad a_3=\frac{\beta}{\mu\lambda_4^2-\mu_0\lambda_3^2}.$$

Similarly

$$\mathbf{w}^{(1)}=b_1grad\varphi_1,\quad \mathbf{w}^{(2)}=b_2grad\varphi_2\quad \mathbf{w}^{(3)}=b_3grad\varphi_3,$$

where

$$b_1=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_1^2},\quad b_2=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_2^2},\quad b_3=\frac{k_3}{k_6\lambda_5^2-k_7\lambda_3^2}.$$

Thus

$$\begin{gathered}\mathbf{u}=a_1grad\varphi_1+a_2grad\varphi_2+a_3grad\varphi_3+\mathbf{u}^{(4)}=\sum_{j=1}^{3}a_jgrad\varphi_j+\mathbf{u}^{(4)},\\ \mathbf{w}=b_1grad\varphi_1+b_2grad\varphi_2+b_3grad\varphi_3+\mathbf{w}^{(5)}=\sum_{j=1}^{3}b_jgrad\varphi_j+\mathbf{w}^{(5)},\\ \theta=\varphi_1+\varphi_2+\varphi_3=\sum_{j=1}^{3}\varphi_j,\\ (\Delta+\lambda_j^2)\varphi_j=0,\quad j=1,2,3,\quad (\Delta+\lambda_4^2)\mathbf{u}^{(4)}=0,\\ div\mathbf{u}^{(4)}=0,\quad (\Delta+\lambda_5^2)\mathbf{w}^{(5)}=0,\quad div\mathbf{w}^{(5)}=0,\end{gathered} \tag{13}$$

Now let us prove that if the vector $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)=0$, then $\varphi_1=\varphi_2=\varphi_3=0$, $\mathbf{u}^{(4)}=\mathbf{w}^{(5)}=0$. It follows from (13) that

$$div[a_1grad\varphi_1+a_2grad\varphi_2+a_3grad\varphi_3+\mathbf{u}^{(4)}]=0,$$

$$div[b_1grad\varphi_1+b_2grad\varphi_2+b_3grad\varphi_3+\mathbf{w}^{(5)}]=0,$$

$$\varphi_1(\mathbf{x})+\varphi_2(\mathbf{x})+\varphi_3(\mathbf{x})=0.$$

From these equations we obtain

$$a_1\lambda_1^2\varphi_1 + a_2\lambda_2^2\varphi_2 + a_3\lambda_3^2\varphi_3 = 0,$$

$$b_1\lambda_1^2\varphi_1 + b_2\lambda_2^2\varphi_2 + b_3\lambda_3^2\varphi_3 = 0,$$

$$\varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}) = 0.$$

The determinant of this system is

$$D_1 = \frac{\beta k_3\mu k_6\lambda_4^2\lambda_5^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)(k_6\mu_0\lambda_5^2-k_7\mu\lambda_4^2)}{(\rho\omega^2-\mu_0\lambda_1^2)(\rho\omega^2-\mu_0\lambda_2^2)(\rho\omega^2-\mu_0\lambda_3^2)(k_8-k_7\lambda_1^2)(k_8-k_7\lambda_2^2)(k_8-k_7\lambda_3^2)} \neq 0.$$

Thus we have $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $\mathbf{u}^{(4)} = 0$, $\mathbf{w}^{(5)} = 0$ and the proof is completed.

We introduce the notations. If $\mathbf{g}(\mathbf{x}) = \mathbf{g}(g_1, g_2, g_3)$ and $\mathbf{q}(\mathbf{x}) = \mathbf{q}(q_1, q_2, q_3)$, then by symbols $(\mathbf{g.q})$ and $[\mathbf{g.q}]$ will be denoted scalar product and vector product respectively

$$(\mathbf{g.q}) = \sum_{k=1}^{3} g_k q_k, \quad [\mathbf{g.q}] = (g_2q_3 - g_3q_2, g_3q_1 - g_1q_3, g_1q_2 - g_2q_1),$$

Let us consider the metaharmonic equation

$$(\Delta + \nu^2)\psi = 0, \quad Im\nu \neq 0.$$

For this equation the following statements are valid and we cite them without proof.

Lemma 2. *If the regular vector ψ satisfies the conditions*

$$(\Delta + \nu^2)\psi = 0, \quad Im\nu \neq 0, \quad div\psi = 0,$$

$$(\mathbf{x} \cdot \psi) = 0, \quad \mathbf{x} \in D^+(orD^-),$$

then it can be represented in the form

$$\psi(\mathbf{x}) = [\mathbf{x} \cdot \nabla]h(\mathbf{x}),$$

where

$$(\Delta + \nu^2)h(\mathbf{x}) = 0, \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

In addition if

$$\int_{S(0,a)} h(\mathbf{x})ds = 0,$$

where $S(0,a) \subset D^+(orD^-)$ is an arbitrary spherical surface of radius a, then between the vector ψ and the function h there exists one-to-one correspondence.

Lemma 3. *If the regular vector ψ satisfies the conditions*

$$(\Delta + \lambda^2)\psi = 0, \quad Im\lambda \neq 0 \quad div\psi = 0, \quad \mathbf{x} \in D^+(orD^-),$$

then it can be represented in the form

$$\psi(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x} \cdot \nabla]\varphi_4(\mathbf{x}),$$

where

$$(\Delta + \lambda^2)\varphi_j = 0, \quad j = 3, 4.$$

I*n addition if*

$$\int\limits_{S(0,a)} \varphi_j ds = 0, \quad j = 3, 4,$$

where $S(0, a) \subset D^+(or D^-)$ *is an arbitrary spherical surface of radius* a, *then between the vector* ψ *and the functions* φ_j, $j = 1, .., 4$, *there exists one-to-one correspondence.*

Lemma 2 and Lemma 3 are proved in [13].

Lemma 2 and Lemma 3 lead to the following result.

Theorem 3. *The vector* $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta)$, *is a regular solution of the homogeneous equations (1),(3), in* $D^+(or D^-)$, *if and only if, when it is represented in the form*

$$\begin{aligned} &\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{3} a_j \, grad\varphi_j + \frac{\mu}{\rho\omega^2} \, rot\psi^3(\mathbf{x}), \\ &\mathbf{w}(\mathbf{x}) = \sum_{j=1}^{3} b_j grad\varphi_j + \frac{k_6}{k_8} \, rot\varphi^3(\mathbf{x}), \\ &\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &(\Delta + \lambda_4^2)\psi^3 = 0, \quad div\psi^3 = 0, \\ &(\Delta + \lambda_5^2)\varphi^3 = 0, \quad div\varphi^3 = 0, \\ &\psi^3(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\psi_3(\mathbf{x}) + rot[\mathbf{x} \cdot \nabla]\psi_4(\mathbf{x}), \\ &\varphi^3(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\varphi_4(\mathbf{x}) + rot[\mathbf{x} \cdot \nabla]\varphi_5(\mathbf{x}), \end{aligned} \tag{15}$$

$$\int\limits_{S(0,a)} \psi_j ds = 0, \quad (\Delta + \lambda_4^2)\psi_j = 0, \quad j = 3, 4,$$

$$\int\limits_{S(0,a)} \varphi_j ds = 0, \quad (\Delta + \lambda_5^2)\varphi_j = 0, \quad j = 4, 5,$$

$S(0, a) \subset D^+(or D^-)$ *is an arbitrary spherical surface of radius* a. *Between the vector* $\boldsymbol{U}(\boldsymbol{x}) = (\boldsymbol{u}, \boldsymbol{w}, \theta)$ *and the functions* φ_j, ψ_j $j = 1, .., 4$, *there exists one-to-one correspondence.*

Remark. By virtue of the equality

$$rotrot[x.\nabla]\varphi_4 = -\Delta[x.\nabla]\varphi_4,$$

formula (14) can be written as

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{3} a_j grad\varphi_j - [\mathbf{x}\cdot\nabla]\psi_4(\mathbf{x}) + \frac{\mu}{\rho\omega^2}\, rot[\mathbf{x}\cdot\nabla]\psi_3(\mathbf{x}),$$

$$\mathbf{w}(\mathbf{x}) = \sum_{j=1}^{3} b_j grad\varphi_j - [\mathbf{x}\cdot\nabla]\varphi_5(\mathbf{x}) + \frac{k_6}{k_8}\, rot[\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}), \tag{16}$$

$$\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) + \varphi_3(\mathbf{x}).$$

Below we shall use solution (16) to solve the Dirichlet boundary value problem of steady vibrations for an elastic space with spherical cavity.

4. Some auxiliary formulas

In the sequel we use the following notations: let us introduce the spherical coordinates

$$x_1 = \rho\sin\vartheta\cos\varphi, \quad x_2 = \rho\sin\vartheta\sin\varphi, \quad x_3 = \rho\cos\vartheta,$$

$$y_1 = R_1\sin\vartheta_0\cos\varphi_0, \quad y_2 = R_1\sin\vartheta_0\sin\varphi_0, \quad y_3 = R_1\cos\vartheta_0, \quad y\in S, \tag{17}$$

$$\rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0\le\vartheta\le\pi, \quad 0\le\varphi\le 2\pi \quad 0\le\rho\le R_1.$$

The operator $\dfrac{\partial}{\partial S_k(\mathbf{x})}$ is determined as follows

$$[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(\mathbf{x})} \quad k=1,2,3 \quad \mathbf{x}\in E_3,$$

Simple calculations give

$$\frac{\partial}{\partial S_1(\mathbf{x})} = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} = -cos\varphi ctg\vartheta\frac{\partial}{\partial\varphi} - sin\varphi\frac{\partial}{\partial\vartheta},$$

$$\frac{\partial}{\partial S_2(\mathbf{x})} = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3} = -sin\varphi ctg\vartheta\frac{\partial}{\partial\varphi} + cos\varphi\frac{\partial}{\partial\vartheta},$$

$$\frac{\partial}{\partial S_3(\mathbf{x})} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} = \frac{\partial}{\partial\varphi}.$$

The following identities are true [13]

$$(\mathbf{x}\cdot rotg(\mathbf{x})) = \sum_{k=0}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(\mathbf{x})}, \quad \sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(rot[\mathbf{x}\cdot\nabla]h)_k = 0,$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(rot\mathbf{g}(\mathbf{x}))_k = \rho\frac{\partial}{\partial\rho}div\mathbf{g}(\mathbf{x}) - \sum_{k=0}^{3} x_k\Delta\mathbf{g}_k(\mathbf{x}),$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{g}]_k=\rho^2 div\mathbf{g}(\mathbf{x})-(\mathbf{x}\cdot\mathbf{g}(\mathbf{x}))-\rho\frac{\partial}{\partial\rho}(\mathbf{x}\cdot\mathbf{g}(\mathbf{x})),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot rot\mathbf{g}(\mathbf{x})]_k=-(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=0}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(\mathbf{x})},$$

$$\sum_{k=0}^{3}x_k\frac{\partial}{\partial S_k(\mathbf{x})}=0,\quad \frac{\partial}{\partial S_k(\mathbf{x})}\frac{\partial}{\partial x_k}=\frac{\partial}{\partial x_k}\frac{\partial}{\partial S_k(\mathbf{x})}, \tag{18}$$

$$\sum_{k=0}^{3}\frac{\partial^2}{\partial S_k^2(\mathbf{x})}=\frac{\partial^2}{\partial\vartheta^2}+ctg\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2},\quad \frac{\partial x_k}{\partial S_k}=0,$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}\frac{\partial}{\partial x_k}=0,\quad \frac{\partial g(\rho)Y(\vartheta,\varphi)}{\partial S_k(\mathbf{x})}=g(\rho)\frac{\partial Y(\vartheta,\varphi)}{\partial S_k(\mathbf{x})}.$$

Let

$$(\mathbf{z}\cdot\mathbf{F}^-)=h_1^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}[\mathbf{z}\cdot\mathbf{F}^-]_k=h_2^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}F_k^-=h_3^-(\mathbf{z}),$$

$$(\mathbf{z}\cdot\mathbf{f}^-)=h_4^-(\mathbf{z}),\qquad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}[\mathbf{z}\cdot\mathbf{f}^-]_k=h_5^-(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{z})}f_k^-=h_6^-(\mathbf{z}),\quad f_7^-=h_7^-(\mathbf{z}).$$

Let us assume that f_k. $k=1,..,7$ are sufficiently smooth(differentiable) functions. Let us expand the functions h_k in spherical harmonics

$$h_k^-(\mathbf{z})=\sum_{m=0}^{\infty}h_{km}^-(\vartheta,\varphi),$$

where h_{km}^- is the spherical harmonic of order m :

$$h_{km}^-=\frac{2m+1}{4\pi R_1^2}\int_S P_m(\cos\gamma)h_k^-(\mathbf{y})dS_y,$$

P_m is Legendre polynomial of the m-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma=\frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{m=1}^{3}x_k y_k.$$

From these formulas it follows that if g_m is the spherical harmonic the operator $\frac{\partial}{\partial S_k}$, $k=1,2,3$, does not affect the order of the spherical function:

$$\sum_{k=0}^{3}\frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(\mathbf{x})}=-m(m+1)g_m(\mathbf{x}).$$

The general solutions of the equations $(\Delta+\lambda_k^2)\psi=0,\quad k=1,2,3,4,5,$ in the domain D^- have the form [13]

$$\psi(x)=\sum_{m=0}^{\infty}\Psi_m^{(1)}(\lambda_k\rho)Y_m(\vartheta,\varphi),\quad \rho>R_1, \tag{19}$$

where

$$\Psi_m^{(1)}(\lambda_k\rho)=\frac{\sqrt{R_1}H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac{1}{2}}(\lambda_kR_1)}.$$

5. The Dirichlet BVP for an infinite space with the spherical cavity

The solution of the Dirichlet BVP problem

$$\mathbf{u}^-=\mathbf{F}^-(f_1,f_2,f_3),\quad \mathbf{w}^-=\mathbf{f}^-(f_4,f_5,f_6),\quad \theta^-=f_7^-$$

in the domain D^- is sought in the form (16).

From (16) we get

$$(\mathbf{x}\cdot\mathbf{u})=\sum_{k=1}^{3}a_k\rho\frac{\partial\varphi_k}{\partial\rho}+c_1\sum_{k=1}^{3}\frac{\partial^2\psi_3}{\partial S_k^2(\mathbf{x})},$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{u}]_k=a_1\sum_{k=1}^{3}\frac{\partial^2\varphi_1}{\partial S_k^2(\mathbf{x})}+a_2\sum_{k=1}^{3}\frac{\partial^2\varphi_2}{\partial S_k^2(\mathbf{x})}$$

$$+a_3\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(\mathbf{x})}-c_1(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial^2\psi_3}{\partial S_k^2(\mathbf{x})},$$

$$\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{x})}=\sum_{k=1}^{3}\frac{\partial^2\psi_4}{\partial S_k^2(\mathbf{x})},\quad (\mathbf{x}\cdot\mathbf{w})=\sum_{k=1}^{3}b_k\rho\frac{\partial\varphi_k}{\partial\rho}+c_2\sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(\mathbf{x})}, \tag{20}$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{w}]_k=b_1\sum_{k=1}^{3}\frac{\partial^2\varphi_1}{\partial S_k^2(\mathbf{x})}+b_2\sum_{k=1}^{3}\frac{\partial^2\varphi_2}{\partial S_k^2(\mathbf{x})}$$

$$+b_3\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(\mathbf{x})}-c_2(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(\mathbf{x})},$$

$$\sum_{k=1}^{3}\frac{\partial w_k}{\partial S_k(\mathbf{x})}=\sum_{k=1}^{3}\frac{\partial^2\varphi_5}{\partial S_k^2(\mathbf{x})},\quad \theta=\sum_{k=1}^{3}\varphi_k,\quad c_1=\frac{1}{\lambda_4^2},\quad c_2=\frac{1}{\lambda_5^2}.$$

Suppose the functions $\varphi_m(\mathbf{x}),\quad m=1,2,3,4,5,\quad$ *and* $\quad\psi_j,\quad j=3,4,$ are sought in the form

$$\varphi_k(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_k\rho) Y_{km}(\vartheta,\varphi), \quad k=1,2,3,$$

$$\varphi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_5\rho) Y_{jm}(\vartheta,\varphi), \quad j=4,5, \tag{21}$$

$$\psi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \Psi_m^{(1)}(\lambda_4\rho) Z_{jm}(\vartheta,\varphi), \quad j=3,4, \quad \rho > R_1,$$

where Y_{km}, *and* Z_{jm} are the unknown spherical harmonic of order m,

$$\Psi_m^{(1)}(\lambda_k\rho) = \frac{\sqrt{R_1} H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho} H^{(1)}_{m+\frac{1}{2}}(\lambda_k R_1)}.$$

Remark. The conditions $\int\limits_{S(0,a)} \psi_j ds = 0$, $\quad j=3,4$, $\int\limits_{S(0,a)} \varphi_j ds = 0$, $\quad j=4,5$ in fact mean that

$$Y_{40} = Y_{50} = Z_{30} = Z_{40} = 0.$$

Substituting the expressions of $\varphi_m(x)$, $\quad m=1,2,3,4,5$ and $\psi_j(x)$, $\quad j=3,4$ in (20), we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = \sum_{k=1}^{3}\sum_{m=0}^{\infty} a_k\rho\frac{\partial}{\partial\rho}\Psi_m^{(1)}(\lambda_k\rho)Y_{km} - c_1\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_4\rho)Z_{3m},$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{u}]_k =$$

$$\sum_{m=0}^{\infty} m(m+1)\left\{-\sum_{k=1}^{3} a_k\Psi_m^{(1)}(\lambda_k\rho)Y_{km} + c_1(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_4\rho)Z_{3m},\right\},$$

$$\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{x})} = -\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_4\rho)Z_{4m}, \tag{22}$$

$$(\mathbf{x}\cdot\mathbf{w}) = \sum_{k=1}^{3}\sum_{m=0}^{\infty} b_k\rho\frac{\partial}{\partial\rho}\Psi_m^{(1)}(\lambda_k\rho)Y_{km} - c_2\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_5\rho)Y_{4m},$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathbf{w}]_k =$$
$$\sum_{m=0}^{\infty} m(m+1)\left\{-\sum_{k=1}^{3} b_k\Psi_m^{(1)}(\lambda_k\rho)Y_{km} + c_2(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(1)}(\lambda_5\rho)Y_{4m},\right\},$$

$$\sum_{k=1}^{3}\frac{\partial w_k}{\partial S_k(\mathbf{x})} = -\sum_{m=0}^{\infty} m(m+1)\Psi_m^{(1)}(\lambda_5\rho)Y_{5m}, \quad \theta = \sum_{k=1}^{3}\sum_{m=0}^{\infty}\Psi_m^{(1)}(\lambda_k\rho)Y_{km}(\vartheta,\varphi).$$

Passing to the limit as $\rho \to R_1$ and taking into account boundary conditions for the determination of Y_{mj} and Z_{mj} we obtain the system of algebraic equations

$$\sum_{k=1}^{3} a_k \left[\rho \frac{\partial}{\partial \rho} \Psi_m^{(1)}(\lambda_k \rho)\right]_{\rho=R_1} Y_{km} - c_1 m(m+1) Z_{3m} = h_{1m}^-,$$

$$m(m+1)\left\{-\sum_{k=1}^{3} a_k Y_{km} + c_1 \left[(\rho \frac{\partial}{\partial \rho} + 1)\Psi_m^{(1)}(\lambda_4 \rho)\right]_{\rho=R_1} Z_{3m}\right\} = h_{2m}^-,$$

$$-m(m+1) Z_{4m} = h_{3m}^-,$$

$$\sum_{k=1}^{3} b_k \left[\rho \frac{\partial}{\partial \rho} \Psi_m^{(1)}(\lambda_k \rho)\right]_{\rho=R_1} Y_{km} - c_2 m(m+1) Y_{4m} = h_{4m}^-,$$

$$m(m+1)\left\{-\sum_{k=1}^{3} b_k Y_{km} + c_2 \left[(\rho \frac{\partial}{\partial \rho} + 1)\Psi_m^{(1)}(\lambda_5 \rho)\right]_{\rho=R_1} Y_{4m},\right\} = h_{5m}^-,$$

$$-m(m+1) Y_{5m} = h_{6m}^-, \quad Z_{40} = Y_{40} = Z_{30} = Y_{50} = 0,$$

$$Y_{1m} + Y_{2m} + Y_{3m} = h_{7m}^-, \quad h_{30}^- = h_{60}^- = h_{20}^- = h_{50}^- = 0. \tag{23}$$

By virtue of Theorem 1 we conclude that the system (23) for $m \geq 0$ is uniquely solvable and the functions Y_{jm} and Z_{jm} are possible to express by the known functions h_{jm}^-.

If we take into account the sufficient conditions of convergence of absolutely and uniformly convergent series with respect to the spherical harmonic and the property of functions $\Psi_m^{(1)}(\lambda_k \rho)$ we conclude that the obtained solutions are represented as absolutely and uniformly convergent series.

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ON THE HEXAGONAL QUANTUM BILLIARD

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Abstract. In the paper a planar classical quantum billiard in the hexagonal type areas with the hard wall conditions is considered. The process is described by the Helmholtz Equation in the hexagon and hexagonal rug with the homogeneous boundary conditions. By means of the conformal mapping method the problem is reduced to the elliptic partial differential equation in the rectangle with the homogeneous boundary condition. It is assumed that one parameter of mapping is sufficiently small. In this case the equation is simplified and analyzed. The asymptotic solutions are obtained. The spectrum and the corresponding eigenfunctions are found near the boundary of the hexagon. The wave functions are found in terms of the Bessel's functions. The results are applied for the estimation of the energy levels of electrons in graphene.

Keywords and phrases: Quantum chaos, Helmholtz Equation, Bessel's functions, graphene.

AMS subject classification (2010): 39A14, 35M11, 35Q40, 32H04.

Introduction

Quantum Billiard is a dynamical system, which describes a motion of a free particle inside a closed domain D with a piece-wise smooth boundary S [2, 3, 7-11, 13-17, 19-22]. In this case the Schrödinger Equation for a free particle assumes the form of the Helmholtz Equation and the spectrum of the Helmholtz Equation reflects the energy levels of the particle.

In the paper the following equation with the homogeneous boundary condition, when D is the hexagon, is considered

$$\Delta u(x,y) + \frac{2m}{h^2} E u(x,y) = 0 \qquad u|_S = 0, \tag{1*}$$

where S is a boundary of D, u is the wave function of the particle, $\lambda^2 = \dfrac{2m}{h^2}E$ is the constant to be determined, E is the energy of the particle, m is mass, h is Planck's constant.

In some cases it is more convenient to replace the condition $u|_S = 0$ by the condition [2, 14, 17,19, 20, 22]

$$\iint\limits_D |u|^2 dxdy = 1.$$

The hexagonal type areas are very important, as the atoms of Carbon and its allotropes are arranged in the hexagonal type structures [4, 7, 17, 19, 20] and has a lot of applications in microeletronics. For example, graphene is a one-atom thick sheet of carbon atoms which form a hexagonal structure ([4], see "One atom thick billiard" https//sites.google.com/a/ucr.edu/physics-lau/) and electrons in such structures behave like quantum billiard balls [4, 7, 17, 21].

