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ON SOME SOLUTIONS OF THE SYSTEM OF EQUATIONS OF STEADY VIBRATION IN THE PLANE THERMOELASTICITY THEORY WITH MICROTEMPERATURES

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Abstract. In the present paper the linear 2D theory of thermoelasticity with microtemperatures is considered. The representation of regular solution of the system of equations of steady vibrations in the considered theory is obtained. The fundamental and singular solutions for a governing system of equations of this theory are constructed. Finally, the single-layer, double-layer and volume potentials are presented.

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Introduction

A thermodynamic theory for elastic materials with inner structure the particles of which, in addition to microdeformations, possess microtemperatures was proposed by Grot [1]. Iesan and Quintanilla [2] have formulated the boundary value problems (BVPs) and presented an uniqueness result and a solution of the Boussinesq-Somigliana-Galerkin type. The fundamental solutions of the equations of the 3D theory of thermoelasticity with microtemperatures were constructed by Svanadze [3]. The representations of the Galerkin type and general solutions of the system of equations in this theory were obtained by Scalia, Svanadze and Tracinà [4]. In [5], a wide class of external BVPs of steady vibrations is investigated and Sommerfeld-Kupradze type radiation conditions and the basic properties of thermoelastopotentials are established. Here the uniqueness and existence theorems of regular solutions of the external BVPs are proved using the potential method and the theory of singular integral equations. The fundamental solutions of the equations of the two-dimensional (2D) theory of thermoelasticity with microtemperatures were constracted by Basheleishvili, Bitsadze and Jaiani [6]. The 2D BVPs of statics of the theory of thermoelasticity with microtemperatures are formulated and the uniqueness and existence theorems are presented in [7]. The basic results and extensive review of the theory of elastic materials with microstructure are given in the literature [8].

For investigation, boundary-value problems of the theory of elasticity and thermoelasticity by potential method are necessary to construct fundamental solutions of respective systems of partial differential equations and to establish their basic properties. There are several known methods to construct a fundamental solution of systems of differential equations of the theory of elasticity and thermoelasticity [9-12].

In the present paper the linear 2D theory of thermoelasticity with microtemperatures is considered. The representation of regular solution of the system of equations of steady vibration of the theory of thermoelasticity with microtemperatures is obtained.The fundamental and singular solutions for a governing system of equations of this theory are constructed. Finally, the single-layer, double-layer and volume potentials are presented.

Basic equations

We consider an isotropic elastic material with microtemperatures. Let $D^+(D^-)$ be a bounded (respectively, an unbounded) domain of the Euclidean 2D space E_2 bounded by the contour S. $\overline{D^+} := D^+ \bigcup S$, $D^- := E_2 \backslash \overline{D^+}$. Let $\mathbf{x} := (x_1, x_2) \in E_2$, $\partial \mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$. In 2D space "rot" is defined as a scalar

$$rot\boldsymbol{\phi} := \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2}$$

for a vector $\phi := (\phi_1, \phi_2)$ and as a vector

$$rot\psi := \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right)$$

for a scalar ψ.

The basic system of equations of steady vibrations in the linear 2D theory of thermoelasticity with microtemperatures has the following form [1],[2]

$$\mu\Delta\mathbf{u} + (\lambda + \mu)graddiv\mathbf{u} - \beta grad\theta + \varrho\omega^2\mathbf{u} = -\varrho\mathbf{N}, \tag{1}$$

$$k_6\Delta\mathbf{w} + (k_4 + k_5)graddiv\mathbf{w} - k_3 grad\theta + k_8\mathbf{w} = \rho\mathbf{M}, \tag{2}$$

$$(k\Delta + a_0)\theta + \beta_0 div\mathbf{u} + k_1 div\mathbf{w} = -\rho s, \tag{3}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector, $\mathbf{w} = (w_1, w_2)^T$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0 > 0$) by the natural state (i.e. by the state of the absence of loads), ρ is the reference mass density ($\rho > 0$), $\mathbf{N} = (N_1, N_2)$ is the body force, $\mathbf{M} = (M_1, M_2)$ is first heat source moment vector, s is the heat supply, $a_0 = i\omega a T_0$, $\beta_0 = i\omega\beta T_0$, $k_8 = i\omega b - k_2$, $b > 0$, λ, μ, β, k, k_j, $(j = 1, ..., 6)$, are the constitutive coefficients, Δ is the 2D Laplace operator and ω is the oscillation frequency ($\omega > 0$). The superscript "T" denotes transposition.

We introduce the matrix differential operator

$$\mathbf{A}(\partial\mathbf{x}, \omega) := \| A_{lj}(\partial\mathbf{x}, \omega) \|_{5\times 5},$$

where

$$A_{\alpha\gamma} := \mu\delta_{\alpha\gamma}(\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_\alpha \partial x_\gamma},$$

$$A_{\alpha+2;\gamma+2} := \delta_{\alpha\gamma}(k_6\Delta + k_8) + (k_4 + k_5)\frac{\partial^2}{\partial x_\alpha \partial x_\gamma},$$

$$A_{\alpha,\gamma+2} := A_{\alpha+2,\gamma} = 0, \quad A_{\alpha 5} := -\beta\frac{\partial}{\partial x_\alpha}, \quad A_{\alpha+2;5} := -k_3\frac{\partial}{\partial x_\alpha},$$
$$A_{5\gamma} := \beta_0\frac{\partial}{\partial x_\gamma}, \quad A_{5;\gamma+2} := k_1\frac{\partial}{\partial x_\gamma}, \quad A_{55} := k\Delta + a_0, \quad \alpha,\gamma = 1,2,$$

$\delta_{\alpha\gamma}$ is the Kronecker delta. Then the system (1)-(3) can be rewritten as

$$\mathbf{A}(\partial\mathbf{x},\omega)\mathbf{U} = \mathbf{F}, \tag{4}$$

where

$$\mathbf{U} := (u_1,u_2,w_1,w_2,\theta)^T, \quad \mathbf{F} = (-\varrho\mathbf{N}, \varrho\mathbf{M}, -\varrho s).$$

When $\mathbf{F} = 0$, we have homogeneous system of equations of steady vibrations in the 2D theory of thermoelasticity with microtemperatures

$$\mu\Delta\mathbf{u} + (\lambda+\mu)graddiv\mathbf{u} - \beta grad\theta + \varrho\omega^2\mathbf{u} = 0, \tag{5}$$

$$k_6\Delta\mathbf{w} + (k_4+k_5)graddiv\mathbf{w} - k_3 grad\theta + k_8\mathbf{w} = 0, \tag{6}$$

$$(k\Delta + a_0)\theta + \beta_0 div\mathbf{u} + k_1 div\mathbf{w} = 0. \tag{7}$$

The matrix $\widetilde{\mathbf{A}}(\partial\mathbf{x},\omega) :=\| \widetilde{A}_{lj}(\partial\mathbf{x},\omega) \|_{5\times5} := \mathbf{A}^T(-\partial\mathbf{x},\omega)$, will be called the associated operator to the differential operator $\mathbf{A}(\partial\mathbf{x},\omega)$. Thus, the homogeneous associated system to (4) has the following form

$$\mu\Delta\mathbf{u} + (\lambda+\mu)graddiv\mathbf{u} - \beta_0 grad\theta + \rho\omega^2 u = 0,$$

$$k_6\Delta\mathbf{w} + (k_4+k_5)graddiv\mathbf{w} - k_1 grad\theta + k_8\mathbf{w} = 0,$$

$$(k\Delta + a_0)\theta + k_3 div\mathbf{w} + \beta div\mathbf{u} = 0.$$

We assume that $\mu\mu_0 kk_6k_7 \neq 0$, where $\mu_0 := \lambda + 2\mu,\ k_7 := k_4+k_5+k_6$. Obviously, if the last condition is satisfied, then $\mathbf{A}(\partial\mathbf{x},\omega)$ is the elliptic differential operator.

Representation of regular solutions

Definition. A vector function $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)$ is called regular in $D^-(or\ \ D^+)$ if

$$1. \quad \mathbf{U}\in C^2(D^-)\cap C^1(\bar{D}^-) \quad or \quad (\mathbf{U}\in C^2(D^+)\cap C^1\bar{D}^+),$$

$$2. \quad \mathbf{u} = \sum_{j=1}^{4}\mathbf{u}^{(j)}(\mathbf{x}), \quad \mathbf{w} = \sum_{j=1,2,3,5}\mathbf{w}^{(j)}(\mathbf{x}), \quad \theta = \sum_{j=1}^{3}\theta^{(j)}(\mathbf{x}),$$

$$\begin{gathered} 3. \quad (\Delta+\lambda_j^2)\mathbf{u}^{(j)} = 0, \quad (\Delta+\lambda_l^2)\mathbf{w}^{(l)} = 0, \quad (\Delta+\lambda_m^2)\theta^{(m)} = 0,\\ \mathbf{u}^{(j)} = (u_1^{(j)},u_2^{(j)}), \quad \mathbf{w}^{(l)} = (w_1^{(l)},w_2^{(l)}),\\ j=1,2,3,4, \quad l=1,2,3,5, \quad m=1,2,3 \end{gathered} \tag{8}$$

and

$$\begin{gathered} \left(\frac{\partial}{\partial|\mathbf{x}|} - i\lambda_j\right)u_l^{(j)} = e^{i\lambda_j|\mathbf{x}|}o(|\mathbf{x}|^{-\frac{1}{2}}), \quad j=1,2,3,4, \quad l=1,2,\\ \left(\frac{\partial}{\partial|\mathbf{x}|} - i\lambda_l\right)w_k^{(l)} = e^{i\lambda_l|\mathbf{x}|}o(|\mathbf{x}|^{-\frac{1}{2}}), \quad l=1,2,3,5, \quad k=1,2, \end{gathered} \tag{9}$$

$$\left(\frac{\partial}{\partial|\mathbf{x}|}-i\lambda_m\right)\theta^{(m)}=e^{i\lambda_m|\mathbf{x}|}o(|\mathbf{x}|^{-\frac{1}{2}}),\quad m=1,2,3\quad for\quad |\mathbf{x}|=\sqrt{x_1^2+x_2^2}\quad >>1;$$

where $\lambda_j^2,\quad j=1,2,3$, are roots of equation $D(-\xi)=0$,

$$D(\Delta)=(\mu_0\Delta+\rho\omega^2)k_1k_3\Delta+(k_7\Delta+k_8)[\beta\beta_0\Delta+(\mu_0\Delta+\rho\omega^2)(k\Delta+a_0)]=$$

$$\mu_0kk_7(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)$$

and the constants λ_4^2 and λ_5^2 are determined by the formulas

$$\lambda_4^2:=\frac{\rho\omega^2}{\mu}>0,\quad \lambda_5^2:=\frac{k_8}{k_6}.$$

The quantities $\lambda_j^2,\quad j=1,2,3,5$ are complex numbers and are chosen so as to ensure positivity of their imaginary part, i.e. it is assumed that $Im\lambda_j^2>0$.

Equalities in (9) are Sommerfeld-Kupradze type radiation conditions in the linear theory of thermoelastisity with microtemperatures.

Remark. The equalities (9) imply [5]

$$U_l(\mathbf{x})=e^{i\lambda_j|\mathbf{x}|}O(|\mathbf{x}|^{-\frac{1}{2}})\quad for\quad |\mathbf{x}|>>1,\quad l,j=1,..,5. \tag{10}$$

Theorem 1. *The regular solution $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta)$ of the systems (5)-(7) admits in the domain of regularity a representation*

$$\boldsymbol{U}=(\overset{\mathbf{1}}{\mathbf{u}}+\overset{\mathbf{2}}{\mathbf{u}},\overset{\mathbf{1}}{\mathbf{w}}+\overset{\mathbf{2}}{\mathbf{w}},\theta),$$

where $\overset{\mathbf{1}}{\mathbf{u}},\quad \overset{\mathbf{2}}{\mathbf{u}},\quad \overset{\mathbf{1}}{\mathbf{w}}$ and $\overset{\mathbf{2}}{\mathbf{w}}$ are the regular vectors, satisfying the conditions

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)\overset{\mathbf{1}}{\mathbf{u}}=0,\quad rot\overset{\mathbf{1}}{\mathbf{u}}=0,$$

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)\overset{\mathbf{1}}{\mathbf{w}}=0,\quad rot\overset{\mathbf{1}}{\mathbf{w}}=0,$$

$$(\Delta+\lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}}=0,\quad div\overset{\mathbf{2}}{\mathbf{u}}=0,\quad (\Delta+\lambda_5^2)\overset{\mathbf{2}}{\mathbf{w}}=0,\quad div\overset{\mathbf{2}}{\mathbf{w}}=0,$$

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)\theta=0.$$

Proof. Let $\mathbf{U}=(\mathbf{u},\mathbf{w},\theta)$ be a regular solution of the equations (5)-(7). Taking into account the identity

$$\Delta\mathbf{w}=graddiv\mathbf{w}-rotrot\mathbf{w}, \tag{11}$$

where

$$rotrot\mathbf{w}:=\left(\frac{\partial}{\partial x_2}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right),-\frac{\partial}{\partial x_1}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)\right),$$

from (5),(6) we obtain

$$\mathbf{u}=-\frac{\mu_0}{\rho\omega^2}graddiv\mathbf{u}+\frac{\mu}{\rho\omega^2}rotrot\mathbf{u}+\frac{\beta}{\rho\omega^2}grad\theta,$$

$$\mathbf{w} = -\frac{k_7}{k_8} graddiv\mathbf{w} + \frac{k_6}{k_8} rotrot\mathbf{w} + \frac{k_3}{k_8} grad\theta,$$

Let

$$\overset{\mathbf{1}}{\mathbf{u}} := -\frac{\mu_0}{\rho\omega^2} graddiv\mathbf{u} + \frac{\beta}{\rho\omega^2} grad\theta, \tag{12}$$

$$\overset{\mathbf{2}}{\mathbf{u}} := \frac{\mu}{\rho\omega^2} rotrot\mathbf{u}, \tag{13}$$

$$\overset{\mathbf{1}}{\mathbf{w}} := -\frac{k_7}{k_8} graddiv\mathbf{w} + \frac{k_3}{k_8} grad\theta, \tag{14}$$

$$\overset{\mathbf{2}}{\mathbf{w}} := \frac{k_6}{k_8} rotrot\mathbf{w}. \tag{15}$$

Clearly

$$\mathbf{u} = \overset{\mathbf{1}}{\mathbf{u}} + \overset{\mathbf{2}}{\mathbf{u}}, \quad \mathbf{w} = \overset{\mathbf{1}}{\mathbf{w}} + \overset{\mathbf{2}}{\mathbf{w}} \quad rot\overset{\mathbf{1}}{\mathbf{u}} = 0, \quad rot\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{u}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{w}} = 0. \tag{16}$$

Taking into account the identity $\Delta\overset{\mathbf{2}}{\mathbf{u}} = -rotrot\overset{\mathbf{2}}{\mathbf{u}}$, $\Delta\overset{\mathbf{2}}{\mathbf{w}} = -rotrot\overset{\mathbf{2}}{\mathbf{w}}$, from (13)-(15) we get

$$(\Delta + \lambda_4^2)\overset{\mathbf{2}}{\mathbf{u}} = 0, \quad (\Delta + \lambda_5^2)\overset{\mathbf{2}}{\mathbf{w}} = 0. \tag{17}$$

Applying the operator *div* to equations (5), (6) we obtain

$$\begin{aligned} &(\mu_0\Delta + \rho\omega^2) div\boldsymbol{u} - \beta\Delta\theta = 0, \\ &(k_7\Delta + k_8) div\boldsymbol{w} - k_3\Delta\theta = 0, \\ &(k\Delta + a_0)\theta + k_1 div\boldsymbol{w} + \beta_0 div\boldsymbol{u} = 0, \end{aligned} \tag{18}$$

Rewrite system (18) as follows

$$D(\Delta)\Psi := \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & 0 & -\beta\Delta \\ 0 & k_7\Delta + k_8 & -k_3\Delta \\ \beta_0 & k_1 & k\Delta + a_0 \end{pmatrix} \Psi = 0,$$

where $\Psi = (div\boldsymbol{u}, div\boldsymbol{w}, \theta)^T$. Clearly, $detD = \mu_0 k k_7(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$,

$$\begin{aligned} &(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2) div\boldsymbol{u} = 0, \\ &(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2) div\boldsymbol{w} = 0, \\ &(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\theta = 0. \end{aligned} \tag{19}$$

Applying the operator $(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$ to equations (12), (14) using the last relations we obtain

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\overset{\mathbf{1}}{\mathbf{u}} = 0,$$

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\overset{\mathbf{1}}{\mathbf{w}} = 0,$$

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\theta = 0.$$

The last formulas prove the theorem.

Theorem 2. *The regular solution* $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta)\in C^2(D)$ *of equation* $\boldsymbol{A}(\partial\boldsymbol{x})\,\boldsymbol{U}=0$ *for* $\boldsymbol{x}\in D$, *is represented as the sum*

$$\boldsymbol{u}=\sum_{j=1}^{4}\boldsymbol{u}^{(j)}(\boldsymbol{x}),\quad \boldsymbol{w}=\sum_{j=1,2,3,5}\boldsymbol{w}^{(j)}(\boldsymbol{x}),\quad \theta=\sum_{j=1}^{3}\theta^{(j)}, \tag{20}$$

where D *is a domain in* E_2 *and* $\boldsymbol{u}^{(j)},\boldsymbol{w}^{(j)}$*and* $\theta^{(j)}$ *are regular functions satisfying the following conditions*

$$\begin{aligned}&(\Delta+\lambda_j^2)\boldsymbol{u}^{(j)}=0,\quad (\Delta+\lambda_l^2)\boldsymbol{w}^{(l)}=0,\quad (\Delta+\lambda_m^2)\theta^{(m)}=0,\\ &j=1,2,3,4,\quad l=1,2,3,5,\quad m=1,2,3.\end{aligned} \tag{21}$$

Proof. Applying the operator div to the equations (5) and (6) and taking into account the relations (18) and (19) we obtain

$$\begin{aligned}&(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)\boldsymbol{u}=0,\\ &(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_5^2)\boldsymbol{w}=0,\\ &(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)\theta=0.\end{aligned} \tag{22}$$

We introduce the notations:

$$\begin{aligned}\mathbf{u}^{(j)}&=\left[\prod_{l=1;l\neq j}^{4}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_j^2}\right]\mathbf{u},\quad j=1,2,3,4,\\ \mathbf{w}^{(p)}&=\left[\prod_{l=1,2,3,5}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_p^2}\right]\mathbf{w},\quad l\neq p,\quad p=1,2,3,5,\\ \theta^{(q)}&=\left[\prod_{l=1}^{3}\frac{\Delta+\lambda_l^2}{\lambda_l^2-\lambda_q^2}\right]\theta,\quad l\neq j,\quad j=1,2,3.\end{aligned} \tag{23}$$

By virtue of (23), it follows that

$$\mathbf{u}=\sum_{j=1}^{4}\mathbf{u}^{(j)},\quad \mathbf{w}=\sum_{j=1,2,3,5}\mathbf{w}^{(j)},\quad \theta=\sum_{j=1}^{3}\theta^{(j)}, \tag{24}$$

$$(\Delta+\lambda_j^2)\mathbf{u}^{(j)}=0,\quad (\Delta+\lambda_l^2)\mathbf{w}^{(l)}=0,\quad (\Delta+\lambda_m^2)\theta^{(m)}=0,$$

$$j=1,2,3,4,\quad l=1,2,3,5,\quad m=1,2,3.$$

Thus, the regular in D solution of equation $\boldsymbol{A}(\partial\mathbf{x},\omega)\mathbf{U}=0$ is represented as a sum of functions $\mathbf{u}^{(j)}$, $\mathbf{w}^{(j)}$, $\theta^{(j)}$, which satisfy Helmholtz' equations in D.

Matrix of fundamental solutions

We introduce the matrix differential operator $\mathbf{B}(\partial\mathbf{x})$ consisting of cofactors of elements of the transposed matrix $\mathbf{A}^T$ divided on $\mu\mu_0 k k_6 k_7$

$$\mathbf{B}(\partial\mathbf{x},\omega):=\|\ B_{lj}(\partial\mathbf{x},\omega)\ \|_{5\times5},$$

where

$$B_{\alpha\gamma} := B^*_{11}\delta_{\alpha\gamma} - B^*_{12}\xi_\alpha\xi_\gamma, \quad B_{\alpha+2,\gamma+2} := B^*_{33}\delta_{\alpha\gamma} - B^*_{34}\xi_\alpha\xi_\gamma,$$

$$B_{1\gamma+2} := B^*_{13}\xi_1\xi_\gamma, \quad B_{2\gamma+2} := B^*_{13}\xi_2\xi_\gamma, \quad B_{\alpha 5} := B^*_{15}\xi_\alpha, \quad B_{5\alpha} := B^*_{51}\xi_\alpha,$$

$$B_{5\gamma+2} := B^*_{53}\xi_\gamma, \quad \xi_\alpha := \frac{\partial}{\partial x_\alpha}, \quad \alpha,\gamma = 1,2, \quad B_{55} := B^*_{55},$$

$$B_{3\gamma} := B^*_{31}\xi_1\xi_\gamma, \quad B_{4\gamma} := B^*_{31}\xi_2\xi_\gamma, \quad B_{2+\gamma,5} := B^*_{35}\xi_\gamma,$$

$$B^*_{11} := \frac{1}{\mu}(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_5^2),$$

$$B^*_{12} := \frac{(\Delta+\lambda_5^2)}{kk_7\mu\mu_0}\left\{\beta\beta_0(k_7\Delta+k_8) + (\lambda+\mu)[(k\Delta+a_0)(k_7\Delta+k_8) + k_1k_3\Delta]\right\},$$

$$B^*_{13} := -\frac{\beta k_1}{\mu_0 kk_7}((\Delta+\lambda_4^2)(\Delta+\lambda_5^2), \quad B^*_{15} := \frac{\beta}{\mu_0 kk_7}(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)(k_7\Delta+k_8),$$

$$B^*_{51} := -\frac{\beta_0}{\mu_0 kk_7}(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)(k_7\Delta+k_8), \quad \mu_0 := \lambda+2\mu,$$

$$B^*_{53} := -\frac{hk_1}{\mu_0 kk_7}(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)(\mu_0\Delta+\rho\omega^2),$$

$$B^*_{55} := \frac{1}{\mu_0 kk_7}(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)(\mu_0\Delta+\rho\omega^2)(k_7\Delta+k_8),$$

$$B^*_{31} := -\frac{k_3\beta_0}{\mu_0 kk_7}((\Delta+\lambda_4^2)(\Delta+\lambda_5^2) \quad B^*_{35} := \frac{k_3}{\mu_0 kk_7}(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)(\mu_0\Delta+\rho\omega^2),$$

$$B^*_{33} := \frac{1}{k_6}(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2),$$

$$B^*_{34} := \frac{(\Delta+\lambda_4^2)}{\mu_0 kk_6k_7}\{k_1k_3(\mu_0\Delta+\rho\omega^2) + (k_4+k_5)[(\mu_0\Delta+\rho\omega^2)(k\Delta+a_0) + \beta\beta_0\Delta]\}.$$

Substituting the vector $\mathbf{U}(\mathbf{x}) = \mathbf{B}(\partial\mathbf{x},\omega)\mathbf{\Psi}$ into $\boldsymbol{A}(\partial\mathbf{x},\omega)\mathbf{U} = 0$, where $\mathbf{\Psi}$ is a five-component vector function, we get

$$B(\Delta) = (\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\mathbf{\Psi}.$$

Whence, applying the method developed in [6], after some calculations, the vector Ψ can be represented as

$$\mathbf{\Psi} = \sum_{j=1}^{5} d_j H_0^{(1)}(\lambda_j r), \quad \sum_{j=1}^{5} d_j = 0, \quad \sum_{j=1}^{5} d_j(\lambda_m^2 - \lambda_j^2) = 0, \quad m = 4,5, \tag{25}$$

$$\sum_{j=1}^{5} d_j(\lambda_4^2-\lambda_j^2)(\lambda_5^2-\lambda_j^2) = 0, \quad d_j = \prod_{m=1}^{5}\frac{1}{\lambda_j^2-\lambda_m^2}, \quad j \neq m, \quad j = 1,2,..,5,$$

where $H_0^{(1)}(\lambda_j r)$ are Hankel's functions of the first kind with the index equal to 0 and $r = |x-y|$.

Substituting (25) into $\mathbf{U} = \mathbf{B\Psi}$, we obtain the matrix of fundamental solution, which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) :=\| \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y},\omega) \|_{5\times 5},$$

where

$$\Gamma_{\alpha\gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := \delta_{\alpha\gamma}\frac{H_0^{(1)}(\lambda_4 r)}{\mu} - \frac{\partial^2\Psi_{11}}{\partial x_\alpha \partial x_\gamma}, \quad \Psi_{11} := -\frac{H_0^{(1)}(\lambda_4 r)}{\mu\lambda_4^2}$$
$$+ \sum_{m=1}^{3} \frac{l_m}{\lambda_m^2 \mu_0 k k_7}[(k_8 - k_7\lambda_m^2)(a_0 - k\lambda_m^2) - k_1 k_3 \lambda_m^2] H_0^{(1)}(\lambda_m r)$$
$$\Gamma_{\alpha+2,\gamma+2}(\mathbf{x}\text{-}\mathbf{y},\omega) := \delta_{\alpha\gamma}\frac{H_0^{(1)}(\lambda_5 r)}{k_6} - \frac{\partial^2\Psi_{33}}{\partial x_\alpha \partial x_\gamma}, \quad \Psi_{33} := -\frac{H_0^{(1)}(\lambda_5 r)}{k_6\lambda_5^2}$$
$$+ \sum_{m=1}^{3} \frac{l_m}{\lambda_m^2 \mu_0 k k_7}[(a_0 - k\lambda_m^2)(\rho\omega^2 - \mu_0\lambda_m^2) - \beta\beta_0\lambda_m^2] H_0^{(1)}(\lambda_m r),$$
$$\Gamma_{55}(\mathbf{x}\text{-}\mathbf{y},\omega) := \frac{1}{k k_7 \mu_0}\sum_{m=1}^{3} l_m((\rho\omega^2 - \mu_0\lambda_m^2)(k_8 - k_7\lambda_m^2) H_0^{(1)}(\lambda_m r),$$
$$\Gamma_{\alpha 5}(\mathbf{x}\text{-}\mathbf{y},\omega) := \beta\frac{\partial\psi_{15}}{\partial x_\alpha}, \quad \Gamma_{2+\alpha,5}(\mathbf{x}\text{-}\mathbf{y},\omega) := k_3\frac{\partial\psi_{51}}{\partial x_\alpha}, \quad \alpha,\gamma - 1,2,$$
$$\Gamma_{5\gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := -\beta_0\frac{\partial\psi_{15}}{\partial x_\gamma}, \quad \psi_{15} = \frac{1}{k k_7 \mu_0}\sum_{m=1}^{3} l_m (k_8 - k_7\lambda_m^2) H_0^{(1)}(\lambda_m r),$$
$$\Gamma_{5,2+\gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := -k_1\frac{\partial\psi_{51}}{\partial x_\gamma}, \quad \psi_{51} = \frac{1}{k k_7 \mu_0}\sum_{m=1}^{3} l_m (\rho\omega^2 - \mu_0\lambda_m^2) H_0^{(1)}(\lambda_m r),$$

$$\Gamma_{\alpha,2+\gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := -k_1\beta\frac{\partial^2\psi_{13}}{\partial x_\alpha \partial x_\gamma}, \quad \psi_{13} := \frac{1}{k k_7 \mu_0}\sum_{m=1}^{3} l_m H_0^{(1)}(\lambda_m r),$$
$$\Gamma_{\alpha+2,\gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := -k_3\beta_0\frac{\partial^2\psi_{13}}{\partial x_\alpha \partial x_\gamma}, \quad l_m = d_m(\lambda_4^2 - \lambda_m^2)(\lambda_5^2 - \lambda_m^2), \quad l = 1,2,3,$$
$$\sum_{m=1}^{3} l_m = 0, \quad \sum_{m=1}^{3} l_m\lambda_m^2 = 0, \quad \sum_{m=1}^{3} l_m\lambda_m^4 = 1.$$

We can easily prove the following

Theorem 3. *The elements of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$ has a logarithmic singularity as $\boldsymbol{x}\to\boldsymbol{y}$ and each column of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$, considered as a vector, is a solution of the system $\boldsymbol{A}(\partial\boldsymbol{x},\omega)\boldsymbol{U} = 0$ at every point $\boldsymbol{x}$ if $\boldsymbol{x}\neq\boldsymbol{y}$.*

According to the method developed in [5], we construct the matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega) := \mathbf{\Gamma}^T(-\mathbf{x},\omega)$ and the following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega)$ may be easily verified:

Theorem 4. *Each column of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$, considered as a vector, satisfies the associated system $\widetilde{\boldsymbol{A}}(\partial\boldsymbol{x})\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega) = 0$, at every point $\boldsymbol{x}$ if $\boldsymbol{x}\neq\boldsymbol{y}$ and the elements of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$ have a logarithmic singularity as $\boldsymbol{x}\to\boldsymbol{y}$.*

Matrix of singular solutions

In solving BVPs of the theory of thermoelasticity with microtemperatures by the potential method, besides the matrix of fundamental solutions, some other matrices of singular solutions to equations (5)-(7) are of a great importance. Using the matrix of fundamental solutions, we construct the so-called singular matrices of solutions by means of elementary functions.

We introduce the special generalized stress vector $\overset{\boldsymbol{\tau}}{\mathbf{R}}(\partial\mathbf{x},\mathbf{n})\mathbf{U}$, which acts on the element of the arc with the unit normal $\mathbf{n} = (n_1, n_2)$, where

$$\overset{\boldsymbol{\tau}}{\mathbf{R}}(\partial\mathbf{x},\mathbf{n}) :=\| \overset{\tau}{\mathrm{R}}_{lj} \|_{5\times5},$$

$$\overset{\tau}{\mathrm{R}}_{\alpha\gamma} := \delta_{\alpha\gamma}\mu\frac{\partial}{\partial\mathbf{n}} + (\lambda+\mu)n_\alpha\frac{\partial}{\partial x_\gamma} + \tau_1\mathcal{M}_{\alpha\gamma}, \quad \overset{\tau}{\mathrm{R}}_{\alpha,\gamma+2} \equiv \overset{\tau}{\mathrm{R}}_{\alpha+2,\gamma} \equiv \overset{\tau}{\mathrm{R}}_{\alpha+2,5}$$
$$\equiv \overset{\tau}{\mathrm{R}}_{5\gamma} \equiv 0, \quad \overset{\tau}{\mathrm{R}}_{\alpha5} := -\beta n_\alpha, \quad \overset{\tau}{\mathrm{R}}_{\alpha+2;\gamma+2} := \delta_{\alpha\gamma}k_6\frac{\partial}{\partial\mathbf{n}} + (k_4+k_5)n_\alpha\frac{\partial}{\partial x_\gamma} + \tau_2\mathcal{M}_{\alpha\gamma},$$
$$\overset{\tau}{\mathrm{R}}_{5,\gamma+2} := k_1 n_\gamma, \quad \overset{\tau}{\mathrm{R}}_{55} := k\frac{\partial}{\partial\mathbf{n}}, \quad \mathcal{M}_{\alpha\gamma} := n_\gamma\frac{\partial}{\partial x_\alpha} - n_\alpha\frac{\partial}{\partial x_\gamma}, \quad \alpha,\gamma = 1,2, \tag{26}$$

here $\boldsymbol{\tau} = (\tau_1, \tau_2)$, τ_α, $\alpha = 1,2$, are the arbitrary numbers. If $\tau_1 = \mu$, $\tau_2 = k_5$, we denote the obtained operator by $\mathbf{P}(\partial\mathbf{x},\mathbf{n})$. The operator, which we get from $\overset{\boldsymbol{\tau}}{\mathbf{R}}(\partial\mathbf{x},\mathbf{n})$ for $\tau_1 = \dfrac{\mu(\lambda+\mu)}{\lambda+3\mu}$, $\tau_2 = \dfrac{k_6(k_4+k_5)}{k_4+k_5+2k_6}$, will be denoted by $\mathbf{N}(\partial\mathbf{x},\mathbf{n})$ and the vector $\mathbf{N}(\partial\mathbf{x},\mathbf{n})\mathbf{U}$ will be called the pseudostress vector.

Applying the operator $\overset{\boldsymbol{\tau}}{\mathbf{R}}(\partial\mathbf{x},\mathbf{n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$, we construct the so-called singular matrix of solutions

$$\overset{\boldsymbol{\tau}}{\mathbf{R}}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) := \| \overset{\tau}{\mathrm{M}}_{lj}(\partial\mathbf{x}) \|_{5\times5},$$

where

$$\overset{\tau}{\mathrm{M}}_{\gamma\gamma}(\partial\mathbf{x}) := \frac{\partial H_0^{(1)}(\lambda_4 r)}{\partial n} + (-1)^\gamma(\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2} + n_\gamma\rho\omega^2\frac{\partial\Psi_{11}}{\partial x_\gamma},$$
$$\overset{\tau}{\mathrm{M}}_{12}(\partial\mathbf{x}) := \frac{\tau_1}{\mu}\frac{\partial}{\partial s}H_0^{(1)}(\lambda_4 r) - (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial^2\Psi_{11}}{\partial x_2^2} + \rho\omega^2 n_1\frac{\partial\Psi_{11}}{\partial x_2},$$
$$\overset{\tau}{\mathrm{M}}_{21}(\partial\mathbf{x}) := -\frac{\tau_1}{\mu}\frac{\partial}{\partial s}H_0^{(1)}(\lambda_4 r) + (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial^2\Psi_{11}}{\partial x_1^2} + \rho\omega^2 n_2\frac{\partial\Psi_{11}}{\partial x_1},$$
$$\overset{\tau}{\mathrm{M}}_{1,\gamma+2}(\partial\mathbf{x}) := k_1\beta\left[n_1\varrho\omega^2\frac{\partial\psi_{13}}{\partial x_\gamma} - (\mu+\tau_1)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_\gamma\partial x_2}\right],$$
$$\overset{\tau}{\mathrm{M}}_{2,\gamma+2}(\partial\mathbf{x}) := k_1\beta\left[n_2\varrho\omega^2\frac{\partial\psi_{13}}{\partial x_\gamma} + (\mu+\tau_1)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_\gamma\partial x_1}\right],$$
$$\overset{\tau}{\mathrm{M}}_{15}(\partial\mathbf{x}) := \beta\left[(\tau_1+\mu)\frac{\partial}{\partial x_2}\frac{\partial}{\partial s} - \rho\omega^2 n_1\right]\psi_{15},$$
$$\overset{\tau}{\mathrm{M}}_{25}(\partial\mathbf{x}) := -\beta\left[(\tau_1+\mu)\frac{\partial}{\partial x_1}\frac{\partial}{\partial s} + \rho\omega^2 n_2\right]\psi_{15},$$
$$\overset{\tau}{\mathrm{M}}_{3\alpha}(\partial\mathbf{x}) := k_3\beta_0\left[\frac{n_1}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2\frac{\partial}{\partial x_\alpha}H_0^{(1)}(\lambda_m r) - (\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_2\partial x_\alpha}\right],$$

$$\overset{\tau}{\mathrm{M}}_{35}(\partial\mathbf{x}) := \frac{k_3}{k\mu_0}\sum_{m=1}^{3} l_m(\rho\omega^2-\mu_0\lambda_m^2)\left[-n_1\lambda_m^2+\frac{\tau_2+k_6}{k_7}\frac{\partial}{\partial x_2}\frac{\partial}{\partial s}\right]H_0^{(1)}(\lambda_m r),$$

$$\overset{\tau}{\mathrm{M}}_{4\alpha}(\partial\mathbf{x}) := k_3\beta_0\left[\frac{n_2}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2\frac{\partial}{\partial x_\alpha}H_0^{(1)}(\lambda_m r)+(\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_1\partial x_\alpha}\right],$$

$$\overset{\tau}{\mathrm{M}}_{45}(\partial\mathbf{x}) := -\frac{k_3}{k\mu_0}\sum_{m=1}^{3} l_m(\rho\omega^2-\mu_0\lambda_m^2)\left[n_2\lambda_m^2+\frac{\tau_2+k_6}{k_7}\frac{\partial}{\partial x_1}\frac{\partial}{\partial s}\right]H_0^{(1)}(\lambda_m r),$$

$$\overset{\tau}{\mathrm{M}}_{5\gamma}(\partial\mathbf{x}) := -\frac{\beta_0}{k_7\mu_0}\sum_{m=1}^{3} l_m\left[\frac{k_1k_3}{k}+k_8-k_7\lambda_m^2\right]\frac{\partial^2 H_0^{(1)}(\lambda_m r)}{\partial n\partial x_\gamma},$$

$$\overset{\tau}{\mathrm{M}}_{5,\gamma+2}(\partial\mathbf{x}) := \left[\frac{n_\gamma}{k_6}H_0^{(1)}(\lambda_5 r)-\frac{\partial^2(\Psi_{33}+k\psi_{51})}{\partial x_\gamma\partial n}\right]k_1$$

$$\overset{\tau}{\mathrm{M}}_{55}(\partial\mathbf{x}) := \frac{1}{\mu_0 k_7}\sum_{m=1}^{3} l_m(\rho\omega^2-\mu_0\lambda_m^2)\left[\frac{k_1k_3}{k}+k_8-k_7\lambda_m^2\right]\frac{\partial H_0^{(1)}(\lambda_m r)}{\partial n},$$

$$\overset{\tau}{\mathrm{M}}_{2+\gamma,2+\gamma}(\partial\mathbf{x}) := \frac{\partial H_0^{(1)}(\lambda_5 r)}{\partial n}+(-1)^\gamma(\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial^2\Psi_{33}}{\partial x_1\partial x_2}-n_\gamma\frac{\partial}{\partial x_\gamma}[k_1k_3\psi_{51}-k_8\Psi_{33}]$$

$$\overset{\tau}{\mathrm{M}}_{43}(\partial\mathbf{x}) := -\frac{\tau_2}{k_6}\frac{\partial H_0^{(1)}(\lambda_5 r)}{\partial s}+(\tau_2+k_6)\frac{\partial^2}{\partial x_1^2}\frac{\partial\Psi_{33}}{\partial s}-n_2\frac{\partial}{\partial x_1}[k_1k_3\psi_{51}-k_8\Psi_{33}],$$

$$\overset{\tau}{\mathrm{M}}_{34}(\partial\mathbf{x}) := \frac{\tau_2}{k_6}\frac{\partial H_0^{(1)}(\lambda_5 r)}{\partial s}-(\tau_2+k_6)\frac{\partial^2}{\partial x_2^2}\frac{\partial\Psi_{33}}{\partial s}-n_1\frac{\partial}{\partial x_2}[k_1k_3\psi_{51}-k_8\Psi_{33}]. \tag{27}$$

We prove the following theorem.

Theorem 5. *Every column of the matrix* $\left[\overset{\tau}{\mathbf{R}}(\partial\mathbf{y},\mathbf{n})\mathbf{\Gamma}(\mathbf{y}\text{-}\mathbf{x},\omega)\right]^T$, *considered as a vector, is a solution of the system* $\widetilde{\mathbf{A}}(\partial\mathbf{x},\omega)=0$ *at any point* $\mathbf{x}$ if $\mathbf{x}\neq\mathbf{y}$.

Let

$$\tilde{\boldsymbol{R}}^{\boldsymbol{\tau}}(\partial\mathbf{x},\mathbf{n}) := \begin{pmatrix} \overset{\tau}{\mathrm{R}}_{11} & \overset{\tau}{\mathrm{R}}_{12} & 0 & 0 & -\beta_0 n_1 \\ \overset{\tau}{\mathrm{R}}_{21} & R_{22} & 0 & 0 & -\beta_0 n_2 \\ 0 & 0 & \overset{\tau}{\mathrm{R}}_{33} & \overset{\tau}{\mathrm{R}}_{34} & 0 \\ 0 & 0 & \overset{\tau}{\mathrm{R}}_{43} & \overset{\tau}{\mathrm{R}}_{44} & 0 \\ 0 & 0 & k_3 n_1 & k_3 n_2 & \overset{\tau}{\mathrm{R}}_{55} \end{pmatrix},$$

where $\overset{\tau}{\mathrm{R}}_{\alpha\gamma},\ \overset{\tau}{\mathrm{R}}_{\alpha+2,\gamma+2},\ \overset{\tau}{\mathrm{R}}_{55},\ \alpha,\gamma=1,2$, are given by (26), then

$$\tilde{\boldsymbol{R}}^{\boldsymbol{\tau}}(\partial\mathbf{x},\mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y},\omega) = \|\widetilde{\mathrm{M}}_{lj}^{\tau}(\partial\mathbf{x},)\|_{5\times 5},$$

Here

$$\widetilde{\mathrm{M}}^{\tau}_{\alpha\gamma}(\partial\mathbf{x}) := \overset{\tau}{\mathrm{M}}_{\alpha\gamma}(\partial\mathbf{x}), \quad \widetilde{\mathrm{M}}^{\tau}_{\alpha+2,\gamma+2}(\partial\mathbf{x}) := \overset{\tau}{\mathrm{M}}_{\alpha+2,\gamma+2}(\partial\mathbf{x}), \quad \widetilde{\mathrm{M}}^{\tau}_{55}(\partial\mathbf{x}) := \overset{\tau}{\mathrm{M}}_{55}(\partial\mathbf{x}),$$

$$\widetilde{\mathrm{M}}^{\tau}_{1,\gamma+2}(\partial\mathbf{x}) := k_3\beta_0\left[n_1\rho\omega^2\frac{\partial\psi_{13}}{\partial x_\gamma} - (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_2\partial x_\gamma}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{2,\gamma+2}(\partial\mathbf{x}) := k_3\beta_0\left[n_2\rho\omega^2\frac{\partial\psi_{13}}{\partial x_\gamma} + (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_1\partial x_\gamma}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{15}(\partial\mathbf{x}) := \beta_0\left[-n_1\rho\omega^2\psi_{15} + (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial\psi_{15}}{\partial x_2}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{25}(\partial\mathbf{x}) := -\beta_0\left[n_2\rho\omega^2\psi_{15} + (\tau_1+\mu)\frac{\partial}{\partial s}\frac{\partial\psi_{15}}{\partial x_1}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{3\gamma}(\partial\mathbf{x}) := k_1\beta\left[\frac{n_1}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2\frac{\partial H_0^{(1)}(\lambda_m r)}{\partial x_\gamma} - (\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_2\partial x_\gamma}\right]$$

$$\widetilde{\mathrm{M}}^{\tau}_{4\gamma}(\partial\mathbf{x}) := k_1\beta\left[\frac{n_2}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2\frac{\partial H_0^{(1)}(\lambda_m r)}{\partial x_\gamma} + (\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial^2\psi_{13}}{\partial x_1\partial x_\gamma}\right]$$

$$\widetilde{\mathrm{M}}^{\tau}_{35}(\partial\mathbf{x}) := -k_1\left[\frac{n_1}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2(\rho\omega^2-\mu_0\lambda_m^2)H_0^{(1)}(\lambda_m r) - (\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial\psi_{51}}{\partial x_2}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{45}(\partial\mathbf{x}) := -k_1\left[\frac{n_2}{k\mu_0}\sum_{m=1}^{3} l_m\lambda_m^2(\rho\omega^2-\mu_0\lambda_m^2)H_0^{(1)}(\lambda_m r) + (\tau_2+k_6)\frac{\partial}{\partial s}\frac{\partial\psi_{51}}{\partial x_1}\right],$$

$$\widetilde{\mathrm{M}}^{\tau}_{5\gamma}(\partial\mathbf{x}) := -\frac{\beta}{k_7\mu_0}\sum_{m=1}^{3} l_m\left[k_8 - k_7\lambda_m^2 + \frac{k_1k_3}{k}\right]\frac{\partial^2 H_0^{(1)}(\lambda_m r)}{\partial x_\gamma\partial n},$$

$$\widetilde{\mathrm{M}}^{\tau}_{5,\gamma+2}(\partial\mathbf{x}) := k_3\left[\frac{n_\gamma}{k_6}H_0^{(1)}(\lambda_5 r) - \frac{\partial^2(\psi_{33}+k\psi_{51})}{\partial x_\gamma\partial n}\right].$$

Let $[\widetilde{\mathbf{P}}(\partial\mathbf{y},\mathbf{n})\widetilde{\mathbf{\Gamma}}(\mathbf{y}\text{-}\mathbf{x}),\omega]^T$, be the matrix which we get from $\widetilde{\mathbf{P}}(\partial\mathbf{x},\mathbf{n})\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y},\omega)$ by transposition of the columns and rows and the variables $\mathbf{x}$ and $\mathbf{y}$. The superscript "T" denotes transposition.

We prove the following theorem.

Theorem 6. *Every column of the matrix* $\left[\tilde{\boldsymbol{R}}^{\boldsymbol{\tau}}(\partial\mathbf{y},\mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{y}\text{-}\mathbf{x},\omega)\right]^T$, *considered as a vector, is a solution of the system* $\mathbf{A}(\partial\mathbf{x},\omega)\mathbf{U}=0$ at *any point* $\mathbf{x}$ *if* $\mathbf{x}\neq\mathbf{y}$.

Let $\mathbf{g}$ and $\boldsymbol{\phi}$ be continuous (or Hölder continuous) vectors and S be a closed Lyapunov curve.

We introduce the potential of a single-layer

$$\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y},\omega)\mathbf{g}(\mathbf{y})ds,$$

the potential of a double-layer

$$\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) = \int_S [\tilde{\boldsymbol{R}}^{\boldsymbol{\tau}}(\partial\mathbf{y},\mathbf{n})\mathbf{\Gamma}^T(\mathbf{y}\text{-}\mathbf{x},\omega)]^T\mathbf{g}(\mathbf{y})ds$$

and the potential of volume

$$\mathbf{Z}^{(3)}(\mathbf{x},\boldsymbol{\phi}) = \int\limits_{D^{\pm}} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y},\omega)\boldsymbol{\phi}(\mathbf{y})ds,$$

where $\boldsymbol{\Gamma}$ is the fundamental matrix, $\mathbf{g}$ and $\boldsymbol{\phi}$ are five-component vectors.

The following theorem is valid:

Theorem 7. *The vectors* $\boldsymbol{Z}^{(j)}$ $(j=1,2,)$ *are the solutions of the system*

$$\boldsymbol{A}(\partial\boldsymbol{x},\omega)\,\boldsymbol{U}=0$$

in both the domains D^+ *and* D^- *and the elements of the matrix* $\left[\tilde{\boldsymbol{R}}^{\boldsymbol{\tau}}(\partial\boldsymbol{y},\boldsymbol{n})\boldsymbol{\Gamma}^{\boldsymbol{T}}(\boldsymbol{y}\text{-}\boldsymbol{x},\omega)\right]^T$, *contain a singular part, which is integrable in the sense of the Cauchy principal value. The vector* $\boldsymbol{Z}^{(3)}(\boldsymbol{x},\boldsymbol{\phi})$ *is a solution of the system* $\boldsymbol{A}(\partial\boldsymbol{x},\omega)\boldsymbol{Z}^{(3)}=\boldsymbol{\phi}$.

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NONLINEAR MATHEMATICAL MODEL OF DYNAMICS OF VOTERS OF TWO POLITICAL SUBJECTS

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Abstract. In the present paper the nonlinear mathematical model describing dynamics of voters of pro-governmental and oppositional parties (two selective subjects, coalitions) is offered. In model three objects are considered: governmental and administrative structures influencing by means of administrative resources citizens (first of all in opposition adjusted voters) for the purpose of their attraction on the side of pro-governmental party; citizens with the selective voice, at present supporting opposition party; citizens with the selective voice, at present supporting pro-governmental party. In cases constant or variable (in proportion to number of voters of opposition party) uses of administrative resources the problem of Cauchy's for nonlinear system of the differential equations is solved analytically exactly. Conditions for model parameters at which the opposition party (coalition) will win the next elections are found. The mathematical model except theoretical interest has also important practical value, as both sides (the state structures together with pro-governmental party; opposition party) can use results according to the purposes. It allows the sides, according to the chosen strategy, to select parameters of action and to achieve desirable results for themselves.

Keywords and phrases: Nonlinear mathematical model, pro-governmental party (coalition), administrative resources, opposition party (coalition), elections.

AMS subject classification (2010): 7M10, 97M70.

Introduction

Mathematical modeling and computing experiment the last decades gained all-round recognition in a science as the new methodology which is roughly developing and widely introduced not only in natural-science and technological spheres, but also in economy, sociology, political science and other public disciplines [1 - 5].

In [6 - 8] the mathematical model of political rivalry devoted to the description of fight occurring in imperious elite competing (but not surely antagonistic) political forces, for example, power branches is considered. It is supposed that each of the parties has ideas of "number" of the power which this party would like to have itself, and about "number" of the power which she would like to have for the partner.

Works [9 - 12] are devoted to creation of mathematical model of such social process what administrative management is. The last can be carried out as at macrolevel (for example, the state) and at microlevel (for example, an educational or research institution, an industrial facility, etc.).

A certain interest represents creation of the mathematical model, allowing to define dynamics of voters of political subjects. It is known that in many countries including developed ones, there are two-party systems. Such systems are characterized by the existence of two largest parties which periodically replace each other in power. And, when in power there is one party, the second is the leading party of opposition. However it doesn't mean that except these two parties in the country there are no other parties, simply their influence on political processes is insignificant. In some countries eventually to change of one of the largest parties can come any else, earlier being in a shadow. For example, in Great Britain in the XIX century and at the beginning of the XX century two largest parties were conservative and liberal. In the XX century Liberal party in this tandem replaced labor, however the two-party system remained. The most rigid option of two-party system exists in the USA. Here only republican and democratic parties apply for the power, other parties almost don't play any role. And in the Congress for it more than two hundred year's history other parties almost were never presented. A version of two-party system is the two-block system. Here not largest parties, and party coalitions appear confronting forces. It is caused by that any party unable to achieve sufficient support of voters independently to create the government therefore parties according to the political orientation and ideological installations unite for increase in the influence. Thus such competing coalitions remain almost in invariable structure throughout quite a long time. Such party system developed, for example, in the Netherlands. Such party systems in which as two main competing forces act, on the one hand, party, and, meet another – the party block also. So, in Australia agrarian and liberal parties make the constant union resisting to the Labour party.

In the real work the nonlinear mathematical model describing dynamics of voters of pro-governmental and oppositional parties (two selective subjects, coalitions) is offered. In the model three objects are considered:

1. The state and administrative structures influencing by means of administrative resources citizens (first of all in opposition adjusted voters) for the purpose of their attraction on the party of pro-governmental party.
2. Citizens with the selective voice, at present supporting opposition party.
3. Citizens with the selective voice, at present supporting pro-governmental party.

1. System of the equations and entry conditions

For dynamics description between elections of voters of pro-governmental and oppositional parties (two selective subjects) we offer the following nonlinear mathematical model:

$$\begin{aligned}\frac{dN_1(t)}{dt} &= (\alpha_1(t) - \alpha_2(t))N_1(t)\,N_2(t) - f(t, N_1(t))\\ \frac{dN_2(t)}{dt} &= (\alpha_2(t) - \alpha_1(t))N_1(t)\,N_2(t) + f(t, N_1(t))\end{aligned} \tag{1.1}$$

$$N_1(0) = N_{10}, \quad N_2(0) = N_{20}, \quad N_{10} < N_{20}, \tag{1.2}$$

where $N_1(t), N_2(t)$ are respectively, a number of the voters supporting oppositional and pro-governmental parties at the moment of time t and $t \in [0, T], t = 0$ is the moment of time of the last elections owing to which the party won elections and became pro-governmental $(N_{10} < N_{20})\,; t = T$ is the moment of the following, for example, parliamentary elections (as a rule $T = 4$ years or 1460 days if t will change on days); $a_1(t), a_1(t)$ respectively factors of attraction of votes of oppositional and pro-governmental parties at the moment of time t, connected with the action program, financial and information possibilities (PR technology) of these parties; $f(t, N_1(t))$ is the positive function of the arguments characterizing use of administrative resources, directed on voters of opposition party, for the purpose of their attraction on the party and power preservation that is the purpose of any authorities in power.

In model (1.1), (1.2) it is supposed that total number of voters ($N_{10} + N_{20} = a$) from elections to elections doesn't change (often, in many countries, their change is insignificant in comparison with a total number of voters). Thus, we consider that in a period between elections the number of the dead voters and the voters who for the first time have received a vote are equal (in many countries of 18 years) authorities in power.

This mathematical model doesn't consider falsification of elections in the election day though it is possible to consider and falsification cases, having initially set their percentage value.

Two cases are considered:

1. $\alpha_1(t) = \alpha_1 = const > 0, \alpha_2(t) = \alpha_2 = const > 0, f(t, N_1(t)) = b > 0$ is constant nature of use of administrative resources.

2. $\alpha_1(t) = \alpha_1 = const > 0, \alpha_2(t) = \alpha_2 = const > 0, f(t, N_1(t)) = \beta N_1(t), \beta > 0$ variable nature of use of administrative resources (in proportion to a number of voters of opposition party).