The problem is investigated by means of the conformal mapping and partial differential equation. The Helmholtz Equation (1*) is transformed to the equation of the elliptic type. One parameter of the mapping is chosen sufficiently small, the initial equation is simplified and replaced by the approximate elliptic equation.The wave function and eigenvalues of this equation are found.

Statement of the problem

Let D be the hexagon of the plane $z_0 = x_0+iy_0$, with the vertexes $a_1, a_2, a_3, a_4, a_5, a_6$ ($a_1 = 0,\ Re\, a_4 = 0$), and with the axis of symmetry a_1a_4 (Fig.1). In this area we consider the following problem

Problem 1. To find a real function $u(x_0, y_0)$ in D having second order derivatives, satisfying the equation

$$\Delta u(x_0, y_0) + \lambda^2 u(x_0, y_0) = 0 \tag{1}$$

and the boundary condition

$$u|_S = 0, \tag{2}$$

where λ is the constant to be determined, S is the boundary of D.

By means of the conformal mapping we reduce Problem 1 to the elliptic partial differential equation in the rectangle.

At first we map the area D at the upper half-plane of the complex plane $z = x+iy$, by the Schwartz-Christoffel formula [1, 6, 15, 17] with the following correspondence of points

$$a_1 \leftrightarrow 0, a_2 \leftrightarrow a, a_3 \leftrightarrow b, a_4 \leftrightarrow \infty, a_5 \leftrightarrow -a, a_6 \leftrightarrow -b; a, b > 0;$$

$$f(z) = z_0 = C \int\limits_0^z t^{-1/3}(t^2 - a^2)^{-1/3}(t^2 - b^2)^{-1/3} dt, \tag{3}$$

where C is the definite constant, which is determined from the formula

$$a_3 - a_2 = C \int\limits_a^b t^{-1/3}(t^2 - a^2)^{-1/3}(t^2 - b^2)^{-1/3} dt.$$

Let $z = f(w)$ be the conformal mapping of the rectangle $D_0\{-a_0/2 \le u \le a_0/2;\ 0 \le \nu \le b_0\}$ with the boundary S_0 of the plane $w(w = \xi + i\eta)$, on the upper half-plane of z. This mapping will be given by [1, 6, 15, 17]

$$z = sn\left(\frac{w}{C_0}\right), \tag{4}$$

or

$$w = C_0 \int\limits_0^z (1 - t^2)^{-1/2}(1 - k^2t^2)^{-1/2} dt,$$

with the following correspondence of points

$$0 \leftrightarrow 0, a \leftrightarrow a_0/2, b \leftrightarrow a_0/2 + ib_0, \infty \leftrightarrow ib_0, -a \leftrightarrow -a_0/2 + ib_0, -b \leftrightarrow -a_0/2;\ a_0, b_0 > 0$$

(Fig. 2), where sn is the Jakobi "sinus" with the modulus k, having the periods $2a_0$ and $2b_0, C_0$ is the definite constant which is defined from the tables [15, 18], a_0 will be chosen accordingly in the following.

By the mappings (3), (4) Problem 1 could be reduced to the following problem

Problem 2. To find a real function $u_0(\xi, \eta)$ in D_0 having second order derivatives, satisfying the following equation

$$\Delta u_0(\xi, \eta) + \lambda^2 |f'(w)|^2 u_0(\xi, \eta) = 0, \tag{5}$$

with the boundary condition

$$u_0|_{S_0} = 0,$$

where $u_0(\xi, \eta) = u(f(w))$, and λ is the constant to be determined.

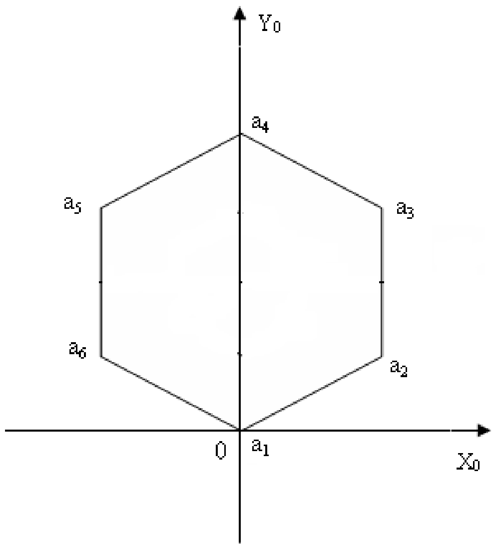

Fig. 1. The hexagonal area

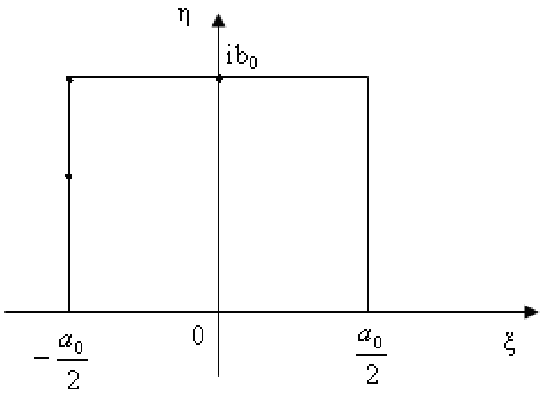

Fig. 2. The image of the hexagon by the mapping $z = f(w)$

Solution of Problem 2

It is obvious that

$$|f'_w(w)|^2 = |f'_z(w)|^2 \cdot |z'_w(w)|^2. \tag{6}$$

If we suppose $a = 1,\ b = 1/k$, from (3), (4), (6) after simple transformations we obtain

$$f'_w(w)^2 = C_1^2 \left(\frac{cn\dfrac{w}{C_0} dn\dfrac{w}{C_0}}{sn\dfrac{w}{C_0}} \right)^{2/3}. \tag{7}$$

where $C_1 = k^{2/3}\frac{C}{C_0}$ and sn, cn, dn are the Jacobi functions [1, 5, 6, 15].

As three parameters of the conformal mapping can be chosen arbitrarily, we can assume that $q = e^{-\pi\chi}$, $(\chi = \frac{2b_0}{a_0})$, is sufficiently small and we can use formulas [5, 6, 15]

$$\begin{aligned} sn(w/C_0) &\approx \sin\gamma(1 + 4q\cos^2\gamma), \\ cn(w/C_0) &\approx \cos\gamma(1 - 4q\sin^2\gamma), \\ dn(w/C_0) &\approx (1 - 8q\sin^2\gamma), \end{aligned} \tag{8}$$

where $\gamma = \frac{\pi w}{a_0 C_0}$. Without loss of generality we can also suppose $q \approx 0$ [1, 5, 6, 15], then the formulas (8) could be simplified and one obtains, (a_0 will be chosen in the following way)

$$\begin{aligned} sn(w/C_0) &\approx \sin\gamma, \\ cn(w/C_0) &\approx \cos\gamma, \\ dn(w/C_0) &\approx 1, \\ k \approx 0,0213,\ b_0 &= \frac{5a_0}{3},\ C_0 \approx \frac{a_0}{3}. \end{aligned} \tag{9}$$

Putting (9) into (7) we can write the approximate formula

$$|f'_w(w)|^2 \approx |C_1|^2 \left(\frac{1+V}{1-V}\right)^{2/3}, \tag{10}$$

where

$$V = \frac{\cos(2\pi\xi/a_0c_0)}{\cosh(2\pi\eta/a_0c_0)},$$

By using (10) Equation (5) may be rewritten as

$$\Delta u_0(\xi,\eta) + \lambda^2|C_1|^2 \left(\frac{1+V}{1-V}\right)^{2/3} u_0(\xi,\eta) = 0. \tag{11}$$

Hence, we obtain the degenerated elliptic equation.

Now, let us choose a_0 in such a way, that $\left(\frac{6\pi\xi}{a_0^2}\right)^4$ and $\left(\frac{6\pi\eta}{a_0^2}\right)^4$ are negligible. Taking into account (9) and

$$\cos\left(\frac{2\pi\xi}{a_0c_0}\right)^2 \approx 1-\frac{1}{2}\left(\frac{6\pi\xi}{a_0^2}\right)^2, \quad cosh\left(\frac{2\pi\eta}{a_0c_0}\right)^2 \approx 1+\frac{1}{2}\left(\frac{6\pi\eta}{a_0^2}\right)^2,$$

from (11) we obtain

$$\frac{\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}}{\left(1+9\frac{\pi^2}{a_0^4}\eta^2-9\frac{\pi^2}{a_0^4}\xi^2\right)^{2/3}}\Delta u_0(\xi,\eta)+\lambda^2|C_1|^2u_0(\xi,\eta)=0. \tag{12}$$

By using the approximate formula

$$\left(1+9\frac{\pi^2}{a_0^4}\eta^2-9\frac{\pi^2}{a_0^4}\xi^2\right)^{-2/3} \approx \left(1-\left(6\frac{\pi^2}{a_0^4}\eta^2-6\frac{\pi^2}{a_0^4}\xi^2\right)+\frac{5}{9}\left(9\frac{\pi^2}{a_0^4}\eta^2-9\frac{\pi^2}{a_0^4}\xi^2\right)^2\right)$$

and neglecting the terms

$$6\frac{\pi^2}{a_0^4}\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2),\ \ 45\left(\frac{\pi^2}{a_0^4}\right)^2\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)^2,$$

from (12) one obtains the approximate equation

$$\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}\Delta u_0(\xi,\eta)+\lambda^2|C_1|^2u_0(\xi,\eta)=0 \tag{13}$$

In our case we have the following estimations

$$\left(\frac{6\pi\xi}{a_0^2}\right)^4 \le \left(\frac{3\pi}{a_0}\right)^4,\ \left(\frac{6\pi\eta}{a_0^2}\right)^4 \le \left(\frac{10\pi}{a_0}\right)^4,$$

$$\Big|6\frac{\pi^2}{a_0^4}\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)Big| \le 150\left(\frac{109}{108}\right)^{2/3}\left(\frac{\pi}{a_0}\right)^{10/3}, \tag{14}$$

$$45\left(\frac{\pi^2}{a_0^4}\right)^2\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)^2 \le 5^5\left(\frac{109}{108}\right)^{2/3}\left(\frac{\pi}{a_0}\right)^{16/3},$$

$$\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3} \le 109^{2/3}\left(\frac{\pi}{2a_0}\right)^{4/3}.$$

For example, if $a_0 = 10^3$, then by (14)

$$\left(\frac{6\pi\xi}{a_0^2}\right)^4 \le 7.9\times10^{-9},\ \left(\frac{6\pi\eta}{a_0^2}\right)^4 \le 9.7\times10^{-7},$$

$$\left|6\frac{\pi^2}{a_0^4}\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)\right|\leq 6.8\times 10^{-7},$$

$$45\left(\frac{\pi^2}{a_0^4}\right)^2\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)^2\leq 1.4\times 10^{-10},$$

$$\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}\leq 4.2\times 10^{-3}.$$

If $a_0=10^4$,then by (14)

$$\left(\frac{6\pi\xi}{a_0^2}\right)^4\leq 7.9\times 10^{-13},\quad \left(\frac{6\pi\xi}{a_0^2}\right)^4\leq 9.7\times 10^{-11},$$

$$\left|6\frac{\pi^2}{a_0^4}\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)\right|\leq 3.2\times 10^{-10},$$

$$45\left(\frac{\pi^2}{a_0^4}\right)^2\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)^2\leq 6.5\times 10^{-16},$$

$$\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}\leq 2\times 10^{-4},$$

If $a_0=10^5$, then

$$\left(\frac{6\pi\xi}{a_0^2}\right)^4\leq 7.9\times 10^{-17},\quad \left(\frac{6\pi\xi}{a_0^2}\right)^4\leq 9.7\times 10^{-15},$$

$$\left|6\frac{\pi^2}{a_0^4}\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)\right|\leq 1.5\times 10^{-13},$$

$$45\left(\frac{\pi^2}{a_0^4}\right)^2\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}(\eta^2-\xi^2)^2\leq 3\times 10^{-21},$$

$$\left(\frac{9\pi^2}{a_0^4}\right)^{2/3}(\eta^2+\xi^2)^{2/3}\leq 9\times 10^{-6}.$$

In the polar coordinates $\xi=r\cos\varphi,\eta=r\sin\varphi$ equation (13) becomes

$$\Delta u_0(r,\varphi)+\frac{1}{r}\frac{\partial u}{\partial r}+\lambda^2|C_1|^2\left(\frac{a_0^4}{9\pi^2}\right)^{2/3}r^{-4/3}u_0(r,\varphi)=0. \tag{15}$$

By the separation of variables $u_0=u_1(r)u_2(\varphi)$ from (15) we obtain

$$\frac{u_1''}{u_1}+\frac{1}{r}\frac{u_1'}{u_1}+\lambda_0^2r^{-4/3}=\beta, \tag{16}$$

$$u_2''+\beta u_2=0,$$

where $\beta\geq 0$ is some constant and

$$\lambda_0^2 = \lambda^2 |C_1|^2 \left(\frac{a_0^4}{9\pi^2} \right)^{2/3}.$$

Suppose ,$\varphi \le \varepsilon_0$, $\varepsilon_0^4 \approx 0$, then for $\beta = 0$, $u_2 = A\varphi$ where A is some constant, which will be calculated from the condition

$$\int_0^{\varepsilon_0} \int_0^{a_0/2} r\, |u|^2\, d\varphi dr = 1. \tag{17}$$

We can rewrite the first equation of (16) in the form

$$u_1'' + \frac{1}{r} u_1' + \lambda_0^2 r^{-4/3} = 0. \tag{18}$$

By the notation $r^{1/3} = t$, equation (18) becomes

$$u_1'' + t^{-1} u_1' + 9\lambda_0^2 u_1 = 0.$$

The solution of this equation is $u_1(t) = I_0(3\lambda_0 t)$ and hence the solution of (18) will be [5, 15]

$$u_1(r) = I_0(3\lambda_0 r^{1/3}), \tag{19}$$

where I_0 is Bessel's function.

Consequently, we can calculate the spectrum of the equation (18)by the boundary condition $I_0(3\lambda_0(\frac{a_0}{2})^{1/3}) = 0$.

By using Maple and formulas (9) one obtains

$$\Big| \int_a^{k^{-1}} t^{-1/3}(t^2 - a^2)^{-1/3}(t^2 - b^2)^{-1/3} dt \Big| = 0.342848,$$

$$|C| = |a_3 - a_2|/0.342848, \quad |C_1| = k^{2/3}\frac{|C|}{C_0} \approx 2^{2/3}10^{-1/3}\frac{|a_3 - a_2|}{a_0}, \tag{20}$$

$$\lambda_n^2 = \frac{\lambda_0^2}{|C_1|}\left(\frac{3\pi}{a_0^2}\right)^{4/3} = (10\pi^2)^{2/3}\frac{c_n^2}{6^{2/3}}\frac{a_0^{-4/3}}{|a_3 - a_2|^2}, \quad n = 1, 2, 3, \ldots$$

where c_n are zeros of Bessel's function I_0[15]

$$c_n \approx \frac{3\pi}{4} + n\pi,$$

$$c_1 \approx 2.4, \; c_2 \approx 5.5, \;\; c_3 \approx 8.7, \;\; c_4 \approx 11.7, \;\; c_5 \approx 14.9, \ldots$$

The constant A will be calculated from the formula (17)

$$\int_0^{\varepsilon_0} \int_0^{a_0/2} r\, |u|^2\, d\varphi dr = A^2\, \varepsilon_0^3/3 \int_0^{a_0/2} r |I_0^2(3\lambda_0 r^{1/3})|^2 dr = 1.$$

Note 1. As we have symmetry, then in the area$D_{b_0-\varepsilon_0} = \{-a_0/2 \le \xi \le a_0/2, b_0 - \varepsilon_0 \le \eta \le b_0\}$the solutions of the Problem 2 will be the similar to the solutions of equation (18).

Now, let us consider (11) in the area D_ε near the line $\xi = 0$ with the conditions

$$\left(\frac{6\pi\xi}{a_0^2}\right)^2 \approx 0, \qquad \iint\limits_{D_\varepsilon} |u|^2 d\xi d\eta = 1, \tag{21}$$

where $D_\varepsilon = \{-\varepsilon \le \xi \le \varepsilon; \quad 0 \le \eta \le b_0\}$, ε is sufficiently small. For example, if $\varepsilon = 10^{-4},\ a_0 = 10^{-3}$, then $\left(\frac{6\pi\xi}{a_0^2}\right)^2 \le 4.10^{-18}$.

By the conditions (21), (11) takes the form

$$th^{4/3}\left(\frac{3\pi\eta}{a_0^2}\right)\Delta u_0(\xi,\eta) + \lambda^2|C_1|^2 u_0(\xi,\eta) = 0. \tag{22}$$

In (22) we can suppose $th^2\left(\frac{3\pi\eta}{a_0^2}\right) \approx \left(\frac{3\pi\eta}{a_0^2}\right)^2$, then the equation (22) may be rewritten as

$$\Delta u_0(\xi,\eta) + \lambda^2|C_1|^2\left(\frac{a_0^2}{3\pi}\right)^{4/3}\eta^{-4/3}u_0(\xi,\eta) = 0. \tag{23}$$

By the separation of variables $u_0(\xi,\eta) = u_1(\xi)u_2(\eta)$ from (23) we obtain

$$\Delta u_1(\xi) + \beta u_1(\xi) = 0, \ \ \beta \ge 0, \tag{24}$$

$$\Delta u_2(\eta) + (\lambda_0^2\eta^{-4/3} - \beta)u_2(\eta) = 0, \tag{25}$$

where

$$\lambda_0^2 = \lambda^2|C_1|^2\left(\frac{a_0^2}{3\pi}\right)^{4/3}. \tag{26}$$

Here we suppose $\beta = 0$, hence (24) gives $u_1 = B(a_0/2 - \xi)$ (B is constant, which will be determined from condition (21)). The solution of (25) will be represented in terms of Bessel's function $I_{3/2}$ [5,15]

$$u_1(\eta) = \sqrt{\eta}I_{3/2}(3\lambda_0\eta^{1/3}), \tag{27}$$

where

$$I_{3/2}(3\lambda_0\eta^{1/3}) = \sqrt{\frac{2}{\pi}}(3\lambda_0)^{-3/2}\eta^{-1/2}\sin(3\lambda_0\eta^{1/3}) - \sqrt{\frac{2}{\pi}}(3\lambda_0)^{-1/2}\eta^{-1/6}\cos(3\lambda_0\eta^{1/3}). \tag{28}$$

(27)and (28) gives

$$u_1(\eta) = \sqrt{\frac{2}{\pi}}(3\lambda_0)^{-3/2}\left[\sin(3\lambda_0\eta^{1/3}) - 3\lambda_0\eta^{1/3}\cos(3\lambda_0\eta^{1/3})\right]$$

The eigenvalues of Problem 2 will be found from the boundary condition

$$\sin(3\lambda_0(b_0)^{1/3}) - 3\lambda_0(b_0)^{1/3}\cos(3\lambda_0(b_0)^{1/3}) = 0,$$

where $b_0 = \frac{5a_0}{3}$.

Consequently, $3\lambda_0(\frac{5a_0}{3})^{1/3}$ will be zeros of Bessel's function $I_{_{3/2}}(3\lambda_0\eta^{1/3})$ and the spectrum of (25)could be determined by using Maple and formulas (20), (26),

$$3\lambda_0(\frac{5a_0}{3})^{1/3} = d_n,$$

$$\lambda_n^2 = \frac{\lambda_0^2}{|C_1|}\left(\frac{3\pi}{a_0^2}\right)^{4/3} = (10\pi^2)^{2/3}\frac{d_n^2}{20^{2/3}}\frac{a_0^{-4/3}}{|a_3 - a_2|^2}, \quad n = 1, 2, 3, \dots \tag{29}$$

where d_n are zeros of Bessel's function $I_{_{3/2}}$[15]

$$d_n \approx \frac{3\pi}{2} + n\pi$$

$$d_1 \approx 4.4934, \ d_2 \approx 7.7252, \ d_3 \approx 10.9041, \ d_4 \approx 14.0662, \ d_5 \approx 17.2208\dots$$

The constant B will be calculated from the formula (21)

$$\iint\limits_{D_\varepsilon} |u|^2 d\xi d\eta = B^2\frac{a_0^2\varepsilon}{2}\int\limits_0^{b_0} \eta\left[I_{3/2}(3\lambda_0\eta^{1/3})\right]^2 d\eta = 1. \tag{30}$$

Note 2. The functions I_0 and $I_{3/2}$ have the following asymptotics [5,15]

$$I_\nu(3\lambda_0 r^{1/3}) \approx \sqrt{\frac{2}{3\pi\lambda_0 r^{1/3}}}\cos\left(3\lambda_0 r^{1/3} - \nu\frac{\pi}{2} - \frac{\pi}{4}\right), \quad \nu = 0, 3/2.$$

According to (13), (15), (19), (20), (23),(27),(29) we conclude.

Conclusion

1. Near the boundary $\eta = 0$ and $\eta = b_0$ the solutions of the Problem 2 are given by

$$u_{n_1}(\xi,\eta) = A_{n_1}\mathrm{arctg}\frac{\eta}{\xi}I_0(3\lambda_0(\eta^2+\xi^2)^{1/3}), \tag{31}$$

where

$$\begin{aligned} \lambda_0^2 &= \lambda_{n_1}^2|C_1|^2\left(\frac{a_0^4}{9\pi^2}\right)^{2/3}, \quad |C_1| \approx 2^{2/3}10^{-1/3}\frac{|a_3 - a_2|}{a_0}, \\ \lambda_{n_1}^2 &= (10\pi^2)^{4/3}\frac{c_{n_1}^2}{6^{2/3}}\frac{a_0^{-4/3}}{|a_3 - a_2|^2}, \quad n_1 = 1, 2, 3, \dots, \end{aligned} \tag{32}$$

λ_{n_1} is the spectrum of Problem 1 and c_{n_1} are zeros of Bessel's function I_0, A_{n_1} are the definite constants

$$A_{n_1}^2 = (3/\varepsilon_0^3)\left(\int\limits_0^{a_0/2} rI_0^2(3\lambda_0 r^{1/3})dr\right)^{-1}. \tag{33}$$

2. Near the line $\xi = 0$ the solutions of Problem 2 will be given by

$$u_{n_2}(\xi,\eta) = B_{n_2(}a_0/2\xi)\sqrt{\eta}I_{3/2}(3\lambda_0\eta^{1/3}), \tag{34}$$

where

$$\begin{aligned} \lambda_0^2 &= \lambda_{n_1}^2|C_1|^2\left(\frac{a_0^4}{9\pi^2}\right)^{2/3}, \quad |C_1| \approx 2^{2/3}10^{-1/3}\frac{|a_3-a_2|}{a_0}, \\ \lambda_{n_2}^2 &= (10\pi^2)^{4/3}\frac{d_{n_2}^2}{20^{2/3}}\frac{a_0^{-4/3}}{|a_3-a_2|^2}, \quad n_2 = 1,2,3,\ldots, \end{aligned} \tag{35}$$

where λ_{n_2} is the spectrum of Problem 1, d_{n_2}, $n_2 = 1,2,3,\ldots$, are zeros of Bassel's function $I_{3/2}$,B_{n_2} are the definite constants

$$B_{n_2}^2 = \left(\frac{2}{a_0^2\varepsilon}\right)\left(\int_0^{b_0}\eta\left[I_{3/2}(3\lambda_0\eta^{1/3})\right]^2 d\eta\right)^{-1}. \tag{36}$$

The energy of the particle will be calculated from the formulas [2,14,16]

$$\begin{aligned} E_{n_1} &= \lambda_{n_1}^2\frac{h^2}{2m} = \frac{4.5\times10^2}{3}(10\pi^2)^{4/3}\frac{c_{n_1}^2}{6^{2/3}}\frac{a_0^{-4/3}}{|a_3-a_2|^2}\times10^{-20}, \quad n_1 = 1,2,3,\ldots, \\ E_{n_2} &= \lambda_{n_2}^2\frac{h^2}{2m} = \frac{4.5\times10^2}{3}(10\pi^2)^{4/3}\frac{d_{n_2}^2}{20^{2/3}}\frac{a_0^{-4/3}}{|a_3-a_2|^2}\times10^{-20}, \quad n_2 = 1,2,3,\ldots, \end{aligned} \tag{37}$$

Below, on Table 1 the numerical results are given for $|a_3-a_2| = 10^{-10}$ by using Maple

$a_0 = 10^4$	ε	λ_0^2		$\|E\|$(eV)
$c_1 = 2.4$	10^{-3}	0.046745	$A \approx \sqrt{6}\times10^{-1}$	0.553961
$d_1 = 4.49$	10^{-6}	0.073319	$B \approx 2\times10^{-8}$	0.8688876

Table 1.