2. Constant nature of use of administrative resources

In this case we have a system of the equations

$$\begin{aligned} \frac{dN_1(t)}{dt} &= (\alpha_1 - \alpha_2) N_1(t) N_2(t) - b \\ \frac{dN_2(t)}{dt} &= (\alpha_2 - \alpha_1) N_1(t) N_2(t) + b \end{aligned} \tag{2.1}$$

depending on ratios between constants of model, the exact solution of a problem of Cauchy's (2.1), (1.2) look like:
a) $\alpha_1 < \alpha_2$

$$\begin{aligned} N_1(t) &= \frac{a}{2} + \frac{p\left(1 + \dfrac{N_{10} - N_{20} - 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pt\right)}{1 + \dfrac{N_{20} - N_{10} + 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pt)} \\ N_2(t) &= \frac{a}{2} - \frac{p\left(1 + \dfrac{N_{10} - N_{20} - 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pt\right)}{1 + \dfrac{N_{20} - N_{10} + 2p}{N_{10} - N_{20} + 2p} \cdot \exp(2(\alpha_2 - \alpha_1)pt)} \end{aligned} \tag{2.2}$$

$$p = \sqrt{\frac{b}{\alpha_2 - \alpha_1} + {a^2}/{4}} > {a}/{2}$$

$$\exp(2(\alpha_2-\alpha_1)pt_1)=\frac{a+2p}{2p-a}\cdot\frac{N_{10}-N_{20}+2p}{N_{20}-N_{10}+2p}>1$$

$$N_2(t_1)=a,\quad N_1(t_1)=0$$

$$t_1=\frac{1}{2(\alpha_2-\alpha_1)p}\ln\left[\frac{a+2p}{2p-a}\cdot\frac{N_{10}-N_{20}+2p}{N_{20}-N_{10}+2p}\right]. \tag{2.3}$$

If $t_1\le T$, then on the following elections the opposition party will have no voters supporting them (exponential aspiration to an one-party regime); if $t_1>T$, then on the following elections opposition party will support only insignificant number of voters (close to a one-party regime)

$$N_1(T)=\frac{a}{2}+\frac{p\left(1+\dfrac{N_{10}-N_{20}-2p}{N_{10}-N_{20}+2p}\cdot\exp(2(\alpha_2-\alpha_1)pT)\right)}{1+\dfrac{N_{20}-N_{10}+2p}{N_{10}-N_{20}+2p}\cdot\exp(2(\alpha_2-\alpha_1)pT)}>0 \tag{2.4}$$

b) $\alpha_1=\alpha_2$

$$N_1(t)=-bt+N_{10}\quad N_2(t)=bt+N_{20}. \tag{2.5}$$

It is clear, that in case of equality of factors of attraction of votes of competing parties, the number of voters of pro-governmental party grows, and oppositional falls, and, if

$$t_2={}^{N_{10}}\!/_{b}\le T,$$

then on the following elections the opposition party will have no voters supporting them (linear aspiration to an one-party regime). If

$$t_2={}^{N_{10}}\!/_{b}>T,$$

then on the following elections opposition party will support only an insignificant number of voters (close to a one - party regime).

$$N_1(T)=-bT+N_{10}>0 \tag{2.6}$$

c) $\alpha_1>\alpha_2$

$$D=\frac{a^2}{4}-\frac{b}{\alpha_1-\alpha_2} \tag{2.7}$$

c1) $D=0$

$$N_1(t)=\frac{a}{2}+\frac{N_{10}-N_{20}}{2+(\alpha_1-\alpha_2)(N_{10}-N_{20})t}, \tag{2.8}$$

$$N_2(t)=\frac{a}{2}+\frac{N_{20}-N_{10}}{2+(\alpha_1-\alpha_2)(N_{10}-N_{20})t}.$$

The decision (2.8) is considered only at a period

$$t\in[0,t_3],t_3=\frac{4N_{10}}{a(\alpha_1-\alpha_2)(N_{20}-N_{10})}>0,N_1(t_3)=0,N_2(t_3)=a. \tag{2.9}$$

Therefore, if

$$t_3 \leq T,$$

then on the following elections the opposition party will have no voters supporting them (hyperbolic aspiration to a one-party regime); if

$$t_3 > T,$$

that at the following elections opposition party will support only an insignificant number of voters (close to a one - party regime)

$$N_1(T) = \frac{a}{2} + \frac{N_{10} - N_{20}}{2 + (\alpha_1 - \alpha_2)(N_{10} - N_{20})T} > 0 \tag{2.10}$$

c2) $D > 0$

$$D = \frac{a^2}{4} - \frac{b}{\alpha_1 - \alpha_2} = q^2, q < {}^a\!/_2 \tag{2.11}$$

$$N_1(t) = \frac{a}{2} + \frac{q\left(1 + \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qt\right)}{1 - \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qt)}, \tag{2.12}$$

$$N_2(t) = \frac{a}{2} - \frac{q\left(1 + \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qt)\right)}{1 - \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qt)}.$$

If the inequality is executed

$$N_{10} < N_{20} < N_{10} + 2q, \tag{2.13}$$

then, at

$$t_4 = \frac{1}{2(\alpha_1 - \alpha_2)q} \ln \frac{N_{20} - N_{10} + 2q}{N_{10} + 2q - N_{20}} \tag{2.14}$$

the ratio takes place

$$N_1(t_4) = N_2(t_4),$$

then, at

$$t > t_4, \quad N_1(t) > N_2(t). \tag{2.15}$$

Therefore, if $t_4 < T$, that the opposition party will win the following elections, a case

$$t_4 = T$$

on the following elections both parties will collect identical quantities of votes, and at

$$t_4 > T,$$

at the following elections at pro - governmental party all the same while will be voters more.

If equality takes place

$$N_{20} = N_{10} + 2q, \tag{2.16}$$

that decision (2.12) will become

$$N_1(t) = N_{10}, \quad N_2(t) = N_{20}, \tag{2.17}$$

i.e. the number of voters of parties doesn't change over time and at the subsequent elections the pro - governmental party will keep the power.

At inequality performance

$$a > N_{20} > N_{10} + 2q$$

$$t_5 = \frac{1}{2(\alpha_1 - \alpha_2)q} \ln \frac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \frac{N_{10} + N_{20} - 2q}{N_{20} + N_{10} + 2q} \tag{2.18}$$

$$N_1(t_5) = 0, N_2(t_5) = a.$$

Therefore, if

$$t_5 \leq T,$$

then at the following elections the opposition party will have no voters supporting them (exponential aspiration to a one - party regime); if

$$t_5 > T,$$

that at the following elections opposition party will support only insignificant number of voters (close to a one - party regime)

$$N_1(T) = \frac{a}{2} + \frac{q\left(1 + \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qT)\right)}{1 - \dfrac{N_{20} - N_{10} + 2q}{N_{20} - N_{10} - 2q} \cdot \exp(-2(\alpha_1 - \alpha_2)qT)} > 0 \tag{2.19}$$

c3) $D < 0$

$$D = \frac{a^2}{4} - \frac{b}{\alpha_1 - \alpha_2} = -r^2, \tag{2.20}$$

$$N_1(t) = \frac{a}{2} - \frac{r\left(\dfrac{N_{20} - N_{10}}{2r} + \tan((\alpha_1 - \alpha_2)rt\right)}{1 - \dfrac{N_{20} - N_{10}}{2r} \tan((\alpha_1 - \alpha_2)rt)} \tag{2.21}$$

$$N_2(t) = \frac{a}{2} + \frac{r\left(\dfrac{N_{20} - N_{10}}{2r} + \tan((\alpha_1 - \alpha_2)rt\right)}{1 - \dfrac{N_{20} - N_{10}}{2r} \tan((\alpha_1 - \alpha_2)rt)}$$

$$t_6 = \frac{1}{(\alpha_1 - \alpha_2)r} arctg \frac{4rN_{10}}{N_{20}^2 - N_{10}^2 + 4r^2}$$
$$N_1(t_6) = 0, N_2(t_6) = a.$$

Therefore, if

$$t_6 \le T,$$

then on the following elections the opposition party will have no voters supporting them (transcendental aspiration to a one-party regime); if $t_6 > T$, that of the following elections opposition party will support only an insignificant number of voters (close to an one-party regime)

$$N_1(T) = \frac{a}{2} - \frac{r\left(\dfrac{N_{20} - N_{10}}{2r} + \tan((\alpha_1 - \alpha_2)rT\right)}{1 - \dfrac{N_{20} - N_{10}}{2r}\tan((\alpha_1 - \alpha_2)rT)} > 0 \tag{2.22}$$

3. Variable nature of use of administrative resources

In this case we have a system of the equations

$$\frac{dN_1(t)}{dt} = (\alpha_1 - \alpha_2)N_1(t)\, N_2(t) - \beta N_1(t), \tag{3.1}$$

$$\frac{dN_2(t)}{dt} = (\alpha_2 - \alpha_1)N_1(t)\, N_2(t) + \beta N_1(t).$$

Depending on ratios between constants of model, the exact solution of a problem of Cauchy's (3.1), (1.2) look like:
a) $\alpha_1 = \alpha_2$

$$N_1(t) = N_{10}e^{-\beta t}, \quad N_2(t) = a - N_{10}e^{-\beta t}. \tag{3.2}$$

From (3.2) it is clear that in case of equality of factors of involvement of voters of competing parties, the number of voters of pro - governmental party grows, and oppositional falls and on the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = N_{10}e^{-\beta T}, \tag{3.3}$$

b) $\alpha_1 \neq \alpha_2, (\alpha_1 - \alpha_2)a = \beta, \alpha_1 > \alpha_2$

$$N_1(t) = \frac{N_{10}}{1 + (\alpha_1 - \alpha_2)N_{10}t},$$
$$N_2(t) = \frac{N_{20} + (\alpha_1 - \alpha_2)aN_{10}t}{1 + (\alpha_1 - \alpha_2)N_{10}t} \tag{3.4}$$

From (3.4) it follows that if a number of voters of pro - governmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (hyperbolic aspiration to a one - party regime).

c)$\alpha_1 \neq \alpha_2, (\alpha_1 - \alpha_2)a \neq \beta$

$$N_1(t) = \frac{((\alpha_2 - \alpha_1)a + \beta)N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)t}}{(\alpha_2 - \alpha_1)N_{20} + \beta + (\alpha_2 - \alpha_1)N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)t}} \tag{3.5}$$

$$N_2(t) = \frac{(\alpha_2 - \alpha_1)aN_{20} + a\beta - \beta N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)t}}{(\alpha_2 - \alpha_1)N_{20} + \beta + (\alpha_2 - \alpha_1)N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)t}}$$

c1) $\alpha_1 < \alpha_2$ From (3.5) it follows that in this case, the number of voters of pro - governmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = \frac{((\alpha_2 - \alpha_1)a + \beta)N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)T}}{(\alpha_2 - \alpha_1)N_{20} + \beta + (\alpha_2 - \alpha_1)N_{10}e^{-((\alpha_2-\alpha_1)a+\beta)T}} \tag{3.6}$$

c2) $\alpha_1 > \alpha_2, 0 < (\alpha_1 - \alpha_2)a < \beta$ From (3.5) it follows that in this case, the number of voters of pro - governmental party grows, and oppositional falls and at the following elections it will support only an insignificant number of voters (exponential aspiration to a one - party regime)

$$N_1(T) = \frac{(\beta - (\alpha_1 - \alpha_2)a)N_{10}e^{-(\beta-(\alpha_1-\alpha_2)a)T}}{\beta - (\alpha_2 - \alpha_1)N_{20} - (\alpha_2 - \alpha_1)N_{10}e^{-(\beta-(\alpha_1-\alpha_2)a)T}} \tag{3.7}$$

c3) $\alpha_1 > \alpha_2, (\alpha_1 - \alpha_2)a > \beta$.

Let's introduce the notation

$$g(t) \equiv (\alpha_1 - \alpha_2)N_{20} - \beta + (\alpha_1 - \alpha_2)N_{10}e^{((\alpha_1-\alpha_2)a-\beta)t}. \tag{3.8}$$

It is easy to show that we have

$$g'(t) > 0, \quad g(0) > 0.$$

Therefore owing to a $g(t)$ function continuity

$$g(t) > 0, \text{for} \quad t > 0$$

If the inequality takes place

$$(\alpha_1 - \alpha_2)a > 2\beta, \tag{3.9}$$

then inequalities are fair

$$(\alpha_1 - \alpha_2)N_{20} > 2\beta, \quad \frac{((\alpha_1 - \alpha_2)N_{20} - \beta)a}{N_{10}((\alpha_1 - \alpha_2)a - 2\beta)} > 1, \tag{3.10}$$

and also an inequality for required functions

$$N_1(t) \geq N_2(t), \quad t \geq t_7, \tag{3.11}$$

$$t_7 = \frac{1}{(\alpha_1 - \alpha_2)a - \beta} \ln \frac{((\alpha_1 - \alpha_2)N_{20} - \beta)a}{N_{10}((\alpha_1 - \alpha_2)a - 2\beta)}.$$

Therefore, if

$$t_7 < T,$$

then the opposition party will win the following elections, a case

$$t_7 = T$$

at the following elections both parties will collect identical quantities of votes, and at

$$t_7 > T,$$

at the following elections at pro - governmental party all the same while will be voters more.

In the case

$$\beta < (\alpha_1 - \alpha_2)a \leq 2\beta,$$
$$N_1(t) < N_2(t), \quad t \geq 0$$

and the opposition party will lose the following elections.

The mathematical model except theoretical interest has also important practical value, as both parties (the state structures in together with pro - governmental party; opposition party) can use results according to the purposes. It allows the parties, according to the chosen strategy, to select parameters of action and to achieve desirable results for them.

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ON THE WELL-POSEDNESS OF A CLASS OF THE OPTIMAL CONTROL PROBLEM WITH DISTRIBUTED DELAY

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Abstract. A theorem of the well-posedness is given for the linear with respect to control optimal problem, when perturbations of the right-hand side of a differential equation and an integrand are small in the integral sense.

Keywords and phrases: Well-posedness of optimal problem, equation with distributed delay, perturbations.

AMS subject classification: 34K35, 34K27, 49J21.

Let $a < t_{01} < t_{02} < t_{11} < t_{12} < b, \theta > 0, \tau > 0$ be given numbers and let R_x^n be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; suppose that $O \subset R_x^n$ is an open set and $U \subset R_u^r$ is a compact and convex set, the $n \times r$ -dimensional matrix-function $f(t, x)$ is continuous on the set $I \times O$ and continuously differentiable with respect to $x \in O$, where $I = [a, b]$. Further, let the scalar function $f^0(t, x, u)$ be continuous on the set $I \times O \times U$ and convex in $u \in U$; let Φ be the set of continuous initial functions $\varphi(t) \in O, t \in [a - \tau, t_{02}]$; let Ω be the set of measurable control functions $u(t) \in U, t \in [a - \theta, b]$.

To each element

$$w = (t_0, t_1, u(\cdot)) \in W = [t_{01}, t_{02}] \times [t_{11}, t_{12}] \times \Omega$$

we assign the differential equation linear with respect to control

$$\dot{x}(t) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} f(t, x(t+s))u(t+\xi)ds \Big\} d\xi, t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi_0(t), t \in [t_0 - \tau, t_0), x(t_0) = x_{00}, \tag{2}$$

where $\varphi_0(\cdot) \in \Phi$ is a given initial function, $x_{00} \in O$ is a given initial vector.

Equation (1) is called a differential equation with distributed delay in phase coordinates and in controls.

Definition 1. Let $w = (t_0, t_1, u(\cdot)) \in W$. A function $x(t) = x(t; w) \in O, t \in [t_0 - \tau, t_1]$ is called solution corresponding to the element w, if the conditions (1) and (2) are fulfilled. Moreover, the function $x(t), t \in [t_0, t_1]$ is absolutely continuous and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Definition 2. An element $w = (t_0, t_1, u(\cdot) \in W$ is admissible if there exists the corresponding solution $x(t) = x(t; w), t \in [t_0 - \tau, t_1]$ and the condition

$$x(t_1) = x_1 \tag{3}$$

is fulfilled. Here $x_1 \in O$ is a given point and also $x_1 \neq x_{00}$.

The set of admissible elements will be denoted by W_0.

Definition 3. An element $w_0 = (t_{00}, t_{10}, u_0(\cdot)) \in W_0$ is called optimal, if

$$J_0 = J(w_0) = \inf_{w \in W_0} J(w), \tag{4}$$

where

$$J(w) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} f^0(t, x(t+s), u(t+\xi))ds \Big\} d\xi, x(t) = x(t; w).$$

Problem (1)-(4) is called an optimal problem with distributed delay. The element w_0 is called the solution of problem (1)-(4).

To formulate the main result we need the following notation: E is the space of vector functions $G(t,x) = (g^0(t,x), g^1(t,x), ..., g^n(t,x))^T$ which satisfy the following conditions: for every $x \in O$ the function $G(t,x)$ is measurable on I; for every $G \in E$ and any compact set $K \subset O$ there exist functions $m_{G,K}(\cdot), L_{G,K}(\cdot) \in L_1(I; R_+), R_+ = [0, \infty)$ such that the inequalities

$$|G(t,x)| \leq m_{G,K}(t), \forall x \in K,$$

$$|G(t,x) - G(t,y)| \leq L_{G,K}(t)|x - y|, \forall (x,y) \in K^2$$

are fulfilled for almost all $t \in I$.

Let $K \subset O$ be a compact set, $C > 0$ is a given number. Denote by W_K the set of perturbations:

$$W_K = \Big\{ G \in E \mid \exists m_{G,K}(\cdot), L_{G,K}(\cdot) \in L_1(I; R_+), \int_I \Big[m_{G,K}(t) + L_{G,K}(t) \Big] dt \leq C \Big\}.$$

Furthermore,

$$V_{\delta,K} = \Big\{ G \in W_K \mid \sup_{(t',t'',x) \in I^2 \times K} \Big| \int_{t'}^{t''} G(s,x)ds \Big| \leq \delta \Big\}, \delta > 0;$$

$$B_{x_{00},\delta} = \Big\{ x_0 \in O \mid |x_0 - x_{00}| \leq \delta \Big\}, B_{\varphi_0,\delta} = \Big\{ \varphi_0(\cdot) \in \Phi | \|\varphi_0 - \varphi\| \leq \delta \Big\},$$

$$\|\varphi_0 - \varphi\| = \max_{t \in [a-\tau, t_{02}]} |\varphi_0(t) - \varphi(t)|.$$

Theorem 1. *Let the following conditions be fulfilled:*

1) $W_0 \neq \emptyset$;

2) *there exists a compact set $K_0 \in O$ such that*

$$x(t; w) \in K_0, t \in [t_0 - \tau, t_1], \forall w \in W_0.$$

Then for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for every

$$\mu = (x_0, \varphi(\cdot), G) \in B_{x_{00},\delta} \times B_{\varphi_0,\delta} \times V_{\delta,K_1}$$

the perturbed optimal control problem

$$\dot{x}(t) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} \Big[f(t, x(t+s))u(t+\xi) + g(t, x(t+s)) \Big] ds \Big\} d\xi, t \in [t_0, t_1],$$

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0), x(t_0) = x_0, x(t_1) \in B_{x_1,\delta},$$

$$J(w;\mu) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} \Big[f^0(t, x(t+s), u(t+\xi)) + g^0(t, x(t+s)) \Big] ds \Big\} d\xi \to \min$$

has the solution $w_0(\mu) = (t_{00}(\mu), t_{10}(\mu), u_0(\cdot;\mu))$. Also,if

$$\mu_i = (x_{0i}, \varphi_i(\cdot), G_i) \in B_{x00,\delta_i} \times B_{\varphi_0,\delta_i} \times V_{\delta_i,K_1}, i = 1, 2, ...,$$

where $\delta_i = \delta(\varepsilon_i), \varepsilon_i \to 0,$ *then*

$$\lim_{i\to\infty} J(w_0(\mu_i);\mu_i) = J_0.$$

Moreover, from the sequence $w_i, i = 1, 2, ...$ *we can choose a subsequence*

$$w_0(\mu_{i_k}) = (t_{00}(\mu_{i_k}), t_{10}(\mu_{i_k}), u_0(\cdot;\mu_{i_k})), k = 1, 2, ...$$

such that

$$\lim_{k\to\infty} t_{00}(\mu_{i_k}) = t_{00}, \lim_{k\to\infty} t_{10}(\mu_{i_k}) = t_{10},$$

$$\lim_{k\to\infty} u_0(t;\mu_{i_k}) = u_0(t), \text{ weakly in } L_1([a-\theta, b]; U)$$

and $w_0 = (t_{00}, t_{10}, u_0(\cdot))$ *is a solution of the problem (1)-(4). Here* $g = (g^1, ..., g^n)^T$, $K_1 \subset O$ *is a compact set containing a certain neighborhood of the compact* K_0.

Some comments.

c1. If the problem (1)-(4) has a unique solution $w_0 = (t_{00}, t_{10}, u_0(\cdot))$, then we have

$$\lim_{i\to\infty} t_{00}(\mu_i) = t_{00}, \lim_{i\to\infty} t_{10}(\mu_i) = t_{10},$$

$$\lim_{i\to\infty} u_0(t;\mu_i) = u_0(t), \text{ weakly in } L_1([a-\theta, b]; U).$$

c2. A theorem analogous to Theorem 1 also is valid for the following optimal control problem

$$\dot{x}(t) = \int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} \Big[f(t, x(t+s))u(t+\xi) + f_1(t, x(t+s)) \Big] ds \Big\} d\xi, t \in [t_0, t_1],$$

$$x(t) = \varphi(t), t \in [t_0 - \tau, t_0), x(t_0) = x_{00}, x(t_1) = x_1,$$

$$\int_{-\theta}^{0} \Big\{ \int_{-\tau}^{0} \Big[f^0(t, x(t+s), u(t+\xi)) + f_1^0(t, x(t+s)) \Big] ds \Big\} d\xi \to \min,$$

where $(f_1^0, f_1)^T \in E$ is a given function.

c3. Theorem 1 is proved by the method given in [1].

c4. Theorems of the continuity of the minimum of the integral functional (well-posedness) with respect to perturbations for various classes of optimal control problems, when perturbations are small in the integral sense, are proved in [1-5]. A theorem on the well-posedness for an nonlinear optimal problem with distributed delay in phase coordinates is proved in [6, 7], with distributed delay in phase coordinates and control-in [8, 9].

c5. Finally, we note that various small values are as a rule ignored in the numerical solutions of optimal problems and therefore it is important to establish the connection between initial and perturbed problem.

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FORECASTING METHOD IN ECONOMICS AND FINANCE

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Abstract. The forecasting problem in economics and finance is considered. A classification of economic forecasting methods is given. Necessary (or corresponding) complexity principle is formulated and the possibilities of practical use of forecasting methods applying to Georgian economy on the basis of current computer systems is demonstrated. the base of current computer systems is shown.

Keywords and phrases: Forecasting in economics and finance, forecasting methods and models, econometric and noneconometric methods, Eviews and Matlab.

AMS subject classification (2010): 91B99.

We won't be mistaken if we say, that an ultimate goal of studying any discipline is receiving the most real forecasting estimates. However, unfortunately it is very difficult to do this in economics and finance. The importance of forecasting is well expressed in the words: "My interest is in the future because I am going to spend the rest of my life there" (C. E. Ketering) [1]. But one thing is the interest and wish, another whether it is possible. The difficulty can be well seen from the following definition (belonging to Evan Esar): "An economist is an expert who will know tomorrow why the things he predicted yesterday didn't happen today"([1]).This is certainly a joke. More seriously this question was considered by a well-known macroeconomist Gr. Mankiw in his most famous textbook in economics, where he says: "Unfortunately with the accounting of modern knowledge of economy, processes flowing in it often are unpredictable", or as famous macroeconomist R. Lukas said: "As consultants, we sometimes try to bend down through ourselves". Thus, forecasting in economics and finance is a very actual, complicated and therefore, very interesting thing.

Scientific forecasts are made by applying logical inference to facts and past experience under the assumption that the future tends to replicate the past. In this way, forecast errors made in the past can be systematically studied, to improve forecast accuracy in the future. The principal technique, used in economic and business forecasting, vary from simple methods to complicated econometric model forecasts. Simple methods are mechanical and ignore the structural relationships of economic systems. Sophisticated methods, which can be empirical, statistical or econometric, are derived from economic theories and statistical inference; and these methods, to a greater extent, incorporate economic causality into the forecasting system. The procedure for making forecasts is similar, no matter what technique is used. It involves building a forecasting device, putting inputs into this device and making a forecast. To an econometrician, the mathematical model is the forecasting device, and judgments along with historical data and inputs. Although, before the device is put into use, it must go trough a series of rigorous economic and statistical tests, to assess its forecasting ability.

The building of such forecasting device is not devoid of a builder's judgments. A forecasting model is greatly influenced by the builder's interpretations of data information, views of economic theories, and preferences for statistical inference techniques. In addition, the construction of a forecast device is also subject to the limitations on time, funds, and the availability of data. Given the objective of the forecast and its limitations, it is the forecaster's judgment to decide how to construct the forecasting model.

Forecasting methods are separated into two groups according to their level of sophistication. Noneconometric forecasts include simple extrapolation, judgmental forecasts, economic indicators and survey forecasts. The econometric techniques these methods require do not go beyond simple and multiple regression analyses.

Econometric forecasts involve the use of a number of advanced econometric techniques and can be classified into three categories, each involving an increased level of sophistication. In a single-equation regression model, the dependent variable to be forecast is explained by a number of explanatory variables in a single equation. The second group consists of methods which are oriented to use a multidimensional econometric models, assuming that initial variants of these models has a structural form (are constructed in accordance with economic theory). The third level of complexity is the time-series (stochastic) models, which are usually empirical.

As to the complexity of using models or methods, here everything depends on the complexity of problem to be solved. Actually, as A. Einstein said: "All must be done as simple as it is possible, but no more". In our opinion it is possible to formulate this idea in a form of "necessary (or corresponding) complexity principle". For illustration of this principle, recall some examples from our issues (of course, we can recall many examples from others issues, but as it is said in a Russian proverb: "our own shirt is closer to the body"!).

Let us begin this following increasing of complexity of mathematical apparatus and models.

Consider, for example, very actual for our economy, Georgian consolidated budget revenues forecasting problem (say, for 2013-2015 years), for incomes expected from tax of profit. Using well known computer system Eviews (Econometric views), we can construct a model of dependence of Gcbr from gdp of the country. The corresponding linear logarithmic model (regression equation) has the form:

$$\mathrm{LOG(GCBTP)} = -13.69992633 + 2.136274694 * \mathrm{LOG(GDP)}, \tag{1}$$

where GCBTP denotes Georgian consolidate budget tax of profit volume (in million GELs), GDP is volume of gdp, LOG is natural logarithm.

As it is clear from the corresponding results, the model has rather high level of accuracy: $R^2 = 0.98$, t-statistics of parameters are rather high, DW-statistic is almost 2, F-statistics is equal to 411, etc.

Besides this, it should be noted that, due to the model (1), the elasticity coefficient of tax of profit, with respect to GDP equals 2.14, i.e. 1% increase of GDP shall cause 2.14% increase of the Georgian tax of profit.

The forecasting problem of this index the model (1) it reduces on finding the forecasting estimation of exogenous variable GDP, for forecasting period.

Finally, concerning GDP's forecasting problem, the semilogarithm trend model of this index has the form:

$$\mathrm{LOG(GDP)} = 8.121026344 + 0.1243837372 * \mathrm{@TREND}, \tag{2}$$

where @TREND denotes artificial time (trend) variable. The accuracy of model (2) is rather high: $R^2 = 0.99$, t-statistics of parameters are very high, F-statistics are equal to 1238, etc.

After all, accounting forecasting estimations from this model, in the model (1) gives forecasting estimations of resulting variable for appropriate period, what's very easy by using Eviews.

Analogously we can forecast the other budget revenues, although sometimes, for achieving appropriate accuracy, one must include trend component in the model.

For example, it can be shown, that Georgian consolidated budget total (own) revenues model (on the base of data of 1995-2011 years) has the form:

$$\mathrm{NSSH} = -902.6632766 + 0.3854412984 * \mathrm{GDP} - 95.58398125 * \mathrm{@TREND},$$

where NSSH denotes the volume of total (own) revenues of Georgian consolidate budget.

However, from the above-considered examples we must not make a conclusion that all forecasting problems can be solved on the basis of such simple models. Consider, for example, Georgian commercial banks total actives forecasting problem basis on dynamics of this index. It can be shown, that based on the months data of 2007.12 – 2010.04, Georgian commercial banks total actives, with rather high accuracy, can be described by following autoregressive and moving average type (ARMA) model (using Eviews):

$$\mathrm{CBA} = 7417816.211 + 69742.51203 * \mathrm{@TREND} + [\mathrm{AR}(2) = 0.502738842, \mathrm{MA}(1)$$

$$= 1.238605111, \mathrm{INITMA} = 2008M02],$$

where $AR(2)$ denotes second order autoregressive term, while $MA(1)$ represents first order moving average (as it is known $MA(1) = u_{t-1}$, where u_{t-1} represents error term of this equation for the previous period).

Although this model is rather accurate, (as it is known) the accuracy of such models will begin to deteriorate as the forecasting period extends. Besides the above, the necessity of use of rather sophisticated models can be caused by technical complexity of problem or specifics of modeling situation or country.

Consider, for example, the capital cost computing problem for investment projects (see [3]). Let us begin again from the very simple example. Consider an investment project which requires initial investment of 100000$ to buy a new special device. By market department's forecasting estimations, the living circle duration of this product is 3 years and the probable incomes from this device at the end of each year will be, correspondingly, $50000, 40000$ and 30000$. Within this conditions, net present value (NPV) for this project can be calculated as follows (see [4]):

$$\mathrm{NPV} = -100 + 50/(1+k) + 40/(1+k)^2 + 30/(1+k)^3 \tag{3}$$

where k (rate of discount) denotes the capital cost for this project.

Clearly for this project there should exist a value of k, say k_0 (internal rate of profitability, IRR), for which NPV of the project equals 0 or project never brings profit nor loss. This means that if $k > k0$, then $NPV < 0$, i.e. project brings loss and if $k < k_0$, then the project brings profit, i.e. $NPV > 0$ or profit is 0. From this it is clear, that if k_0 for project is rather low, the project is not acceptable and it is acceptable for the case when k_0 is sufficiently high. Thus, it is clear, that problem of finding k_0 in this case is reduced to the solution of equation $NPV = 0$, which by (3) means that it is needed to solve a third order equation. On the other hand, to solve such an equation (and more complex ones) is very simple by using modern computer programs, such as Matlab (see, for example [5]). Actually, using this system, the above mentioned problem can be solved by using the single command:

$$fsolve('-100 + 50/(1 + x) + 40/(1 + x)^2 + 30/(1 + x)^{3'}, 0),$$

which gives the value $k_0 = 0.1065$. Thus, if capital cost of this project is lower than 10.65%, the project is profitable, and not otherwise. Now it is clear, that analogously one can find internal value of profitability for projects, which have any living circle duration, i.e. solve the profitability problem for them.

At last a real problem in economics and finance can be so complex, that it will require the application of all above mentioned instruments. For example, consider very actual problem for Georgian economy, optimal tax burden definition problem (see [6-7]). As is known, this problem (in a theory) can be solved using Lafer curve. If we try practical realization this theory for Georgia in the base of data of 1995-2011 years, we receive following classical Lafer's product curve equation

$$X = -34790.71 * q^2 + 48624.40 * q - 942.65, \tag{4}$$

where X denotes value of gdp (in real representation), and q denotes tax burden level on economy. It should be noted that, although statistical characters of coefficients of this equation are not very high, they have "right" signs (i.e. corresponding to the signs of economic theory). Besides this, as a whole, the obtained regression equation is not very unreliable: $R^2 = 0.89$, F-statistics is equal to 52.7, etc. Hence, one can use it for deriving some estimations. If we try to define the optimal tax burden for Georgia on the basis of maximization of (4) we find that from the production point of view optimal tax burden level for Georgia must be near to 70% and such a result is very far from the reality. By this reason (and taking into account the specifics our country!), it may have a sense to create and analyse an alternative (non-classical) variant of product curve(see [7]). One of this non-classical product curve equation for our economy has form:

$$X = (42325.99251 * q - 690.0227668)/(1 + 25.76346016 * q^4). \tag{5}$$

It is remarkable, that this equation has the same level of accuracy as (4), i.e. we can use it instead of the equation (4). However, additional difficulty in this case is that the maximization problem of function (5) is much more complex; however this problem is not very hard to solve using the same Matlab system. In fact, since maximization of X is equivalent to minimization of the function $-X$, by using Matlab we will have:

$$fminbnd('-(42325.99251*x-90.0227668)/(1+25.76346016*x^4)',0,1)ans=0.3428$$

Hence, in this case we obtain that the so called Lafer's first type point for Georgian economy tax burden is the 34% and this corresponds much better to the real situation. Besides this, from (5), considering the relation

$$q = T/X,$$

where T denotes Georgian consolidate budget tax revenues (in real representation), one can built fiscal curves following non-classical variant, for our country:

$$T = (42325.99251 * q - 690.0227668) * q/(1 + 25.76346016 * q^4). \tag{6}$$

From the equation (6) one can find also an estimation of the tax burden level corresponding tax revenues maximum (Lafer's second type point). Actually, in this case, maximization of function T (i.e. minimization of function $-T$) on the basis of Matlab, gives:

$$fminbnd('-x*(42325.99251*x-90.0227668)/(1+25.76346016*x^4)',0,1)ans=0.4481$$

Hence, on the basis of 1996-2011 years data, achieving maximal tax revenues level of Georgian consolidate budget requires 44.8% tax burden. It's obvious that, this is maximal level of tax burden for Georgian economy. Moreover, as we have mentioned this above, real tax burden on our economy must not exceed Lafer's first type point, i.e. 34%. Hence, for the solution of this problem we are forced to use such computer systems as Eviews and Matlab. Note that we did not say anything about more complicated direction in forecasting, which suggests to use models of so called nonlinear dynamics (for example Samuelson-Hicks models, etc.) and which is of course very perspective.

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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR ONE CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS TAKING INTO ACCOUNT DELAY PERTURBATION

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Abstract. In the present paper, for the quasilinear functional differential equation with the discontinuous initial condition we formulate the theorems on the continuous dependence of the solution, on perturbations of the initial moment, the variable delay entering in the phase coordinates, the initial vector, the initial functions and the nonlinear term of right-hand side. The discontinuous initial condition means that the values of the initial function and trajectory, generally, do not coincide at the initial moment.

Keywords and phrases: Neutral functional differential equation; well-posedness of the Cauchy problem; discontinuous initial condition.

AMS subject classification (2010): 39A05.

Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T means transpose; let $I = [a, b] \subset \mathbb{R}^1_t$ be a finite interval, let $O \subset \mathbb{R}^n_x$ be a open set; let D be the set of continuously differentiable functions $\tau(t)$ satisfying the conditions: $\tau(t) < t,\ \dot{\tau}(t) > 0$ with

$$\inf\{\tau(a) : \tau \in D\} = \hat{\tau} < \infty, ||\tau|| = \sup\{|\tau(t)| : t \in I\}.$$

Let E_φ be the space of piecewise-continuous functions $\varphi : I_1 = [\hat{\tau}, b] \to \mathbb{R}^n_x$, with finitely many discontinuity points of the first kind, $||\varphi|| = \sup\{|\varphi(t)| : t \in I_1\}$; let $\Phi_1 = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ be the set of initial functions with $cl\varphi(I_1) \subset O$; let Φ_2 be the set of bounded measurable functions $h : I_1 \to \mathbb{R}^n_x,\ ||h|| = \sup\{|h(t)| : t \in I_1\}$.

Let E_f be the space of functions $f : I \times O^2 \to \mathbb{R}^n_x$ satisfying the following conditions: the function $f(\cdot, x, y) : I \to \mathbb{R}^n_x$ is measurable for each fixed $(x, y) \in O^2$; for an arbitrary compact set $K \subset O$ and for $f \in E_f$ there exist functions $m_{f,K}(\cdot),\ L_{f,K}(\cdot) \in L(I, [0, \infty))$, such that for almost all $t \in I$ the following inequalities are fulfilled

$$| f(t, x, y) |\leq m_{f,K}(t), \quad \forall(x, y) \in K^2,$$

$$| f(t, x_1, y_1) - f(t, x_2, y_2) |\leq L_{f,K}(t)(| x_1 - x_2 | + | y_1 - y_2 |),$$
$$\forall(x_i, y_i) \in K^2, \ \ i = 1, 2.$$

To each element $\mu = (t_0, \tau, x_0, \varphi, h, f) \in \Lambda = I \times D \times O \times \Phi_1 \times \Phi_2 \times E_f$ we put in correspondence the quasilinear neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t))) \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \dot{x}(t)=h(t),\ t\in[\hat{\tau},t_0),\quad x(t_0)=x_0. \tag{2}$$

Here $A(t)$ is a given continuous matrix function with dimension $n\times n;\sigma\in D$ is a fixed function.

The condition (2) is said to be the discontinuous initial condition since generally $x(t_0)\neq\varphi(t_0)$.

Definition 1. Let $\mu=(t_0,\tau,x_0,\varphi,h,f)\in\Lambda,\ t_0\in[a,b)$. A function $x(t)=x(t;\mu)\in O,\ t\in[\hat{\tau},t_1],\ t_1\in(t_0,b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to element μ and defined on the interval $[\hat{\tau},t_1]$ if it satisfies condition (2) and it is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

If t_1-t_0 is a sufficiently small number, then the unique solution always corresponds to μ.

To formulate the main results, we introduce the following sets:

$$W(K,\alpha_1)=\Big\{\delta f\in E_f:\exists m_{\delta f,K},L_{\delta f,K}\in L(I,[0,\infty)),$$

$$\int_I[m_{\delta f,K}(t)+L_{\delta f,K}(t)]dt\leq\alpha_1\Big\},$$

where $K\subset O$ is a compact set and $\alpha_1>0$ is a given number independent of δf;

$$V_{K,\delta}=\Big\{\delta f\in E_f:\Big|\int_{s_1}^{s_2}\delta f(t,x,y)dt\Big|\leq\delta,\ \forall(s_1,s_2,x,y)\in I^2\times K^2\Big\},$$

$$B(t_{00};\delta)=\{t_0\in I:|t_0-t_{00}|<\delta\},\ \ B(x_{00};\delta)=\{x_0\in O:|x_0-x_{00}|<\delta\},$$
$$V(\tau_0;\delta)=\{\tau\in D:||\tau-\tau_0||<\delta\},V(\varphi_0;\delta)=\{\varphi\in\Phi_1:||\varphi-\varphi_0||<\delta\},$$
$$V(h_0;\delta)=\{h\in\Phi_2:||h-h_0||<\delta\},$$

where $t_{00}\in I,\ x_{00}\in O$ are fixed points; $\tau_0\in D,\ \varphi_0\in\Phi_1,\ h_0\in\Phi_2$ are fixed functions.

Theorem 1. *Let $x_0(t)=x(t;\mu_0)$, where $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,h_0,f_0)\in\Lambda$, is the solution defined on $[\hat{\tau},t_{10}]$, $t_{10}<b$; let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $cl\varphi_0(I_1)\cup x_0([t_{00},t_{10,}])$. Then the following assertions hold:*

1. *there exist numbers $\delta_i>0,\ i=0,1$, such that, to each element*

$$\mu\in V(\mu_0;K_1,\delta_0,\alpha_1)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)\times B(x_{00};\delta_0)\times V(\varphi_0;\delta_0)$$

$$\times V(h_0;\delta_0)\times\Big[f_0+W(K_1,\alpha_1)\cap V_{K_1,\delta_0}\Big]$$

we put in correspondence the solution $x(t;\mu)$ defined on the interval $[\hat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\mu)\in intK_1,\ t\in[\hat{\tau},t_{10}+\delta_1]$;

2. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that for any $\mu\in V(\mu_0;K_1,\delta_2,\alpha_1)$ the following inequality holds:*

$$|x(t;\mu)-x(t;\mu_0)|\leq\varepsilon,\ \ \forall t\in[s_1,t_{10}+\delta_1],\ \ s_1=\max\{t_{00},t_0\};$$

3. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0]$ *such that for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha_1)$ *the following inequality holds:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \le \varepsilon.$$

In the space $E_\mu - \mu_0$, where $E_\mu = \mathbb{R}^1_t \times D \times \mathbb{R}^n_x \times \Phi_1 \times \Phi_2 \times E_f$ introduce the set of variation:

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta h, \delta f) \in E_\mu - \mu_0 : |\,\delta t_0\,| \le \alpha_2, |\,\delta\tau\,| \le \alpha_2,$$

$$|\,\delta x_0\,| \le \alpha_2, \|\delta\varphi\|_1 \le \alpha_2, \|\delta h\|_1 \le \alpha_2, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,\ |\lambda_i| \le \alpha_2,\ i = \overline{1,k}\Big\},$$

where $\alpha_2 > 0$ is a fixed number, $\delta f_i \in E_f,\ i = \overline{1,k}$, are fixed functions.

The following theorem is a simple consequence of theorem 1.

Theorem 2. *Let* $x_0(t) = x(t;\mu_0)$ *be the solution defined on* $[\hat{\tau}, t_{10}],\ t_{i0} \in (a,b),\ i = 0, 1;$ *let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $cl\varphi_0(I_1)\cup x_0([t_{00}, t_{10,}])$. *Then the following assertions hold:*

4. *there exist numbers* $\varepsilon_1 > 0,\ \delta_1 > 0$, *such that, for an arbitrary* $(\varepsilon, \mu) \in [0, \varepsilon_1] \times \Im$ *the element* $\mu_0 + \varepsilon\delta\mu \in \Lambda$, *we put in correspondence the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ *and satisfying the condition* $x(t; \mu_0 + \varepsilon\delta\mu) \in intK_1,\ t \in [\hat{\tau}, t_{10} + \delta_1]$;

5. *the following relations hold:*

$$\lim_{\varepsilon\to 0} \sup\{|x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : t \in [s_1, t_{10} + \delta_1]\} = 0, \quad s_1 = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\};$$

$$\lim_{\varepsilon\to 0} \int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)|dt = 0$$

uniformly for $\delta\mu \in \Im$.

Now let us formulate the theorem on the continuous dependence of the solution for an equation whose righthand side depends on the control. Let $U_0 \subset \mathbb{R}^r_u$ be an open set and let Ω be the set of measurable functions $u(t) \in U_0,\ t \in I$, satisfying the condition: $clu(I)$ is a compact set in $\mathbb{R}^r_u$ and $clu(I) \subset U_0$.

To each element $\rho = (t_0, \tau, x_0, \varphi, h, u) \in \Lambda_1 = [a, b) \times D \times O \times \Phi_1 \times \Phi_2 \times \Omega$ we assign the control neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + g(t, x(t), x(\tau(t)), u(t)) \tag{3}$$

with the initial condition (2). Here the function $g(t, x, y, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x, y, u) \in O^2 \times U_0$ the function $g(\cdot, x, y, u) : I \to \mathbb{R}^n_u$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist functions $m_{K,U}, L_{K,U} \in L(I, [0, \infty))$ such that for almost all $t \in I$

$$|g(t, x, y, u)| \le m_{K,U}(t), \quad \forall (x, y, u) \in K^2 \times U,$$

$$|g(t,x_1,y_1,u_1)-g(t,x_2,y_2,u_2)|\leq L_{K,U}(t)\Big[|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\Big],$$

$$\forall(x_1,x_2,y_1,y_2,u_1,u_2,)\in K^4\times U^2.$$

Definition 2. Let $\rho=(t_0,\tau,x_0,\varphi,h,u)\in\Lambda_1$. A function $x(t)=x(t;\rho)\in O,\ t\in[\hat{\tau},t_1],\ t_1\in(t_0,b]$, is called a solution of equation (3) with the initial condition (2) or a solution corresponding to element ρ and defined on the interval $[\hat{\tau},t_1]$, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (3) almost everywhere on $[t_0,t_1]$.

Theorem 3. *Let $x_0(t)=x(t;\rho_0)$, where $\rho_0=(t_{00},\tau_0,x_{00},\varphi_0,h_0,u_0)\in\Lambda_1$, be a solution defined on $[\hat{\tau},t_{10}]$, $t_{10}<b$; let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $cl\varphi_0(I_1)\cup x_0([t_{00},t_{10,}])$. Then the following assertions hold:*

6. *there exist numbers $\delta_i>0,\ i=0,1$, such that, to each element $\rho\in\hat{V}(\rho_0;\delta_0)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)\times B(x_{00};\delta_0)\times V(\varphi_0;\delta_0)\times V(h_0;\delta_0)\times V(u_0;\delta_0)$ corresponds the solution $x(t;\rho)$ defined on the interval $[\hat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\rho)\in intK_1$; here $V(u_0;\delta_0)=\{u\in\Omega:||u-u_0||<\delta\}$;*

7. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that for any $\rho\in\hat{V}(\rho_0;\delta_0)$ the following inequality holds:*

$$|x(t;\rho)-x(t;\rho_0)|\leq\varepsilon,\ \ \forall t\in[s_1,t_{10}+\delta_1],\ \ s_1=\max\{t_{00},t_0\};$$

8. *for an arbitrary $\varepsilon>0$ there exists a number $\delta_3=\delta_3(\varepsilon)\in(0,\delta_0]$ such that for any $\rho\in\hat{V}(\rho_0;\delta_0)$ the following inequality holds:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1}|x(t;\rho)-x(t;\rho_0)|dt\leq\varepsilon.$$

Some comments. Theorems analogous to Theorem 1-3, without perturbation of variable delay, for various classes of functional differential equations are proved in [1-3]. In Theorem 1 perturbations of the nonlinear term of right-hand side of equation (1) are small in the integral sense. Theorems 1-3 play an important role in proving necessary optimality conditions and variation formulas of solution [1,4-7].

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NEUMANN TYPE INTERIOR BOUNDARY VALUE PROBLEM OF THERMOELASTOSTATICS FOR HEMITROPIC SOLIDS

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Abstract. The purpose of this paper is investigation of the three-dimensional interior Neumann type boundary value problem of the theory of thermoelastostatics for hemitropic solids. Hemitropic solids belong to the class of Cosserat type continua and the corresponding system of partial differential equations generates a 7×7 nonselfadjoint matrix elliptic operator. The uniqueness and existence results are studied by the potential method and the theory of singular integral equations. The boundary integral operators associated with the layer potentials are analyzed and on the basis of the results obtained we derive the explicit necessary and sufficient conditions for the interior Neumann type boundary value problem to be solvable. We show that solutions are representable in the form of the single layer potential.

Keywords and phrases: Elasticity theory, elastic hemitropic materials, potential theory, uniqueness theorems, existence theorems.

AMS subject classification (2010): 35J57, 74A60, 74G30, 74F05.

1. Introduction

In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model continua with a complex inner structure whose material particles have 6 degree of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Mathematical models describing the so called hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2] (for historical notes see also [3], [4], [19], and the references therein).

Hemitropic solids are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

In the present paper we deal with the model of micropolar elasticity for hemitropic solids when the thermal effects are taken into consideration.

In the mathematical theory of hemitropic thermoelasticity there are introduced the asymmetric force stress tensor and couple stress tensor, which are kinematically related with the asymmetric strain tensor, torsion (curvature) tensor and the temperature function via the constitutive equations. All these quantities along with the heat flux vector are expressed in terms of the components of the displacement and microrotation vectors and the temperature function. In turn, the displacement and microrotation vectors and the temperature distribution function satisfy a coupled complex system of second order partial differential equations. When the mechanical and thermal characteristics (displacements, microrotations, temperature, body force, body couple vectors, and heat source) do not depend on the time variable t we have the differential equations of statics. These equations generate a strongly elliptic, formally nonselfadjoint 7×7 matrix differential operator.

The Dirichlet, Neumann and mixed type boundary value problems (BVP) for the so called *pseudo oscillation case* with complex frequency parameter, which are related to the dynamical equations via the Laplace transform, are well investigated for homogeneous bodies of arbitrary shape (see [14], [15], [17], [18], [13], [16] and the references therein).

The main goal of the present paper is investigation of the interior Neumann type boundary value problem of statics of thermoelasticity for hemitropic solids. In the case of static problems there arise significant difficulties which need a special consideration.

Here we develop the boundary integral equations method to obtain the existence and uniqueness results in Hölder ($C^{k,\alpha}$) functional spaces. We reduce the Neumann type BVP to the equivalent system of normally solvable singular integral equations. We construct explicitly the null spaces of the corresponding singular integral operator and its adjoint one, and on the basis of the results obtained we derive necessary and sufficient conditions for the interior Neumann type BVP to be solvable.

2. Problems setting, Green's formulas and uniqueness theorems

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega^+ =: S \in C^{1,\kappa}$ with $0 < \kappa \leq 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. The outward unit normal vector to S at the point $x \in S$ we denote by $n(x) = (n_1(x), n_2(x), n_3(x))$. We assume that the domains Ω^+ are filled with a hemitropic elastic continua.

The basic governing homogeneous equations of the theory of thermoelastostatics for hemitropic materials read as (see [19])

$$
\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + (\chi+\nu)\Delta\omega(x) \\
&\qquad + (\delta+\chi-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x) - \eta\operatorname{grad}\vartheta(x) = 0, \\
&(\chi+\nu)\Delta u(x) + (\delta+\chi-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x) + (\gamma+\varepsilon)\Delta\omega(x) \\
&\qquad + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x) - \zeta\operatorname{grad}\vartheta(x) - 4\alpha\omega(x) = 0, \\
&\kappa'\Delta\vartheta(x) = 0,
\end{aligned}
\tag{2.1}
$$

where $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ are the displacement vector and the microrotation vector respectively, ϑ is the temperature distribution function, α, β, γ, δ, λ, μ, ν, χ, ε, η, ζ and κ' are the material constants, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, the symbol $(\cdot)^\top$ denotes transposition.

The matrix differential operator generated by these equations is not formally self-adjoint and has the form

$$
L(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) & L^{(5)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) & L^{(6)}(\partial) \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\Delta \end{bmatrix}_{7\times 7}, \tag{2.2}
$$

where

$$
\begin{aligned}
&L^{(1)}(\partial) := (\mu+\alpha)\Delta I_3 + (\lambda+\mu-\alpha)Q(\partial),\\
&L^{(2)}(\partial) = L^{(3)}(\partial) := (\chi+\nu)\Delta I_3 + (\delta+\chi-\nu)Q(\partial) + 2\alpha R(\partial),\\
&L^{(4)}(\partial) := [(\gamma+\varepsilon)\Delta - 4\alpha]I_3 + (\beta+\alpha-\varepsilon)Q(\partial) + 4\nu R(\partial),\\
&L^{(5)}(\partial) := -\eta\nabla^\top, \quad L^{(6)}(\partial) := -\zeta\nabla^\top,\\
&R(\partial) := [\,-\varepsilon_{pqj}\partial_j\,]_{3\times 3}, \quad Q(\partial) := [\,\partial_k\partial_j\,]_{3\times 3}.
\end{aligned}
\tag{2.3}
$$

Here and in what follows ε_{pqj} denotes the permutation (Levi-Civitá) symbol and I_k stands for the $k\times k$ unit matrix . Throughout the paper repeated indices indicate summation from one to three if not otherwise stated.

Denote by $L^*(\partial) := L^\top(-\partial)$ the operator formally adjoint to $L(\partial)$. Moreover, let $\widetilde{L}(\partial)$ denote the operator corresponding to the equilibrium equations of hemitropic elastostatics when thermal effects are not taken into consideration (see [14])

$$
\widetilde{L}(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6\times 6}, \tag{2.4}
$$

where $L^{(k)}(\partial)$ are defined in (2.3). Note that $\widetilde{L}(\partial)$ is formally selfadjoint, i.e., $\widetilde{L}(\partial) = \widetilde{L}^*(\partial) = \widetilde{L}^\top(-\partial)$.

The force stress tensor $\{\tau_{pq}\}_{3\times 3}$ and the couple stress tensor $\{\mu_{pq}\}_{3\times 3}$ in the linear theory of hemitropic thermoelasticity read as follows (the constitutive equations) [18]

$$
\begin{aligned}
\tau_{pq} = \tau_{pq}(U) :=\; & (\mu+\alpha)\partial_p u_q + (\mu-\alpha)\partial_q u_p + \lambda\delta_{pq}\operatorname{div} u + \delta\,\delta_{pq}\operatorname{div}\omega\\
& + (\varkappa+\nu)\partial_p\omega_q + (\varkappa-\nu)\partial_q\omega_p - 2\alpha\varepsilon_{pqk}\omega_k - \delta_{pq}\,\eta\,\vartheta,\\
\mu_{pq} = \mu_{pq}(U) :=\; & \delta\,\delta_{pq}\operatorname{div} u + (\varkappa+\nu)\Big[\partial_p u_q - \varepsilon_{pqk}\omega_k\Big] + \beta\,\delta_{pq}\operatorname{div}\omega\\
& + (\varkappa-\nu)\Big[\partial_q u_p - \varepsilon_{qpk}\omega_k\Big] + (\gamma+\varepsilon)\partial_p\omega_q + (\gamma-\varepsilon)\partial_q\omega_p - \delta_{pq}\zeta\vartheta,
\end{aligned}
$$

where $U = (u,\omega,\vartheta)^\top$, δ_{pq} is the Kronecker delta.

The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a unite normal vector $n = (n_1, n_2, n_3)$, are expressed as

$$
\tau^{(n)} = \big(\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)}\big)^\top, \qquad \mu^{(n)} = \big(\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)}\big)^\top,
$$

where

$$
\tau_q^{(n)} = \tau_{pq} n_p, \qquad \mu_q^{(n)} = \mu_{pq} n_p, \quad q = 1,2,3.
$$

Introduce the generalized stress operators associated with the differential operators $L(\partial)$ and $\widetilde{L}(\partial)$ (cf. [14], [17], [18])

$$\mathcal{P}(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}, \tag{2.5}$$

$$\mathcal{P}^*(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & [0]_{3\times 1} \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}, \tag{2.6}$$

where

$$T^{(j)}=[T^{(j)}_{pq}]_{3\times 3},\quad j=\overline{1,4},\quad n=(n_1,n_2,n_3),$$
$$T^{(1)}_{pq}(\partial,n)=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,$$
$$T^{(2)}_{pq}(\partial,n)=(\chi+\nu)\delta_{pq}\partial_n+(\chi-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\varepsilon_{pqk}n_k,$$
$$T^{(3)}_{pq}(\partial,n)=(\chi+\nu)\delta_{pq}\partial_n+(\chi-\nu)n_q\partial_p+\delta n_p\partial_q,$$
$$T^{(4)}_{pq}(\partial,n)=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q-2\nu\varepsilon_{pqk}n_k.$$

Here $\partial_n=\partial/\partial n$ denotes the usual normal derivative.