Note 1. As $f(w)$ is a holomorphic function, we can continue it through the sides a_2a_3 and a_6a_5. Hence, we obtain the quantum billiard in the hexagonal rug (Fig.3). Consequently, for this problem equation (5) will be valid. So, the solutions will be the same as for the hexagon and given by formulas (31),(32), (33),(34),(35),(36), (37). The boundary conditions will depend on the number of cells in the rug.

Also, we can continue $f(w)$ through the sides a_3a_4,a_4a_5 and a_6a_1, a_1a_2. So we obtain billiard in the hexagonal flower (Fig. 4), where energy levels of particles will be calculated by formula (37).

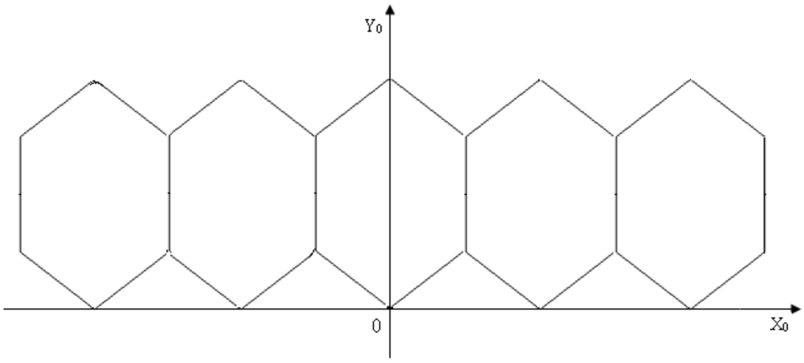

Fig. 3. The hexagonal rug

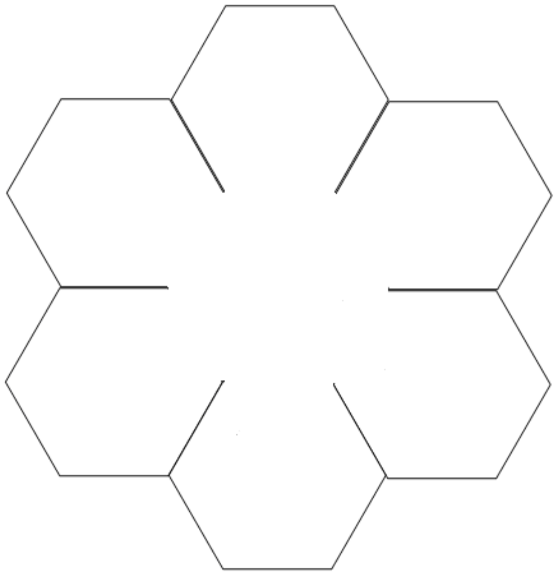

Fig. 4. Hexagonal flower

Note 2. Let us consider a half of the hexagon $D' = a_1a_2a_3a_4$ (Fig.1). For this area we can consider the following problem

Problem 3. To find a real function $u(x_0, y_0)$ in D' having second order derivatives, satisfying the equation

$$\Delta u(x_0, y_0) + \lambda^2 u(x_0, y_0) = 0,$$

and the boundary conditions

$$u\Big|_{a_1a_4} = 0, \quad u\Big|_{a_2a_3} = 0,$$

where λ is the constant to be determined.

The function $f(w)$ map the area D' at the rectangle D'_0 with the vertexes $(0,0)$, $(a_0/2, 0)$, $(a_0/2, b_0), (0, b_0)$. We can continue $f(w)$ through the sides a_1a_2 and a_3a_4 (step by step)and obtain the mapping of the hexagon with the hexagonal hall at the rectangle $D'_0 = \{0 \leq \xi \leq a_0/2; 0 \leq \eta \leq 6b_0\}$ (Fig. 5). So we can consider the billiard in the hexagon with the hexagonal hall. In this cases equation (11) will be valid. for the area $D'_\varepsilon = \{0 \leq \xi \leq a_0/2; 0 \leq \eta \leq \varepsilon\}$the equation (11) may be rewritten as

$$\Delta u_0(\xi, \eta) + \lambda^2 |C_1|^2 \left(\frac{a_0^2}{3\pi}\right)^{4/3} \xi^{-4/3} u_0(\xi, \eta) = 0,$$

This equation can be solved in analogy with (23) with the boundary condition

$$I_{3/2}(3\lambda_0(a_0/2)^{1/3}) = 0.$$

Near the line $\eta = 0$ we obtain the following solutions

$$u_{n_2}(\xi,\eta) = B_{n_2}\sqrt{\xi}I_{3/2}(3\lambda_0\xi^{1/3}), \quad n_2 = 1,2,3,\ldots,$$

where λ_0 and B_{n_2} are given by (35) and (36).

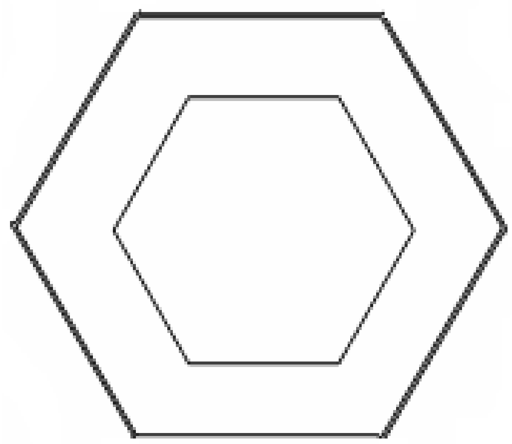

Fig. 5. Hexagon with the hexagonal hall

Note 3. By using the solutions of Problem 2 it is easy to obtain the solutions of the same problem for the particle trapped in 3D potential box of the hexagonal configuration $D \times \{0 \leq \zeta \leq c_0\}$. This problem can be solved in analogy with Problem 2 and the solutions will be given by

$$U = \sqrt{\frac{2}{c_0}}u_n(\xi,\eta)sin\frac{\pi n_1}{c_0}, \quad n, n_1 = 1,2,3,\ldots,$$

where $u_n(\xi,\eta)$ are given by (31),(32) or (34),(35) and corresponding energy eigenvalues are given by

$$E_n = \lambda_n^2\frac{h^2}{2m}\frac{n_1^2}{c_0^2}, \quad n, n_1 = 1,2,3,\ldots$$

Note 4. Problem 1 could also be applied for the description of the growth of the single crystal of hexagonal configuration [12].

Discussion. The complete system of solutions of Problem 2 will be found if equation (11) or the equation

$$u_1'' + t^{-1}u_1' + 9(\lambda_0^2 - \beta t^4)u_1 = 0.$$

is solved globally.

Example. Now we consider the electron transport in graphene and find energy levels of the electron. "As an emergent electronic material and model system for condensed-matter physics, graphene and its electrical transport properties have become a subject of intense focus. By performing low-temperature transport spectroscopy on single-layer and bilayer graphene, we observe ballistic propagation and quantum interference of multiply reflected waves of charges from normal electrodes and multiple Andreev reflections from superconducting electrodes, thereby realizing quantum billiards in which scattering only occurs at the boundaries."("Phase-Coherent Transport in Graphene Quantum Billiards" (Science, Vol. 317, Issue 5844, Pages 1530-1533, 2007).

Graphen is a one-atom thick sheet of carbon atoms arranged in hexagonal rings in which scattering occurs at the boundaries. Hence, we can apply our results (Fig.3). The width of the side of the hexagonal cell is about 0.14×10^{-10} [17, 21].As we have billiard in the hexagonal rug, we can use formulas (31), (32), (33). Here we suppose, that the rug has 7 cells and by using Maple we have obtained the following result (Table 2)

a_0	d	ε	λ_0^2	A	$\|E\|$(eV)
10^4	2.4	10^{-6}	0.046745	$\sqrt{2} \times 10^{-6}$	0.553961

Table 2.

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OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS WITH SEVERAL DELAY ARGUMENTS

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Abstract. In the paper the following difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0$$

is considered, where $m \in N$, the functions $p_i : N \to R_+$, $\tau_i : N \to N$, $\tau_i(k) \le k-1$, $\lim\limits_{k\to+\infty} \tau_i(k) = +\infty$ $(i = 1, \dots, m)$ are defined on the set of natural numbers and the difference operator is defined by $\Delta u(k) = u(k+1) - u(k)$. New oscillation criteria of all solutions to these equation are established.

Keywords and phrases: Oscillation, proper solution, difference equations with several delay.

AMS subject classification (2010): 34K11.

1. Introduction

Consider the difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0, \tag{1.1}$$

where $m \ge 1$ is a natural number, $p_i : N \to R_+$, $\tau_i : N \to N$, $(i = 1, \dots, m)$, are functions defined on the set $N = \{1, 2, \dots\}$ and $\Delta u(k) = u(k+1) - u(k)$. Everywhere below it is assumed that

$$\lim_{k\to+\infty} \tau_i(k) = +\infty, \quad \tau_i(k) \le k-1. \tag{1.2}$$

For each $n \in N$ denote $N_n = \{n, n+1, \dots\}$.

Definition 1.1. Let $n \in N$. We will call a function $u : N \to R$ a proper solution of equation (1.1) on the set N_n, if it satisfies (1.1) on N_n and $\sup\{|u(i)| : i \ge k\} > 0$ for any $k \in N_n$.

Definition 1.2. We say that a proper solution $u : N_n \to R$ of equation (1.1) is oscillatory if for any $k \in N$ there exist $n_1, n_2 \in N_k$ such that $u(n_1) \cdot u(n_2) \le 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. Equation (1.1) is said to be oscillatory, if any of its proper solutions is oscillatory.

The problem of oscillation of solutions of linear difference equation (1.1) for $m = 1$, has been studied by several authors, see [1,2] and references therein.

As to investigation of the analogous problem for equation of type (1.1) $(m > 1)$, to our knowledge for them there have not been obtained results analogous to those known for equation (1.1), where $m = 1$. Analogous results for first order differential equations with several delay see [3,4].

2. Sufficient conditions for oscillation

Denote

$$\psi_1(k) = 1, \quad \psi_s(k) = \left(\prod_{\ell=1}^{m} \prod_{j=\tau_\ell(k)}^{k} \left[1 + m \Big(\prod_{\ell=1}^{m} p_\ell(j) \Big)^{\frac{1}{m}} \psi_{s-1}(j) \right] \right)^{\frac{1}{m}} \tag{2.1}$$
$$k \in N, \quad s = 2, 3, \ldots$$

Theorem 2.1. *Let there exist $k_0 \in N$ and nondecreasing functions $\sigma_i : N \to N$ $(i = 1, \ldots, m)$ such that*

$$1 + \tau_i(k) \le \sigma_i(k) \le k \quad \textit{for} \quad k \in N \quad (i = 1, \ldots, m) \tag{2.2}$$

and

$$\limsup_{k \to +\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left[1 + m \Big(\prod_{\ell=1}^{m} p_\ell(j) \Big)^{\frac{1}{m}} \psi_{k_0}(j) \right] \right)^{\frac{1}{m}} > \frac{1}{m^m},$$

then equation (1.1) *is oscillatory, where ψ_{k_0} is given by* (2.1) *when $k = k_0$.*

Corollary 2.1. Let there exist nondecreasing functions $\sigma_i : N \to R$ such that

$$\limsup_{k \to +\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left[1 + m \Big(\prod_{\ell=1}^{m} p_\ell(j) \Big)^{\frac{1}{m}} \right] \right)^{\frac{1}{m}} > \frac{1}{m^m},$$

then equation (1.1) is oscillatory.

Corollary 2.2. Let there exist nondecreasing functions $\sigma_i : N \to R$ such that condition (2.2) is fulfilled and

$$\limsup_{k \to +\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \right)^{\frac{1}{m}} > \frac{1}{m^m},$$

then equation (1.1) is oscillatory.

Theorem 2.2. *Let there exist nondecreasing functions $\sigma_i : N \to N$ such that* (2.2) *is fulfilled,*

$$\limsup_{k \to +\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \prod_{j=\tau_j(s)}^{\sigma_i(k)-1} \Big(\prod_{\ell=1}^{m} p_\ell(j) \Big)^{\frac{1}{m}} \right)^{\frac{1}{m}} > 0 \tag{2.3}$$

and

$$\liminf_{k \to +\infty} \prod_{j=\tau_\ell(k)}^{k} \left(\prod_{i=1}^{m} p_i(j) \right)^{\frac{1}{m}} = \alpha_\ell > 0 \quad (\ell = 1, \ldots, m). \tag{2.4}$$

Moreover, if for some $\ell \in \{1, \dots, m\}$

$$\lim_{k\to+\infty} \big(k - \tau_\ell(k)\big) = +\infty, \tag{2.5}$$

then equation (1.1) *is oscillatory.*

Theorem 2.3. *Let there exist nondecreasing functions* $\sigma_i : N \to N$, *such that* (2.2) *and* (2.3) *hold. If moreover,*

$$\liminf_{k\to+\infty} \big(k - \tau_\ell(k)\big) = n_\ell \in N \quad (\ell = 1, \dots, m) \tag{2.6}$$

and

$$\prod_{\ell=1}^{m} \alpha_\ell > \frac{1}{n_0^m} \left(\frac{n_0}{n_0 + m}\right)^{n_0+m}, \tag{2.7}$$

where α_ℓ ($\ell = 1, \dots, m$) *are given by* (2.4) *and* $n_0 = \sum\limits_{\ell=1}^{m} n_\ell$. *Then equation* (1.1) *is oscillatory.*

Theorem 2.4. *Let* $\tau_i : N \to N$ ($i = 1, \dots, m$) *be nondecreasing functions, let* (2.6) *and* (2.7) *be fulfilled and*

$$\liminf_{k\to+\infty} \sum_{i=\tau_j(k)}^{k-1} p_j(i) > 0 \quad (j = 1, \dots, m). \tag{2.8}$$

Then equation (1.1) *is oscillatory, where* α_ℓ *is given by* (2.4).

Theorem 2.5. *Let there exist nondecreasing functions* $\sigma_i : N \to N$ *such that* (2.2), (2.3) *and let* (2.6) *be fulfilled. Moreover, if* $m \le \sum\limits_{\ell=1}^{m} n_\ell$ *and*

$$\prod_{\ell=1}^{m} \alpha_\ell > (2\sqrt{m})^{-\sum\limits_{\ell=1}^{m}(n_\ell+1)}, \tag{2.9}$$

then equation (1.1) *is oscillatory, where*

$$\alpha_\ell = \liminf_{k\to+\infty} \prod_{j=\tau_\ell(k)}^{k} \left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{2m}} \quad (\ell = 1, \dots, m). \tag{2.10}$$

Theorem 2.6. *Let* $\tau_i : N \to N$ *be nondecreasing functions and* (2.6), (2.8) *and let* (2.9) *be fulfilled. Then equation* (1.1) *is oscillatory, where* α_ℓ *is given by* (2.10).

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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF n-TH ORDER GENERALIZED EMDEN-FOWLER DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT

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Abstract. In the paper the following differential equation

$$u^{(n)}(t) + p(t)\left|u(\tau(t))\right|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0$$

is considered, where $n \geq 3$, $p \in L_{\mathrm{loc}}(R_+; R_-)$, $\mu \in C(R_+; (0, +\infty))$, $\tau \in C(R_+; R_+)$, $\tau(t) \leq t$ for $t \in R_+$ and $\lim_{t\to+\infty} \tau(t) = +\infty$. We say that the equation is "almost linear" if the condition $\lim_{t\to+\infty} \mu(t) = 1$ is fulfilled, while if $\limsup_{t\to+\infty} \mu(t) \neq 1$ or $\liminf_{t\to+\infty} \mu(t) \neq 1$, then the equation is an essentially nonlinear differential equation. In case of "almost linear" and essentially nonlinear differential equations to have Property **A** have been extensively studied [1–5]. In the paper new sufficient conditions are established for a general class of essentially nonlinear functional differential equations to have Property **B**.

Keywords and phrases: Property **B**, oscillation, functional differential equation.

AMS subject classification (2010): 34K11.

1. Introduction

This work deals with the investigation of oscillatory properties of solutions of a functional-differential equation of the form

$$u^{(n)}(t) + p(t)\left|u(\tau(t))\right|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \tag{1.1}$$

where

$$\begin{gathered} p \in L_{\mathrm{loc}}(R_+; R_-), \quad \mu \in C(R_+; (0, +\infty)), \\ \tau \in C(R_+; R_+), \quad \tau(t) \leq t \quad \text{and} \quad \lim_{t\to+\infty} \tau(t) = +\infty. \end{gathered} \tag{1.2}$$

It will always be assumed that the condition

$$p(t) \leq 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0, +\infty)$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive, $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality $\overline{u}^{(n)}(t) + p(t)|\overline{u}(\tau(t))|^{\mu(t)} \operatorname{sign} \overline{u}(\tau(t)) = 0$ holds almost everywhere for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even, and either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2. We say that equation (1.1) has Property **B** if any of its proper solutions is either oscillatory or satisfies either (1.4) or

$$\big|u^{(i)}(t)\big| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even and either is oscillatory or satisfies (1.5), when n is odd.

Definition 1.3. We say that equation (1.1) is almost linear if the condition $\lim\limits_{t\to+\infty} \mu(t)$ $= 1$ holds, while if $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$ or $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$, then we say that the equation is an essentially nonlinear differential equation.

Oscillatory properties of almost linear and essentially nonlinear differential equation with advanced argument are studied well enough in [1–5]. For Emden-Fowler equations with deviating arguments, essential contribution was made in [6–9]. In the present paper for the generalized differential equation with delay argument, sufficient conditions are established for equation (1.1) to have Property **B**. Analogously results for Property **A**, see [10].

2. Essentially nonlinear differential equation with property B

The following notations will be used throughout the work

$$\begin{gathered}\alpha = \inf\big\{\mu(t) : t \in R_+\big\}, \quad \beta = \sup\big\{\mu(t) : t \in R_+\big\}, \\ \tau_{(-1)}(t) = \sup\big\{s \geq 0,\ \tau(s) \leq t\big\}, \quad \tau_{(-k)} = \tau_{(-1)} \circ \tau_{(-(k-1))}, \quad k = 2, 3, \dots\end{gathered} \tag{2.1}$$

Clearly $\tau_{(-1)}(t) \geq t$ and $\tau_{(-1)}$ is nondecreasing and coincides with the inverse of τ when the latter exists.

Let $\alpha \in [1, +\infty)$, $\gamma \in (1, +\infty)$, $\ell \in \{1, \dots, n-2\}$ and $t_* \in R_+$. Denote

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell!\, \exp\bigg\{\gamma_\ell(\alpha) \int_{\tau_{(-1)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi\, ds\bigg\}, \tag{2.2}$$

$$\begin{aligned}\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell! + \frac{1}{(n-\ell)!} \int_{\tau_{(-i)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \times \\ \times \Big(\frac{1}{\ell!}\, \rho^{\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\Big)^{\mu(\xi)} |p(\xi)| d\xi\, ds \quad (i = 2, 3, \dots),\end{aligned} \tag{2.3}$$

$$\gamma_\ell(\alpha) = \begin{cases} \gamma & \text{if} \quad \alpha > 1, \\ \dfrac{1}{\ell!\,(n-\ell)!} & \text{if} \quad \alpha = 1. \end{cases} \tag{2.4}$$

In the section, when $\alpha > 1$, we derive sufficient conditions for functional differential equation (1.1) to have Property **B**.

Proposition 2.1. *Let $\alpha > 1$, conditions* (1.2) *and* (1.3) *be fulfilled and for any $\ell \in \{1,\dots,n\}$ with $\ell+n$ even, the conditions*

$$\int_0^{+\infty} t^{n-\ell}\big(c,\tau^{\ell-1}(t)\big)^{\mu(t)}|p(t)|dt = +\infty \ \ \textit{for} \ \ c \in (0,1] \tag{$2.5_{\ell,c}$}$$

and

$$\int_0^{+\infty} t^{n-\ell-1}\big(\tau(t)\big)^{\ell\,\mu(t)}|p(t)|dt = +\infty \ \ \textit{for} \ \ \ell \in \{1,\dots,n-2\} \tag{2.6_ℓ}$$

be fulfilled. Moreover, let for any large $t_* \in R$, for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$

$$\int_{\tau_{(-k)}(t_*)}^{+\infty}\int_s^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}\Big(\frac{1}{\ell!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\Big)^{\mu(\xi)-\delta}|p(\xi)|d\xi\,ds = +\infty. \tag{2.7_ℓ}$$

Then equation (1.1) *has Property* **B**, *where α is defined by first condition of* (2.1) *and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by* (2.2)–(2.4).

Proposition 2.1′. *Let $\alpha > 1$, $\beta < +\infty$, conditions* (1.2) *and* (1.3) *be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even, conditions* ($2.5_{\ell,1}$) *and* (2.6_ℓ) *hold. Moreover, let for some $k \in N$, $\gamma \in (1,+\infty)$ and $\delta \in (1,\alpha]$ condition* (2.7_ℓ) *be fulfilled. Then equation* (1.1) *has Property* **B**, *where α and β are defined by* (2.1) *and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by* (2.2)–(2.4).

Theorem 2.1. *Let $\alpha > 1$, conditions* (1.2), (1.3), ($2.5_{1,c}$) *and*

$$\liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} > 0 \tag{2.8}$$

be fulfilled. Moreover, let for some $\delta \in (1,\alpha]$ the conditions

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta}|p(\xi)|d\xi\,ds = +\infty, \tag{2.9}$$

when n is odd and

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-3-\delta}(\tau(\xi))^{\delta+\mu(\xi)}|p(\xi)|d\xi\,ds = +\infty, \tag{2.10}$$

when n is even, be fulfilled. Then equation (1.1) *has Property* **B**, *where α is defined by the first condition of* (2.1).