In addition, let us introduce the "pure hemitropic boundary stress operator" associated with the differential operator $\widetilde{L}(\partial)$

$$T(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix}_{6\times 6} \tag{2.7}$$

with $T^{(j)}(\partial,n)$ defined in (2.4).

For a vector $U=(u,\omega,\vartheta)^\top$ the seven vector $\mathcal{P}(\partial,n)\,U$ has the following physical sense: the first three components

$$T^{(1)}(\partial,n)u+T^{(2)}(\partial,n)\omega-\eta\, n^\top\,\vartheta=\big(\tau^{(n)}_1,\,\tau^{(n)}_2,\,\tau^{(n)}_3\big)^\top$$

correspond to the thermo-mechanical stress vector, the second triplet

$$T^{(3)}(\partial,n)u+T^{(4)}(\partial,n)\omega-\zeta\, n^\top\,\vartheta=\big(\mu^{(n)}_1,\,\mu^{(n)}_2,\,\mu^{(n)}_3\big)^\top$$

corresponds to the thermo-mechanical couple stress vector, while the seventh component $\kappa'\partial_n\vartheta$ corresponds to the normal component of the heat flux vector.

For regular vector-functions

$$U=(u,\omega,\vartheta)^\top,\, U'=(u',\omega',\vartheta')^\top\in[C^2(\Omega^+)]^7\cap[C^1(\overline{\Omega^+})]^7,$$

the following Green's formula holds [18]

$$\int_{\Omega^+}\big[\,U'\cdot L(\partial)U - L^*(\partial)U'\cdot U\big]\,dx = \int_{\partial\Omega^+}\big[\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+ - \{\mathcal{P}^*(\partial,n)U'\}^+\cdot\{U\}^+\big]\,dS, \tag{2.8}$$

where the operator $L(\partial)$ is defined in (2.2) and $L^*(\partial) = L^\top(-\partial)$ is the operator formally adjoint to $L(\partial)$, while $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are given by (2.5) and (2.6); the symbols $\{\cdot\}^\pm$ denote one sided limits on S from $\Omega^\pm$ respectively, while the central dot denotes scalar product of two vectors in Euclidean space $\mathbb{R}^n$.

3. Problem formulation and uniqueness theorem

The Neumann type interior boundary value problem $(N)^+$ is formulated as follows: *Find a regular vector-function* $U \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ *satisfying the differential equation*

$$L(\partial)U(x) = 0, \qquad x \in \Omega^+ \tag{3.1}$$

and the Neumann type boundary condition on S

$$\big\{\mathcal{P}(\partial,n)U(x)\big\}^+ = F(x), \qquad x \in S, \tag{3.2}$$

where $F = (F_1, F_2, ..., F_7)^\top \in \big[C(S)\big]^7$ *is a given vector-function.*
The following uniqueness theorem holds true.

Theorem 3.1. *A general solution to the homogeneous Neumann type interior boundary value problem reads as*

$$U_0 = (\widetilde{\Psi}, 0)^\top + \vartheta_0\,(u_0, \omega_0, 1)$$

where $\widetilde{\Psi}$ *is a generalized rigid displacement vector,*

$$\widetilde{\Psi}(x) = ([a\times x] + b, a)^\top \tag{3.3}$$

with $a = (a_1, a_2, a_3)^\top$ *and* $b = (b_1, b_2, b_3)^\top$ *being arbitrary three dimensional constant vectors,* ϑ_0 *is an arbitrary constant, while the vector-functions* $u_0 = (u_{01}, u_{02}, u_{03})^\top$ *and* $\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03})^\top$ *are such that the six dimensional vector-function* $\widetilde{V}_0 = (u_0, \omega_0)^\top$ *solves the following boundary value problem*

$$\begin{aligned} &\widetilde{L}(\partial)\widetilde{V}_0(x) = 0, \qquad x \in \Omega^+, \\ &\{T(\partial,n)\widetilde{V}_0\}^+ = (\eta\, n(x), \zeta\, n(x))^\top, \qquad x \in S. \end{aligned} \tag{3.4}$$

Here η *and* ζ *are material parameters involved in the basic system* (2.1) *and the operators* $\widetilde{L}(\partial)$ *and* $T(\partial,n)$ *are defined in* (2.4) *and* (2.7).

Proof. Form the structure of the operators (2.2) and (2.5) it is easy to see that for the temperature function ϑ the corresponding boundary value problem can be separated, which reads as

$$\Delta\vartheta(x) = 0, \qquad x \in \Omega^+,$$

$$\left\{\frac{\partial\vartheta(x)}{\partial n}\right\}^{+} = 0, \qquad x \in S.$$

A general solution to this problem is a constant function,

$$\vartheta(x) = \vartheta_0 = const, \qquad x \in \Omega^+,$$

where ϑ_0 is an arbitrary real constant.

Therefore a general solution to the homogeneous Neumann type boundary value problem has the following form: $U = (u, \omega, \vartheta_0)^\top = (\widetilde{U}, \vartheta_0)^\top$ with $\widetilde{U} = (u, \omega)^\top$. Consequently, in view of (2.2), (2.4), (2.5), and (2.7), the vector $\widetilde{U}$ solves the following nonhomogeneous boundary value problem

$$\begin{aligned} &\widetilde{L}(\partial)\,\widetilde{U}(x) = 0, \qquad x \in \Omega^+, \\ &\{T(\partial, n)\,\widetilde{U}(x)\}^+ = \widetilde{F}_0(x), \qquad x \in S, \end{aligned} \tag{3.5}$$

where

$$\widetilde{F}_0(x) = \vartheta_0\,(\eta\, n(x), \zeta\, n(x))^\top, \qquad x \in S. \tag{3.6}$$

Recall that $n(x)$ is the outward unit normal vector at the point $x \in S$, while η and ζ are the material parameters. Thus $\widetilde{U}$ is a solution to the nonhomogeneous interior Neumann type boundary value problem for hemitropic model, when the thermal effects are not taken into consideration. In the reference [18] it is shown that the condition

$$\int\limits_S \widetilde{F}_0(x) \cdot \widetilde{\Psi}(x)\, dS = 0 \tag{3.7}$$

is necessary and sufficient for the problem (3.5)–(3.6) to be solvable. Here $\widetilde{\Psi}$ is a generalized rigid displacement vector define in (3.3).

With the help of the relations

$$[a \times x] \cdot n = [x \times n] \cdot a, \qquad \int\limits_S n_k(x) dS = 0, \qquad \int\limits_S [x_j n_k(x) - x_k n_j(x)] dS = 0, \quad k, j = 1, 2, 3,$$

and the Gauss divergence theorem, it is easy to verify that conditions (3.7) for the vector (3.6) hold true,

$$\begin{aligned} \int\limits_S \vartheta_0(\eta\, n(x), \zeta n(x))^\top \cdot ([a \times x] + b, a)^\top dS &= \vartheta_0 \int\limits_S \Big\{\eta\,\big(n \cdot [a \times x] + n \cdot b\big) + \zeta\, n \cdot a\Big\}\, dS \\ = \vartheta_0\,\eta \int\limits_S [x \times n] \cdot a\, dS &= \vartheta_0\,\eta \sum_{k=1}^{3} a_k \int\limits_S [x \times n]_k dS = 0. \end{aligned}$$

Consequently, the boundary value problem (3.5) is solvable for arbitrary constant ϑ_0 and solutions are defined modulo the vector $\widetilde{\Psi}$ given by (3.3). Denote by $\widetilde{V}_0 := (u_0, \omega_0)^\top$

with $u_0=(u_{01},u_{02},u_{03})^\top$ and $\omega_0=(\omega_{01},\omega_{02},\omega_{03})^\top$ some particular solution of problem (3.4) which coincide with problem (3.5) for $\vartheta_0=1$. Then it follows that $\vartheta_0\widetilde{V}_0$ represents a particular solution of problem (3.5), while a general solution of the same problem reads as $\widetilde{U}=\vartheta_0\widetilde{V}_0+\widetilde{\Psi}$. Whence we deduce that the vector $U=(\widetilde{U},\vartheta_0)^\top=\vartheta_0(u_0,\omega_0,1)^\top+(\widetilde{\Psi},0)$ is a general solution to the homogeneous interior Neumann type problem which completes the proof. □

Remark 3.2. Introduce the system of vector-functions $\{\Phi^{(k)}(x)\}_{k=1}^7$, where

$$\begin{aligned} &\Phi^{(1)}=(0,-x_3,x_2,1,0,0,0)^\top, && \Phi^{(2)}=(x_3,0,-x_1,0,1,0,0)^\top,\\ &\Phi^{(3)}=(-x_2,x_1,0,0,0,1,0)^\top, && \Phi^{(4)}=(1,0,0,0,0,0,0)^\top,\\ &\Phi^{(5)}=(0,1,0,0,0,0,0)^\top, && \Phi^{(6)}=(0,0,1,0,0,0,0)^\top,\\ &\Phi^{(7)}=(u_0,\omega_0,1)^\top. \end{aligned} \tag{3.8}$$

Here the vector $(u_0,\omega_0)^\top$ is a particular solution of the nonhomogeneous problem (3.4) existence of which is shown in the above presented proof of Theorem 3.1 It is easy to check that the vectors (3.8) are linearly independent in Ω^+ and each of them is a solution to the homogeneous interior Neumann type problem (3.1)–(3.2) with $F=0$. Moreover, from Theorem 3.1 it follows that a general solution to the homogeneous interior Neumann type problem is representable as

$$U(x)=\sum_{k=1}^{7}C_k\Phi^{(k)}(x),$$

where C_k are arbitrary real constants, while $\Phi^{(k)}(x)$ are defined in (3.8).

In our analysis below, we need uniqueness results for the exterior boundary value problems for the operators $L(\partial)$ and $L^*(\partial)$ in special spaces of vector-functions which are bounded at infinity. To this end let us introduce the following definitions.

Definition 3.3. A vector-function $U=(u,\omega,\vartheta)^\top$ is said to belong to the class $Z(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

$$\text{(i)}\quad u(x)=\mathcal{O}(1),\quad \omega(x)=\mathcal{O}(|x|^{-2}),\quad \vartheta(x)=\mathcal{O}(|x|^{-1})\quad \text{as}\quad |x|\to\infty,$$

$$\text{(ii)}\quad \lim_{R\to\infty}\frac{1}{4\pi R^2}\int\limits_{\Sigma(0,R)}u(x)\,d\Sigma(0,R)=0,$$

where $\Sigma(0,R)$ is a sphere centered at the origin and radius R.

Definition 3.4. A vector-function $U^*=(u^*,\omega^*,\vartheta^*)^\top$ is said to belong to the class $Z^*(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

$$\text{(i)}\quad u^*(x)=\mathcal{O}(|x|^{-1}),\quad \omega^*(x)=\mathcal{O}(|x|^{-2}),\quad \vartheta^*(x)=\mathcal{O}(1)\quad \text{as}\quad |x|\to\infty, \tag{3.9}$$

$$\text{(ii)}\quad \lim_{R\to\infty}\frac{1}{4\pi R^2}\int\limits_{\Sigma(0,R)}\vartheta^*(x)\,d\Sigma(0,R)=0. \tag{3.10}$$

4. Layer potentials and general integral representations

The matrix of fundamental solutions $\Gamma(x-y) = [\Gamma_{kj}(x-y)]_{7\times 7}$ associated with the operator $L(\partial)$ can be constructed explicitly in terms of standard functions (see Appendix). It is a solution of the distributional equation $L(\partial_x)\Gamma(x-y) = I_7\delta(x-y)$, where $\delta(x-y)$ is Dirac's delta distribution. Let us introduce the single layer and double layer potentials

$$V(g)(x) = V_S(g)(x) := \int_S \Gamma(x-y)\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

$$W(g)(x) = W_S(g)(x) := \int_S \left[\mathcal{P}^*\big(\partial_y, n(y)\big)\Gamma^\top(x-y)\right]^\top g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

where $g = (g_1, g_2, ..., g_7)^\top$ and $h = (h_1, h_2, ..., h_7)^\top$ are density vector-functions defined on S, while the boundary operator $\mathcal{P}^*\big(\partial, n\big)$ is defined in (2.6).

Further, we introduce the "adjoint" layer potentials associated with the operator $L^*(\partial)$,

$$V^*(g)(x) := \int_S \Gamma^*(x-y\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus\ S, \tag{4.1}$$

$$W^*(g)(x) := \int_S \left[\mathcal{P}\big(\partial_y, n(y)\big)\left[\Gamma^*(x-y)\right]^\top\right]^\top g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus\ S, \tag{4.2}$$

where $\Gamma^*(x-y) := \Gamma^\top(y-x)$ is a fundamental matrix of the operator $L^*(\partial)$, the boundary operator $\mathcal{P}\big(\partial, n\big)$ is defined in (2.5), and $g = (g_1, g_2, ..., g_7)^\top$ and $h = (h_1, h_2, ..., h_7)^\top$ are density vector-functions defined on S.

Theorem 4.1. *Let $S \in C^{1,\kappa}$ with $0 < \kappa \leqslant 1$ and vector-functions $U \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ and $U^* \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ be regular solutions of the equations $L(\partial)U = 0$ and $L^*(\partial)U^* = 0$ in Ω^+ respectively. Then the following integral representation formulas hold*

$$W\big(\{U\}^+\big)(x) - V\big(\{\mathcal{P}U\}^+\big)(x) = \begin{cases} U(x), & x \in \Omega^+, \\ 0, & x \in \Omega^-, \end{cases} \tag{4.3}$$

$$W^*\big(\{U^*\}^+\big)(x) - V^*\big(\{\mathcal{P}^*U^*\}^+\big)(x) = \begin{cases} U^*(x), & x \in \Omega^+, \\ 0, & x \in \Omega^-. \end{cases} \tag{4.4}$$

Proof. It is standard and follows from Green's formula (2.8). □

The mapping properties of the above introduced layer potentials V, W, V^*, and W^* can be established by standard arguments applied, e.g., in the references [9], [10], [6], [12], [14].

Theorem 4.2. *The single and double layer potentials $V(g)$ and $W(g)$ solve the homogeneous equation $L(\partial)U = 0$ in $\mathbb{R}^3 \setminus S$, belong to the class $Z(\Omega^-)$ and the following* *operators*

$$V\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(\overline{\Omega^\pm})]^7,$$
$$W\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(\overline{\Omega^\pm})]^7,$$

are continuous provided $S \in C^{k+1,\kappa}$, *where* $k \geqslant 0$ *is an integer and* $0 < \sigma < \kappa \leqslant 1$.

Proof. It can be found in [7]. □

Lemma 4.3. *The single and double layer potentials* $V^*(g)$ *and* $W^*(g)$ *solve the homogeneous equation* $L^*(\partial)U^* = 0$ *in* $\mathbb{R}^3 \setminus S$, *belong to the class* $Z^*(\Omega^-)$, *and the following operators*

$$V^*\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(\overline{\Omega^\pm})]^7,$$
$$W^*\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(\overline{\Omega^\pm})]^7$$

are continuous provided $S \in C^{k+1,\kappa}$, *where* $k \geqslant 0$ *is an integer number and* $0 < \sigma < \kappa \leqslant 1$.

Proof. It can be found in [8]. □

Theorem 4.4. *Let* $S \in C^{1,\kappa}$, $g \in [C^{0,\sigma}(S)]^7$ *and* $h \in [C^{1,\sigma}(S)]^7$ *with* $0 < \sigma < \kappa \leqslant 1$. *Then the following relations hold true:*

$$\begin{aligned}
&\{V(g)(x)\}^\pm = V(g)(x) = \mathcal{H}g(x),\\
&\{\mathcal{P}(\partial_x, n(x))V(g)(x)\}^\pm = [\mp 2^{-1} I_7 + \mathcal{K}\,]\, g(x),\\
&\{W(g)(x)\}^\pm = [\pm 2^{-1} I_7 + \mathcal{N}\,]\, g(x),\\
&\{\mathcal{P}(\partial_x, n(x))W(h)(x)\}^+ = \{\mathcal{P}(\partial_x, n(x))W(h)(x)\}^- = \mathcal{L}h(x), \quad S \in C^{2,\kappa},
\end{aligned}$$

where $\mathcal{H}$ *is a weakly singular integral operator,* $\mathcal{K}$ *and* $\mathcal{N}$ *are singular integral operators, while* $\mathcal{L}$ *is a singular integro-differential operator*

$$\begin{aligned}
&\mathcal{H}g(x) := \int_S \Gamma(x-y) g(y) dS_y,\\
&\mathcal{K}g(x) := \int_S \big[\mathcal{P}\big(\partial_x, n(x)\big)\Gamma(x-y)\big] g(y) dS_y,\\
&\mathcal{N}g(x) := \int_S \big[\mathcal{P}^*\big(\partial_y, n(y)\big)\Gamma^\top(x-y)\big]^\top g(y) dS_y,\\
&\mathcal{L}h(x) := \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{P}\big(\partial_z, n(x)\big) \int_S \big[\mathcal{P}^*\big(\partial_y, n(y)\big)\Gamma^\top(z-y)\big]^\top h(y) dS_y.
\end{aligned}$$

Proof. It can be found in [7]. □

Theorem 4.5. *Let* $k \geqslant 0$ *be integers, and* $S \in C^{k+1,\kappa}$ *with* $0 < \sigma < \kappa \leqslant 1$. *Then the following operators are continuous*

$$\begin{aligned}
&\mathcal{H}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(S)]^7, &\qquad &\mathcal{K}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7,\\
&\mathcal{N}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7, &\qquad &\mathcal{L}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k-1,\sigma}(S)]^7.
\end{aligned}$$

Moreover, the operators

$$\pm 2^{-1} I_7 + \mathcal{K}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7, \qquad \pm 2^{-1} I_7 + \mathcal{N}\ :\ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(S)]^7$$

are elliptic singular integral operators with index equal to zero. The principal homogenous symbol matrices of the operators $-\mathcal{H}$ *and* $\mathcal{L}$ *are positive definite.*

The operators $\mathcal{H}$, $\pm\frac{1}{2}I_7+\mathcal{K}$, $\pm\frac{1}{2}I_7+\mathcal{N}$ *and* $\mathcal{L}$ *are pseudodifferential operators with zero index and of order* -1, 0, 0, *and* 1, *respectively.*

Moreover, the following operator equalities hold true:

$$\mathcal{N}\mathcal{H}=\mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{N}=\mathcal{K}\mathcal{L}, \quad \mathcal{H}\mathcal{L}=-4^{-1}\,I_7+\mathcal{N}^2, \qquad \mathcal{L}\mathcal{H}=-4^{-1}\,I_7+\mathcal{K}^2.$$

Proof. It can be found in [18]. □

Remark 4.6. Let $S\in C^{2,\kappa}$ and $0<\sigma<\kappa\leqslant 1$. The integral operator

$$\mathcal{H}\ :\ [C^{0,\sigma}(S)]^7\to[C^{1,\sigma}(S)]^7$$

is invertible and

$$[\mathcal{H}]^{-1}\ :\ [C^{1,\sigma}(S)]^7\to[C^{0,\sigma}(S)]^7$$

is a pseudodifferential operator of order 1, more precisely, it is a singular integro-differential operator (cf., [18]).

Now we prove the counterpart of Theorem for exterior unbounded domains.

Theorem 4.7. *Let* $S\in C^{1,\kappa}$ *with* $0<\kappa\leqslant 1$ *and vector-functions* $U\in[C^1(\overline{\Omega^+})]^7\cap[C^2(\Omega^-)]^7\cap Z(\Omega^-)$ *and let* $U^*\in[C^1(\overline{\Omega^-})]^7\cap[C^2(\Omega^+)]^7\cap Z^*(\Omega^-)$ *be regular solutions of the equations* $L(\partial)U=0$ *and* $L^*(\partial)U^*=0$ *in* Ω^- *respectively. Then the following integral representation formulas hold*

$$-W\big(\{U\}^-\big)(x)+V\big(\{\mathcal{P}U\}^-\big)(x)=\begin{cases} U(x), & x\in\Omega^-,\\ 0, & x\in\Omega^+,\end{cases} \tag{4.5}$$

$$-W^*\big(\{U^*\}^-\big)(x)+V^*\big(\{\mathcal{P}^*U^*\}^-\big)(x)=\begin{cases} U^*(x), & x\in\Omega^-,\\ 0, & x\in\Omega^+.\end{cases} \tag{4.6}$$

Proof. Formula (4.5) is derived in [7]. To prove (4.6) we proceed as follows. Let U^* be as in the theorem and let us write the integral representation formula (4.4) for a bounded domain $\Omega^-_R:=\Omega^-\cap B(0,R)$, where R is a sufficiently large positive number, $B(0,R):=\{x\in\mathbb{R}^3\ :\ |x|<R\}$ is a ball centered at the origin and radius R, such that $\overline{\Omega^+}\subset B(0,R)$,

$$U^*(x)=-W^*_S(\{U^*\}^-_S)+V^*_S(\{\mathcal{P}^*U^*\}^-_S)+\Phi^*_R(x), \qquad x\in\Omega^-_R, \tag{4.7}$$

$$0=-W^*_S(\{U^*\}^-_S)+V^*_S(\{\mathcal{P}^*U^*\}^-_S)+\Phi^*_R(x), \qquad x\in\Omega^+\cup\big[\mathbb{R}^3\setminus\overline{B(0,R)}\big]; \tag{4.8}$$

here V^*_S and W^*_S are the single and double layer potentials defined in (4.1) and (4.2), while

$$\Phi^*_R(x):=W^*_{\Sigma_R}\big(\{U^*\}^+_{\Sigma_R}\big)(x)-V^*_{\Sigma_R}\big(\{\mathcal{P}^*U^*\}^+_{\Sigma_R}\big)(x) \tag{4.9}$$

with $V^*_{\Sigma_R}$ and $W^*_{\Sigma_R}$ being again the single and double layer potentials with the integration surface $\Sigma_R=\partial\,B(0,R)$.

From equality (4.9) it follows that

$$L^*(\partial)\Phi^*_R(x) = 0, \qquad x \notin \Sigma_R. \tag{4.10}$$

Moreover, from (4.7) and (4.8) we have

$$\begin{aligned}\Phi^*_R(x) &= U^*(x) + W^*_S\big(\{U^*\}^-_S\big) - V^*_S\big(\{\mathcal{P}^*U^*\}^-_S\big), \qquad x \in \Omega^-_R,\\ \Phi^*_R(x) &= W^*_S\big(\{U^*\}^-_S\big) - V^*_S\big(\{\mathcal{P}^*U^*\}^-_S\big), \qquad x \in \Omega^+ \cup \big[\mathbb{R}^3 \setminus \overline{B(0,R)}\,\big].\end{aligned}$$

This implies that for sufficiently large numbers $R_1 < R_2$,

$$\Phi^*_{R_1}(x) = \Phi^*_{R_2}(x) \quad \text{for} \quad |x| < R_1 < R_2. \tag{4.11}$$

Therefore, for arbitrary $x \in \mathbb{R}^3$ the following limit exists

$$\Phi^*(x) := \lim_{R\to\infty} \Phi^*_R(x) = \begin{cases} U^*(x) + W^*_S\big(\{U^*\}^-_S\big)(x) - V^*_S\big(\{\mathcal{P}^*U^*\}^-_S\big)(x), & x \in \Omega^-,\\ W^*_S\big(\{U^*\}^-_S\big)(x) - V^*_S\big(\{\mathcal{P}^*U^*\}^-_S\big)(x), & x \in \Omega^+.\end{cases} \tag{4.12}$$

Consequently,

$$L^*(\partial)\Phi^*(x) = 0, \qquad x \in \Omega^+ \cup \Omega^-.$$

On the other hand, from (4.11) we get

$$\Phi^*(x) = \lim_{R\to\infty} \Phi^*_R(x) = \Phi^*_{R_1}(x) \tag{4.13}$$

for arbitrary $x \in \mathbb{R}^3$ with $R_1 > |x|$ and $\overline{\Omega^+} \subset B(0,R_1)$. From (4.9) and (4.10) then we conclude

$$L^*(\partial)\Phi^*(x) = 0, \qquad x \in \mathbb{R}^3. \tag{4.14}$$

At the same time, from (4.12) we have

$$\Phi^* \in Z^*(\mathbb{R}^3), \tag{4.15}$$

since $U^* \in Z^*(\Omega^-)$ and $W^*_S, V^*_S \in Z^*(\Omega^-)$ due to Lemma 4.3.

From the relations (4.14) we deduce that $\Phi^*(x) = 0$, for all $x \in \mathbb{R}^3$. Indeed, from the relations (4.14)-(4.15) by the Fourier transform we get

$$L^*(-i\xi)\,\widehat{\Phi^*}(\xi) = 0, \qquad \xi \in \mathbb{R}^3,$$

where $\widehat{\Phi^*}(\xi)$ is a generalized vector-function that belongs to the Schwartz space of tempered distributions. Since the determinant $\det L^*(-i\xi)$ is nonsingular for $\xi \in \mathbb{R}^3 \setminus \{0\}$ (see [18]), it follows that the support of the distribution $\widehat{\Phi^*}(\xi)$ is the origin $\xi = 0$. Consequently, $\widehat{\Phi^*}$ is a linear combination of the Dirac distribution and its derivatives,

$$\widehat{\Phi^*}(\xi) = \sum_{|\alpha|\leqslant M} C_\alpha \delta^{(\alpha)}(\xi),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, C_α are constant seven dimensional vectors, M is a nonnegative integer, while $\delta^{(\alpha)}$ stands for the α-th order derivative of δ. Therefore the vector-function $\Phi^*(x)$ is a polynomial in x,

$$\Phi^*(x) = \sum_{|\alpha| \leqslant M} C_\alpha x^\alpha, \qquad x \in \mathbb{R}^3.$$

Further, since $\Phi^* \in Z^*(\mathbb{R}^3)$, in accordance with (3.9) and (3.10), we finally conclude $\Phi^*(x) = 0$ for $x \in \mathbb{R}^3$. Now, passing to the limit in (4.7) as $R \to \infty$ and keeping in mind (4.13), we arrive at the general integral representation formula (4.6). □

Further we characterize the jump relations for the adjoint layer potentials (for details see [8]).