Theorem 2.1′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions* (1.2), (1.3), ($2.5_{1.1}$), (2.6_1) *and* (2.8) *be fulfilled. Moreover, let for some $\delta \in (1,\alpha)$, when n is odd (n is even) condition* (2.9) ((2.10)) *holds. Then equation* (1.1) *has Property* **B**, *where α and β are given by* (2.1).

Remark 2.1. In Theorem 2.1 condition ($2.5_{1,c}$) cannot be replaced by condition ($2.5_{1,1}$). Indeed, let $n \geq 3$, $c \in (0,1)$, $c_1 \in (c,1)$,

$$\mu(t) = n\log_{\frac{1}{c_1}} t, \ \ p(t) = -\frac{c\,n!}{t^{1+n}}\,c^{-\mu(t)}\bigg(t^{n-1} + \frac{(-1)^n}{t}\bigg)^{-\mu(t)} \ \ \text{and} \ \ \tau(t) \equiv t.$$

It is obvious that condition $(2.5_{1,1})$ is fulfilled, but for large t, equation (1.1) has the solution $u(t) = c(t^{n-1} + \frac{(-1)^n}{t})$. Therefore, equation (1.1) has the solution u, satisfying the condition $\lim\limits_{t\to+\infty} u^{(n-1)}(t) = c(n-1)!$, that is equation (1.1) does not have Property **B**.

Theorem 2.2. *Let $\alpha > 1$, let conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *and* (2.8) *be fulfilled and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-3}\tau(s)|p(s)|ds > 0. \tag{2.11}$$

Moreover, let for some $\delta \in (1, \alpha]$ and $\gamma > 0$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta+\gamma(\mu(\xi)-\delta)}|p(\xi)|d\xi\, ds = +\infty. \tag{2.12}$$

Then equation (1.1) *has Property* **B**, *where α is defined by the first condition of* (2.1).

Theorem 2.2′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions* (1.2), (1.3), $(2.5_{1.1})$, (2.6_1), (2.8) *and* (2.11) *be fulfilled. Moreover, if for some $\delta \in (1, \alpha]$ and $\gamma > 0$, condition* (2.12) *holds, then equation* (1.1) *has Property* **B**, *where α and β are given by* (2.1).

Theorem 2.3. *Let $\alpha > 1$, conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1), (2.8) *and* (2.11) *be fulfilled. Moreover, if there exists $m \in N$ such that*

$$\liminf_{t\to+\infty} \frac{\tau^m(t)}{t} > 0, \tag{2.13}$$

then equation (1.1) *has Property* **B**, *where α is given by the first condition of* (2.1).

Theorem 2.3′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions* (1.2), (1.3), $(2.5_{1.1})$, (2.6_1), (2.8), (2.11) *and for some $m \in N$ condition* (2.13) *be fulfilled. Then equation* (1.1) *has Property* **B**, *where α and β are given by* (2.1).

Theorem 2.4. *Let $\alpha > 1$, conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.6_{n-1}) *and*

$$\limsup_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} < +\infty \tag{2.14}$$

be fulfilled. Moreover, if for some $\delta \in (1, \alpha]$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta}(\tau(\xi))^{\delta+(n-3)\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{2.15}$$

then equation (1.1) *has Property* **B**, *where α is given by the first condition of* (2.1).

Theorem 2.4′. *Let $\alpha > 1$ and $\beta < +\infty$, conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}) *and* (2.14) *be fulfilled. Moreover, if for some $\delta \in (1, \alpha]$ condition* (2.15) *holds, then equation* (1.1) *has Property* **B**, *where α and β are given by* (2.1).

Theorem 2.5. *Let $\alpha > 1$, conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.7_{n-1}) *and* (2.14) *be fulfilled and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} (\tau(s))^{1+(n-3)\mu(s)}|p(s)|ds > 0. \tag{2.16}$$

Moreover, if for some $\delta \in (1, \alpha]$ *and* $\gamma > 0$

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta} (\tau(\xi))^{\delta+(n-3)\mu(\xi)+\gamma(\mu(\xi)-\delta)} |p(\xi)| d\xi \, ds = +\infty, \tag{2.17}$$

then equation (1.1) *has Property* **B**, *where* α *is given by* (2.1).

Theorem 2.5′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}), (2.14) *and* (2.16) *be fulfilled and for some* $\delta \in (1, \alpha)$ *and* $\gamma > 0$ *condition* (2.17) *holds. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 2.6 *Let* $\alpha > 1$, *conditions* (1.2), (1.3), $(2.5_{n-1,c})$, (2.6_{n-1}), (2.14) *and* (2.17) *be fulfilled. Moreover, if for some* $m \in N$ *condition* (2.13) *holds, then equation* (1.1) *has Property* **B**, *where* α *is given by* (2.1).

Theorem 2.6′. *Let* $\alpha > 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{n-1,1})$, (2.6_{n-1}) *and* (2.17) *be fulfilled. Moreover, if for some* $m \in N$ *condition* (2.13) *holds, then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

3. Quasi-linear differential equations with property B

In the section we define sufficient conditions for functional differential equations (1.1), when $\alpha = 1$, to have Property **B**.

Proposition 3.1 *Let* $\alpha = 1$, *conditions* (1.2) *and* (1.3) *be fulfilled and for any* $\ell \in \{1, \dots, n+1\}$ *with* $\ell + n$ *even, conditions* $(2.5_{\ell,c})$ *and* (2.6_ℓ) *hold. Let moreover, for any large* $t_* \in R_+$ *and for some* $k \in N$

$$\limsup_{t\to+\infty} \frac{1}{t} \int_{\tau_{(-k)}(t_*)}^{t} \int_s^{+\infty} \xi^{n-\ell-1} \big(\tau(\xi)\big)^{(\ell-1)\mu(t)} \times \\ \times \left(\frac{1}{\ell!} \rho^{(1)}_{k,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi \, ds > 0. \tag{3.1_ℓ}$$

Then equation (1.1) *has Property* **B**, *where* α *is given by the first condition of* (2.1).

Proposition 3.1′. *Let* $\alpha = 1$ *and* $\beta < +\infty$, *conditions* (1.2) *and* (1.3) *be fulfilled and for any* $\ell \in \{1, \dots, n\}$ *with* $\ell+n$ *even, conditions* $(2.5_{\ell,1})$ *and* (2.6_ℓ) *hold. Moreover, let for any large* $t_* \in R_+$ *and for some* $k \in N$, *condition* (3.1_ℓ) *holds. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 3.1 *Let* $\alpha = 1$, *conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *and* (2.8) *be fulfilled and*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2} |p(\xi)| d\xi \, ds > 0. \tag{3.2}$$

Then equation (1.1) *has Property* **B**, *where* α *is defined by first condition of* (2.1).

Theorem 3.1′. *Let* $\alpha = 1$ *and* $\beta < +\infty$, *conditions* (1.2), (1.3), $(2.5_{1,1})$, (2.6_1), (2.8) *and* (3.2) *be fulfilled. Then equation* (1.1) *has Property* **B**, *where* α *and* β *are given by* (2.1).

Theorem 3.2 *Let* $\alpha = 1$, *conditions* (1.2), (1.3), $(2.5_{1,c})$, (2.6_1) *be fulfilled. Let moreover*

$$\liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} > 1 \tag{3.3}$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-3}\tau(s))ds > (n-1)!. \tag{3.4}$$

Then for equation (1.1) *to have Property* **B** *it is sufficient that*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\int_s^{+\infty}\xi^{n-2}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\,ds > 0. \tag{3.5}$$

Theorem 3.2′. *Let* $\alpha = 1$ *and* $\beta < +\infty$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{1,1})$*,* (2.6_1)*,* (3.3) *and* (3.4) *be fulfilled. Then equation* (1.1) *has Property* **B***, it is sufficient that condition* (3.5) *holds.*

Theorem 3.3 *Let* $\alpha = 1$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{n-1,c})$*,* (2.6_{n-2}) *be fulfilled. Moreover, if the conditions*

$$\liminf_{t\to+\infty}\frac{(\tau(t))^{\mu(t)}}{t} < 1 \tag{3.6}$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty}(\tau(s))^{1+(n-3)\mu(s)}|p(s)|ds > 2(n-2)! \tag{3.7}$$

are fulfilled, then for equation (1.1) *to have Property* **B** *it is sufficient that*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\int_s^{+\infty}\xi(\tau(\xi))^{(n-3)\mu(\xi)}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\,ds > 0. \tag{3.8}$$

Theorem 3.3′. *Let* $\alpha = 1$ *and* $\beta < +\infty$*, conditions* (1.2)*,* (1.3)*,* $(2.5_{n-1,1})$*,* (2.6_{n-1})*,* (3.6) *and* (3.7) *be fulfilled. Then for equation* (1.1) *to have Property* **B***, it is sufficient that condition* (3.8) *holds.*

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EFFECTIVE SOLUTION OF THE BASIC MIXED BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN

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Abstract. By the method N. Muskhelishvili an explisit solution to the basic mixed boundary value problem for homogeneous equation of statics of the linear theory of elastic mixture for a circular domain is obtained.

Keywords and phrases: Basic mixed boundary value problem, elastic mixture theory, equation of statics, nodal points.

AMS subject classification (2010): 74E35, 74E20, 74G05.

1. Introduction

The basic plane boundary value problem and the basic mixed boundary value problem in a simple connected domain for homogeneous equation of statics of the linear theory of elastic mixture, by analogues of general Kolosov-Muskhelishvili representation have been investigated in [3] and [2], respectively.

By the method M. Muskhelishvili an explicit solution of the basic mixed boundary value problem for homogeneous equation of statics of the linear theory of elastic mixture for an half-plane was obtained in [5].

In the present work we studied an analogous problem which in the case of the plane theory of elasticity has been studied by N. Muskhelishvili [4, §123]. To solve the problem we use the formulas due to Kolosov-Muskhelishvili and the method described in [4,5].

1. Some auxiliary formulas and operators

The homogeneous equation of static of the linear theory of elastic mixtures in a complex form is of the type [3]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{1.1}$$

where $z = x_1 + ix_2$, $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}\right)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$,

$u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements,

$$K = -\frac{1}{2}\ em^{-1}, \qquad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \qquad m^{-1} = \frac{1}{\triangle_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$

$$\triangle_0 = m_1 m_3 - m_2^2, \qquad m_k = e_k + \frac{1}{2}\ e_{3+k}, \quad e_1 = a_2/d_2, \quad e_2 = -c/d_2,$$

$$e_3 = a_1/d_2, \quad d_2 = a_1a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5,$$
$$e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1, \quad e_3 + e_6 = a/d_1, \quad d_1 = ab - c_0^2,$$
$$a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho,$$
$$b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4, \quad \rho = \rho_1 + \rho_2,$$
$$d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5},$ are elastic modules characterizing mechanical properties of the mixture, ρ_1 and ρ_2 are partial densities of the mixture. It will be assumed that the elastic constants $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$, and partial rigid densities ρ_1 and ρ_2 satisfy the conditions (inequalities) [1].

In [3] M.O. Basheleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2}\; ez\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right], \tag{1.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$are arbitrary analytic vector-functions,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \; B = \mu e, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$\Delta_0 = dem > 0, \quad \Delta_1 = det\mu > 0, \quad \Delta_2 = det(A - 2E) > 0, \quad A_1 + A_3 - 2 = B_1 + B_3,$
$A_2 + A_4 - 2 = B_2 + B_4, \quad \dfrac{\partial}{\partial S(x)} = -n_2\dfrac{\partial}{\partial x_1} + n_1\dfrac{\partial}{\partial x_2}, \quad n = (n_1, n_2)^T$ unit vector of the outer normal, $(Tu)_p, \quad p = \overline{1,4}$ are stress components, $Tu = ((Tu)_1, (Tu)_2, (Tu)_3, (Tu)_4)^T$,[1,6].

Now we note that, from (1,2) we have

$$2\mu\frac{\partial U}{\partial S(x)} = \frac{\partial}{\partial S(x)}\left[A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\right]. \tag{1.4}$$

Formulas (1,2), (1,3) and (1,4) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixtures.

2. Statement of the mixed problem and scheme of its solution

In the present work we study an analogous problem which in the case of the plane theory of elasticity has been studied by N. Muskhelishvili [4, §123]. For the solution of the problem use will be made of the generalized Kolosov-Muskhelishvili's formula and the method developed in [4,5].

Let us assume that an elastic mixture occupies the circular domain $D^+ = \{z : |z| < 1\}$ bounded by the circumference $L = \{z : |z| = 1,\}$ and let $L_j = a_jb_j, \quad j = \overline{1,n}, \quad (a_{j+1} \neq b_j, \quad a_{n+1} \equiv a_1)$, be arcs separately lying on it, note that the points $a_1, b_1, a_2, b_2, ..., a_n, b_n$ follow each other in the positive direction on L.

Suppose that $L^{'} = \bigcup\limits_{j=1}^{n} L_j$ and $L^{''}$ is the remaining part of L.

Definition 2.1. The vector $u = (u^{'}, u^{''})^T = (u_1, u_2, u_3, u_4)^T$ is called regular if (see[2]) (i) $u_p \in C^2(D^+) \bigcap C(D^+ \bigcup L)$, $p = \overline{1,4}$;
$(ii)(Tu)_p$, $(p = \overline{1,4})$, is continuously extendable at every point of L from D^+ except perhaps the points a_j and b_j, $j = \overline{1,n}$;
(iii) near the points a_j and b_j, $j = \overline{1,n}$ $(Tu)_p$, $p = \overline{1,4}$ admit estimate of the type $|(Tu)_p| < const|z - \alpha_0|^{-\beta}$, $0 \le \beta < 1$, $z \in D^+$ $(\alpha_0 = a_j; \quad \alpha_0 = b_j, \quad j = \overline{1,n})$, $p = \overline{1,4}$.

We consider the mixed boundary value problem. Define an elastic equilibrium of the plate D^+ if

$$U^+(t) = f^0(t), \qquad t \in L^{'}, \qquad [TU(t)]^+ = 0, \quad t \in L^{''}, \qquad (2,1)_{f,0}$$

where $f^0 = (f_1^0, f_2^0)$ is a given complex vector-function on $L^{'}$, $(f^0(t))^{'} \in H)$. Using the Green formula [1] it is easy to prove.

Theorem 2.1. *The homogeneous mixed boundary value problem* $(2.1)_0$ *admits only a trivial solution.*

Below instead of conditions $(2.1)_{f,0}$ we consider its following equivalent conditions

$$2\mu \left(\frac{\partial U(t)}{\partial S(t)} \right)^+ = \frac{\partial f(t)}{\partial S(t)}, \quad t \in L^{'}, \quad (TU(t))^+ = 0, \quad t \in L^{''}, \qquad (2.1)^{'}$$

where $f(t) = 2\mu f^0(t)$.

Let $t = e^{i\theta}$ $0 \le \theta \le 2\pi$. Then $\dfrac{\partial}{\partial S(t)} = \dfrac{d}{d\theta} = \dfrac{d}{dt}\dfrac{d}{d\theta} = ie^{i\theta}\dfrac{d}{dt}$.

Now note that, on the basis of analogous Kolosov-Muskhelishvili's formulas (1.4) and (1.3) our problem is reduced to finding two analytic vector-functions $\phi(z) = \varphi^{'}(z)$ and $\Psi(z) = \psi^{'}(z)$ in D^+ by the boundary conditions (see $(2.1)^{'}$)

$$\begin{aligned} &[A\phi(t) + B\overline{\phi(t)} - B\overline{t\phi'(t)} - 2\mu\overline{t^2\Psi(t)}]^+ = f^{'}(t), \qquad t \in L^{'}, \\ &[(A - 2E)\phi(t) + B\overline{\phi(t)} - B\overline{t\phi'(t)} - 2\mu\overline{t^2\Psi(t)}]^+ = 0, \qquad t \in L^{''}. \end{aligned} \qquad (2.2)$$

Consider the vector-function

$$(A - 2E)\phi(z) = -B\overline{\phi(\frac{1}{\overline{z}})} + B\frac{1}{z}\overline{\phi'(\frac{1}{\overline{z}})} + 2\mu\frac{1}{z^2}\overline{\Psi(\frac{1}{\overline{z}})}. \qquad (2.3)$$

From (2.3) it follows the equation (2.3) define $\phi(z)$ as an analytic vector-function toward z in the domain $|z| > 1$, and to $\dfrac{1}{\overline{z}}$ in the $|z| < 1$.

Due to the above formula we find that

$$2\mu\Psi(z) = (A - 2E)\frac{1}{z^2}\overline{\phi(\frac{1}{\overline{z}})} + B\frac{1}{z^2}\phi(z) - B\frac{1}{z}\phi^{'}(z). \qquad (2.4)$$

If follows from (2.4) that the vector-function $\Psi(z)$ is definite in the entire $z = x_1 + ix_2$ plane by means of the $\phi(z)$.

Note also that if

$$\phi_j(z) = A_0^{(j)} + A_1^{(j)} z + A_2^{(j)} z^2 + ..., \quad |z| < 1, \quad j = 1, 2,$$

$$\phi_j(z) = B_0^{(j)} + B_1^{(j)} \frac{1}{z} + B_2^{(j)} \frac{1}{z^2} + ..., \quad |z| > 1, \quad j = 1, 2,$$

then due to $A_1 + A_3 - 2 = B_1 + B_3, \quad A_2 + A_4 - 2 = B_2 + B_4, \quad (see[2])$, we can conclude that, (see(2.4)), $\Psi(z)$ to be analytic in the entire plane $z = x_1 + ix_2$ with the point $z = 0$ it is sufficient that the conditions

$$(A_0^{(1)}, A_0^{(2)})^T + (\overline{B_0^{(1)}}, \overline{B_0^{(2)}})^T = 0, \quad (B_1^{(1)}, B_1^{(2)})^T = 0 \tag{2.5}$$

be fulfilled.

In view of (2.3) the boundary conditions (2.2) can be written as:

$$\phi^+(t) - A^{-1}(A - 2E)\phi^-(t) = A^{-1} f^{'}(t) = h(t), \quad t \in L^{'}, \quad h = (h_1, h_2)^T, \tag{2.6}$$

$$\phi^+(t) - \phi^-(t) = 0, \quad t \in L^{''}. \tag{2.7}$$

From (2.7) it follows that the vector-function $\phi(z)$ is analytic in the entire plane $z = x_1 + ix_2$ cutting to the $L^{'}$.

To solve problem (2.6) we rewrite condition (2.6) as

$$\begin{pmatrix} 1 \\ y \end{pmatrix} \phi^+(t) - \frac{2\Delta_0\Delta_1 - A_4 + A_3 y}{2\Delta_0\Delta_1} \begin{pmatrix} 1 \\ y \end{pmatrix} \phi^-(t) = \begin{pmatrix} 1 \\ y \end{pmatrix} h(t), \quad t \in L^{'}, \tag{2.8}$$

where y is an arbitrary real constant. We define the unknown y by the equation

$$y = \frac{A_2 + y(2\Delta_0\Delta_1 - A_1)}{2\Delta_0\Delta_1 - A_4 + A_3 y}, \qquad or \qquad A_3 y^2 + (A_1 - A_4)y - A_2 = 0.$$

Note that $0 < A_1 + A_4 < 4$, $A_1 + A_4 - 4\Delta_0\Delta_1 > 0$ and $(A_1 + A_4)^2 - 16\Delta_0\Delta_1 > 0)$ $(see[2])$.

On the basis of (2,8) representation we can conclude that a bounded at infinity solution of problem (2.6) is given by the formula (see [4 123])

$$\phi(z) = \frac{1}{y_2 - y_1} \begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix} \left[\frac{\aleph(z)}{2\pi i} \int_{L'} \frac{[\aleph^+(t)]^{-1} R(t) dt}{t - z} + \aleph(z) P_n(z) \right] \tag{2.10}$$

where y_1 and y_2 are the roots of equation (2.9),

$$R(t) = \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \end{bmatrix} h(t), \qquad \aleph(z) = \begin{bmatrix} \aleph_1(z) & 0 \\ 0 & \aleph_2(z) \end{bmatrix},$$

$$\aleph_j(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2} - i\beta_j} (z - b_k)^{-\frac{1}{2} + i\beta_j}, \qquad \beta_j = \frac{ln|M_j|}{2\pi},$$

$$M_j = \frac{1}{4\Delta_0\Delta_1}\left[4\Delta_0\Delta_1 - A_1 - A_4 - (-1)^j\sqrt{(A_1+A_4)^2 - 16\Delta_0\Delta_1}\right] < 0$$

$$P_n(z) = (P_{n_1}(z), P_{n_2}(z))^T, \qquad P_{n_j}(z) = \sum_{q=0}^{n} C_q^{(j)} z^{n-q}, \quad j=1,2.$$

To define $C_q^{(j)}$, $\quad j=1,2, \quad q=\overline{0,n}$, we use the following conditions (see [4, 123], $(2.1)'$ and (2.5))

$$2\mu \int\limits_{b_k a_{k+1}} d\begin{bmatrix} u_1+iu_2 \\ u_3+iu_4 \end{bmatrix} = f(a_{k+1}) - f(b_k), \qquad \phi(0) + \overline{\phi(\infty)} = 0. \tag{2.11}$$

If we take into account (2.6), (2.7) and (2.10) for determining the unknown vectors $(C_q^1, C_q^2)^T$, $\quad q=\overline{0,n}$, from (2.11) we obtain the following system of equations:

$$2\int\limits_{b_k a_{k+1}} \phi_0(t_0)dt_0 + \sum_{q=0}^{n} N_{kq}\begin{pmatrix} C_q^{(1)} \\ C_q^{(2)} \end{pmatrix} = f(a_{k+1}) - f(b_k), \tag{2.12}$$

$$\begin{pmatrix} \overline{C}_0^{(1)} \\ \overline{C}_0^{(2)} \end{pmatrix} + \aleph(0)\begin{pmatrix} Cn^{(1)} \\ Cn^{(2)} \end{pmatrix} + \frac{\aleph(0)}{2\pi i}\int\limits_{L'} [\aleph^+(t)]^{-1}\frac{R(t)dt}{t} = 0. \tag{2.13}$$

where (see (2.10))

$$\phi_0(t) = \frac{1}{y_0 - y_1}\begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix}\frac{\aleph(t_0)}{2\pi i}\int\limits_{L'} [\aleph^+(t)]^{-1}\frac{R(t)dt}{t-t_0},$$

$$N_{kq} = \frac{2}{y_2 - y_1}\begin{bmatrix} y_2 & -y_1 \\ -1 & 1 \end{bmatrix}\int\limits_{b_k a_{k+1}} \aleph(t)t^{n-q}dt.$$

Now note that, on the basis of the uniqueness theorem (see Theorem 2.1) for (2.1) mixed problem, we can conclude that the (2.12) and (2.13) system is solvable for $C_q^{(1)}$, $\quad q=\overline{0,n}, \quad j=1,2$.

Having found $C_q^{(1)}$, $\quad q=\overline{0,n}, \quad j=1,2$ we can be define $\phi(z)$, hence $\Psi(z)$, $\varphi(z)$ and $\psi(z)$. Finally by (1.2) we obtain the solution of the mixed $(2.1)_{f,0}$ problem.

The mixed boundary value problem considered in the paper, for domain outside the circle, can be solved in a similar way.

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VARIATION FORMULAS OF SOLUTION FOR A CLASS OF CONTROLLED NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION CONSIDERING DELAY FUNCTION PERTURBATION AND THE CONTINUOUS INITIAL CONDITION

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Abstract. Variation formulas of solution are obtained for linear with respect to prehistory of the phase velocity (quasi-linear) controlled neutral functional-differential equation with variable delays. The effects of delay function perturbation and continuous initial condition are detected in the variation formulas.

Keywords and phrases: Neutral controlled functional-differential equation,variation formula of solution, effect of delay function perturbation,continuous initial condition.

AMS subject classification (2010): 34K38, 34K40, 34K27.

Let $I = [a, b]$ be a finite interval and let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T is the sign of transposition. Suppose that $O \subset \mathbb{R}^n_x$ and $U_0 \subset \mathbb{R}^r_u$ are open sets. Let the n-dimensional function $f(t, x, y, u)$ satisfy the following conditions: for almost all $t \in I$, the function $f(t, \cdot) : O^2 \times U_0 \to \mathbb{R}^n_x$ is continuously differentiable; for any $(x, y, u) \in O^2 \times U_0$, the functions $f(t, x, y, u), f_x(\cdot), f_y(\cdot), f_u(\cdot)$ are measurable on I; for arbitrary compacts $K \subset O, U \subset U_0$ there exists a function $m_{K,U}(\cdot) \in L(I, [0, \infty))$, such that for any $(x, y, u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t, x, y, u) | + | f_x(\cdot) | + | f_y(\cdot) | + | f_u(\cdot) | \leq m_{K,U}(t).$$

Further, let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t), t \in I$, satisfying the conditions:

$$\tau(t) < t, \dot{\tau}(t) > 0, \inf\{\tau(a) : \tau \in D\} := \hat{\tau} > -\infty.$$

Let Φ be the set of continuously differentiable initial functions $\varphi(t) \in O, t \in I_1 = [\hat{\tau}, b]$ and let $\Omega = \{u \in E_u : clu(I) \subset U_0\}$ be the set of control functions, where E_u is the space of bounded measurable functions $u : I \to \mathbb{R}^r_u$ and $u(I) = \{u(t) : t \in I\}$

To each element $\mu = (t_0, \tau, \varphi, u) \in \Lambda = [a, b) \times D \times \Omega$ we assign the quasi-linear controlled neutral functional-differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t)), u(t)) \qquad (1)$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \qquad (2)$$

where $A(t)$ is a given continuous matrix function with dimension $n \times n; \sigma \in D$ is a fixed delay function.

Definition 1. Let $\mu = (t_0, \tau, \varphi, u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a given element and let $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\hat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

Let us introduce the set of variations

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta u) : \mid \delta t_0 \mid \le \alpha, \parallel \delta\tau \parallel \le \alpha,$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \mid \lambda_i \mid \le \alpha, i = \overline{1,k}, \parallel \delta u \parallel \le \alpha \Big\}.$$

Here

$$\delta t_0 \in \mathbb{R}, \delta\tau \in D - \tau_0, \|\delta\tau\| = \sup\{|\delta\tau(t)| : t \in I\}, \delta u \in \Omega - u_0$$

and

$$\delta\varphi_i \in \Phi - \varphi_0, i = \overline{1,k}$$

are fixed functions, $\alpha > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1] \times V$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and there corresponds the solution $x(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ ([1],Theorem 3).

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0 + \varepsilon\delta\mu) - x_0(t), \forall (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1] \times V.$$

Theorem 1. *Let the following conditions hold:*

1) *The function $f_0(z), z = (t, x, y) \in I \times O^2$ is bounded, where $f_0(t, x, y) = f(t, x, y, u_0(t))$;*

2) *There exists the limit*

$$\lim_{z \to z_0} f_0(z) = f_0^-, z \in (a, t_{00}] \times O^2$$

where $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that

$$\Delta x(t;\varepsilon\delta\mu) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$ and

$$\delta x(t;\delta\mu) = Y(t_{00}-;t)[\dot{\varphi}_0(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^-]\delta t_0 + \beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu) = \Psi(t_{00};t)\delta\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$+\int_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s)\dot{\delta\varphi}(s)ds+\int_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds$$

$$+\int_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)]ds; \tag{5}$$

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon}=0 \; uniformly \; for \;\; (t,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times V^-,$$

$Y(s;t)$ and $\Psi(s;t)$ are $n\times n$-matrix functions satisfying the system

$$\begin{cases}\Psi_s(s;t)=-Y(s;t)f_{0x}[t]-Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s),\\ Y(s;t)=\Psi(s;t)+Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s), s\in[t_{00}-\delta_2,t]\end{cases}$$

and the condition

$$\Psi(s;t)=Y(s;t)=\begin{cases}H, s=t,\\ \Theta, s>t;\end{cases}$$

$$f_{0x}[s]=f_{0x}(s,x_0(s),x_0(\tau_0(s)));$$

$\gamma_0(s)$ is the inverse function of $\tau_0(t)$, $\varrho(s)$ is the inverse function of $\sigma(t)$, H is the identity matrix and Θ is the zero matrix.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t), t\in[t_{00},t_{10}+\delta_2]$, and the expression (4) is called the variation formula.

The addend

$$\int_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds$$

in formula (5) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression

$$Y(t_{00}-;t)[\dot{\varphi}_0(t_{00})-A(t_{00})\dot{\varphi}_0(\sigma(t_{00}))-f_0^-]\delta t_0$$

is the effect of continuous initial condition (2) and perturbation of the initial moment t_{00}.

The expression

$$\Psi(t_{00};t)\delta\varphi(t_{00})+\int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$+\int_{\sigma(t_{00})}^{t_{00}} Y(\varrho(s);t)A(\varrho(s))\dot{\varrho}(s)\dot{\delta\varphi}(s)ds$$

in formula (5) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$\int_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)]ds$$

in formula (5) is the effect of perturbation of the control function $u_0(t)$.

Variation formulas of solution for various classes of neutral functional differential equations without perturbation of delay are given in [2-4]. The variation formula of solution plays the basic role in proving the necessary conditions of optimality and under sensitivity analysis of mathematical models [5-8]. Finally we note that the variation formula allows to obtain an approximate solution of the perturbed equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau_0(t) + \varepsilon\delta\tau(t)), u_0(t) + \varepsilon\delta u(t))$$

with the perturbed initial condition

$$x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), t \in [\hat{\tau}, t_{00} + \varepsilon\delta t_0].$$

In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ it follows from (3) that

$$x(t; \mu_0 + \varepsilon\delta\mu) \approx x_0(t) + \varepsilon\delta x(t; \delta\mu).$$

Theorem 2. *Let the following conditions hold:*
1) *The function $f_0(z), z \in I \times O^2$ is bounded;*
2) *There exists the limit*

$$\lim_{z \to z_0} f_0(z) = f_0^+, z \in [t_{00}, b) \times O^2$$

Then for each $\hat{t}_0 \in (t_{00}, t_{10})$ there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^+$, where $V^+ = \{\delta\mu \in V : \delta t_0 \geq 0\}$, formula (3) holds, where

$$\delta x(t; \delta\mu) = Y(t_{00}+; t)(\dot{\varphi}(t_{00}) - A(t_{00})\dot{x}(\sigma(t_{00})) - f_0^+)\delta t_0 + \beta(t; \delta\mu).$$

The following assertion is a corollary to Theorems 1 and 2.

Theorem 3. *Let the assumptions of Theorems 1 and 2 be fulfilled. Moreover, $f_0^- = f_0^+ := \hat{f}_0$ and $t_{00} \notin \{\sigma(t_{10}), \sigma^2(t_{10})), ...\}$. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V$ formula (3) holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(A(t_{00})\dot{x}(\sigma(t_{00})) - \hat{f}_0)\delta t_0 + \beta(t; \delta\mu).$$

All assumptions of Theorem 3 are satisfied if the function $f_0(t, x, y)$ is continuous and bounded. Clearly, in this case $\hat{f}_0 = f_0(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.

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ON THE EXISTENCE OF AN OPTIMAL ELEMENT IN QUASI-LINEAR NEUTRAL OPTIMAL PROBLEMS

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Abstract. For an optimal control problem involving neutral differential equation, whose right-hand side is linear with respect to prehistory of the phase velocity, existence theorems of optimal element are proved. Under element we imply the collection of delay parameters and initial functions, initial moment and vector, control and finally moment.

Keywords and phrases: Neutral differential equation, neutral optimal problem, optimal element, existence theorem.

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1. Formulation of main results

Let R^n_x be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T is the sign of transposition, let $a < t_{01} < t_{02} < t_{11} < t_{12} < b, 0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2$ be given numbers with $t_{11} - t_{02} > \max\{\tau_2, \sigma_2\}$; suppose that $O \subset R^n_x$ is a open set and $U \subset R^r_u$ is a compact set, the function $F(t, x, y, u) = (f^0(t, x, y, u), f^1(t, x, y, u), ...,$ $f^n(t, x, y, u))^T$ is continuous on the set $I \times O^2 \times U$ and continuously differentiable with respect to x and y, where $I = [a, b]$; further, let Φ and Δ be sets of measurable initial functions $\varphi(t) \in K_0, t \in [\hat{\tau}, t_{02}]$ and $\varsigma(t) \in K_1, t \in [\hat{\tau}, t_{02}]$, respectively, where $\hat{\tau} = a - \max\{\tau_2, \sigma_2\}, K_0 \subset O$ is a compact set, $K_1 \subset R^n_x$ is a convex and compact set ; let Ω be a set of measurable control functions $u(t) \in U, t \in I$ and let $g^i(t_0, t_1, \tau, \eta, x_0, x_1), i = \overline{0, l}$ be continuous scalar functions on the set $[t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times X_0 \times O$, where $X_0 \subset O$ is a compact set.

To each element $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W = [t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times X_0 \times \Phi \times \Delta \times \Omega$ we assign the quasi-linear neutral differential equation

$$\dot{x}(t) = A(t)\dot{x}(t - \sigma) + f(t, x(t), x(t - \tau), u(t)), t \in [t_0, t_1] \tag{1.1}$$

with the initial condition

$$x(t) = \varphi(t), \dot{x}(t) = \varsigma(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0, \tag{1.2}$$

where $A(t) = (a^i_j(t)), i, j = \overline{1, n}, t \in I$ is a given $n \times n$ -dimensional continuous matrix function, $f = (f^1, ..., f^n)^T$.

Remark 1.1. The symbol $\dot{x}(t)$ on the interval $[\hat{\tau}, t_0)$ is not connected with derivative of the function $\varphi(t)$.

Definition 1.1. Let $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W$. A function $x(t) = x(t; w) \in O, t \in [\hat{\tau}, t_1]$, is called a solution corresponding to the element w, if it satisfies condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1.1) almost everywhere (a.e.) on $[t_0, t_1]$.

Definition 1.2. An element $w = (t_0, t_1, \tau, \sigma, x_0, \varphi, \varsigma, u) \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$ satisfying the condition

$$g(t_0, t_1, \tau, \sigma, x_0, x(t_1)) = 0, \tag{1.3}$$

where $g = (g^1, ..., g^l)$.

We denote the set of admissible elements by W_0. Now we consider the functional

$$J(w) = g^0(t_0, t_1, \tau, \sigma, x_0, x(t_1)) +$$

$$\int_{t_0}^{t_1} \Big[a_0(t)\dot{x}(t-\sigma) + f^0(t, x(t), x(t-\tau), u(t))\Big] dt, w \in W_0,$$

where $x(t) = x(t; w)$, and $a_0(t) = (a_0^1(t), ..., a_0^n(t)), t \in I$ is a given continuous function.

Definition 1.3. An element $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, \varphi_0, \varsigma_0, u_0) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{1.4}$$

The problem (1.1)-(1.4) is called the quasi-linear neutral optimal problem.

Theorem 1.1. *There exists an optimal element w_0 if the following conditions hold:*

1.1. $W_0 \neq \emptyset$;

1.2. There exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_0$

$$x(t; w) \in K_2, t \in [\hat{\tau}, t_1];$$

1.3. The sets

$$P(t, x) = \big\{F(t, x, y, u) : (y, u) \in K_0 \times U\big\}, (t, x) \in I \times O$$

and

$$P_1(t, x, y) = \big\{F(t, x, y, u) : u \in U\big\}, (t, x, y) \in I \times O^2$$

are convex.

Remark 1.2. Let K_0 and U be convex sets, and

$$F(t, x, y, u) = B(t, x)y + C(t, x)u.$$

Then the condition 1.3 of Theorem 1.1 holds.

Theorem 1.2. *There exists an optimal element w_0 if the conditions 1.1 and 1.2 of Theorem 1.1 hold, moreover the following conditions are fulfilled:*

1.4. The function $f(t, x, y, u)$ has a form

$$f(t, x, y, u) = D(t, x)y + E(t, x)u;$$

1.5. The sets K_0 and U are convex and for each fixed $(t, x) \in I \times O$ the function $f^0(t, x, y, u)$ is convex in $(y, u) \in K_0 \times U$.

The proof of existence of optimal delay parameters, initial functions and initial moment is the essential novelty in this work. Theorems of existence for optimal control problems involving various functional differential equations with fixed delay , initial function and moment are given in [1-5].

2. Auxiliary assertions

To each element $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi = [t_{01}, t_{02}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times O \times \Phi \times \Delta \times \Omega$ we will set in correspondence the functional differential equation

$$\dot{q}(t) = A(t)h(t_0, \varsigma, \dot{q})(t - \sigma) + f(t, q(t), h(t_0, \varphi, q)(t - \tau), u(t)) \tag{2.1}$$

with the initial condition

$$q(t_0) = x_0, \tag{2.2}$$

where the operator $h(t_0, \varphi, q)(t)$ is defined by the formula

$$h(t_0, \varphi, q)(t) = \begin{cases} \varphi(t), t \in [\hat{\tau}, t_0), \\ q(t), t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi$. A function $q(t) = q(t; \mu) \in O, t \in [r_1, r_2]$, where $r_1 \in [t_{01}, t_{02}], r_2 \in [t_{11}, t_{12}]$, is called a solution corresponding to the element μ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, and it satisfies condition (2.2) and is absolutely continuous on the interval $[r_1, r_2]$ and satisfies equation (2.1) a.e. on $[r_1, r_2]$.

Let $K_i \subset O, i = 3, 4$ be compact sets and K_4 contains a certain neighborhood of the set K_3.

Theorem 2.1. *Let $q_i(t) \in K_3, i = 1, 2, ...,$ be a solution corresponding to the element $\mu_i = (t_{0i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in \Pi, i = 1, 2, ...,$ respectively,defined on the interval $[t_{0i}, t_{1i}]$, where $t_{1i} \in [t_{11}, t_{12}]$. Moreover,*

$$\lim_{i \to \infty} t_{0i} = t_{00}, \lim_{i \to \infty} \sigma_i = \sigma_0, \lim_{i \to \infty} t_{1i} = t_{10}. \tag{2.4}$$

Then there exist numbers $\delta > 0$ and $M > 0$ such that for a sufficiently large i_0 the solution $\psi_i(t)$ corresponding to the element $\mu_i, i \geq i_0$, respectively, is defined on the interval $[t_{00} - \delta, t_{10} + \delta] \subset I$. Moreover,

$$\psi_i(t) \in K_4, | \dot{\psi}_i(t) | \leq M, t \in [t_{00} - \delta, t_{10} + \delta]$$

and

$$\psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}] \subset [t_{00} - \delta, t_{10} + \delta].$$

Proof. Let $\varepsilon > 0$ be so small that a closed ε -neighborhood of the set $K_3 : K_3(\varepsilon) = \{x \in O : \exists \hat{x} \in K_3, |x - \hat{x}| \leq \varepsilon\}$ is contained $intK_4$. There exists a compact set $Q \subset R^n_x \times R^n_y$

$$K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)] \subset Q \subset K_4 \times [K_0 \cup K_4]$$

and a continuously differentiable function $\chi : R_x^n \times R_y^n \to [0,1]$ such that

$$\chi(x,y) = \begin{cases} 1, (x,y) \in Q, \\ 0, (x,y) \notin K_4 \times [K_0 \cup K_4] \end{cases} \tag{2.5}$$

(see [6]). For each $i = 1, 2, ...$ the differential equation

$$\dot{\psi}(t) = A(t)h(t_{0i}, \varsigma_i, \dot{\psi})(t - \sigma_i) + \phi(t, \psi(t), h(t_{0i}, \varphi_i, \psi)(t - \tau_i), u_i(t)),$$

where

$$\phi(t, x, y, u) = \chi(x, y) f(t, x, y, u),$$

with the initial condition

$$\psi(t_{0i}) = x_{0i},$$

has the solution $\psi_i(t)$ defined on the interval I (see proof of Theorem 4.1,[7]). Since

$$(q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i)) \in K_3 \times [K_0 \cup K_3] \subset Q, t \in [t_{0i}, t_{1i}],$$

(see (2.3)), therefore

$$\chi(q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i)) = 1, t \in [t_{0i}, t_{1i}],$$

(see (2.5)),i.e.

$$\phi(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i), u_i(t)) = f(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(t - \tau_i), u_i(t)),$$

$$t \in [t_{0i}, t_{1i}].$$

By the uniqueness

$$\psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}]. \tag{2.6}$$

There exists a number $M > 0$ such that

$$| \dot{\psi}_i(t) | \leq M, t \in I, i = 1, 2, ... \tag{2.7}$$

Indeed, first of all we note that

$$| \phi(t, \psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t - \tau_i), u_i(t)) | \leq \sup \Big\{ | \phi(t, x, y, u) | : t \in I, x \in K_4,$$

$$y \in K_4 \cup K_0, u \in U \Big\} := N_1, i = 1, 2, ...$$

It is not difficult to see that for sufficiently large i_0 we have

$$\left[\frac{b - t_{0i}}{\sigma_i} \right] = \left[\frac{b - t_{00}}{\sigma_0} \right] := d, i \geq i_0,$$

where $[\alpha]$ means the integer part of a number α, i.e.

$$t_{0i} + d\sigma_i \leq b < t_{0i} + (d+1)\sigma_i.$$

If $t \in [a, t_{0i} + \sigma_i)$ then

$$| \dot{\psi}_i(t) | = | A(t)\varsigma_i(t - \sigma_i) + \phi(t, \psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t - \tau_i), u_i(t)) |$$

$$\leq \| A \| N_2 + N_1 := M_1,$$

where

$$\| A \| = \sup \Big\{ \mid A(t) \mid : t \in I \Big\}, N_2 = \sup \Big\{ \mid \xi \mid : \xi \in K_1 \Big\}.$$

Let $t \in [t_{0i} + \sigma_i, t_{0i} + 2\sigma_i)$ then

$$| \dot{\psi}_i(t) | \leq \| A \| \| \dot{\psi}_i(t - \sigma_i) | + N_1 \leq \| A \| M_1 + N_1 := M_2$$

Continuing this process we obtain

$$| \dot{\psi}_i(t) | \leq \| A \| M_{j-1} + N_1 := M_j, t \in [t_{0i} + (j-1)\sigma_i, t_{0i} + j\sigma_i), j = 3, ..., d.$$

Moreover, if $t_{0i} + d\sigma_i < b$ then we have

$$| \dot{\psi}_i(t) | \leq M_{d+1}, t \in [t_{0i} + d\sigma_i, b].$$

It is clear that for $M = \max\{M_1, ..., M_{d+1}\}$ the condition (2.7) is fulfilled.

Further, there exists a number $\delta_0 > 0$ such that for an arbitrary $i = 1, 2..., [t_{0i} - \delta_0, t_{1i} + \delta_0] \subset I$ and the following conditions hold

$$|\psi_i(t_{0i}) - \psi_i(t)| \leq \int_t^{t_{0i}} \Big[|A(s)h(t_{0i}, \varsigma_i, \dot{\psi}_i)(s - \sigma_i)|$$

$$+ |\phi(s, \psi_i(s), h(t_{0i}, \varphi_i, \psi_i)(s - \tau_i), u_i(s))| ds \leq \varepsilon, t \in [t_{0i} - \delta_0, t_{0i} \Big],$$

$$|\psi_i(t) - \psi_i(t_{1i})| \leq \int_{t_{1i}}^{t} \Big[|A(s)h(t_{0i}, \xi_i, \dot{\psi}_i)(s - \sigma_i)|$$

$$+ |\phi(s, \psi_i(s), h(t_{0i}, \varphi_i, \psi_i)(s - \tau_i), u_i(s))| \Big] ds \leq \varepsilon, t \in [t_{1i}, t_{1i} + \delta_0].$$

These inequalities, taking into account $\psi_i(t_{0i}) \in K_3$ and $\psi_i(t_{1i}) \in K_3$, (see (2.6)), yield

$$(\psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t - \tau_i)) \in K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)], t \in [t_{0i} - \delta_0, t_{1i} + \delta_0],$$

i.e.

$$\chi(\psi_i(t), h(t_{0i}, \varphi_i, \psi_i)(t - \tau_i)) = 1, t \in [t_{0i} - \delta_0, t_{1i} + \delta_0], i = 1, 2, ...,$$

Thus, $\psi_i(t)$ satisfies equation (2.1) and the conditions $\psi_i(t_{0i}) = x_{0i}, \psi_i(t) \in K_4, t \in [t_{0i} - \delta_0, t_{1i} + \delta_0]$, i.e. $\psi_i(t)$ is the solution corresponding to the element μ_i and defined on the interval $[t_{0i} - \delta_0, t_{1i} + \delta_0] \subset I$. Let $\delta \in (0, \delta_0)$, according to (2.4) for a sufficiently large i_0 we have

$$[t_{0i} - \delta_0, t_{1i} + \delta_0] \supset [t_{00} - \delta, t_{10} + \delta] \supset [t_{0i}, t_{1i}], i \geq i_0.$$

Consequently, $\psi_i(t), i \geq i_0$ solutions are defined on the interval $[t_{00}-\delta, t_{10}+\delta]$ and satisfy the conditions: $\psi_i(t) \in K_4, \mid \dot{\psi}_i(t) \mid \leq M, t \in [t_{00}-\delta, t_{10}+\delta]; \psi_i(t) = q_i(t), t \in [t_{0i}, t_{1i}]$, (see (2.6),(2.7)).