Theorem 4.8. *Let $S \in C^{1,\kappa}$, $g \in [C^{0,\sigma}(S)]^7$ and $h \in [C^{1,\sigma}(S)]^7$ with $0 < \sigma < \kappa \leqslant 1$. Then for all points $x \in S$ the following relations hold true:*

$$\{V^*(g)(x)\}^\pm = V^*(g)(x) = \mathcal{H}^* g(x), \tag{4.16}$$

$$\{\mathcal{P}^*(\partial_x, n(x))V^*(g)(x)\}^\pm = [\mp 2^{-1} I_7 + \mathcal{K}^*]g(x), \tag{4.17}$$

$$\{W^*(g)(x)\}^\pm = [\pm 2^{-1} I_7 + \mathcal{N}^*]g(x), \tag{4.18}$$

$$\{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^+ = \{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^- = \mathcal{L}^* h(x), \quad S \in C^{2,\kappa}, \tag{4.19}$$

where the operators $\mathcal{H}^*$, $\mathcal{K}^*$, $\mathcal{N}^*$, and $\mathcal{L}^*$ are pseudodifferential operators of order -1, 0, 0, and 1, respectively, and are defined by the formulas

$$\mathcal{H}^* g(x) := \int_S \Gamma^*(x-y) g(y) dS_y, \tag{4.20}$$

$$\mathcal{K}^* g(x) := \int_S \big[\mathcal{P}^*\big(\partial_x, n(x)\big)\Gamma^*(x-y)\big] g(y) dS_y, \tag{4.21}$$

$$\mathcal{N}^* g(x) := \int_S \big[\mathcal{P}\big(\partial_y, n(y)\big)\big[\Gamma^*(x-y)\big]^\top\big]^\top g(y) dS_y, \tag{4.22}$$

$$\mathcal{L}^* h(x) := \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{P}^*\big(\partial_z, n(x)\big) \int_S \big[\mathcal{P}\big(\partial_y, n(y)\big)\big[\Gamma^*(z-y)\big]^\top\big]^\top g(y) dS_y. \tag{4.23}$$

The following equalities hold in appropriate function spaces:

$$\mathcal{N}^* \mathcal{H}^* = \mathcal{H}^* \mathcal{K}^*, \qquad \mathcal{L}^* \mathcal{N}^* = \mathcal{K}^* \mathcal{L}^*,$$
$$\mathcal{H}^* \mathcal{L}^* = -4^{-1} I_7 + [\mathcal{N}^*]^2, \qquad \mathcal{L}^* \mathcal{H}^* = -4^{-1} I_7 + [\mathcal{K}^*]^2.$$

Proof. It can be found in [8]. □

Lemma 4.9. *Let $S \in C^{2,\kappa}$ and $0 < \sigma < \kappa \leqslant 1$. The integral operator*

$$\mathcal{H}^* \;:\; [C^{0,\sigma}(S)]^7 \to [C^{1,\sigma}(S)]^7$$

is invertible and

$$[\mathcal{H}^*]^{-1} \;:\; [C^{1,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7$$

is a pseudodifferential operator of order 1, more precisely, it is a singular integro-differential operator.

Proof. It is word for word of the proof of Theorem 6.6 in [18]. □

In our analysis below we need also the following auxiliary assertion which is proved in [8].

Theorem 4.10. *Let $S \in C^{2,\kappa}$ and $0 < \sigma < \kappa \leqslant 1$. The null spaces of the singular integral operators*

$$2^{-1}I_7 + \mathcal{K}^* \ : \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$
$$2^{-1}I_7 + \mathcal{N}^* \ : \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$

are trivial, while the null spaces of the singular integral operators

$$-2^{-1}I_7 + \mathcal{K}^* \ : \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$
$$-2^{-1}I_7 + \mathcal{N}^* \ : \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7,$$

have the dimension equal to 7. Moreover, the vectors

$$\begin{gathered}
\Psi^{(1)}(x) = (0, -x_3, x_2, 1, 0, 0, 0)^\top, \qquad \Psi^{(2)}(x) = (x_3, 0, -x_1, 0, 1, 0, 0)^\top, \\
\Psi^{(3)}(x) = (-x_2, x_1, 0, 0, 0, 1, 0)^\top, \qquad \Psi^{(4)}(x) = (1, 0, 0, 0, 0, 0, 0)^\top, \\
\Psi^{(5)}(x) = (0, 1, 0, 0, 0, 0, 0, 0)^\top, \qquad \Psi^{(6)}(x) = (0, 0, 1, 0, 0, 0, 0)^\top, \\
\Psi^{(7)}(x) = (0, 0, 0, 0, 0, 0, 0, 1)^\top,
\end{gathered} \tag{4.24}$$

restricted onto the surface S, $\{\Psi^{(k)}(x),\ x \in S\}_{k=1}^{k=7}$, represent a basis of the null space of the operator $[-2^{-1}I_7 + \mathcal{N}^*]$, while the system of vectors $\{g^{(k)}(x),\ x \in S\}_{k=1}^{k=7}$ with

$$g^{(k)} = [\mathcal{H}^*]^{-1}\Psi^{(k)}, \qquad k = \overline{1,7},$$

represents a basis of the null space of the operator $[-2^{-1}I_7 + \mathcal{K}^*]$.

5. Reduction to integral equations and existence theorems

We look for a solution to the interior Neumann type boundary value problem in the form of the single layer potential

$$U(x) = V(g)(x) = \int_S \Gamma(x-y)\, g(y)\, dS_y, \qquad x \in \Omega^+, \tag{5.1}$$

where $g \in [C^{0,\sigma}(S)]^7$ is an unknown density vector-function. Evidently, the vector-function (5.1) automatically satisfies the differential equation (3.1), while the boundary condition (3.2) leads to the following singular integral equation

$$-2^{-1}g(x) + \mathcal{K}\, g(x) = F(x), \qquad x \in S, \tag{5.2}$$

where the operator $\mathcal{K}$ is defined by (4.10). Due to Theorem 4.5 the operator $[-2^{-1}I_7 + \mathcal{K}]$ is an elliptic singular integral operator of normal type, i.e., its symbol matrix is non-degenerate and for the equation (5.2) the Fredholm theorems hold.

To analyse the solvability of equation (5.2) we need to investigate the null spaces of the operator $[-2^{-1}I_7 + \mathcal{K}]$ and its adjoint one.

First we study $\ker[-2^{-1}I_7 + \mathcal{K}]$. To this end let us consider the homogeneous equation

$$-2^{-1}g(x) + \mathcal{K}\,g(x) = 0, \qquad x \in S. \tag{5.3}$$

In what follows we show that (5.3) possesses only seven independent solutions, i.e.,

$$\dim \ker[-2^{-1}I_7 + \mathcal{K}] = 7.$$

Indeed, let $g_0 \in \ker[-2^{-1}I_7 + \mathcal{K}]$ and consider the single layer potential $V(g_0)$. It is evident that $V(g_0)$ solves the homogeneous Neumann type interior boundary value problem (3.1)–(3.2) with $F = 0$. Therefore in view of Remark 3.2, the following representation

$$V(g_0)(x) = \sum_{k=1}^{7} C_k\,\Phi^{(k)}(x), \qquad x \in \Omega^+, \tag{5.4}$$

holds with appropriately chosen constants C_k. Here the vector-functions $\Phi^{(k)}$, $k = \overline{1,7}$, are defined in (3.8). Theorem and the relation (5.4) imply

$$\{V(g_0)(x)\}^+ = \mathcal{H}(g_0)(x) = \sum_{k=1}^{7} C_k\,\Phi^{(k)}(x), \qquad x \in S,$$

where the integral operator $\mathcal{H}$ is defined by (4.9). By the invertibility of the operator $\mathcal{H}$ (see Remark 4.6, we deduce

$$g_0(x) = \sum_{k=1}^{7} C_k\,\mathcal{H}^{-1}\Phi^{(k)}(x), \qquad x \in S.$$

Further, since the system $\{\Phi^{(k)}(x)\}_{k=1}^{7}$ is linearly independent in Ω^+ , the same system is linearly independent on S as well. Indeed, if there are constants $b_k, k = \overline{1,7}$, such that $\sum_{k=1}^{7} |b_k| \neq 0$ and

$$\sum_{k=1}^{7} b_k\,\Phi^{(k)}(x) = 0, \qquad x \in S,$$

then it follows that the vector-function

$$U(x) := \sum_{k=1}^{7} b_k\,\Phi^{(k)}(x), \qquad x \in \Omega^+,$$

solves the interior Dirichlet type problem in Ω^+ and due to the uniqueness Theorem 2.2 in [7], we conclude $U(x) = 0$, $x \in \Omega^+$, which contradicts to the linear independency of the system $\{\Phi^{(k)}(x)\}_{k=1}^{7}$ in Ω^+.

Let us now prove that the system

$$\{\mathcal{H}^{(-1)}\Phi^{(k)}(x)\}_{k=1}^{7}, \qquad x \in S,$$

is also linearly independent. Indeed, let there be constants d_k, $k=\overline{1,7}$, such that $\sum_{k=1}^{7}|d_k|\neq 0$ and

$$\sum_{k=1}^{7} d_k\,\mathcal{H}^{-1}\Phi^{(k)}(x)=0,\qquad x\in S.$$

Applying the operator $\mathcal{H}$ to this equation we get

$$\sum_{k=1}^{7} d_k\,\Phi^{(k)}(x)=0,\qquad x\in S,$$

which contradicts the linear independency of the system $\{\Phi^{(k)}(x)\}_{k=1}^{7}$ on S.

Further, let us introduce the notation

$$g^{(k)}(x):=\mathcal{H}^{-1}\Phi^{(k)}(x),\qquad\qquad x\in S. \tag{5.5}$$

It is evident that the system $\{g^{(k)}(x)\}_{k=1}^{7}$ is linearly independent, implying that

$$\dim\,\ker[-2^{-1}I_7+\mathcal{K}]\geqslant 7.$$

On the other hand, from the above arguments it follows that the system $\{g^{(k)}(x)\}_{k=1}^{7}$ is a basis of the null space $\ker[-2^{-1}I_7+\mathcal{K}]$, i.e., any solution to the homogeneous equation (5.3) is representable in the form

$$g_0=\sum_{k=1}^{7} C_k\,g^{(k)}(x),\qquad x\in S$$

with some constants C_k. Thus we have proven the following assertion.

Theorem 5.1. *Let $S\in C^{2,\alpha}$ with $0<\alpha\leqslant 1$. The dimension of the null space of the singular integral operator $[-2^{-1}I_7+\mathcal{K}]$ equals to seven and the system $\{\mathcal{H}^{-1}\Phi^{(k)}(x)\}_{k=1}^{7}$, $x\in S$, is its basis, where $\Phi^{(k)}$, $k=\overline{1,7}$, are given in* (3.8)*. Moreover, if the nonhomogeneous equation* (5.2) *is solvable and g^* is its particular solution, then the vector*

$$g=g^*+\sum_{k=1}^{7} C_k\,g^{(k)}$$

with $g^{(k)}$ given by (5.5) *and C_k being arbitrary constants, solves the same nonhomogeneous equation.*

To derive the necessary and sufficient conditions for the nonhomogeneous equation (5.2) to be solvable, we need to analyze the null space of the corresponding adjoint operator $[-2^{-1}\,I_7+\widetilde{\mathcal{K}}]$, where $\widetilde{\mathcal{K}}$ is the operator adjoint to $\mathcal{K}$ in the sense of the space $[L_2(S)]^7$, i.e., $(\mathcal{K}g,\varphi)_{[L_2(S)]^7}=(g,\widetilde{\mathcal{K}}\varphi)_{[L_2(S)]^7}$ for all $g,\varphi\in[L_2(S)]^7$.

From the following chain of equalities

$$(\mathcal{K}g,\varphi)_{[L_2(S)]^7}=\int_S\Big(\int_S\mathcal{P}\big(\partial_x,n(x)\big)\Gamma(x-y)g(y)dS_y\Big)\varphi(x)dS_x$$

$$
\begin{aligned}
&= \int_S \Big(\int_S \mathcal{P}\big(\partial_x, n(x)\big)\Gamma(x-y)g(y)\varphi(x)dS_x\Big) dS_y \\
&= \int_S \Big(\int_S g(y)\big[\mathcal{P}\big(\partial_x, n(x)\big)\Gamma(x-y)\big]^\top \varphi(x)dS_x\Big) dS_y \\
&= \int_S g(y)\Big(\int_S \big[\mathcal{P}\big(\partial_x, n(x)\big)\Gamma(x-y)\big]^\top \varphi(x)dS_x\Big) dS_y \\
&= \int_S g(x)\Big(\int_S \big[\mathcal{P}\big(\partial_y, n(y)\big)\Gamma(y-x)\big]^\top \varphi(y)dS_y\Big) dS_x,
\end{aligned}
$$

and taking into account that $\Gamma(y-x) = [\Gamma^*(x-y)]^\top$, we get

$$
\widetilde{\mathcal{K}}\,\varphi(x) = \int_S \big[\mathcal{P}\big(\partial_y, n(y)\big)\big(\Gamma^*(x-y)\big)^\top\big]^\top \varphi(y)dS_y, \qquad x \in S,
$$

whence it follows that the operator $\widetilde{\mathcal{K}}$ coincides with the operator $\mathcal{N}^*$ defined in (4.22), i.e., $\mathcal{N}^* = \widetilde{\mathcal{K}}$. Therefore the following assertion immediately follows from Theorem 4.10.

Theorem 5.2. *Let $S \in C^{2,\alpha}$ with $0 < \alpha \leqslant 1$. The null space of the operator $[-2^{-1}I_7 + \widetilde{\mathcal{K}}]$ is seven dimensional and the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7$, $x \in S$, with $\Psi^{(k)}$, $k = \overline{1,7}$ defined in* (4.24)*, represents its basis.*

Now we are in the position to formulate the main existence results which directly follow from Theorems 5.1 and 5.2.

(see, e.g., [6. Ch. IV],[11])

Theorem 5.3. *Let $S \in C^{2,\alpha}$ and $F \in C^{0,\sigma}(S)$ with $0 < \sigma < \alpha \leqslant 1$. For solvability of the nonhomogeneous equation* (5.2) *the necessary and sufficient conditions read as follows*

$$
(F, \Psi^{(k)})_{[L_2(S)]^7} \equiv \int_S F(x) \cdot \Psi^{(k)}(x)\, dS = 0, \qquad k = \overline{1,7}, \tag{5.6}
$$

where the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7$, $x \in S$, is defined in (4.24)*.*

Proof. It immediately follows from the general theory of singular integral equations (see, e.g., [6. Ch. IV], [11]). since the operator $[-2^{-1}I_7 + \mathcal{K}]$ is of normal type with index equal to zero and the system of vector-functions $\{\Psi^{(k)}(x)\}_{k=1}^7$, $x \in S$, defined in (4.24) represents the basis of the null space of the adjoint operator $[-2^{-1}I_7 + \widetilde{\mathcal{K}}]$. Therefore for a given right hand side vector-function F the nonhomogeneous equation (5.2) is solvable if and only if the orthogonality conditions (5.6) are satisfied. □

Theorem 5.4. *Let $S \in C^{2,\alpha}$ and $F \in C^{0,\sigma}(S)$ with $0 < \sigma < \alpha \leqslant 1$. The nonhomogeneous Neumann type boundary value problem* (3.1)–(3.2) *is solvable if and only if the boundary vector-function F satisfies the orthogonality conditions* (5.6)*.*

Moreover, a solution U to the interior Neumann type boundary value problem is representable by the single layer potential (5.1)*, where the density vector-function g is defined by the singular integral equation* (5.2)*. The solution vector U is defined modulo a linear combination*

$$
U^{(*)}(x) = \sum_{k=1}^{7} C_k\, \Phi^{(k)}(x), \qquad x \in \Omega^+,
$$

where C_k are arbitrary constants and $\Phi^{(k)}$, $k=\overline{1,7}$, are defined in (3.8).

Proof. It directly follows from Theorems 5.1, 5.2, and 5.3. □

6. Appendix

6.1 Particular solutions the problem (3.4)

Unlike the classical thermoelasticity theory, explicit construction of a particular solution $\widetilde{V}_0=(u_0,\omega_0)^\top$ of the problem (3.4) in Ω^+ is problematic. If the condition

$$\frac{\eta}{2\mu+3\lambda}=\frac{\zeta}{2\chi+3\delta}$$

is satisfied, then for an arbitrary domain Ω^+ a particular solution to the problem (3.4) reads as

$$\widetilde{V}_0=\frac{\eta}{2\mu+3\lambda}(x,0)^\top=\frac{\eta}{2\mu+3\lambda}(x_1,x_2,x_3,0,0,0)^\top.$$

If the domain Ω^+ is a sphere $B(0,R)$ centered at the origin and radius R, then a particular solution $\widetilde{V}_0=(u_0,\omega_0)^\top$ to the problem (3.4) can be constructed without any restriction of material parameters and reads as follows [18]

$$u_0(x)=A_1x^\top-A_2(\delta+2\chi)\frac{dg_0(r)}{dr}\widetilde{n}(x),\qquad \omega_0(x)=A_2(\lambda+2\mu)\frac{dg_0(r)}{dr}\widetilde{n}(x),$$

where

$$x=(x_1,x_2,x_3),\quad r=|x|,\quad \widetilde{n}(x)=\frac{x^\top}{r},\quad g_0(r)=\frac{J_{1/2}(i\lambda_1 r)}{\sqrt{r}},\quad \lambda_1\lambda^2=\frac{4\alpha(\lambda+2\mu)}{d_2},$$

$$A_1=\frac{4\eta}{RD}\Big\{\big[\chi(\delta+2\chi)-\gamma(\lambda+2\mu)\big]\frac{dg_0(R)}{dR}+\alpha(\lambda+2\mu)Rg_0(R)\Big\}-\frac{4\zeta(\mu\delta-\lambda\chi)}{RD}\frac{dg_0(R)}{dR},$$

$$A_2=\frac{\zeta(3\lambda+2\mu)-\eta(3\delta+2\chi)}{D},$$

$$D=\big\{(3\lambda+2\mu)\big[\chi(\delta+2\chi)-\gamma(\lambda+2\mu)\big]+(3\delta+2\chi)(\lambda\chi-\mu\delta)\big\}\frac{4}{R}\frac{dg_0(R)}{dR}$$
$$+4\alpha(\lambda+2\mu)(3\lambda+2\mu)g_0(R),$$

$$d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\chi)^2>0.$$

Here $J_{1/2}(i\lambda_1 r)$ is the Bessel function of the first order. Note that the vector $\widetilde{n}(x)$ for $x\in\partial B(0,R)$ coincides with the exterior normal vector at the point $x\in\partial B(0,R)$.

6.2 Fundamental solution

The fundamental matrix of the operator of elastostatics $L(\partial)$, which solves the distributional matrix differential equation $L(\partial_x)\Gamma(x-y)=I_7\,\delta(x-y)$ with Dirac's delta distribution $\delta(x-y)$, reads as (for details see [18], [5])

$$\Gamma(x)=\begin{bmatrix}[\Gamma^{(1)}_{pq}(x)]_{3\times3} & [\Gamma^{(2)}_{pq}(x)]_{3\times3} & [\Gamma^{(5)}_{pq}(x)]_{3\times1}\\ [\Gamma^{(3)}_{pq}(x)]_{3\times3} & [\Gamma^{(4)}_{pq}(x)]_{3\times3} & [\Gamma^{(6)}_{pq}(x)]_{3\times1}\\ [\Gamma^{(7)}_{pq}(x)]_{1\times3} & [\Gamma^{(8)}_{pq}(x)]_{1\times3} & \Gamma^{(9)}(x)\end{bmatrix}_{7\times7}$$

$$
=\frac{1}{4\pi}\begin{bmatrix} \widetilde{\Psi}_1(x)I_3 & \widetilde{\Psi}_2(x)I_3 & [0]_{3\times 1} \\ \widetilde{\Psi}_3(x)I_3 & \widetilde{\Psi}_4(x)I_3 & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \widetilde{\Psi}_5(x) \end{bmatrix}_{7\times 7}
$$

$$
-\frac{1}{4\pi}\begin{bmatrix} Q(\partial)\widetilde{\Psi}_6(x) & Q(\partial)\widetilde{\Psi}_7(x) & [0]_{3\times 1} \\ Q(\partial)\widetilde{\Psi}_8(x) & Q(\partial)\widetilde{\Psi}_9(x) & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & 0 \end{bmatrix}_{7\times 7}
$$

$$
+\frac{1}{4\pi}\begin{bmatrix} R(\partial)\Psi_{10}(x) & R(\partial)\Psi_{11}(x) & \nabla^\top\Psi_{14}(x) \\ R(\partial)\Psi_{12}(x) & R(\partial)\Psi_{13}(x) & \nabla^\top\Psi_{15}(x) \\ [0]_{1\times 3} & [0]_{1\times 3} & 0 \end{bmatrix}_{7\times 7},
$$

where

$$
\Psi_1(x) = -\frac{\gamma+\varepsilon}{d_1|x|} - \frac{1}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big\{4[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) + 4\nu(\alpha\chi-\mu\nu)]
$$

$$
+ d_1(\gamma+\varepsilon)\lambda_1^2 + \frac{16\alpha^2\mu}{\lambda_j^2}\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|},
$$

$$
\Psi_2(x) = \Psi_3(x) = \frac{\chi+\nu}{d_1|x|} + \frac{1}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big\{4\alpha[\mu(\chi+\nu) + 2(\alpha\chi-\mu\nu)]
$$

$$
+ d_1(\chi+\nu)\lambda_j^2\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|},
$$

$$
\Psi_4(x) = -\frac{\mu+\alpha}{d_1|x|} - \frac{\mu+\alpha}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j(d_1\lambda_j^2+4\alpha\mu)\frac{e^{i\lambda_j|x|}-1}{|x|},
$$

$$
\Psi_5(x) = -\frac{1}{\kappa'|x|},
$$

$$
\Psi_6(x) = -\frac{(\lambda+\mu)|x|}{2\mu(\lambda+2\mu)} + \frac{(\delta+2\chi)^2 d_2}{4\alpha(\lambda+2\mu)^2}\frac{e^{-\lambda_1|x|}-1}{|x|} + \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\gamma+\varepsilon}{d_1}
$$

$$
+ \frac{4}{d_1^2\lambda_j^2}[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) + 4\nu(\alpha\chi-\mu\nu)] + \frac{16\alpha^2\mu}{d_1^2\lambda_j^4}\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|},
$$

$$
\Psi_7(x) = \Psi_8(x) = -\frac{\delta+2\chi}{4\alpha(\lambda+2\mu)}\frac{e^{-\lambda_1|x|}-1}{|x|} - \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\chi+\nu}{d_1}
$$

$$+\frac{4\alpha}{d_1^2\lambda_j^2}\big[\mu(\chi+\nu)+2(\alpha\chi-\mu\nu)\big]\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|},$$

$$\Psi_9(x)=\frac{1}{4\alpha}\frac{e^{-\lambda_1|x|}-1}{|x|}+\frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\,\frac{\mu+\alpha}{d_1^2}\Big(d_1+\frac{4\alpha\mu}{\lambda_j^2}\Big)\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{10}(x)=\frac{4}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big[\nu d_1+(\gamma+\varepsilon)(\alpha\chi-\mu\nu)+\frac{4\alpha^2\chi}{\lambda_j^2}\Big]\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{11}(x)=\Psi_{12}(x)=\frac{2}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big[2(\chi+\nu)(\mu\nu-\alpha\chi)-\alpha d_1$$

$$-\frac{4\alpha^2\mu}{\lambda_j^2}\Big]\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{13}(x)=\frac{4(\mu+\alpha)(\alpha\chi-\mu\nu)}{d_1^2(\lambda_2^2-\lambda_3^2)}\,\frac{e^{i\lambda_2|x|}-e^{i\lambda_3|x|}}{|x|},$$

$$\Psi_{14}(x)=\frac{1}{\kappa'}\Big\{-\frac{\eta|x|}{2(\lambda+2\mu)}+\big[\zeta(\lambda+2\mu)-\eta(\delta+2\chi)\big]\frac{\delta+2\chi}{4\alpha(\lambda+2\mu)^2}\frac{e^{-\lambda_1|x|}-1}{|x|}\Big\}\,,$$

$$\Psi_{15}(x)=\frac{\eta(\delta+2\chi)-\zeta(\lambda+2\mu)}{4\kappa'\alpha(\lambda+2\mu)}\,\frac{e^{-\lambda_1|x|}-1}{|x|};$$

where

$$d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2,\quad d_2:=(\lambda+2\,\mu)\,(\beta+2\,\gamma)-(\delta+2\,\varkappa)^2,$$

$$d_3:=(\mu+\alpha)\,(\mathcal{I}\,\sigma^2-4\alpha)+(\gamma+\varepsilon)\,\varrho\sigma^2+4\alpha^2,\quad \lambda_1^2=\frac{4\alpha(\lambda+2\mu)}{d_2}>0,$$

$$\lambda_{2,3}^2=\frac{4}{d_1^2}\Big\{2(\mu\nu-\alpha\chi)^2-\alpha\mu d_1\pm i2(\mu\nu-\alpha\chi)\sqrt{\big(\mu+\alpha[\alpha(\mu\gamma-\chi^2)+\mu(\alpha\varepsilon-\nu^2)]\big)}\Big\}\,.$$

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SOLUTION OF THE BASIC PLANE BOUNDARY VALUE PROBLEMS OF STATICS OF THE ELASTIC MIXTURE FOR A MULTIPLY CONNECTED DOMAIN BY THE METHOD OF D. SHERMAN

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Abstract. In the present work we consider the basic plane boundary value problems of statics of the linear theory of elastic mixture for a multiply connected finite domain, when on the boundary a displacement vector (the first problem) and a stress vector (the second problem) are given.

For the solution of the problem we use the generalized Kolosov-Muskhelishvili formulas and the method of D. Sherman.

Keywords and phrases: Elastic mixtures, boundary value problems, generalized Kolosov-Muskhelishvili's representation, Method D. Sherman.

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1. Introduction

The construction and the intensive investigation of the mathematical models of elastic mixtures arise by the wide use of composites into practice. The diffusion and shift models of the linear theory of elastic mixtures are presented by several authors.

In [1,3.4] for a simply connected finite and infinite domain the basic plane boundary value problems of statics of the elastic mixture theory are considered when on the boundary a displacement vector (the first problem), a stress vector (the second problem); differences of partial displacements and the sum of stress vector components (the third problem) are given.

In [1] two-dimensional boundary value problems of statics are investigated by potential method and the theory of singular integral equations.

In [3] by applying the general Kolosov-Muskhelishvili representations from ([2]) these problems are splitted and reduced to the first and the second boundary value problem for an elliptic equation which structurally coincides with an equation of statics of an isotropic elastic body.

In [4] using potentials with complex densities the solutions of basic plane boundary value problems of statics are reduced to solution of Fredholm linear integral equation of second kind.

In [5] the basic mixed boundary value problem of equation of statisc of the elastic mixture theory is considered in a simply connected domain when the displacement vector is given on one part of the boundary and the stress vector on the remaing part.

In [7] three - dimensional boundary value problems of two isotropic elastic medea are investigated by means of the potential method. The uniqueness and existence theorems for the statics, steady oscillations and dynamical problems are proved.

In the present work in the case of the plane theory of elastic mixture for a multiply connected finite domain we study the problems the variant of which in the case of the plane theory of elasicity has been solved by N. Muskhelishvili, owing to the method of D. Sherman [6, §102]

For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvuli's formula [2,4] and the method D. Sherman developed in [6; §102].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixture in the complex form is written as [4]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + K \frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0 \tag{2.1}$$

where $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$,

$$K = -\tfrac{1}{2} l m^{-1},\ l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix},\ m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1},$$

$m_k = l_k + \frac{1}{2}\, l_{3+k}$, $\quad k = 1,2,3$, $l_1 = a_2/d_2$, $\quad l_2 = -c/d_2$, $\quad l_3 = a_1/d_2$, $a_1 = \mu_1 - \lambda_5$, $a_2 = \mu_2 - \lambda_5$, $c = \mu_3 + \lambda_5$, $d_2 = a_1a_2 - c^2$, $l_1 + l_4 = b/d_1$, $l_2 + l_5 = -c_0/d_1$, $l_3 + l_6 = a/d_1$, $a = a_1 + b_1$, $b = a_2 + b_2$, $c_0 = c + d$, $b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho$, $b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho$, $d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho$, $\alpha_2 = \lambda_3 - \lambda_4$, $\rho = \rho_1 + \rho_2$, $d_1 = ab - c^2$.