Theorem 2.2.([8]). *Let $p(t,u) \in R_p^m$ be a continuous function on the set $I \times U$ and let the set*

$$P(t) = \{p(t,u) : u \in U\}$$

be convex and

$$p_i(\cdot) \in L_1(I),\ p_i(t) \in P(t) \text{ a.e. on } I, i = 1, 2, ...$$

Moreover,

$$\lim_{i\to\infty} p_i(t) = p(t) \text{ weakly on } I.$$

Then

$$p(t) \in P(t) \text{ a.e. on } I$$

and there exists a measurable function $u(t) \in U, t \in I$ such that

$$p(t, u(t)) = p(t) \text{ a.e. on } I.$$

3. Proof of Theorem 1.1

Let

$$w_i = (t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in W_0, i = 1, 2, ...$$

be a minimizing sequence,i.e.

$$\lim_{i\to\infty} J(w_i) = \hat{J} = \inf_{w\in W_0} J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty} t_{0i} = t_{00}, \lim_{i\to\infty} t_{1i} = t_{10}, \lim_{i\to\infty} \tau_i = \tau_0, \lim_{i\to\infty} \sigma_i = \sigma_0, \lim_{i\to\infty} x_{0i} = x_{00}.$$

The set $\Delta \subset L_1([\hat{\tau}, t_{02}])$ is weakly compact (see Theorem 2.2), therefore we assume that

$$\lim_{i\to\infty} \varsigma_i(t) = \varsigma_0(t), \text{ weakly in } t \in [\hat{\tau}, t_{02}]. \tag{3.1}$$

Introduce the following notation:

$$x_i^0(t) = \int_{t_{0i}}^{t} \Big[a_0(s)\dot{x}_i(s-\sigma_i) + f^0(s, x_i(s), x_i(s-\tau_i), u_i(s))\Big] ds,$$

$$x_i(t) = x(t; w_i), \rho_i(t) = (x_i^0(t), x_i(t))^T, t \in [t_{0i}, t_{1i}].$$

Obviously,

$$\begin{cases} \dot{\rho}_i(t) = \hat{A}(t)\dot{x}_i(t-\sigma_i) + F(t, x_i(t), x_i(t-\tau_i), u_i(t)), t \in [t_{0i}, t_{1i}], \\ x_i(t) = \varphi_i(t), t \in [\hat{\tau}, t_{0i}), \rho_i(t_{0i}) = (0, x_{0i})^T, \\ \dot{x}_i(t) = \varsigma_i(t), t \in [\hat{\tau}, t_{0i}), \end{cases}$$

where $\hat{A}(t) = (a_0(t)\ A(t))^T$. It is clear that

$$J(w_i) = g^0(t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, x_i(t_{1i})) + x_i^0(t_{1i}).$$

To each element $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi$ we will set in correspondence the functional differential equation

$$\dot{z}(t) = \hat{A}(t)h(t_0, \varsigma, \dot{v})(t-\sigma) + F(t, v(t), h(t_0, \varphi, v)(t-\tau), u(t)),$$

with the initial condition

$$z(t_0) = z_0 = (0, x_0)^T,$$

where $z(t) = (v^0(t), v(t))^T \in R_z^{1+n}$.

It is easy to see that

$$\begin{cases} \dot{\rho}_i(t) = \hat{A}(t)h(t_{0i}, \varsigma_i, \dot{x}_i)(t-\sigma_i) + F(t, x_i(t), h(t_{0i}, \varphi_i, x_i)(t-\tau_i), u_i(t)), t \in [t_{0i}, t_{1i}], \\ \rho_i(t_{0i}) = (0, x_{0i})^T \end{cases}$$

(see (2.3)). Thus, $\rho_i(t)$ is the solution corresponding to $\mu_i = (t_{0i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in \Pi$ and defined on the interval $[t_{0i}, t_{1i}]$. Since $x_i(t) \in K_2$, therefore in a similar way (see the proof of Theorem 2.1) we prove that $|\ \dot{x}_i(t)\ | \le N_3, t \in [t_{0i}, t_{1i}], i = 1, 2, ..., N_3 > 0$. Further, there exists a compact $H_1 \subset H = \{z = (v^0, v)^T : v^0 \in R_{v^0}^1, v \in O\} \subset R_z^{1+n}$ such that $\rho_i(t) \in H_1, t \in [t_{0i}, t_{1i}]$.

Let $H_2 \subset H$ be a compact set containing a certain neighborhood of the set H_1. By Theorem 2.1 there exists a number $\delta > 0$ such that for a sufficiently large i_0 the solutions $z_i(t) = z(t; \mu_i), i \ge i_0$ are defined on the interval $[t_{00} - \delta, t_{10} + \delta] \subset I$ and the following conditions hold

$$\begin{cases} z_i(t) \in H_2, |\ \dot{z}_i(t)\ | \le M, t \in [t_{00} - \delta, t_{10} + \delta], \\ z_i(t) = \rho_i(t) = (x_i^0(t), x_i(t))^T, t \in [t_{0i}, t_{1i}], i \ge i_0. \end{cases} \tag{3.2}$$

Thus, there exist numbers $N_4 > 0$ and $N_5 > 0$ such hat

$$\begin{cases} |F(t, v_i(t), h(t_{0i}, \varphi_i, v_i)(t-\sigma_i), u_i(t))| \le N_5, \\ |\ h(t_{0i}, \varsigma_i, \dot{v}_i)(t-\eta_i)\ | \le N_4, t \in [t_{00} - \delta, t_{10} + \delta], i \ge i_0. \end{cases} \tag{3.3}$$

The sequence $z_i(t) = (v_i^0(t), v_i(t))^T, t \in [t_{00} - \delta, t_{10} + \delta], i \ge i_0$ is uniformly bounded and equicontinuous. By the Arzela-Ascoli lemma, from this sequence we can extract a subsequence, which will again be denoted by $z_i(t), i \ge i_0$, that

$$\lim_{i \to \infty} z_i(t) = z_0(t) = (v_0^0(t), v_0(t))^T \text{ uniformly in } [t_{00} - \delta, t_{10} + \delta].$$

Further, from the sequence $\dot{z}_i(t), i \ge i_0$, we can extract a subsequence, which will again be denoted by $\dot{z}_i(t), i \ge i_0$, that

$$\lim_{i \to \infty} \dot{z}_i(t) = \gamma(t) \text{ weakly in } [t_{00} - \delta, t_{10} + \delta],$$

(see (3.2)). Obviously,

$$z_0(t)=\lim_{i\to\infty}z_i(t)=\lim_{i\to\infty}[z_i(t_{00}-\delta)+\int_{t_{00}-\delta}^{t}\dot z_i(s)ds]$$

$$=z_0(t_{00}-\delta)+\int_{t_{00}-\delta}^{t}\gamma(s)ds.$$

Thus, $\dot z_0(t)=\gamma(t)$ i.e.

$$\lim_{i\to\infty}\dot z_i(t)=\dot z_0(t) \text{ weakly in } [t_{00}-\delta,t_{10}+\delta].$$

Further, we have

$$z_i(t)=z_{0i}+\int_{t_{0i}}^{t}\Big[\hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)+F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))\Big]ds$$

$$=z_{0i}+\vartheta_{1i}(t)+\vartheta_{2i}+\theta_{1i}(t)+\theta_{2i},t\in[t_{00},t_{10}],i\ge i_0,$$

where

$$z_{0i}=(0,x_{0i})^T,\vartheta_{1i}(t)=\int_{t_{00}}^{t}\hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)ds,$$

$$\theta_{1i}(t)=\int_{t_{00}}^{t}F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))ds,$$

$$\vartheta_{2i}=\int_{t_{0i}}^{t_{00}}\hat A(s)h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)ds,$$

$$\theta_{2i}=\int_{t_{0i}}^{t_{00}}F(s,v_i(s),h(t_{0i},\varphi_i,v_i)(s-\tau_i),u_i(s))ds.$$

Obviously, $\vartheta_{2i}\to 0$ and $\theta_{2i}\to 0$ as $i\to\infty$.

First of all we transform the expression $\vartheta_{1i}(t)$ for $t\in[t_{00},t_{10}]$. For this purpose, we consider two cases. Let $t\in[t_{00},t_{00}+\sigma_0]$, we have

$$\vartheta_{1i}(t)=\vartheta_{1i}^{(1)}(t)+\vartheta_{1i}^{(2)}(t),$$

where

$$\vartheta_{1i}^{(1)}(t)=\int_{t_{00}}^{t}\hat A(s)h(t_{00},\varsigma_i,\dot v_i)(s-\sigma_i)ds,\vartheta_{1i}^{(2)}(t)=\int_{t_{00}}^{t}\vartheta_{1i}^{(3)}(s)ds,$$

$$\vartheta_{1i}^{(3)}(s)=\hat A(s)\Big[h(t_{0i},\varsigma_i,\dot v_i)(s-\sigma_i)-h(t_{00},\varsigma_i,\dot v_i)(s-\sigma_i)\Big].$$

It is clear that

$$|\ \vartheta_{1i}^{(2)}(t)\ |\le\int_{t_{00}}^{t_{10}}|\ \vartheta_{1i}^{(3)}(s)\ |\ ds,t\in[t_{00},t_{10}]. \tag{3.4}$$

Suppose that $t_{0i}+\sigma_i>t_{00}$ for $i\ge i_0$. According to (2.3)

$$\vartheta_{1i}^{(3)}(s)=0,s\in[t_{00},t_{0i}^{(1)})\cup(t_{0i}^{(2)},t_{1i}],$$

where

$$t_{0i}^{(1)} = \min\{t_{0i} + \sigma_i, t_{00} + \sigma_i\}, t_{0i}^{(2)} = \max\{t_{0i} + \sigma_i, t_{00} + \sigma_i\}$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(2)} - t_{0i}^{(1)}) = 0,$$

therefore,

$$\lim_{i\to\infty}\vartheta_{1i}^{(2)}(t) = 0, \text{ uniformly in } t \in [t_{00}, t_{10}], \tag{3.5}$$

(see (3.3)). For $\vartheta_{1i}^{(1)}(t), t \in [t_{00}, t_{00} + \sigma_0]$ we get

$$\vartheta_{1i}^{(1)}(t) = \int_{t_{00}-\sigma_i}^{t-\sigma_i} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds = \vartheta_{1i}^{(4)}(t) + \vartheta_{1i}^{(5)}(t),$$

where

$$\vartheta_{1i}^{(4)}(t) = \int_{t_{00}-\sigma_0}^{t-\sigma_0} \hat{A}(s+\sigma_0)\varsigma_i(s)ds, \vartheta_{1i}^{(5)}(t) = \int_{t_{00}-\sigma_0}^{t-\sigma_0} \Big[\hat{A}(s+\sigma_i) - \hat{A}(s+\sigma_0)\Big]\varsigma_i(s)ds$$

$$+\int_{t_{00}-\sigma_i}^{t_{00}-\sigma_0} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds + \int_{t-\sigma_0}^{t-\sigma_i} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds.$$

Obviously,

$$\lim_{i\to\infty}\vartheta_{1i}^{(5)}(t) = 0 \text{ uniformly in } t \in [t_{00}, t_{00} + \sigma_0]$$

and

$$\lim_{i\to\infty}\vartheta_{1i}^{(1)}(t) = \lim_{i\to\infty}\vartheta_{1i}^{(4)}(t) = \int_{t_{00}-\sigma_0}^{t-\sigma_0} \hat{A}(s+\sigma_0)\varsigma_0(s)ds$$

$$= \int_{t_{00}}^{t} \hat{A}(s)\varsigma_0(s-\sigma_0)ds, t \in [t_{00}, t_{00} + \sigma_0] \tag{3.6}$$

(see (3.1)).

Let $t \in [t_{00} + \sigma_0, t_{10}]$ then

$$\vartheta_{1i}^{(1)}(t) = \vartheta_{1i}^{(1)}(t_{00} + \sigma_0) + \vartheta_{1i}^{(6)}(t),$$

where

$$\vartheta_{1i}^{(6)}(t) = \int_{t_{00}+\sigma_0}^{t} \hat{A}(s)h(t_{0i}, \varsigma_i, \dot{v}_i)(s-\sigma_i)ds.$$

Further,

$$\vartheta_{1i}^{(6)}(t) = \int_{t_{00}+\sigma_0}^{t} \hat{A}(s)h(t_{00}, \varsigma_i, \dot{v}_i)(s-\sigma_i)ds + \int_{t_{00}+\sigma_0}^{t} \vartheta_{1i}^{(3)}(s)ds = \vartheta_{1i}^{(7)}(t) + \vartheta_{1i}^{(8)}(t).$$

It is clear that

$$\lim_{i\to\infty}\vartheta_{1i}^{(8)}(t) = 0 \text{ uniformly in } t \in [t_{00} + \sigma_0, t_{10}],$$

(see (3.5)). For $\vartheta_{1i}^{(7)}(t), t \in [t_{00}+\sigma_0, t_{10}]$ we have

$$\vartheta_{1i}^{(7)}(t) = \int_{t_{00}+\sigma_0-\sigma_i}^{t-\sigma_i} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds = \vartheta_{1i}^{(9)}(t) + \vartheta_{1i}^{(10)}(t),$$

where

$$\vartheta_{1i}^{(9)}(t) = \int_{t_{00}}^{t-\sigma_0} \hat{A}(s+\sigma_0)\dot{v}_i(s)ds, \vartheta_{1i}^{(10)}(t) = \int_{t_{00}+\sigma_0-\sigma_i}^{t_{00}} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds$$

$$+\int_{t-\sigma_0}^{t-\sigma_i} \hat{A}(s+\sigma_i)h(t_{00}, \varsigma_i, \dot{v}_i)(s)ds + \int_{t_{00}}^{t-\sigma_0} \Big[\hat{A}(s+\sigma_i) - \hat{A}(s+\sigma_0)\Big]\dot{v}_i(s)ds.$$

Obviously,

$$\lim_{i\to\infty} \vartheta_{1i}^{(10)}(t) = 0 \text{ uniformly in } t \in [t_{00}+\sigma_0, t_{10}]$$

and

$$\lim_{i\to\infty} \vartheta_{1i}^{(1)}(t) = \lim_{i\to\infty} \vartheta_{1i}^{(1)}(t_{00}+\sigma_0) + \lim_{i\to\infty} \vartheta_{1i}^{(6)}(t) = \int_{t_{00}}^{t_{00}+\sigma_0} \hat{A}(t)\varsigma_0(t-\sigma_0)dt$$

$$+\lim_{i\to\infty} \vartheta_{1i}^{(9)}(t) = \int_{t_{00}}^{t_{00}+\sigma_0} \hat{A}(t)\varsigma_0(t-\sigma_0)dt + \int_{t_{00}}^{t-\sigma_0} \hat{A}(s+\sigma_0)\dot{v}_0(s)ds$$

$$= \int_{t_{00}}^{t_{00}+\sigma_0} \hat{A}(t)\varsigma_0(t-\sigma_0)dt + \int_{t_{00}+\sigma_0}^{t} \hat{A}(s)\dot{v}_0(s-\sigma_0)ds. \tag{3.7}$$

Now we transform the expression $\theta_{1i}(t)$ for $t \in [t_{00}, t_{10}]$. We consider two cases again . Let $t \in [t_{00}, t_{00}+\tau_0]$, we have

$$\theta_{1i}(t) = \theta_{1i}^{(1)}(t) + \theta_{1i}^{(2)}(t),$$

$$\theta_{1i}^{(1)}(t) = \int_{t_{00}}^{t} F(s, v_i(s), h(t_{00}, \varphi_i, v_i)(s-\tau_i), u_i(s))ds, \theta_{1i}^{(2)}(t) = \int_{t_{00}}^{t} \theta_{1i}^{(3)}(s)ds,$$

$$\theta_{1i}^{(3)}(s) = F(s, v_i(s), h(t_{0i}, \varphi_i, v_i)(s-\tau_i), u_i(s)) - F(s, v_i(s), h(t_{00}, \varphi_i, v_i)(s-\tau_i), u_i(s)).$$

It is clear that

$$\mid \theta_{1i}^{(2)}(t) \mid \le \int_{t_{00}}^{t_{10}} \mid \theta_{1i}^{(3)}(s) \mid ds, t \in [t_{00}, t_{10}]. \tag{3.8}$$

Suppose that $t_{0i}+\tau_i > t_{00}$ for $i \ge i_0$. According to (2.3)

$$\theta_{1i}^{(3)}(s) = 0, s \in [t_{00}, t_{0i}^{(3)}) \cup (t_{0i}^{(4)}, t_{1i}],$$

where

$$t_{1i}^{(3)} = \min\{t_{0i}+\tau_i, t_{00}+\tau_i\}, t_{1i}^{(4)} = \max\{t_{0i}+\tau_i, t_{00}+\tau_i\}.$$

Since

$$\lim_{i\to\infty} (t_{0i}^{(4)} - t_{0i}^{(3)}) = 0$$

therefore,

$$\lim_{i\to\infty}\theta_{1i}^{(2)}(t)=0 \text{ uniformly in } t\in[t_{00},t_{10}], \tag{3.9}$$

(see (3.3)). For $\theta_{1i}^{(1)}(t), t\in[t_{00},t_{00}+\tau_0]$, we have

$$\theta_{1i}^{(1)}(t)=\int_{t_{00}-\tau_i}^{t-\tau_i}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds$$

$$=\theta_{1i}^{(4)}(t)+\theta_{1i}^{(5)}(t), i\ge i_0,$$

where

$$\theta_{1i}^{(4)}(t)=\int_{t_{00}-\tau_0}^{t-\tau_0}F(s+\tau_0,v_0(s+\tau_0),\varphi_i(s),u_i(s+\tau_i))ds,$$

$$\theta_{1i}^{(5)}(t)=\int_{t_{00}-\tau_i}^{t-\tau_i}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds$$

$$-\int_{t_{00}-\tau_0}^{t-\tau_0}F(s+\tau_0,v_0(s+\tau_0),\varphi_i(s),u_i(s+\tau_i))ds.$$

For $t\in[t_{00},t_{00}+\tau_0]$ we obtain

$$\theta_{1i}^{(5)}(t)=\int_{t_{00}-\tau_i}^{t_{00}-\tau_0}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds$$

$$+\int_{t_{00}-\tau_0}^{t-\tau_0}[F(s+\tau_i,v_i(s+\tau_i),\varphi_i(s),u_i(s+\tau_i))-F(s+\tau_0,v_0(s+\tau_0),\varphi_i(s),u_i(s+\tau_i))]ds$$

$$+\int_{t-\tau_0}^{t-\tau_i}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds.$$

Suppose that $|\ \tau_i-\tau_0\ |\le\delta$ as $i\ge i_0$. According to condition (3.3) and

$$\lim_{i\to\infty}F(s+\tau_i,v_i(s+\tau_i),y,u)=F(s+\tau_0,v_0(s+\tau_0),y,u)$$

uniformly in $(s,y,u)\in[t_{00}-\tau_0,t_{00}]\times K_0\times U$, we have

$$\lim_{i\to\infty}\theta_{1i}^{(5)}(t)=0 \text{ uniformly in } t\in[t_{00},t_{00}+\tau_0].$$

From the sequence $F_i(s)=F(s+\tau_0,v_0(s+\tau_0),\varphi_i(s),u_i(s+\tau_i)), i\ge i_0, t\in[t_{00}-\tau_0,t_{00}]$, we extract a subsequence, which will again be denoted by $F_i(s), i\ge i_0$, such that

$$\lim_{i\to\infty}F_i(s)=F_0(s) \text{ weakly in the space } L_1([t_{00}-\tau_0,t_{00}]),$$

(see (3.3)). It is not difficult to see that

$$F_i(s)\in P(s+\tau_0,v_0(s+\tau_0)), s\in[t_{00}-\tau_0,t_{00}].$$

By Theorem 2.2

$$F_0(s)\in P(s+\tau_0,v_0(s+\tau_0)) \text{ a.e.} s\in[t_{00}-\tau_0,t_{00}]$$

and on the interval $[t_{00}-\tau_0, t_{00}]$ there exist measurable functions $\varphi_{01}(s) \in K_0, u_{01}(s) \in U$ such that

$$F_0(s) = F(s+\tau_0, v_0(s+\tau_0), \varphi_{01}(s), u_{01}(s)) \text{ a.e.} s \in [t_{00}-\tau_0, t_{00}].$$

Thus ,

$$\lim_{i\to\infty}\theta_{1i}^{(1)}(t) = \lim_{i\to\infty}\theta_{1i}^{(4)}(t) = \int_{t_{00}-\tau_0}^{t-\tau_0} F_0(s)ds$$

$$= \int_{t_{00}-\tau_0}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), \varphi_{01}(s), u_{01}(s))ds$$

$$= \int_{t_{00}}^{t} F(s, v_0(s), \varphi_{01}(s-\tau_0), u_{01}(s-\tau_0))ds, t \in [t_{00}, t_{00}+\tau_0]. \tag{3.10}$$

Let $t \in [t_{00}+\tau_0, t_{10}]$ then

$$\theta_{1i}^{(1)}(t) = \theta_{1i}^{(1)}(t_{00}+\tau_0) + \theta_{1i}^{(6)}(t), t \in [t_{00}+\tau_0, t_{10}],$$

where

$$\theta_{1i}^{(6)}(t) = \int_{t_{00}+\tau_0}^{t} F(s, v_i(s), h(t_{0i}, \varphi_i, v_i)(s-\tau_i), u_i(s))ds.$$

Further,

$$\theta_{1i}^{(6)}(t) = \theta_{1i}^{(7)}(t) + \theta_{1i}^{(8)}(t),$$

$$\theta_{1i}^{(7)}(t) = \int_{t_{00}+\tau_0}^{t} F(s, v_i(s), h(t_{00}, \varphi_i, v_i))(s-\tau_i), u_i(s))ds, \theta_{1i}^{(8)}(t) = \int_{t_{00}+\tau_0}^{t} \theta_{1i}^{(3)}(s)ds.$$

It is clear that

$$\lim_{i\to\infty}\theta_{1i}^{(8)}(t) = 0 \text{ uniformly in } t \in [t_{00}+\tau_0, t_{10}],$$

(see (3.8),(3.9)). For the expression $\theta_{1i}^{(7)}(t), t \in [t_{00}+\tau_0, t_{10}]$ we have

$$\theta_{1i}^{(7)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$= \theta_{1i}^{(9)}(t) + \theta_{1i}^{(10)}(t), i \geq i_0,$$

where

$$\theta_{1i}^{(9)}(t) = \int_{t_{00}}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), v_0(s), u_i(s+\tau_i))ds,$$

$$\theta_{1i}^{(10)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t-\tau_i} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$- \int_{t_{00}}^{t-\tau_0} F(s+\tau_0, v_0(s+\tau_0), v_0(s), u_i(s+\tau_i))ds.$$

Clearly, for $t \in [t_{00}+\tau_0, t_{10}]$ we get

$$\theta_{1i}^{(10)}(t) = \int_{t_{00}+\tau_0-\tau_i}^{t_{00}} F(s+\tau_i, v_i(s+\tau_i), h(t_{00}, \varphi_i, v_i)(s), u_i(s+\tau_i))ds$$

$$+\int_{t_{00}}^{t-\tau_0}[F(s+\tau_i,v_i(s+\tau_i),v_i(s),u_i(s+\tau_i))-F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_i(s+\tau_i))]ds$$

$$+\int_{t-\tau_0}^{t-\tau_i}F(s+\tau_i,v_i(s+\tau_i),h(t_{00},\varphi_i,v_i)(s),u_i(s+\tau_i))ds.$$

According to condition (3.3) and

$$\lim_{i\to\infty}F(s+\tau_i,v_i(s+\tau_i),v_i(s),u)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u)$$

uniformly in $(s,u)\in[t_{00},t_{10}-\tau_0]\times U$, we obtain

$$\theta_{1i}^{(10)}(t)=0 \text{ uniformly in } t\in[t_{00}+\tau_0,t_{10}].$$

From the sequence $F_i(s)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_i(s+\tau_i)),i\geq i_0,t\in[t_{00},t_{10}-\tau_0]$, we extract a subsequence, which will again be denoted by $F_i(s),i\geq i_0$, such that

$$\lim_{i\to\infty}F_i(s)=F_0(s) \text{ weakly in the space } L_1([t_{00},t_{10}-\tau_0]).$$

It is not difficult to see that

$$F_i(s)\in P_1(s+\tau_0,v_0(s+\tau_0),v_0(s)),s\in[t_{00},t_{10}-\tau_0].$$

By Theorem 2.2

$$F_0(s)\in P_1(s+\tau_0,v_0(s+\tau_0),v_0(s)),a.e.s\in[t_{00},t_{10}-\tau_0]$$

and on the interval $[t_{00},t_{10}-\tau_0]$ there exists a measurable function $u_{02}(s)\in U$ such that

$$F_0(s)=F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_{02}(s)) \text{ a.e.} s\in[t_{00},t_{10}-\tau_0].$$

Thus,

$$\lim_{i\to\infty}\theta_{1i}^{(1)}(t)=\lim_{i\to\infty}\theta_{1i}^{(1)}(t_{00}+\tau_0)+\lim_{i\to\infty}\theta_{1i}^{(9)}(t)=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}}^{t-\tau_0}F_0(s)ds=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}}^{t-\tau_0}F(s+\tau_0,v_0(s+\tau_0),v_0(s),u_{02}(s))ds=\int_{t_{00}}^{t_{00}+\tau_0}F(s,v_0(s),\varphi_{01}(s-\tau_0),u_{01}(s-\tau_0))ds$$

$$+\int_{t_{00}+\tau_0}^{t}F(s,v_0(s),v_0(s-\tau_0),u_{02}(s-\tau_0))ds,t\in[t_{00}+\tau_0,t_{10}], \tag{3.11}$$

(see (3.10)).

Introduce the following notation

$$\varphi_0(s)=\begin{cases}\hat{\varphi},s\in[\hat{\tau},t_{00}-\tau_0)\cup(t_{00},t_{02}],\\ \varphi_{01}(s),s\in[t_{00}-\tau_0,t_{00}],\end{cases}$$

$$u_0(s) = \begin{cases} \hat{u}, s \in [a, t_{00}) \cup (t_{10}, b], \\ u_{01}(s - \tau_0), s \in [t_{00}, t_{00} + \tau_0], \\ u_{02}(s - \tau_0), s \in (t_{00} + \tau_0, t_{10}], \end{cases}$$

where $\hat{\varphi} \in K_0$ and $\hat{u} \in U$ are fixed points;

$$x_0(t) = \begin{cases} \varphi_0(t), t \in [\hat{\tau}, t_{00}), \\ v_0(t), t \in [t_{00}, t_{10}]; \end{cases}$$

$$\dot{x}_0(t) = \varsigma_0(t), t \in [\hat{\tau}, t_{00}),$$

(see Remark 1.1),

$$x_0^0(t) = v^0(t), t \in [t_{00}, t_{10}].$$

Clearly, $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, \varphi_0, \varsigma_0, u_0) \in W$. Taking into account (3.6),(3.7),(3.10) and (3.11) we obtain

$$x_0^0(t) = \int_{t_{00}}^{t} \Big[a_0(s)\dot{x}_0(s - \sigma_0) + f^0(s, x_0(s), x_0(s - \tau_0), u_0(s))\Big]ds, t \in [t_{00}, t_{10}],$$

$$x_0(t) = x_{00} + \int_{t_{00}}^{t} \Big[A(s)\dot{x}_0(s - \sigma_0) + f(s, x_0(s), x_0(s - \tau_0), u_0(s))\Big]ds, t \in [t_{00}, t_{10}].$$

It is not difficult to see that

$$\lim_{i\to\infty}(x_i^0(t_{1i}), x_i(t_{1i}))^T = \lim_{i\to\infty}\rho_i(t_{1i}) = \lim_{i\to\infty} z_i(t_{1i}))$$

$$= \lim_{i\to\infty}[z_i(t_{1i}) - z_i(t_{10})] + \lim_{i\to\infty}[z_i(t_{10}) - z_0(t_{10})] + z_0(t_{10}) = z_0(t_{10})$$

$$= (v^0(t_{10}), v_0(t_{10}))^T = (x_0^0(t_{10}), x_0(t_{10}))^T \in H,$$

(see (3.2)). Consequently,

$$0 = \lim_{i\to\infty} g(t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, x_i(t_{1i})) = g(t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, x_0(t_{10})),$$

i.e. the element w_0 is admissible and $x_0(t) = x(t; w_0), t \in [\hat{\tau}, t_{10}]$.

Further, we have

$$\hat{J} = \lim_{i\to\infty}[g^0(t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, x_i(t_{1i})) + x_i^0(t_{1i})] = g(t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, x_0(t_{10}))$$

$$+x_0^0(t_{10}) = J(w_0).$$

Thus, w_0 is an optimal element.

4. Proof of Theorem 1.2

First of all we note that the sets $\Delta \subset L_1([\hat{\tau}, t_{02}])$ and $\Omega \subset L_1(I)$ are weakly compacts (see Theorem 2.2). Let

$$w_i = (t_{0i}, t_{1i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i) \in W_0, i = 1, 2, ...$$

be a minimizing sequence,i.e.

$$\lim_{i\to\infty} J(w_i) = \hat{J} = \inf_{w\in W_0} J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty} t_{0i} = t_{00}, \lim_{i\to\infty} t_{1i} = t_{10}, \lim_{i\to\infty} \tau_i = \tau_0, \lim_{i\to\infty} \sigma_i = \sigma_0, \lim_{i\to\infty} x_{0i} = x_{00},$$

$$\begin{cases} \lim_{i\to\infty} \varphi_i(t) = \varphi_0(t), \text{weakly on } [\hat{\tau}, t_{02}], \\ \lim_{i\to\infty} \varsigma_i(t) = \varsigma_0(t), \text{weakly on } [\hat{\tau}, t_{02}], \\ \lim_{i\to\infty} u_i(t) = u_0(t) \text{ weakly on } I. \end{cases} \tag{4.1}$$

(see (3.1)).

To each element $\mu = (t_0, \tau, \sigma, x_0, \varphi, \varsigma, u) \in \Pi$ we will set in correspondence the functional differential equation

$$\dot{\zeta}(t) = A(t)h(t_0, \varsigma, \dot{\zeta})(t-\sigma) + C(t, \zeta(t))h(t_0, \varphi, \zeta)(t-\tau) + D(t, \zeta(t))u(t)$$

with the initial condition

$$\zeta(t_0) = x_0$$

It is easy to see that for $x_i(t) = x(t; w_i)$ we have

$$\begin{cases} \dot{x}_i(t) = A(t)h(t_0, \varsigma, \dot{x}_i)(t-\sigma_i) + C(t, x_i(t))h(t_{0i}, \varphi_i, x_i)(t-\tau_i) + \\ D(t, x_i(t))u_i(t), t \in [t_{0i}, t_{1i}], \\ x_i(t_{0i}) = x_{0i}. \end{cases}$$

Thus, $x_i(t) \in K_2$ is the solution corresponding to $\mu_i = (t_{0i}, \tau_i, \sigma_i, x_{0i}, \varphi_i, \varsigma_i, u_i)$ and defined on the interval $[t_{0i}, t_{1i}]$. Let $\hat{K}_2 \subset O$ be a compact set containing a certain neighborhood of the set K_2. By Theorem 2.1 there exists a number $\delta > 0$ such that for a sufficiently large i_0 the solutions $\zeta_i(t) = \zeta(t; \mu_i), i \geq i_0$ are defined on the interval $[t_{00} - \delta, t_{10} + \delta] \subset I$ and

$$\zeta_i(t) \in \hat{K}_2, t \in [t_{00} - \delta, t_{10} + \delta],\ \zeta_i(t) = x_i(t), t \in [t_{0i}, t_{1i}], i \geq i_0.$$

After this (see the proof of Theorem 1.1) we prove in the standard way that

$$\lim_{i\to\infty} \zeta_i(t) = \zeta_0(t) \text{ uniformly in } t \in [t_{00} - \delta, t_{10} + \delta],$$

and

$$\lim_{i\to\infty} \dot{\zeta}_i(t) = \dot{\zeta}_0(t) \text{ weakly on } t \in [t_{00} - \delta, t_{10} + \delta],$$

where $\zeta_0(t)$ is the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, \sigma_0, x_{00}, \varphi_0, \varsigma_0, u_0)$, defined on the interval $[t_{00} - \delta, t_{10} + \delta]$ and satisfying the condition $\zeta_0(t_{00}) = x_{00}$. Moreover,

$$\lim_{i\to\infty} x_i(t_{1i}) = \lim_{i\to\infty} \zeta_i(t_{1i}) = \lim_{i\to\infty} [\zeta_i(t_{1i}) - \zeta_i(t_{10})]$$

$$+\lim_{i\to\infty}[\zeta_i(t_{10})-\zeta_0(t_{10})]+\zeta_0(t_{10})=\zeta_0(t_{10}),$$

Hence,

$$0=\lim_{i\to\infty} g(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))=g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},\zeta_0(t_{10})).$$

Introduce the following notation

$$x_0(t)=\begin{cases}\varphi_0(t),t\in[\hat\tau,t_{00}),\\ \zeta_0(t),t\in[t_{00},t_{10}]\end{cases} \tag{4.2}$$

$$\dot x_0(t)=\varsigma_0(t),t\in[\hat\tau,t_{00}), \tag{4.3}$$

(see Remark 1.1).

Clearly the function $x_0(t)$ is the solution corresponding to the element $w_0=(t_{00},t_{10},\tau_0,\sigma_0,x_{00},\varphi_0,\varsigma_0,u_0)\in W$ and satisfying the condition

$$g(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))=0,$$

i.e. $w_0\in W_0$.

Now we prove optimality of the element w_0. We have,

$$\lim_{i\to\infty} g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10})),$$

$$\int_{t_{0i}}^{t_{1i}} a_0(t)\dot x_i(t-\sigma_i)dt=\int_{t_{0i}}^{t_{1i}} a_0(t)h(t_{1i},\xi_i,\dot\zeta_i)(t-\sigma_i)dt,$$

$$\int_{t_{0i}}^{t_{1i}} f^0(t,x_i(t),x_i(t-\tau_i),u_i(t))dt=\int_{t_{0i}}^{t_{1i}} f^0(t,\zeta_i(t),h(t_{0i},\varphi_i,\zeta_i)(t-\tau_i),u_i(t))dt.$$

In a similar way (see proof of Theorem 1.1) it can be proved that

$$\int_{t_{0i}}^{t_{1i}} a_0(t)h(t_{1i},\varsigma_i,\dot\zeta_i)(t-\eta_i)dt=\varrho_{1i}+\varrho_{2i}+\varrho_{3i}$$

$$\int_{t_{0i}}^{t_{1i}} f^0(t,\zeta_i(t),h(t_{0i},\varphi_i,\zeta_i)(t-\tau_i),u_i(t))dt=\gamma_{1i}+\gamma_{2i}+\gamma_{3i},$$

where

$$\varrho_{1i}=\int_{t_{00}-\sigma_0}^{t_{00}} a_0(t+\sigma_0)\xi_i(t)dt,\ \varrho_{2i}=\int_{t_{00}}^{t_{10}-\sigma_0} a_0(t+\sigma_0)\dot v_i(t)dt$$

$$\gamma_{1i}=\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi_i(t),u_i(t+\tau_i))dt,$$

$$\gamma_{2i}=\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u_i(t+\tau_i))dt$$

and

$$\lim_{i\to\infty}\varrho_{3i}=0,\ \lim_{i\to\infty}\gamma_{3i}=0.$$

The functionals

$$\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi(t),u(t))dt,\ (\varphi,u)\in\Delta\times\Omega$$

and

$$\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u(t))dt, u\in\Omega$$

are lower semicontinuous (see [3]).

It is not difficult to see that, if

$$\lim_{i\to\infty} u_i(t)=u_0(t) \text{ weakly on } I$$

then

$$\lim_{i\to\infty} u_i(t+\tau_i)=u_0(t+\tau_0) \text{ weakly on } [t_{00}-\tau_0,t_{10}-\tau_0],$$

(see (4.1)). Using the latter and above given relations, we get

$$\hat J=\lim_{i\to\infty} J(w_i)=\lim_{i\to\infty}[g^0(t_{0i},t_{1i},\tau_i,\sigma_i,x_{0i},x_i(t_{1i}))+\varrho_{1i}+\varrho_{2i}+\varrho_{3i}$$

$$+\gamma_{1i}+\gamma_{2i}+\gamma_{3i}]=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))+\lim_{i\to\infty}[\varrho_{1i}+\varrho_{2i}]$$

$$+\lim_{i\to\infty}[\gamma_{1i}+\gamma_{2i}]\geq g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))+\int_{t_{00}-\sigma_0}^{t_{00}} a_0(t+\sigma_0)\zeta_0(t)dt$$

$$+\int_{t_{00}}^{t_{10}-\sigma_0} a_0(t+\sigma_0)\dot\zeta_0(t)dt+\int_{t_{00}-\tau_0}^{t_{00}} f^0(t+\tau_0,\zeta_0(t+\tau_0),\varphi_0(t),u_0(t+\tau_0))dt$$

$$+\int_{t_{00}}^{t_{10}-\tau_0} f^0(t+\tau_0,\zeta_0(t+\tau_0),\zeta_0(t),u_0(t+\tau_0))dt=g^0(t_{00},t_{10},\tau_0,\sigma_0,x_{00},x_0(t_{10}))$$

$$+\int_{t_{00}}^{t_{10}}\Big[a_0(t)\dot x_0(t-\sigma_0)+f^0(t,x_0(t),x_0(t-\tau_0),u_0(t))\Big]dt=J(w_0),$$

(see (4.2),(4.3)). Here, by definition of $\hat J$, the inequality is impossible. The optimality of the element w_0 has been proved.

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THE BOUNDARY VALUE PROBLEMS IN THE FULL COUPLED THEORY OF ELASTICITY FOR PLANE WITH DOUBLE POROSITY WITH A CIRCULAR HOLE[1]

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Abstract. The purpose of this paper is to consider two-dimensional version of the full coupled theory of elasticity for solids with double porosity and to solve explicitly the Dirichlet and Neumann BVPs of statics in the full coupled theory for an elastic plane with a circular hole. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series. The questions on the uniqueness of a solutions of the problems are established.

Keywords and phrases: Double porosity, explicit solution,elastic plane with circular hole, absolutely and uniformly convergent series.

AMS subject classification (2010): 74F10, 74G30, 74G05, 74G10.

Introduction

Many geothermal fields are naturally fractured systems. Classic double porosity models the flow between matrix and fractures, under the hypothesis that petrophysical properties are uniform in each medium. Fractures have the largest permeability and drive the fluid toward the wells. The matrix, with smaller permeability, only acts as a source of fluid for the fractures. Double porosity models can be classified as special cases of this general theoretical concept, applicable to all class reservoirs. The matrix blocks surrounded by fractures can have several geometries and any size. Fractures have very little storage, but provide the high permeability conduits to drive the fluid toward the wells. Matrix blocks have higher porosity and constitute the largest storage, but have smaller permeability, acting only as a source of stationary fluid for the fractures.

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e.a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [2] gave detailed physical interpretations of the phenomenological

[1] This paper dedicated to our teacher to the 85^{th} birth anniversary of professor Mikheil Basheleishvili

coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [3] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [4] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [2],[3],[4] and the references cited therein). The basic results and the historical information on the theory of porous media were summarized by Boer [5].

However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [5,9]. In [10] the full coupled linear theory of elasticity for solids with double porosity is considered. Four spatial cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established. The fundamental solution of quasi-static equations of the linear theory elasticity for double porosity solids is constructed and basic properties are established in [11]. In [12-15] the explicit solutions of the problems of porous elastostatics for an elastic circle and for the plane with a circular hole are constructed, the uniqueness theorems for regular solutions are proved and the numerical results are given for boundary value problems. Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-plane and half-space are considered in [16,17].

The purpose of this paper is to consider two-dimensional version of the full coupled theory of elasticity for solids with double porosity and to solve explicitly the Dirichlet and Neumann BVPs of statics in the full coupled theory for an elastic plane with a circular hole. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series. The questions on the uniqueness of a solutions of the problems are established.

Basic equations and boundary value problems

Let D be a plane with a circular hole. Let R be the radius of a circle with the boundary S centered at point $O(0,0)$. Let us assume that the domain D is filled with an isotropic material with double porosity.

The system of homogeneous equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows [6,10]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1}$$

$$\begin{aligned} &(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0, \\ &(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0, \end{aligned} \tag{2}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, are constitutive coefficients, $k_j = \dfrac{\kappa_j}{\mu'}$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the 2D Laplace operator. Throughout this article it is assumed that $\beta_1^2 + \beta_2^2 > 0$, and the superscript "T" denotes transposition.

Introduce the definition of a regular vector-function.

Definition. A vector-function $\mathbf{U(x)} = (u_1, u_2, p_1, p_2)$ defined in the domain D is called regular if it has integrable continuous second derivatives in D, and $\mathbf{U(x)}$ itself and its first order derivatives are continuously extendable at every point of the boundary of D, i.e., $\mathbf{U(x)} \in C^2(D) \bigcap C^1(\overline{D})$; $\mathbf{x} \in D$, $\mathbf{x} = (x_1, x_2)$. Note that in the domain D the vector $\mathbf{U(x)}$ additionally has to satisfy certain conditions at infinity.

Note that system (2) would be considered separately. Further we assume that p_j is known, when $\mathbf{x} \in D$.

Supposing

$$\begin{pmatrix} p_1(\mathbf{x}) \\ p_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} k_2\Delta - \gamma & -(k_{12}\Delta + \gamma) \\ -(k_{21}\Delta + \gamma) & k_1\Delta - \gamma \end{pmatrix} \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2)$ is a four times differentiable vector function, we can write the system (2) as

$$(\Delta + \lambda_1^2)\Delta\psi_j(\mathbf{x}) = 0. \tag{3}$$

With the help of (3) we find the solution of system (2) in the form

$$p_1(\mathbf{x}) = \varphi(\mathbf{x}) + A_1\varphi_1(\mathbf{x}), \quad p_2(\mathbf{x}) = \varphi(\mathbf{x}) + \varphi_1(\mathbf{x}), \tag{4}$$

where

$$\Delta\varphi = 0, \quad (\Delta + \lambda_1^2)\varphi_1 = 0, \quad A_1 = \frac{\gamma - k_{12}\lambda_1^2}{\gamma + k_1\lambda_1^2} = -\frac{k_2 + k_{12}}{k_1 + k_{21}},$$

$$\lambda_1 = i\sqrt{\frac{\gamma k_0}{k_1k_2 - k_{12}k_{21}}} = i\lambda_0,\ i = \sqrt{-1},\ k_0 = k_1 + k_2 + k_{12} + k_{21};$$

$$k_1 > 0, \quad k_2 > 0, \quad \gamma > 0, \quad k_1k_2 - k_{12}k_{21} > 0, \quad k_0 > 0.$$

Let us substitute the expression $\beta_1p_1 + \beta_2p_2$ into (1) and let us search the particular solution of the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu)graddiv\mathbf{u} = grad[(\beta_1 + \beta_2)\varphi + (A_1\beta_1 + \beta_2)\varphi_1].$$

It is well-known that a general solution of the last equation is presented in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}), \tag{5}$$

where $\mathbf{v}(\mathbf{x})$ is a general solution of the equation

$$\mu\Delta\mathbf{v} + (\lambda + \mu)graddiv\mathbf{v} = 0, \tag{6}$$

and $\mathbf{v}_0(\mathbf{x})$ is a particular solution of the nonhomogeneous equation

$$\mathbf{v}_0(\mathbf{x}) = \frac{1}{\lambda+2\mu} grad \left[(\beta_1+\beta_2)\varphi_0 - \frac{\beta_1 A_1 + \beta_2}{\lambda_1^2}\varphi_1 \right], \tag{7}$$

where φ_0 is a biharmonic function $\Delta\Delta\varphi_0 = 0$ and $\Delta\varphi_0 = \varphi$, $\Delta\varphi = 0$.

So it remains to study the problem of finding the functions $p_j(\mathbf{x})$, $j = 1, 2$.

We consider only the exterior boundary value problems. The interior one can be treated quite similarly.

The basic BVPs in the full coupled linear equilibrium theory of elasticity for materials with double porosity are formulated as follows.

The Dirichlet BVP problem. Find a regular solution $\mathbf{U}(\mathbf{u}, p_1, p_2)$ to systems (1) and (2) for $\mathbf{x} \in D$ satisfying the following boundary conditions:

$$\mathbf{u} = \mathbf{f}(\mathbf{z}), \quad p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S; \tag{8}$$

Note that for the domain D the vector $\mathbf{U(x)}$ additionally has to satisfy the following decay conditions at infinity

$$\mathbf{U}(\mathbf{x}) = o(1), \quad \frac{\partial \mathbf{U}(\mathbf{x})}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2, \tag{9}$$

where o(.) and O(.) are Landau's notion.

The Neumann BVP problem. Find a regular solution $\mathbf{U}(\mathbf{u}, p_1, p_2)$ to systems (1) and (2) for $\mathbf{x} \in D$ satisfying the following boundary conditions:

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U(z)} = \mathbf{f}(\mathbf{z}), \quad \frac{\partial}{\partial n}p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad \frac{\partial}{\partial n}p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S, \tag{10}$$

where $\mathbf{f}(\mathbf{z})$, and $f_j(\mathbf{z})$, $j = 3, 4$, are known functions, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector on S at $\mathbf{z}$ and $\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U}$ is the stress vector in the considered theory

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{U} = \mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{11}$$

$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u}$ is the stress vector in the classical theory of elasticity,

$$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) = \mu\frac{\partial}{\partial \mathbf{n}}\mathbf{u}(\mathbf{x}) + \lambda \mathbf{n} div \mathbf{u}(\mathbf{x}) + \mu\sum_{i=1}^{2} n_i(\mathbf{x}) grad u_i(\mathbf{x}).$$

Vector $\mathbf{U}(\mathbf{x})$ additionally has to satisfy the following decay conditions at infinity

$$\mathbf{U}(\mathbf{x}) = O(1), \quad \frac{\partial \mathbf{U}(\mathbf{x})}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2. \tag{12}$$

The uniqueness theorems

For a regular solutions of the Dirichlet and the Neumann BVPs in D Green's formulas:

$$\int_D [E(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div\mathbf{u}]d\mathbf{x} = -\int_S \mathbf{u}\mathbf{P}(\partial\mathbf{y},\mathbf{n})\mathbf{U}d_yS,$$

$$\int_D \{\gamma(p_1-p_2)^2 + (k_{12}+k_{21})gradp_1 gradp_2\}d\mathbf{x}$$
$$+\int_D \left\{k_1(gradp_1)^2 + k_2(gradp_2)^2\right\}d\mathbf{x} = -\int_S \mathbf{p}\mathbf{P}^{(1)}(\partial\mathbf{y},\mathbf{n})\mathbf{p}d_yS, \qquad (13)$$

are valid, where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)(divu)^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2.$$

$$\mathbf{P}^{(1)}(\partial\mathbf{x},\mathbf{n})\mathbf{p} = \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix}\frac{\partial\mathbf{p}}{\partial\mathbf{n}}, \qquad \mathbf{p} = (p_1,p_2).$$

For positive definiteness of the potential energy the inequalities $\mu > 0$, $\lambda+\mu > 0$ are necessary and sufficient.