ρ_1 and ρ_2 appearing in (2.2) are the partial densities, and $\mu_1, \mu_2, \mu_3, \lambda_p$, $p = \overline{1,5}$ are real constants characterizing physical properties of the elastic mixture and satisfying certain inequalities [1] and [7].

Let D^+ be a bounded two-dimensional domain (surrounded by the curve S) and let D^- be the complement of $\bar{D}^+ = D^+ US$. We assume that $S \in C^{k+\beta}$, $k = 1,2$, $0 < \beta \le 1$.

A vector $u = (u', u'')^T = (u_1, u_2, u_3, u_4)^T$ is said to be regular in $D^+[D^-]$ if $u_k \in C^2(D^+) \bigcap C^1(\bar{D^+})$ $[u_k \in C^2(D^-) \bigcap C^1(\bar{D^-})]$ and the second order derivatives of u_k are summable in $D^+[D^-]$, in the case of the domain D^- we assume, in addition the following conditions at infinity

$$u_k(x) = 0\,(1), \qquad |x|^2 \frac{\partial u_k}{\partial x_j} = 0(1), \qquad j = 1,2; \qquad k = \overline{1,4},$$

to be fulffiled with $|x|^2 = x_1^2 + x_2^2$.

In [2] M. Basheleishvili obtained the following representations

$$U = (U_1, U_2)^T = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2}\, l\ \ z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.3}$$

$$\begin{aligned} TU &= [(TU)_1, (TU)_2]^T = [(Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3]^T \\ &= \frac{\partial}{\partial s(x)}[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}], \end{aligned} \tag{2.4}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)$ are arbitrary analytic vector-functions,

$$A=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}=2\mu m,\ \mu=\begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix},\ m=\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix},$$

$$B=\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}=\mu l,\ E=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

are known matrices and (see [5])

$$A_1+A_3-2=B_1+B_3,\quad A_2+A_4-2=B_2+B_4, \tag{2.5}$$

$\det\ m>0, \det\ \mu>0,\ \det(A-2E)>0.$

$\dfrac{\partial}{\partial S(x)}=n_1\dfrac{\partial}{\partial x_2}-n_2\dfrac{\partial}{\partial x_1},\quad n=(n_1,n_2)^T$ is a unit vector of the outer normal

$(Tu)_p,\ p=\overline{1,4}$ are the components of stresses [2]

$(Tu)_1=r'_{11}n_1+r'_{21}n_2,\quad (Tu)_2=r'_{12}n_1+r'_{22}n_2,$

$(Tu)_3=r''_{11}n_1+r''_{21}n_2,\quad (Tu)_4=r''_{12}n_1+r''_{22}n_2,$

$$\tau^{(1)}=\begin{pmatrix} r'_{11} \\ r''_{11} \end{pmatrix}=\begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}-2\mu\frac{\partial}{\partial x_2}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix},$$

$$\tau^{(2)}=\begin{pmatrix} r'_{22} \\ r''_{22} \end{pmatrix}=\begin{bmatrix} a & c_0 \\ c_0 & b \end{bmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}-2\mu\frac{\partial}{\partial x_1}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix},$$

$$\eta^{(1)}=\begin{pmatrix} \eta'_{21} \\ \eta''_{21} \end{pmatrix}=-\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}+2\mu\frac{\partial}{\partial x_1}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix},$$

$$\eta^{(2)}=\begin{pmatrix} r'_{12} \\ r''_{12} \end{pmatrix}=\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}+2\mu\frac{\partial}{\partial x_2}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}. \tag{2.6}$$

$\theta'=divu',\quad \theta''=divu'',\quad \omega'=rotu',\quad \omega''=rotu''.$

By virtue of (2.2) and (2.6) we obtain lengthy but elementary calculations.

$$\tau=\tau^{(1)}+\tau^{(2)}=2(2E-A-B)Re\ \ \varphi'(z),$$

$$\tau^{(1)}-\tau^{(2)}-i\eta=2[B\overline{z}\varphi''(z)+2\mu\psi'(z)],\quad \eta=\eta_1+\eta_2, \tag{2.7}$$

$\det(2E-A-B)>0$ (see [2]).

Formulas (2.3), (2.4) and (2.7) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixture.

Also note that

$$X + iY = i[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_S \tag{2.8}$$

is the principal vector of stresses applied on S.

For our purpose let us rewrite formulas (2.4) in a more convenient form. Namely, for the stress vector we have

$$(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)} = F + \nu, \tag{2.4$'$}$$

where $\nu = (\nu_1, \nu_2)^T$ is an arbitrary complex vector,

$$F = (F_1, F_2)^T = \int_{z_0}^{z} TU ds,$$

here the integral is taken over any smooth arc within D^+ connecting an arbitrary fixed point z_0 with a variable point z of D^+.

Multiplying (2.4)$'$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}\overline{dt}$ and integrating over S. Owing to (2.5) we obtain

$$\begin{pmatrix} B_1 + B_3 \\ B_2 + B_4 \end{pmatrix} \int_S [\varphi(t)\overline{dt} - \overline{\varphi(t)}dt] = \int_S \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t)\overline{dt}. \tag{2.9}$$

From (2.9) we have $Re \int_S F(t)\overline{dt} = 0$.

Below we will need the following Greens formulas [1] and [4]

$$\int_{D^\pm} E(u,u)dx = \pm\ I_m \int_S U\overline{TU}ds, \tag{2.10}$$

where $E(u,u)$ is the positively defined quadratic form, the equation

$E(u,u) = 0$ admits a solution $u = (u', u'')^T$, $u' = (u_1, u_2)^T = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$,

$$u'' = (u_3, u_4)^T = a'' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{2.11}$$

where a' and a'' are arbitrary real constant vectors, and b' is an arbitrary real constant.

Let G^+ be a finite multiply connected domain bounded by the contours $L_1, L_2, L_3, ...,$ L_p, L_{p+1}, the last of which contains all the others, $L_j \in C^{1,\beta}$ $0 < \beta \leq 1, j = \overline{1, p+1}$. In this case the boundary of G^+ is $L = \bigcup\limits_{j=1}^{p+1} L_j$; note that the contours $L_j (j \leq p)$ are oriented clockwise, while L_{p+1} is oriented counterclockwise. Let $G_j (j = \overline{1,p})$ be a finite two-dimensional domain bounded by the contour L_j, $j = \overline{1,p}$. By G_{p+1} we denote an infinite domain bounded by the contour L_{p+1}. $G' = \bigcup\limits_{j=1}^{p+1} G_j$, and $G^- = R^2 \setminus \bigcup\limits_{j+1}^{p} G_p$.

Note that in a domain G^+ components of the partial displasements and stress vectors are one-valued functions.

Repeating word by word the reasoning developed in [6 §35], owing to formulas (2.7)-(2.8) we obtain that (2.3) represent one-valued vector-function in the domain G^+, when

$$\varphi(z)=\sum_{k=1}^{p}\gamma_k ln(z-z_k)+\varphi^*(z) \tag{2.12}$$

$$\psi(z)=\sum_{k=1}^{p}\gamma_k' ln(z-z_k)+\psi^*(z) \tag{2.13}$$

where z_k is an arbitrary point in $G_k, k=\overline{1,p}$

$$\gamma_k=-\frac{X_k+iY_k}{4\pi},\quad \gamma_k'=-\frac{m(X_k-iY_k)}{4\pi},$$

$X_k+iY_k=i[(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi}(t)]_{L_k}$; $\varphi^*(z)$ and $\psi^*(z)$ are holomorphic vector-functions in G^+.

Finally note that the formula $(2.10)^+$ is valid for domain G^+

$$\int_{G^+}E(u,u)dx=I_m\int_L U\overline{TU}ds$$

$$=I_m\int_L[m\varphi(t)+\frac{1}{2}\ \ lt\overline{\varphi'(t)}+\overline{\psi(t)}]d[(A-2E)\overline{\varphi(t)}+B\bar{t}\varphi'(t)+2\mu\psi(t)]. \tag{2.14}$$

3. Solution of the first boundary valu problem for the finite multiply connected domain

Let G^+ be a finite multiply connected domain (see section 2). The first boundary value problem is formulated as follows: Find in the domain G^+ a vector $U(x)$ which belongs to the class $C^2(G^+)\bigcap C^{(1,\alpha)}(\overline{G^+})$ is a solution of equation (2.1.) and satisfying the following condition

$$U^+(t_0)=f(t_0)\quad on\quad L,\quad -(I)_f^+\quad problem;$$

where $f\in C^{1,\alpha}(L),\ \ L\in C^{(2,\beta)}, 0<\alpha<\beta\leq 1$ is a given complex vector-function.

Using the Green formula (2.14) it is easy to prove.

Theorem 3.1. *The homogeneous problem $(I)_0^+$, has no nontrivial regular solution.*

By virtue of (2.3) it is obvious that the $(I)_f^+$ problem can be reduced to a problem of defining two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in G^+ using the boundary condition

$$U^+(t_0)=m\varphi((t_0)+\frac{1}{2}\quad lt_0\overline{\varphi'(t_0)}+\overline{\psi(t_0)}=f(t_0), on\ \ L. \tag{3.1}$$

Let us look for analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the form (see (2.12) and (2.13))

$$\varphi(z)=\frac{m^{-1}}{2\pi i}\int_L\frac{g(t)dt}{t-z}+\sum_{j=1}^{p}m^{-1}q_j\ \ ln(z-z_j), \tag{3.2}$$

$$\psi(z)=\frac{1}{2\pi i}\int_L\frac{\overline{g(t)}dt}{t-z}-\frac{K}{2\pi i}\int_L\frac{g(t)\overline{dt}}{t-z}+$$

$$+\frac{K}{2\pi i}\int_L \frac{\bar{t}g(t)dt}{(t-z)^2}+\sum_{j=1}^{p}\overline{q_j}\ \ ln(z-z_j), \tag{3.3}$$

where $z_j = x_{1j} + ix_{2j}$ is a arbitrary point in $G_j,\ j = \overline{1,p}, z = (x_1 + ix_2) \in G^+$, $g = (g_1, g_2)^T$ is the unknown complex vector to the Hölder class and has the integrable derivative, and $q_j = (q_{j1}, q_{j2})^T$ is an arbitrary constant vector, $(j = \overline{1,p})$.

We tie the unknown constant vector q_j and the unknown vector g by the relation

$$q_j = \int_{L_j} g(t)ds, \qquad j = \overline{1,p}. \tag{3.4}$$

Substituting (3.2) and (3.3) into (2.3.) we have by (3.4) that

$$U(x) = \frac{1}{2\pi i}\int_L g(t)dln\frac{t-z}{\bar{t}-\bar{z}} + \frac{K}{2\pi i}\int_L \overline{g(t)}d\frac{t-z}{\bar{t}-\bar{z}}$$

$$+\sum_{j=1}^{p}\left[2ln|z-z_j|\int_{L_j} g(t)ds - K\frac{z}{\bar{z}-\overline{z_j}}\int_{L_j} g(t)ds\right]. \tag{3.5}$$

Passing to the limit in (3.5) $G^+ \ni z \to t_0 \in L$ and using boundary condition (3.1.) to define the vector g we obtain the following integral equation of Sherman type

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t)dln\frac{t-t_0}{\bar{t}-\overline{t_0}} + \frac{K}{2\pi i}\int_L \overline{g(t)}d\frac{t-t_0}{\bar{t}-\overline{t_0}} + \sum_{j=1}^{p}[2ln|t_0-z_j|$$

$$-K\frac{t_0}{\overline{t_0}-\overline{z_j}}\ \ ]\int_{L_j} g(t)ds = f(t_0), \quad t_0 \in L. \tag{3.6}$$

Since $f \in C^{1,\alpha}(L), \qquad L \in C^{2,\beta} \quad (0 < \alpha < \beta \le 1)$, therefore from (3.6) it follows (see [4]) $g \in C^{1,\alpha}(L)$.

Let us show now that equation (3.6) is always solvable. For this it is suficient that the homogeneous equation corresponding to (3.6) has only a trivial solution. Denote the homogeneous equation (which we do not write) by $(3.6.)^0$ and assume that it has a solution different from zero which is denoted by g_0. Compose the complex potentials $\varphi_0(z)$ and $\psi_0(z)$ using (3.2) and (3.3.), where g is replased by g_0. We have

$$U_0(t_0) = m\varphi_0(t_0) + \frac{1}{2}\ \ lt_0\overline{\varphi'(t_0)} + \overline{\psi_0(t_0)} = 0, \qquad t_0 \in L. \tag{3.7}$$

Due to Theorem 3.1. we obtain $\quad u_0(x) = 0, \quad x \in G^+,\ $ hence (see [5])

$$\varphi_0(z) = \nu; \quad \psi_0(z) = -m\bar{\nu}, \tag{3.8}$$

where $\nu = (\nu_1, \nu_2)^T$ is an arbitrary constant vector.

Now note that since vector-functions $\varphi_0(z)$ and $\psi_0(z)$ are one-valued in G^+ therefore by (3.2.) - (3.4.) and (3.8.) we can write

$$\varphi_0(z) = \frac{m^{-1}}{2\pi i}\int_L \frac{g_0 dt}{t-z} = \nu, \qquad z \in G^+,$$

$$\psi_0(z) = \frac{1}{2\pi i}\int_L \frac{\overline{g_0(t)}dt}{t-z} + \frac{K}{2\pi i}\int_L \frac{\bar{t}g_0'(t)dt}{t-z} = -m\bar{\nu}, \qquad z \in G^+, \tag{3.9}$$

$$q_j^0 = \int_{L_j} g_0(t)ds \quad j = \overline{1,p}. \tag{3.10}$$

Consider the following vector-functions:

$$i\varphi^*(t) = m^{-1}g_0(t) - \nu;; \qquad i\psi^*(t) = \overline{g_0(t)} + K\bar{t}g_0'(t) + m\bar{\nu}. \tag{3.11}$$

By virtue of (3.9.) we obtain

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)dt}{t-z} = 0, \qquad \frac{1}{2\pi i}\int_L \frac{\psi^*(t)dt}{t-z} = 0, \quad \forall z \in G^+.$$

Hence we conclude, that (see [6, §74]) the vector-functions $\varphi^*(t)$ and $\psi^*(t)$ are the boundary values of the vector functions $\varphi^*(z)$ and $\psi^*(z)$ which are holomorphic in the domains $G_1, G_2, G_3, ..., G_p, G_{p+1}$ and $\varphi^*(\infty) = 0$, $\psi^*(\infty) = 0$.

After eliminating $g_0(t)$; in (3.11.), we obtain

$$m\overline{\varphi^*(t_0)} + \frac{1}{2}\, l\, \overline{t_0}\varphi^{*\acute{}}(t_0) + \psi^*(t_0) = -2im\bar{\nu}, \quad on \quad L_j, \quad j = \overline{1,p+1}.$$

By (2.3.) this condition correspoinds to the first boundary value problem of statics in the elastic mixture theory the domain G_j, $j = \overline{1,p+1}$, when at the body boundary the displacement vector is equal to constants $-2im\bar{\nu}$.

Using the uniqueness theorem for the domain G_j, $j = \overline{1,p+1}$ (see [4]) we have

$$\varphi^*(z) = c_j, \quad \psi^*(z) = -im\bar{\nu} - m\bar{c}_j, in \quad G_j, \; j = \overline{1,p+1},$$

where $c_j = (c_{j1}, c_{j2})^\tau$, $(j = \overline{1,p+1})$, is an arbitrary constant complex vector.

Since in the domain G_{P+1} $\varphi^*(\infty) = \psi^*(\infty) = 0$ therefore $\nu = 0$ and $C_{p+1} = 0$. Hence $\varphi^*(z) = c_j$, $\psi^*(z) = -m\bar{c}_j$, in G_j $j = \overline{1,p}$, $\varphi^*(z) = \psi^*(z) = 0$ in G_{p+1}.

In that case (3.11) implies

$$m^{-1}g_0(t) = ic_j \quad on \quad L_j, \quad j = \overline{1,p} \quad and \quad g_0(t) = 0 \quad on \quad L_{P+1}. \tag{3.12}$$

Now on the basis of (3.10) we obtain that every $c_j = 0$, hence $g_0(t) = 0$.

Consequently the homogeneous equation corresponding to (3.6) has no nontrivial solution. This means that (3.6) has a unique solution. Substituting g in (3.5), we get a solution of the first boundary value problem.

The existence of solution of the first boundary value problem can also be proved when domain G is an infinite multiply-connected domain

4. Solution of the second boundary value problem for the finite multiply connected domain

Let G^+ be a finite multiply connected domain (see section 2). The origin is assumed to lie in the domain G_{P+1}.

The second boundary value problem is investigated with the vector

$TU = ((Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3)^T$ given on the boundary where $(Tu)_k$, $k = \overline{1,4}$ are the components of stresses (see (2.6).)

Using the Green formula (2.14) it easy to prove.

Theorem 4.1. *The general solution of the second homogeneous boundary value problem, in G^+ is represented by the formula*

$$U = a^0 + i\varepsilon^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z,$$

where $z = x_1 + ix_2$, $a^0 = (a_1^0, a_2^0)^T$ is an arbitrary complex constant vector, and ε^0 is an arbitrary constant.

The latter formula expresses a rigid displacement of the body.

It is assumed that the principal vector and the principal moment of external forces are equal to zero on every contour $L_j (j = \overline{1,p})$. Moreover for solvability of the problem we also assume that the principal vector of external forces is equal to zero on L_{P+1}.

By virtue of (2.4) and (2.4)$'$ it is obvious that the second plane boundary value problem can be reduced to a problem of defining two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in G^+ using the boundary condition

$$(A - 2E)\varphi(t_0) + Bt_0\overline{\varphi'(t_0)} + 2\mu\overline{\psi(t_0)} - \nu_k = F(t_0),$$
$$on \quad L_k, \quad k = \overline{1,p+1}, \tag{4.1}$$

where $F = (F_1F_2)^T \in C^{1,\alpha}(L_k)$, $L_k \in C^{2,\beta}, 0 < \alpha < \beta \leq 1$ is a given vector-function. $\nu_k = (\nu_{k1}, \nu_{k2})^T$, $(k = \overline{1,p+1})$ is a constant vector. Note that the constants $\nu_1, \nu_2, \nu_3, ..., \nu_p, \nu_{p+1}$ are not given in advance and defined while solving the problem, if we fix one of them. Below we will assume that $\nu_{p+1} = 0$.

In (4.1) $\varphi(t_0), \varphi'(t_0)$ and $\psi(t_0)$ denote the boundary values on $L_k, k = \overline{1,p+1}$, of the vector-functions $\varphi(z), \varphi'(z)$ and $\psi(z)$ respectively.

In the sequel we will be assume that

$$Re \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t)d\bar{t} = 0. \tag{4.2}$$

Note that (see [6], [4]) condition (4.2) expresses the principal vector and the principal moment of external forces are equal to zero.

The analytic vector-functions $\varphi(z)$ and $\psi(z)$ sought for in the domain G^+ have the form

$$\varphi(z) = \frac{(A-2E)^{-1}}{2\pi i} \int_L \frac{g(t)dt}{t-z} + \sum_{j=1}^{P} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j}{z - z_j}, \tag{4.3}$$

$$\psi(z) = (2\mu)^{-1} \left[\frac{1}{2\pi i} \int_L \frac{\overline{g(t)}dt}{t-z} + \frac{H}{2\pi i} \int_L \frac{g(t)\overline{dt}}{t-z} - \frac{H}{2\pi i} \int_L \frac{\bar{t}g(t)dt}{(t-z)^2} \right.$$
$$\left. + \sum_{j=1}^{P} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j}{z - z_j} \right] \tag{4.4}$$

where $H = B(A-2E)^{-1}$ is a known matrix, $z_j = x_{1j} + x_{2j}$ is an arbitrary fixed point in $G_j, (j = \overline{1,p}), g = (g_1, g_2)^T$ is a complex unknown vector-function, M_j is a real constant. Then we tie the unknown constant M_j and unknown vector-function g by the relation

$$M_j = i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \int_{L_j} (g(t)\overline{dt} - \overline{g(t)}dt), \quad j = \overline{1,p}. \tag{4.5}$$

Taking into account (4.3) and (4.4) in (4.1) after some calculations for the determination of the vector g we obtain the following equation of Sherman type

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t) dln\frac{t-t_0}{\overline{t}-\overline{t_0}} - \frac{H}{2\pi i}\int_L \overline{g(t)} d\frac{t-t_0}{\overline{t}-\overline{t_0}}$$
$$+\sum_{j=1}^{p}\left[(A-2E)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{t_0 - z_j} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{\overline{t_0}-\overline{z_j}} - B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j t_0}{(\overline{t_0}-\overline{z_j})^2}\right]$$
$$-\nu_k = F(t_0), \quad on \quad L_k, \quad k = \overline{1,p+1}, \tag{4.6}$$

where $\nu_k, \quad k = \overline{1,p}$ are an arbitrary constant vector, $\nu_{p+1} = 0$, and $M_j, \quad j = \overline{1,p}$ are given by (4.5).

We tie the unknown constant vector ν_k and the unknown vector-function g by the relation

$$\nu_k = -\int_{L_k} g(t)ds, \quad k = \overline{1,p}. \tag{4.7}$$

If now in the left-hand side of the second integral equation in (4.6) under the vector ν_k is meant the expression (4.7) then this equation will transform into *a* equation containing no unknown except vector g.

To investigate equation (4.6) it's advisable to consider, instead of (4.6) the equation

$$g(t_0) + \frac{1}{2\pi i}\int_L g(t) dln\frac{t-t_0}{\overline{t}-\overline{t_0}} - \frac{H}{2\pi i}\int_L \overline{g(t)} d\frac{t-t_0}{\overline{t}-\overline{t_0}}$$
$$+\sum_{j=1}^{p}\left[(A-2E)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{t_0 - z_j} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j}{\overline{t_0}-\overline{z_j}} - B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{M_j t_0}{(\overline{t_0}-\overline{z_j})^2}\right]$$
$$+\frac{1}{4\pi i}\begin{pmatrix} 1 \\ 1 \end{pmatrix} M_{p+1}\left(\frac{1}{t_0} + \frac{1}{\overline{\overline{t_0}}} - \frac{t}{\overline{\overline{t_0^2}}}\right) - \nu_k = F(t_0),$$
$$on \quad L_k. \quad k = 1, \overline{p+1}, \tag{4.8}$$

where

$$M_{P+1} = -i\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\varphi'(\xi_0) - \overline{\varphi'(\xi_0)}), \tag{4.9}$$

$\xi_0 = \xi_1^0 + i \ \xi_2^0$ is a fixed point in G^+.

Now note that, by means of analytic vector-functions $\varphi(z)$ and $\psi(z)$ (which are defined by (4.3) and (4.4)) equation (4.8) can be rewritten as

$$(A-2E)\varphi(t_0) + Bt_0\overline{\varphi'(t_0)} + 2\mu\overline{\psi(t_0)} + \frac{1}{4\pi i}\begin{pmatrix} 1 \\ 1 \end{pmatrix} M_{p+1}\left(\frac{1}{t_0} + \frac{1}{\overline{\overline{t_0}}} - \frac{t}{\overline{\overline{t_0^2}}}\right)$$

$$-\nu_j = F(t_0) \ \ on \quad L_j, \quad j = \overline{1, p+1}, \tag{4.8$'$}$$

where $\varphi(t_0), \varphi^{'}(t_0)$ *and* $\psi(t_0)$ are boundary values on L_j of the analytic vector-functions $\varphi(t_0), \varphi^{'}(t_0)$ *and* $\psi(t_0)$ respectively.

Multiplying (4.8)$'$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overline{dt_0}$ and integrating over L. Owing to (2.5) we obtain

$$\begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix} [\varphi(t_0)\overline{dt_0} - \overline{\varphi(t_0)}dt] + \frac{M_{p+1}}{4\pi i} \int_L \left[\frac{\overline{dt_0}}{t_0} + \frac{dt_0}{\overline{t_0}} \right] + M_{p+1}$$

$$= \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t_0)\overline{dt_0}.$$

Since M_{p+1} represents a real constant, (see (4.9)), therefore by virtue of (4.2) from the last equalities we find that

$$M_{p+1} = Re \int_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} F(t_0)\overline{dt_0} = 0. \tag{4.10}$$

From (4.10) it follows that the principal vector and the principal moment of eternal forces are equal to zero (see (4.2)), then any solution g of equation (4.8) is simultaneously a solution of the initial equation (4.6).

Let us prove that equation (4.8) is always solvable. To this end it is sufficient to show that the homogeneous equation corresponding to (4.8)has only the trivial solution.Assume the contrary, let g_0 be its solution.Denote the corresponding complex potentials by $\varphi_0(z)$ and $\psi_0(z)$. By virtue of (4.3)-(4.5) and (4.7) we obtain

$$\varphi_0(z) = \frac{(A-2E)^{-1}}{2\pi i} \int_L \frac{g_0(t)dt}{t-z} + \sum_{j=1}^{p} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j^0}{z-z_j}, \tag{4.11}$$

$$\psi_0(z) = \frac{(2\mu)^{-1}}{2\pi i} \int_L \frac{\overline{g_0(t)}dt}{t-z} - \frac{(2\mu)^{-1}H}{2\pi i} \int_L \frac{\overline{t}g_0'(t)dt}{t-z}$$

$$+(2\mu)^{-1} \sum_{j=1}^{p} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{M_j^0}{z-z_j}, \tag{4.12}$$

$$\nu_j^0 = -\int_{L_j} g_0(t)ds, \ M_j^0 = i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \int_{L_j} (g_0(t)\overline{dt} - \overline{g_0(t)}dt), \ j = \overline{1, p}. \tag{4.13}$$

Obviously the condition

$$M_{p+1}^0 = -i \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\varphi_0'(\xi_0) - \overline{\varphi_0'}(\xi_0)) = 0 \tag{4.14}$$

is fulfilled.

Finally note that, it is easy to see that analytic vector-functions, i.e.complex potentials, $\varphi_0(z)$ and $\psi_0(z)$ satisfy the condition

$$(A-2E)\varphi_0(t_0)+Bt_0\overline{\varphi'(t_0)}+2\mu\overline{\psi_0(t_0)}-\nu_j^0=0, on L_j, j=\overline{1,p+1}, \nu_{p+1}^0=0. \quad (4.15)$$

In that case condition (4.15) corresponds to the boundary condition

$$(TU_0(t_0))^+=0, t_0\in L,$$

where U_0 is obtained from (2.3), if instead of $\varphi(z)$ and $\psi(z)$ we take $\varphi_0(z)$ and $\psi_0(z)$.