Now let us prove the following theorems.

Theorem 1. *The Dirichlet boundary value problem has at most one regular solution in the infinite domain D.*

Proof: Let the first BVP have in the domain D two regular solutions $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$. Denote $\mathbf{U} = \mathbf{U}^{(1)} - \mathbf{U}^{(2)}$. The vectors $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ in the domain D must satisfy the condition (9); In this case formula (13) is valid and $\mathbf{U}(\mathbf{x}) = C$, $\mathbf{x} \in D$, where C is a constant vector. But $\mathbf{U}$ on the boundary satisfies the condition $\mathbf{U} = 0$, which implies that $C = 0$ and $\mathbf{U}(\mathbf{x}) = 0$, $\mathbf{x} \in D$.

Theorem 2. *The regular solution of the Neumann boundary value problem $\boldsymbol{U} = const$ in the infinite domain D.*

Proof: For the exterior second homogeneous boundary value problem the vector $\mathbf{U}$ must satisfy condition at infinite (12). In this case, the formulas (13) are valid for a regular $\mathbf{U}$. Using these formulas, we obtain

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \quad p_1 = p_2 = const, \quad \mathbf{x} \in D,$$

where c_1, c_2, ε are constants. Bearing in mind (12), we have $\varepsilon = 0$, and

$$u_1 = c_1, \ u_2 = c_2, \ p_1 = p_2 = const, \quad \mathbf{x} \in D.$$

Explicit solution of the Dirichlet BVP for a plane with circular hole

A solution of system (2) with boundary conditions $p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S$ is sought in the form (5), where the functions φ and φ_1 are unknown in D. On the basis of boundary conditions we reformulate the problem in question as follows

$$\varphi(\mathbf{z}) = h(\mathbf{z}), \quad \varphi_1(\mathbf{z}) = h_1(\mathbf{z}), \quad \mathbf{z} \in S, \tag{14}$$

where

$$\begin{aligned} h &= \frac{1}{k_0}[(k_1 + k_{21})f_3 + (k_2 + k_{12})f_4], \\ h_1 &= \frac{1}{k_0}(k_1 + k_{21})(f_4 - f_3). \end{aligned} \tag{15}$$

Obviously the function φ is solution of the equation $\Delta\varphi = 0$ and it is represented in the form of the following series ([19], p. 281)

$$\varphi(\mathbf{x}) = \sum_{k=0}^{\infty} \left(\frac{R}{\rho}\right)^k (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{16}$$

where

$$\mathbf{x}(x_1, x_2) = (\rho, \psi), \quad \rho^2 = x_1^2 + x_2^2, \quad \mathbf{Y}_k = (A_k, B_k),$$

$$\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi), \quad \mathbf{Y}_0 = (A_0, 0), \quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta,$$

$$A_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta d\theta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta d\theta.$$

The regular metaharmonic function φ_1 in the domain D can be written as follows ([18], p. 99)

$$\varphi_1(\mathbf{x}) = \sum_{k=0}^{\infty} K_k(\lambda_0\rho)(\mathbf{Z}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{17}$$

where $K_k(\lambda_0\rho)$ is a modified Hankel's function of an imaginary argument, with the index k.

$K_k(\lambda_0\rho) \to 0, \quad \rho \to \infty; \quad \boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi); \quad \mathbf{Z}_k = (C_k, D_k)$; $\mathbf{Z}_0 = (C_0, 0)$; $C_0, \; C_k, \; D_k$ are the unknown quantities.

The function $h_1(z)$ in (15) can be represented in a Fourier series. Keeping in mind (17) and boundary conditions (14) we obtain the values of C_k and D_k

$$C_0 = \frac{1}{2\pi K_0(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)d\theta, \qquad C_k = \frac{1}{\pi K_k(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)\cos k\theta d\theta, \tag{18}$$

$$D_k = \frac{1}{\pi K_k(\lambda_0 R)} \int_0^{2\pi} h_1(\theta) \sin k\theta d\theta.$$

If we substitute the values of φ and φ_1 into (4), we find the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ in D.

A solution $\mathbf{v}(\mathbf{x}) = (v_1, v_2)$ of homogeneous equation (6) is sought in the form [14]

$$\begin{aligned} v_1(\mathbf{x}) &= \frac{\partial}{\partial x_1}[\Phi_1 + \Phi_2] - \frac{\partial \Phi_3}{\partial x_2}, \\ v_2(\mathbf{x}) &= \frac{\partial}{\partial x_2}[\Phi_1 + \Phi_2] + \frac{\partial \Phi_3}{\partial x_1}, \end{aligned} \tag{19}$$

where Φ_1, Φ_2 and Φ_3 are scalar functions,

$$\begin{aligned} &\Delta\Phi_1 = 0, \quad \Delta\Delta\Phi_2 = 0, \quad \Delta\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_1}\Delta\Phi_2 - \mu\frac{\partial}{\partial x_2}\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_2}\Delta\Phi_2 + \mu\frac{\partial}{\partial x_1}\Delta\Phi_3 = 0. \end{aligned} \tag{20}$$

Taking into account (5) and boundary conditions (8), we can write

$$\mathbf{v}(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}), \tag{21}$$

where $\mathbf{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \mathbf{v}_0(\mathbf{z})$ is the known vector; $\varphi(z)$ and $\varphi_1(z)$ are defined by equalities (14). On the basis of equation $\Delta\varphi_0 = \varphi$ the function φ_0 is represented in the following form

$$\varphi_0(x) = \frac{R^2}{4} \sum_{k=2}^{\infty} \frac{1}{1-k} \left(\frac{R}{\rho}\right)^{k-2} (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{22}$$

where $\mathbf{Y}_k$ is defined by (16).

In view of (20) we can represent the harmonic function Φ_1, biharmonic functions Φ_2 and Φ_3 in the form

$$\begin{aligned} \Phi_1 &= \sum_{k=0}^{\infty} \left(\frac{R}{\rho}\right)^k (\mathbf{X}_{k1} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_2 &= \sum_{k=0}^{\infty} R^2 \left(\frac{R}{\rho}\right)^{k-2} (\mathbf{X}_{k2} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_3 &= \frac{R^2(\lambda + 2\mu)}{\mu} \sum_{k=0}^{\infty} \left(\frac{R}{\rho}\right)^{k-2} (\mathbf{X}_{k2} \cdot \mathbf{s}_k(\psi)), \end{aligned} \tag{23}$$

where $\mathbf{X}_{ki} = (X_{ki1}, X_{ki2})$, $k = 1, 2$ are the unknown two-component vectors, $\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi)$, $\mathbf{s}_k = (-\sin k\psi, \cos k\psi)$. Using the formulas

$$\frac{\partial}{\partial x_1} = n_1 \frac{\partial}{\partial \rho} - \frac{n_2}{\rho}\frac{\partial}{\partial \psi}, \quad \frac{\partial}{\partial x_2} = n_2 \frac{\partial}{\partial \rho} + \frac{n_1}{\rho}\frac{\partial}{\partial \psi}$$

the boundary conditions (21) are rewritten in the form

$$v_n(\mathbf{z}) = \Psi_n(\mathbf{z}), \quad v_s(\mathbf{z}) = \Psi_s(\mathbf{z}), \quad \mathbf{z} \in S, \tag{24}$$

where v_n and $\Psi_n(\mathbf{z})$ are the normal components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively; v_s and $\Psi_s(\mathbf{z})$ are the tangent components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively. Substituting the equalities (19),(23) into (24), we get

$$\begin{aligned}
&v_n = \frac{\partial}{\partial \rho}(\Phi_1 + \Phi_2) - \frac{1}{\rho}\frac{\partial}{\partial \psi}\Phi_3,\\
&v_s = \frac{1}{\rho}\frac{\partial}{\partial \psi}(\Phi_1 + \Phi_2) + \frac{\partial}{\partial \rho}\Phi_3,\\
&\Psi_n = n_1\Psi_1 + n_2\Psi_2, \quad \Psi_s = -n_2\Psi_1 + n_1\Psi_2,\\
&\mathbf{n} = (n_1, n_2), \quad \mathbf{s} = (-n_2, n_1), \quad n_1 = \frac{x_1}{\rho}, \quad n_2 = \frac{x_2}{\rho}.
\end{aligned} \tag{25}$$

Let us expand the functions Ψ_n *and* Ψ_s in Fourier series, that Fourier coefficients are γ_k and δ_k :

$$\begin{aligned}
&\boldsymbol{\gamma}_0 = (\gamma_{01}, 0), \quad \boldsymbol{\gamma}_k = (\gamma_{k1}, \gamma_{k2}), \quad \boldsymbol{\delta}_0 = (\delta_{01}, 0), \quad \boldsymbol{\delta}_k = (\delta_{k1}, \delta_{k2}),\\
&\gamma_{01} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)d\theta, \quad \delta_{01} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)d\theta,\\
&\gamma_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\cos k\theta d\theta, \quad \delta_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\cos k\theta d\theta,\\
&\gamma_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\sin k\theta d\theta, \quad \delta_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\sin k\theta d\theta.
\end{aligned} \tag{26}$$

If we substitute (25) into (24), then obtained into (26), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations whose solution is written in the following form:

$$X_{01i} = \frac{\gamma_{0i}R}{2}, \quad X_{k1i} = \frac{R(\gamma_{ki} + \delta_{ki})}{2k(\lambda + 3\mu)}[2\mu + (\lambda + \mu)k] - \frac{\gamma_{ki}R}{k},$$

$$X_{02i}=\frac{\delta_{0i}R\mu}{2},\quad X_{k2i}=\frac{(\gamma_{ki}+\delta_{ki})\mu}{2R(\lambda+3\mu)},\quad i=1,2,\quad k=1,2,....$$

Thus the solution of the Dirichlet boundary problem is represented by the sum (5) in which $\mathbf{v}(\mathbf{x})$ is defined by means of formula (19), $\mathbf{v}_0(\mathbf{x})$ by formula (7), $\varphi_0(\mathbf{x})$ by formula (22) and $\varphi_1(\mathbf{x})$ by formulas (17) and (18). It can be proved that if the functions $\mathbf{f}$ and $f_j,\quad j=3,4$ satisfy the following conditions on S

$$\mathbf{f}\in C^3(S),\quad f_j\in C^3(S),\quad j=3,4,$$

then the resulting series are absolutely and uniformly convergent.

Explicit solution of the Neumann BVP for a plane with circular hole

We sought the solution of the Neumann BVP in the form (4), where the functions φ and φ_1 are unknown in the domain D. Taking into account formulas (4), the boundary conditions can be rewritten as

$$\frac{\partial\varphi(\mathbf{z})}{\partial R}=h(\mathbf{z}),\qquad \frac{\partial\varphi_1(\mathbf{z})}{\partial R}=h_1(\mathbf{z}),\quad \mathbf{z}\in S. \tag{27}$$

$h(\mathbf{z})$ and $h_1(\mathbf{z})$ are given by (15), where $f_3=\dfrac{\partial p_1}{\partial R},\quad f_4=\dfrac{\partial p_2}{\partial R}$.

Thus for the unknown harmonic function φ we obtain the Neumann problem, the solution that is represented in the form of series ([19],p.282)

$$\varphi(\mathbf{x})=c_1-\sum_{k=1}^{\infty}\frac{R}{k}\left(\frac{R}{\rho}\right)^k(\mathbf{Y}_k\cdot\boldsymbol{\nu}_k(\psi)), \tag{28}$$

where c_1 is an arbitrary constant; $\mathbf{Y}_k=(A_k,B_k)$,

$$A_k=\frac{1}{\pi}\int_0^{2\pi}h(\theta)\cos k\theta d\theta,\quad B_k=\frac{1}{\pi}\int_0^{2\pi}h(\theta)\sin k\theta d\theta.$$

The metaharmonic function $\varphi_1(\mathbf{x})$ in the domain D can be written as (17), where $\mathbf{Z}_k=(C_k,D_k)$; $C_0,\ C_k,\ D_k$ are the unknown quantities. Keeping in mind (15) and boundary conditions (27), we obtain the values of $Z_0,\ C_k$ and D_k

$$C_0=\frac{1}{2\pi\lambda_0K_0'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)d\theta,\qquad C_k=\frac{1}{\pi\lambda_0K_k'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)\cos k\theta d\theta, \tag{29}$$

$$D_k=\frac{1}{\pi\lambda_0K_k'(\lambda_0R)}\int_0^{2\pi}h_1(\theta)\sin k\theta d\theta,$$

where

$$K_k'(\xi)=\frac{\partial K_k(\xi)}{\partial\xi},\quad \frac{\partial K_k(\lambda_0\rho)}{\partial\rho}=\lambda_0K_k'(\lambda_0\rho),\quad K_k'(\lambda_0R)\neq0,\quad k=0,1,2,....$$

Taking into account (10) the boundary condition (9) for $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}$ can be rewritten as

$$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)(z)} = \mathbf{\Omega(z)}, \quad \mathbf{z} \in S, \tag{30}$$

where

$$\mathbf{\Omega}(\ \mathbf{z}) = \mathbf{f(z)} + \mathbf{n(z)}[a\varphi_1(\mathbf{z}) + b\varphi(\mathbf{z})] - \mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})$$

is the known vector, $\mathbf{\Omega} = (\Omega_1, \Omega_2)$; φ is defined by (28) and φ_1 - formulas (17) and (18); $\quad a = \beta_1 + \beta_2, \quad b = A_1\beta_1 + \beta_2$.

Let us rewrite the boundary conditions (30) in the form

$$\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n = \Omega_n(\mathbf{z}), \quad \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s = \Omega_s(\mathbf{z}), \tag{31}$$

where $\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n$ and $\Omega_n(\mathbf{z})$ are the normal components of the vectors $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}$ and $\mathbf{\Omega}(\mathbf{z})$ respectively; $\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s$ and $\Omega_s(\mathbf{z})$ are the tangent components of the vectors $\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)})$ and $\mathbf{\Omega}(\mathbf{z})$ respectively.

$$\begin{aligned}
&\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_n = (\lambda+\mu)\left[\frac{\partial v_n(\mathbf{z})}{\partial \rho}\right]_{\rho=R} + \frac{\lambda}{R}\frac{\partial v_s(\mathbf{z})}{\partial \psi},\\
&\left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v(z)}\right]_s = \mu\left[\frac{\partial v_s(\mathbf{z})}{\partial \rho}\right]_{\rho=R} + \frac{\mu}{R}\frac{\partial v_n(\mathbf{z})}{\partial \psi};\\
&\Omega_n(\mathbf{z}) = f_n(\mathbf{z}) + a\varphi_1(\mathbf{z}) + b\varphi(\mathbf{z}) - \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right]_n,\\
&\Omega_s(\mathbf{z}) = f_s(\mathbf{z}) - \left[\mathbf{T}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right)\mathbf{v}_0(\mathbf{z})\right]_s, \quad \mathbf{z} \in S.
\end{aligned} \tag{32}$$

v_n and v_s are defined from (25), $\mathbf{v}_0$ is defined by means of formula (7), where function $\varphi_0(x)$ is the solution of equation $\Delta\varphi_0 = \varphi$ and represented in the form [14]

$$\varphi_0(\mathbf{x}) = \frac{-R^3}{4}\sum_{k=2}^{\infty}\frac{1}{k(1-k)}\left(\frac{R}{r}\right)^{k-2}(\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)),$$

Y_k are defined in (28); c_1 is an arbitrary constant.

Let us expand the functions Ω_n *and* Ω_s in Fourier series, those Fourier coefficients are $\gamma_k = (\gamma_{k1}, \gamma_{k2})$ and $\delta_k = (\delta_{k1}, \delta_{k2})$. Taking into account the formulas (25),(23) and (32), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations

$$k[\lambda+2\mu(k+1)]X_{k1i}+$$

$$\left\{(\lambda+2\mu)(1-k)(2-k+\frac{\lambda+2\mu}{\mu}k)-\lambda kR^2\left[k+\frac{\lambda+2\mu}{\mu}(2-k)\right]\right\}X_{k2i}=\gamma_{ki}R^2,$$

$$-k(1+2k)X_{k1i}+R^2\left[k(3-2k)+\frac{\lambda+2\mu}{\mu}(k^2-3k+2)\right]X_{k2i}=\frac{\delta_{ki}R^2}{\mu},$$
$$i=1,2;\quad k=1,2,...,$$

where γ_{ki} and δ_{ki} are the Fourier coefficients of normal and tangential components of the vector $\mathbf{\Omega}(\mathbf{z})$ respectively.

We assume that the functions $\mathbf{f}$ and f_j, $(j=3,4)$ satisfies the following conditions on S

$$\mathbf{f}\in C^2(S),\quad f_j\in C^2(S),\quad j=3,4.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

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თერმოდრეკადობის თეორიის დირიხლეს ამოცანის ეფექტური ამოხსნა სივრცისათვის სფერული ღრუთი მიკროტემპერატურის გათვალისწინებით

ლ. ბიწაძე

მიღებულია სამგანზომილებიანი მდგრადი რხევის განტოლებათა სისტემის რეგულარული ამონახსნის ზოგადი წარმოდგენის ფორმულა, რომლის საფუძველზე ამოხსნილია დირიხლეს სასაზღვრო ამოცანა სივრცისათვის სფერული ღრუთი. ამონახსნები წარმოდგენილია აბსოლუტური და თანაბრად კრებადი მწკრივების სახით.

ჰექსაგონალური ქვანტური ბილიარდის შესახებ

ნ. ხატიაშვილი

განხილულია ბრტყელი კლასიკური ქვანტური ბილიარდი ჰექსაგონალური ტიპის არეებში. ეს პროცესი აღწერილია ჰელმჰოლცის განტოლებით ერთგვაროვანი სასაზღვრო პირობებით ჰექსაგონისთვის და ჰექსაგონალური ხალიჩისთვის.

კონფორმულ ასახვათა მეთოდით ეს პრობლემა რედუცირებულია ელიფსური ტიპის დიფერენციალურ განტოლებაზე მართკუთხედში ერთგვაროვანი სასაზღვრო პირობებით. მიღებულია მისი ასიმპტოტური ამონახსნები ცხადი სახით.

შედეგები გამოყენებულია გრაფენში ელექტრონების ენერგეტიკული დონეების შესაფასებლად.

რხევადობის კრიტერიუმი დაგვიანებულ არგუმენტებიანი სხვაობიანი განტოლებებისათვის

რ. კოპლატაძე

პირველი რიგის დაგვიანებულ არგუმენტიანი სხვაობიანი განტოლებებისათვის დადგენილია წესიერი ამონახსნების რხევადობის საკმარისი პირობები.

დაგვიანებულ არგუმენტიანი n- რიგის ემდენ-ფაულერის განზოგადოებული დიფერენციალური განტოლებების ამონახსნების ოსცილაციური თვისებების შესახებ

რ. კოპლატაძე

შესწავლილია მაღალი რიგის დაგვიანებულ არგუმენტიანი ემდენ-ფაულერის განზოგადოებული დიფერენციალური განტოლებების ამონახსნების თვისებები. კერძოდ, დადგენილია საკმარისი პირობები იმისა, რომ მოცემულ განტოლებას გააჩნდეს B თვისება.

დრეკად ნარევთა თეორიის სტატიკის ძირითადი შერეული სასაზღვრო ამოცანის ეფექტური ამოხსნა წრიულ არეში

კ. სვანაძე

დრეკად ნარევთა წრფივი თეორიის სტატიკის ერთგვაროვანი განტოლებისათვის წრეში, ნ. მუსხელიშვილის მეთოდით, ეფექტურადაა ამოხსნილი ძირითადი შერეული ამოცანა.

ამონახსნის ვარიაციის ფორმულები ერთი კლასის სამართი ნეიტრალური ფუნქციონალურ-დიფერენციალური განტოლებისათვის ცვლადი დაგვიანების ფუნქციის შეშფოთებისა და უწყვეტი საწყისი პირობის გათვალისწინებით

თ. თადუმაძე, ნ. გორგოძე

სიჩქარის წინა ისტორიის მიმართ წრფივი სამართი ნეიტრალური ფუნქციონალურ-დიფერნციალური განტოლებისათვის ცვლადი დაგვიანებით მიღებულია ამონახსნის ვარიაციის ფორმულები. ვარიაციის ფორმულებში გამოვლენილია დაგვიანების ფუნქციის შეშფოთებისა და უწყვეტი საწყისი პირობის ეფექტები.

ოპტიმალური ელემენტის არსებობის შესახებ კვაზი-წრფივ ნეიტრალურ ოპტიმალურ ამოცანებში

თ. თადუმაძე, ა. ნაშავი

დამტკიცებულია ოპტიმალური ელემენტის არსებობის თეორემები ოპტიმალური მართვის ამოცანისათვის, რომელიც შეიცავს ნეიტრალურ დიფერენციალურ განტოლებას, რომლის მარჯვენა მხარე წრფივია ფაზური სიჩქარის წინაისტორიის მიმართ. ელემენტის ქვეშ იგულისხმება საწყისი და საბოლოო მომენტების, დაგვიანების პაარამეტრების, საწყისი ვექტორის, საწყისი ფუნქციებისა და მართვის ერთობლიობა.

დრეკადობის ბმული თეორიის სასაზღვრო ამოცანები ორგვარი ფოროვნების მქონე სიბრტყისათვის წრიული ხვრელით

ი. ცაგარელი , ლ. ბიწაძე

აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით, ამოხსნილია დრეკადობის ბმული თეორიის სტატიკის დირიხლეს და ნეიმანის ტიპის სასაზღვრო ამოცანები ორგვარი ფოროვნობის მქონე სიბრტყისათვის წრიული ხვრელით. დადგენილია ამონახსნის ერთადერთობა.

C O N T E N T S

ავტორთა საყურადღებოდ!

ჟურნალში გამოსაქვეყნებელი სტატია უნდა წარმოადგინოთ ინგლისურად, ელექტრონული და ამობეჭდილი სახით, აკრეფილი TEX (Plain, LATEX, AMSTEX, AMSLATEX) პროგრამის გამოყენებით.

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  Hartman P. Ordinary differential equations. *John Wiley & Sons, New York-London-Sydney,* 1964.
  Hartman P. A differential equation with non-unique solutions. *Amer. Math. Monthly*, **70 (**1963) , 255-259.
  Vekua, I. N. A class of statically definable problems of the theory of shells. (Russian)
  Sakharth. SSR Mecn. Akad. Moambe **83**, 2 (1976), 273-276.
- ♦ ინფორმაცია ავტორის შესახებ: გვარი და სახელი, დაწესებულება და მისამართი, ელ-ფოსტა;
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  Hartman P. Ordinary differential equations, *John Wiley & Sons, New York-London-Sydney,* 1964.
  Hartman P. A differential equation with non-unique solutions, *Amer. Math. Monthly*, **70 (**1963), 255-259.
  Vekua, I. N. A class of statically definable problems of the theory of shells. (Russian)
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