Now note that on the basis of uniqueness of Theorem 4.1. we can conclude that solution of the problem (4.15) in the case

$$\nu_j^0=0, j=\overline{1,p+1}, \quad (4.16)$$

is given by

$$U_0=m\varphi_0(z)+\frac{1}{2}\,l\,z\overline{\varphi_0'(z)}+\overline{\psi_0(z)},$$

where

$$\varphi_0(z)=i\varepsilon Rz+(A-2E)^{-1}\gamma, \psi_0(z)=-(2\mu)^{-1}\overline{\gamma}. \quad (4.17)$$

Here R is an arbitrary real constant, $\gamma=(\gamma_1,\gamma_2)^T$ is an arbitrary constant complex vector, and $\varepsilon=(\varepsilon_1,\varepsilon_2)^T$ is the real vector defined by, (see[5]),

$$\varepsilon_1=\frac{1}{\triangle_2}[A_2-H_0(2-A_4)], \quad \varepsilon_2=\frac{1}{\triangle_2}(2-A_1-H_0A_3). \quad (4.18)$$

$$H_0=\frac{A_2(\mu_2+\mu_3)-(2-A_1)(\mu_1+\mu_3)}{(2-A_4)(\mu_2+\mu_3)-A_3(\mu_1+\mu_3)}; \triangle_2=det(A-2E)>0.$$

Due to (4.17) and (4.14) we arrive at

$$\varphi_0(z)=(A-2E)^{-1}\gamma,\ \psi_0(z)=-(2\mu)^{-1}\overline{\gamma},\ z\in G^+. \quad (4.19)$$

Finally comparing (4.11), (4.12) and (4.19) we obtain

$$\gamma=\frac{1}{2\pi i}\int_L\frac{g_0(t)dt}{t-z}+(A-2E)\sum_{j=1}^{p}\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{z-z_j}, \quad (4.20)$$

$$-\overline{\gamma}=\frac{1}{2\pi i}\int_L\frac{\overline{g_0(t)}dt}{t-z}-\frac{H}{2\pi i}\int_L\frac{\bar{t}g_0'(t)dt}{t-z}+\sum_{j=1}^{p}B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{z-z_j}. \quad (4.21)$$

Introduce the notation

$$i\varphi^*(t)=(A-2E)^{-1}g_0(t)+\sum_{j=1}^{p}\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}-(A-2E)^{-1}\gamma, \quad (4.22)$$

$$i\psi^*(t)=(2\mu)^{-1}\overline{g_0(t)}-(2\mu)^{-1}H\bar{t}g_0'(t)$$
$$+(2\mu)^{-1}\sum_{j=1}^{p}B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}+(2\mu)^{-1}\overline{\gamma}\,. \quad (4.23)$$

By (4.20) and (4.21) we obtain

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)dt}{t-z}=0,\ \frac{1}{2\pi i}\int_L \frac{\psi^*(t)dt}{t-z}=0, \forall z\in G^+. \tag{4.24}$$

From (4.24) we have, (see [6, §74]) the vector-functions (4.22) and (4.23) are the boundary value of the vector-functions $\varphi^*(z)$ and $\psi^*(z)$ which are holomorphic in the domains $G_1, G_2, ..., G_{p+1}$ and $\varphi^*(\infty)=\psi^*(\infty)=0$.

After eliminating $g_0(t)$ in (4.22) and (4.23) we obtain

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=i\sum_{j=1}^{p}[(A-2E)\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{\bar{t}-\bar{z}}$$

$$-B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0}{t-z_j}+B\begin{pmatrix}1\\1\end{pmatrix}\frac{M_j^0\bar{t}}{(t-z_j)^2}]-2i\overline{\gamma}, \quad on \quad L. \tag{4.25}$$

Multiplying (4.25) by $\begin{pmatrix}1\\1\end{pmatrix}dt$ and integrating over $L_k, k=\overline{1,p}$. Owing to (2.5) we obtain

$$\begin{pmatrix}B_1+B_3\\B_2+B_4\end{pmatrix}\int_{L_k}[\overline{\varphi^*(t)}dt-\varphi^*(t)\overline{dt}]$$

$$=i\sum_{j=1}^{p}\begin{pmatrix}B_1+B_3\\B_2+B_4\end{pmatrix}M_j^0\begin{pmatrix}1\\1\end{pmatrix}\int_{L_k}\left[\frac{dt}{\bar{t}-\overline{z_i}}+\frac{\overline{dt}}{t-z_i}\right]-4\pi M_k^0, k=\overline{1,p}.$$

Since $M_k^0, (k=\overline{1,p})$ are real constants (see (4.13)) therefore from the last relation it follows

$$M_k^0=0, (k=\overline{1,p}) \tag{4.26}$$

Thus, we have

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=-2i\overline{\gamma}, on L_k, k=\overline{1,p+1.}$$

By $(2.4)'$ this condition corresponds to the second boundary value problem of statics in the domains $G_1, G_2, G_3, ..., G_p$, and G_{p+1}, when the boundaries are free form external forces.

By virtue of uniqueness theorem [1] for domain G_{p+1} and the fact that $\varphi^*(\infty)=\psi^*(\infty)=0$, we find that $\varphi^*(z)=\psi^*(z)=0, in\ G_{p+1}$, then $\gamma=0$.

Due to the above reasoning we can write

$$(A-2E)\overline{\varphi^*(t)}+B\bar{t}\varphi^{*\prime}(t)+2\mu\psi^*(t)=0, on\ L_k,\ k=\overline{1,p}.$$

Using the uniqueness theorem for the problem $(II)_0^+$, (see [1]), in the domain $G_k,\ k=\overline{1,p}$ we find that

$$\varphi^*(z)=iR_k\varepsilon z+(A-2E)^{-1}C_k,$$

$$\psi^*(z)=-(2\mu)^{-1}\overline{C_k} \quad z\in G_k, \quad k=\overline{1,p}, \tag{4.27}$$

where R_k is an arbitrary real constant, $C_k = (C_{k1}, C_{k2})^T$ is an arbitrary complex constant vector and $\varepsilon = (\varepsilon_1, \varepsilon_2)^T$ is a real vector defined by (4.18).

From (4.27) it follows, (see (4.22), (4.23) and (4.26)) that

$$g_0(t) = -R_k \varepsilon t + i(A - 2E)^{-1} C_k \quad on \quad L_k, \ k = \overline{1, p},$$

further since $\varphi^*(z) = \psi^*(z) = 0$ in G_{p+1}, therefore

$$g_0(t) = 0 \quad on \quad L_{p+1}.$$

Finally, note that from (4.9), (4.26), (4.7) and (4.16) it follows that $R_k = C_k = 0$ for every k, hence $g_0(t) = 0$ on L.

Thus, we proved that the homogeneous equation correspond to equation (4.8) has no solution different from zero.

Therefore equation (4.8) has one and only one solution $g = (g_1, g_2)^T$. Further note that $g \in C^{o,\alpha}(L)$.

On substituting value $g = (g_1, g_2)^T$ info formula (4.3) and (4.4) we find the analytic vector-functions $\varphi(z)$ and $\psi(z)$.

Having found the vector-functions $\varphi(z)$ and $\psi(z)$ by virtue of (2.3) we obtain a solution of the second boundary value problem provided that the requirement for the principal vector and the principal moment of external forces to be equal to zero is fulfilled. Displacement U is defined to within rigind displacement, while stresses are defined precisely.

The existence of solution of the second boundary value problem can also be proved when domain G is an infinite multiply-connected domain.

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ON THE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE ON FINDING EQUISTRONG HOLES IN A SQUARE

Svanadze K.

Abstract. In the present work we consider one inverse problem of statics in the linear theory of elastic mixture for a square which is weakened by four unknown equal holes, whose boundaries are free from external forces, and the sides of the square are under the action of absolutely rigid punches of rectilinear base.

Unknown boundaries of the holes are found under the condition that tangential normal stress takes on them one and the same constant value.

Keywords and phrases: Elastic mixture, equistrong holes, Keldish-Sedov and Riemann-Hilbert problems, Kolosov-Muskhelishvili type formulas.

AMS subject classification (2010): 74B05.

1^0 The homogeneous equation of statics of the linear theory of elastic mixture in the complex form is written as [1]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + K \frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0, \qquad U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1}$$

where $u_p, \quad p = \overline{1,4}$ are components of the displacement vector,

$$z = x_1 + ix_2, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad K = -\frac{1}{2}em^{-1},$$

$$e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1}. \quad m_k = e_k + \frac{1}{2}e_{3+k},$$

the $e_q, \ q = \overline{1,6}$ are expressed in terms of the elastic mixture [1].

In [1] M. Basheleishvili obtained the representations:

$$2\mu U = 2\mu(u_1 + iu_2, u_3 + iu_4)^T = A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}, \tag{2}$$

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \begin{pmatrix} r'_{12}n_1 + r'_{22}n_2 - i(r'_{11}n_1 + r'_{21}n_2) \\ r''_{12}n_1 + r''_{22}n_2 - i(r''_{11}n_1 + r''_{21}n_2) \end{pmatrix}$$

$$= \frac{\partial}{\partial s(x)}((A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}), \tag{3}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $(TU)_p$, $p = \overline{1,4}$, are the components of stress vector,

$$\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}, \qquad n = (n_1, n_2)^T \qquad is \quad unit \quad vector$$

$$A = 2\mu m, \qquad \mu = \begin{bmatrix} \mu_1\mu_3 \\ \mu_3\mu_2 \end{bmatrix}, \quad B = \mu e, \quad E = \begin{bmatrix} 10 \\ 01 \end{bmatrix},$$ μ_1, μ_2 and μ_3 are elastic constants [1].

Let us now consider the vectors:

$$U_n = (u_1 n_1 + u_2 n_2; u_3 n_1 + u_4 n_2)^T, \quad U_s = (u_2 n_1 - u_1 n_2; u_4 n_1 - u_3 n_2)^T,$$

$$\sigma_n = \begin{pmatrix} (TU)_1 n_1 + (TU)_2 n_2 \\ (TU)_3 n_1 + (TU)_4 n_2 \end{pmatrix}, \sigma_s = \begin{pmatrix} (TU)_2 n_1 - (TU)_1 n_2 \\ (TU)_4 n_1 - (TU)_3 n_2 \end{pmatrix},$$

$$\sigma_t = \begin{pmatrix} [r'_{21} n_1 - r'_{11} n_2; r'_{22} n_1 - r'_{12} n_2]^T s \\ \left[r''_{21} n_1 - r''_{11} n_2; r''_{22} n_1 - r''_{12} n_2\right]^T s \end{pmatrix}. \tag{4}$$

Here $n = (n_1, n_2)^T = (\cos\alpha \quad sin\alpha)^T$, $\quad s = (-n_2, n_1)^T = (-\sin\alpha, \cos\alpha))^T$, and $\alpha(t)$ is an angle between the outer normal to the contour L of the point t and ox_1 axis. Let us call the vector (4) tangential normal stress vector in the linear theory of elastic mixture.

Elementary calculations result in [4]

$$\sigma_n + \sigma_t = (2E - A - B)\ Re\varphi'(t), \quad t \in L, \tag{5}$$

$$\sigma_n + 2\mu\left(\frac{\partial U_s}{\partial s} + \frac{U_n}{\rho_0}\right) + i\left[\sigma_s - 2\mu\left(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0}\right)\right] = 2\varphi'(t) \ \ t \in L, \tag{6}$$

$$[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_L = -i\int\limits_L e^{i\alpha}(\sigma_n + i\sigma_s)ds, \tag{7}$$

where $\dfrac{1}{\varrho_0}$ is the curvature of the curve L at the point t.

2^0 in the work, in the case of the linear theory of elastic mixtures we study the problem analogous to that solved in [2]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili formula and the method developed in [2] and [4].

Let an isotropic elastic mixture occupy on the plane $z = x_1 + ix_2$ a multiply connected domain G, which is square with vertices lying on the coordinate axes weakened by four unknown equal holes. The holes are intersected by the square diagonals and are symmetric both with respect to these diagonals and to the straight lines connecting middle points of the opposite square sides. The boundaries of the holes are assumed to be free from external loads, the square sides are under the action of absolutely rigid punches of rectilinear base, and concentrated forces $P = (p_1, p_2)^T$ are applied to the middle points of the punches.

Assume that the vector σ_s is equal to zero on the entire boundary G, also $\sigma_n = 0$ on the unknown part of the boundary G. Further note that the vector U_n takes on sides square constant value. Suppose also that the surfaces of the bodies are assumed to be absolutely smooth, and hence the frictional force will be neglected.

The problem is formulated as follows: Find unknown holes and stressed state of the square under the condition that the tangential normal stress σ_t at the hole boundaries takes constant value. Let $\sigma_t = -K^0, \quad K^0 = (K_1^0, K_2^0) = const.$

Since the problem is axially symmetric, we consider a curvilinear pentagon $A_1A_2A_3$ A_4A_5 (Figure 1).

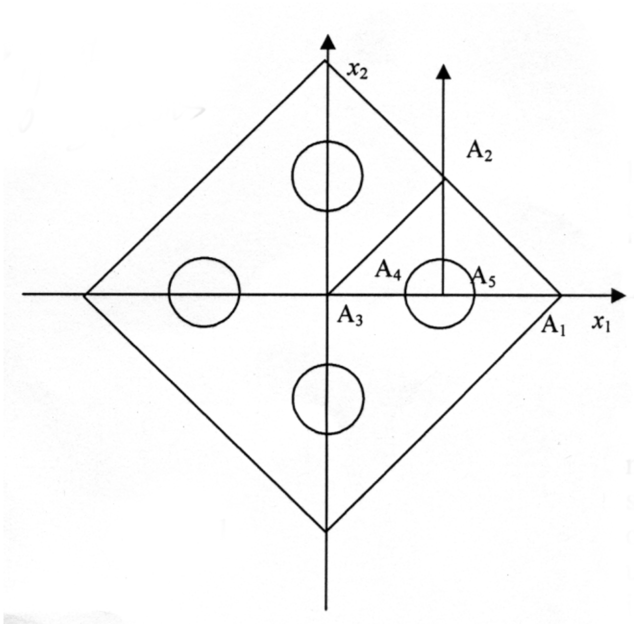

Figure 1:

Introduce the notation $A_kA_{k+1} = \Gamma_k, \quad k = 1, 2, 3, \quad \Gamma_4 = A_5A_1, \quad \Gamma = \bigcup_{k=1}^{4} \Gamma_k$. Let us denote the arc A_4A_5 by Γ_5 and the domain occupied by the curvilinear pentagon by D. Let $2d^0$ be the square diagonal.

On the basis of analogous Kolosov-Muskhelishvilis formulas (5)-(7) our problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in D by the boundary condtions:

$$Re\varphi'(t) = \frac{1}{2}(A + B - 2E)^{-1}K^0, \quad t \in \Gamma_5, \quad Im\varphi'(t) = 0, \quad t \in \Gamma, \tag{8}$$

$$(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = q^0, \ \ t \in \Gamma_5, \quad q_0 = const, \tag{9}$$

$$Ree^{-i\alpha(t)}[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}] = C(t), \quad t \in \Gamma, \tag{10}$$

where $\alpha(t)$ is the size of the angle made by the normal and the ox_1 axis,

$$C(t) = \int_{A_1}^{t} \sigma_n(t_0)\sin(\alpha(t_0) - \alpha(t))ds_0, \quad t \in \Gamma, \quad If \qquad t \in \Gamma_j,$$

then

$$C(t) = 0, \quad t \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4, \qquad C(t) = \frac{1}{2}P, \quad t \in \Gamma_2.$$

The conditions (8) are the vector-form of the Keldysh-Sedov problem for the domain D. It is proved that

$$\varphi(z) = \frac{1}{2}(A + B - 2E)^{-1}K^0 z + (A - 2E)^{-1}l^0,$$

$$z \in D,\ l^0 = const,\ Iml^0 = 0. \tag{11}$$

$$If \quad t \in \Gamma_k, \quad k = \overline{1,4}, \quad then \quad Re(e^{-i\alpha_k}t) = Re(e^{-i\alpha_k}A_k), \quad t \in \Gamma_k, \quad k = \overline{1,4},$$

$$\alpha_1 = \frac{\pi}{4}, \quad \alpha_2 = \frac{3}{4}\pi, \quad \alpha_3 = \alpha_4 = \frac{3}{2}\pi.$$

Taking into accound equality (11), we can rewrite the boundary conditions (9) and (10) as follows:

$$\frac{1}{2}K^0 t + 2\mu\overline{\psi(t)} = q^0 - l^0, \quad t \in \Gamma_5,$$

$$2\mu Re(e^{-i\alpha(t)}\overline{\psi(t)}) = -\begin{cases} Ree^{-i\alpha(t)}\left(\frac{1}{2}K^0 t + l^0\right), t \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4, \\ \\ Ree^{-i\alpha(t)}\left(\frac{1}{2}K^0 t + l^0\right) - \frac{1}{2}P, t \in \Gamma_2. \end{cases} \tag{12}$$

Further note that

$$Re(e^{-i\alpha(t)}t) = \frac{\sqrt{2}}{2}d^0, t \in \Gamma_1, Re(e^{-i\alpha(t)}t) = 0, t \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{13}$$

Let the function $z = w(\zeta), \zeta = \xi_1 + i\xi_2$ map conformaly domain D onto semi-circle $|\zeta| < 1, Im\zeta > 0.$ In addition, we may assume that the arc $A_4A_5$is mapped onto the diameter $(-1,1); A_4 \to \beta_4 = -1, A_5 \to \beta_5 = 1, A_2 \to \beta_2 = i.$ We map two points A_1 and A_3 onto the unknown points β_1 and β_3.

If we introduce

$$W(\zeta) = \begin{cases} \frac{1}{2}K^0 w(\zeta), |\zeta| < 1, Im\zeta > 0, \\ \\ -2\mu\overline{\psi_0(\overline{\zeta})} + q^0 - l^0, |\zeta| < 1, I_m\zeta < 0, \psi_0(\zeta) = \psi(w(\zeta)), \end{cases} \tag{14}$$

then the boundary value problems (12)-(13) (see [2]) are reduced to the Riemann-Hilbert problem for the circle $|\zeta| < 1$

$$Re\left((e^{-i\alpha(\sigma)}W(\sigma)) = f(\sigma), \sigma \in \gamma, Re(e^{-i\alpha(\sigma)}W(\sigma))\right) = f^0(\sigma), \sigma \in \gamma^0, \tag{15}$$

where $\gamma=\bigcup\limits_{k=1}^{4}\gamma_k, \gamma_k=\omega^{-1}(\Gamma_k), k=\overline{1,4}$ and γ^0 is the mirror image of γ with respect to the diameter (-1,1).

A solution of the problem (15) can be represented in the form [3] and [2]

$$W(\zeta)=\frac{\aleph(\zeta)}{2\pi i}\int_{\gamma\bigcup\gamma^0}\frac{\zeta+\sigma}{\sigma-\zeta}\,\frac{F(\sigma)}{\sigma\aleph(\sigma)}d\sigma, F(\sigma)=\begin{cases} f(\sigma), \sigma\in\gamma,\\ f^0(\sigma), \sigma\in\gamma^0.\end{cases}$$

$$\aleph(\zeta)=\exp\left(\frac{1}{4\pi i}\int\limits_{\gamma\bigcup\gamma^0}\frac{\zeta+\sigma}{\sigma-\zeta}\,\frac{2i\alpha(\sigma)d\sigma}{\sigma}\right)=\frac{\aleph_1(\zeta)}{\sqrt{\aleph_1(0)}},$$

$$\aleph_1(\zeta)=\sqrt[4]{\frac{\zeta-\beta_2}{\zeta-\beta_1}\left(\frac{\zeta-\beta_3}{\zeta-\beta_2}\right)^3\left(\frac{\zeta-\overline{\beta_3}}{\zeta-\beta_3}\right)^2\left(\frac{\zeta-\overline{\beta_2}}{\zeta-\overline{\beta_3}}\right)^3\frac{\zeta-\overline{\beta_1}}{\zeta-\overline{\beta_2}}\left(\frac{\zeta-\beta_1}{\zeta-\overline{\beta_1}}\right)^2}.$$

Having known $W(\zeta)$ we can define $\psi_0(\zeta)$ and $\omega(\zeta)$ by (14) and the stressed state of the body and the boundaries of unknown holes.

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ი. ვეკუას სახელობის გამოყენებითი
მათემატიკის ინსტიტუტის სემინარის
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თერმოდრეკადობის ბრტყელი თეორიის მდგრადი რხევის განტოლებათა სისტემის ზოგიერთი ამონახსნის შესახებ მიკროტემპერატურის გათვალისწინებით

ლ. ბიწაძე

მიღებულია ზოგადი ამონახსნის წარმოდგენის ფორმულა. კვადრატურებში აგებულია ფუნდამენტურ და სინგულარულ ამონახსნთა მატრიცები.

ორი პოლიტიკური სუბიექტის ამომრჩეველთა დინამიკის არაწრფივი მათემატიკური მოდელი

თ. ჩილაჩავა

შემოთავაზებულია არაწრფივი მათემატიკური მოდელი, რომელიც აღწერს სახელისუფლებო და ოპოზიციური პარტიების (ორი საარჩევნო სუბიექტი, კოალიცია) ამომრჩეველთა დინამიკას. მოდელში განიხილება სამი ობიექტი: სახელისუფლებო და ადმინისტრაციული სტრუქტურები, რომლებიც ადმინის-ტრაციული რესურსების მეშვეობით ზემოქმედებენ ამომრჩეველზე (უპირველეს ყოვლისა ოპოზიციურად განწყობილზე) რათა გადაიბირონ ისინი სახელი-სუფლებო პარტიის მხარეზე; ამომრჩევლები, რომლებიც მოცემულ მომენტში სახელისუფლებო პარტიის მხარდამჭერია; ამომრჩევლები, რომლებიც მოცემულ მომენტში ოპოზიციური პარტიის მხარდამჭერია. მუდმივი და ცვლადი (ოპოზიციური პარტიის ამომრჩეველთა რაოდენობის პროპორციულად) ადმინისტრაციული რესურსების გამოყენების შემთხვევაში კოშის ამოცანა არაწრფივი დიფერენციალურ განტოლებათა სისტემისათვის ამოხსნილია ანალიზურად ზუსტად. ნაპოვნია პირობები მოდელის კონსტანტებზე, როდესაც ოპოზიციური პარტია (კოალიცია) მოიგებს მორიგ არჩევნებს. მათემატიკურ მოდელს თეორიული ინტერესის გარდა გააჩნია პრაქტიკული მნიშვნელობა, რადგანაც ორივე მხარეს (სახელმწიფო სტრუქტურები სახელმწიფო პარტიასთან ერთად; ოპოზიციური პარტია) შეუძლია გამოიყენონ შედეგები თავისი მიზნების შესაბამისად. მოდელის შედეგები აძლევს საშუალებას მხარეებს,

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ოპტიმალური მართვის ერთი კლასის განაწილებულ დაგვიანების შემცველი ამოცანის კორექტულობის შესახებ

ფ. დვალიშვილი

მოყვანილია თეორემა მართვის მიმართ წრფივი ოპტიმალური ამოცანის კორექტულობის შესახებ, როცა განტოლების მარჯვენა მხარისა და ინტეგრანდის შეშფოთებები მცირეა ინტეგრალური აზრით .

პროგნოზირების მეთოდები ეკონომიკასა და ფინანსებში

ა. გაბელაია

განხილულია პროგნოზირების პრობლემა ეკონომიკასა და ფინანსებში. მოცემულია ეკონომიკური პროგნოზირების მეთოდების კლასიფიკაცია. ჩამოყა-ლიბებულია „აუცილებელი (ანუ შესაბამისი) სირთულის პრინციპი“ და ნაჩვენებია პროგნოზირების მოდელების პრაქტიკული გამოყენების შესაძლებ-ლობები თანამედროვე კომპიუტერული პროგრამის ბაზაზე, საქართველოს ეკონომიკის მაგალითზე.

თერმოდრეკადობის თეორიის ნეიმანის ტიპის შიგა სასაზღვრო ამოცანის გამოკვლევა ჰემიტროპული სხეულებისათვის

დ. ივანიძე, დ. ნატროშვილი

ჰემიტროპული სხეულებისათვის, პოტენციალთა მეთოდისა და სინგულარული ინტეგრალური განტოლებების თეორიის გამოყენებით, გამოკვლეულია თერმოდრეკადობის თეორიის ნეიმანის ტიპის შიგა სასაზღვრო ამოცანის ამონახსნის ერთადერთობისა და არსებობის საკითხი. გაანალიზებულია სასაზღვრო ინტეგრალური ოპერატორის თვისებები, ცხადი სახითაა ამოწერილი შესაბამისი ნულ სივრცეები და მიღებული შედეგების საფუძველზე დადგენილია ნეიმანის ტიპის შიგა სასაზღვრო ამოცანის ამოხსნადობის აუცილებელი და საკმარისი პირობები. ნაჩვენებია, რომ ამონახსნები წარმოდგენადია მარტივი ფენის პოტენციალის სახით.

დრეკად ნარევთა ბრტყელი თეორიის სტატიკის ძირითადი სასაზღვრო ამოცანების ამოხსნა მრავლადბმულ არეში დ. შერმანის მეთოდით

კ. სვანაძე

განზოგადებული კოლოსოვ-მუსხელიშვილის ფორმულებისა და დ. შერმანის მეთოდის გამოყენებით, სასრულ მრავლადბმულ არეს შემთხვევაში, ამოხსნილია დრეკად ნარევთა წრფივი თეორიის სტატიკის ძირითადი ბრტყელი სასაზღვრო ამოცანები, როცა საზღვარზე მოცემულია გადაადგილების ვექტორი (პირველი ამოცანა) და ძაბვის ვექტორი (მეორე ამოცანა).

დრეკად ნარევთა თეორიის სტატიკის ამოცანა კვადრატში თანაბრადმტკიცე ხვრელების მოძებნის შესახებ

კ. სვანაძე

გამოკვლეულია დრეკად ნარევთა წრფივი თეორიის სტატიკის ერთი შებრუნებული ამოცანა, ოთხი უცნობი ტოლი სიდიდის ხვრელით შესუსტებული დრეკადი კვადრატისათვის, როდესაც ხვრელების საზღვარი თავისუფალია გარე დატვირთვისაგან, ხოლო კვადრატის გვერდებზე მოქმედებს სწორფუძიანი აბსოლუტურად ხისტი შტამპები. შტამპების შუა წერტილებზე მოდებულია შეყურსული ძალები. მოძებნილია ხვრელების უცნობი საზღვარი იმ პირობით, რომ მასზე ტანგეციალური ნორმალური ძაბვა ღებულობს ერთი და იგივე მუდმივ მნიშვნელობას.

კოშის ამოცანის კორექტულობის შესახებ ერთი კლასის ნეიტრალური ფუნქციონალურ დიფერენციალური განტო-ლებისთვის დაგვიანების შეშფოთების გათვალისწინებით

ნ. გორგოძე

კვაზი წრფივი ნეიტრალური ფუნქციონალურ დიფერენციალური განტოლე-ბისთვის მოყვანილია თეორემები ამონახსნის საწყისი მონაცემებისა და განტოლების მარჯვენა მხარის შეშფოთებებზე უწყვეტად დამოკიდებულების შესახებ. საწყისი მონაცემების ქვეშ იგულისხმება საწყისი მომენტის, ფაზურ კოორდინატებში შემავალი ცვლადი დაგვიანების, საწყისი ვექტორისა და საწყისი ფუნქციების ერთობლიობა.

CONTENTS