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ON LINEAR BOUNDARY VALUE PROBLEMS FOR MULTIDIMENSIONAL REGULAR DIFFERENCE SYSTEMS

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Abstract. The Green's type theorems are established for unique solvability of linear boundary value problems for multidimensional systems of linear regular difference equations. Moreover, a successive approximation method is given for the construction of the solution of the difference system under the Cauchy condition.

Keywords and phrases: Linear systems of regular difference equations, linear boundary value problems, unique solvability, the Green's type theorem, generalized ordinary differential equations.

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1. Statement of the problem and formulation of the results

This work is dedicated to the investigation of the solvability question of the regular difference system

$$\Delta y(k-1) = G_1(k)y(k-1) + G_2(k)y(k) + g(k) \quad (k = 1, 2, ...) \tag{1.1}$$

under the general boundary value problem

$$\mathcal{L}(y) \equiv \sum_{i=1}^{\infty} L(k)y(k) = c_0, \tag{1.2}$$

where $G_j \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$, $\mathcal{L} : BV_{\cup}(\mathbb{N}_0\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator, and $g \in E(\mathbb{N}_0, \mathbb{R}^n)$ are respectively, discrete matrix and vector functions, and $c_0 \in \mathbb{R}^n$. In this work the Green's type theorem is proved for the unique solvability of the problem (1.1),(1.2) in the case when $G_j \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ and $g(k) \in E(\mathbb{N}_0, \mathbb{R}^n)$ are, respectively, so called regular matrix and vector functions on the set $\mathbb{N}_0$(see below). Moreover, successive approximations methods is investigated for constructing the solution for the Cauchy problem for the system (1.1). For investigating this problem we use the theory of so called generalized ordinary differential equations [1]. Analogous questions for the finite difference system are investigated in [1,2].

Along with the problem (1.1),(1.2) we consider the corresponding homogeneous problem

$$\Delta y(k-1) = G_1(k)y(k-1) + G_2(k)y(k) \quad (k = 1, 2, ...), \tag{1.1$_0$}$$

$$\mathcal{L}(y) = 0. \tag{1.2$_0$}$$

Throughout the paper, the following notation and definitions will be used.

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$.

$\mathbb{R} =]-\infty, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in$ are, respectively, a closed and an open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ – matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max\left\{\sum_{i=1}^{n} |x_{ij}| : j = 1, \dots, m\right\}.$$

If $X = (x_{ij})_{i,j=1}^{n,m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$O_{n\times m}$ is the zero $n \times m$-matrix.

$\mathbb{R}_{+}^{n\times m} = \{(x_{i,j})_{i,j=1}^{n,m} : x_{i,j} \geq 0 \quad (i = 1, \dots, n; j = 1, \dots, m)\}$.

$\mathbb{R}^{n} = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_{+}^{n} = \mathbb{R}_{+}^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} is the matrix, inverse to X; $\det X$ is the determinant of X; and $r(X)$ is the spectral radius of X.

I_n is the identity $n \times n$-matrix.

$E(\mathbb{N}_0, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $Y : \mathbb{N}_0 \to \mathbb{R}^{n\times m}$.

Δ is the difference operator of the first order, i.e.,

$$\Delta Y(k-1) = Y(k) - Y(k-1) \ \text{ for } \ Y \in E(\mathbb{N}_0, \mathbb{R}^{n\times m}) \ \ (k = 1, 2, ...).$$

We say that the discrete matrix function $X \in E(\mathbb{N}_0, \mathbb{R}^{l\times m})$ has the bounded total variation on the set $\mathbb{N}_0$ if

$$\sum_{k=1}^{\infty} \|\Delta X(k-1)\| < +\infty.$$

In this case we assume

$$\|X\|_v = \sum_{k=1}^{\infty} \|\Delta X(k-1)\|.$$

By $BV_v(\mathbb{N}_0; \mathbb{R}^{n\times m})$ we denote the Banach space of all discrete matrix-functions $E(\mathbb{N}_0, \mathbb{R}^{n\times m})$ with the norm $\|.\|_v$.

The inequalities between the matrices are understood componentwise.

A matrix function is said to be continuous, integrable, nondecreasing, etc., if such is every its component.

Under a solution of the difference problem (1.1),(1.2) we understand a matrix function $y \in BV_v(\mathbb{E}_0, \mathbb{R}^n)$ satisfying difference system (1.1) (i.e., the equality (1.1) for every $k \in \mathbb{N}$) and the boundary condition (1.2).

Below we show that, in the regular case, i.e., when discrete matrix G_1 and G_2 and vector g functions are regular, every discrete vector-function $y \in E(\mathbb{N}_0, \mathbb{R}^n)$ satisfying difference system (1.2) belongs to $BV_v(\mathbb{E}_0, \mathbb{R}^n)$, as well. So that the definition of solutions of system (1.1) given above, is natural for the regular case.

The discrete matrix-function $X \in E(\mathbb{N}_0, \mathbb{R}^{n\times m})$ is said to be regular if

$$\sum_{k=1}^{\infty} \|X(k)\| < +\infty.$$

Definition 1.1. The system (1.1) is called regular if the matrix-and vector functions G_1, G_2 and g are regular, i.e., (1.3)

$$\sum_{k=1}^{\infty} \|G_j(k)\| < +\infty \quad (j = 1, 2) \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} \|g(k)\| < +\infty. \tag{1.4}$$

We will assume that system (1.1) is regular. Moreover, we assume that the matrix function $L \in E(\mathbb{N}_0, \mathbb{R}^{n\times n})$ is regular, too.

Let Y be the fundamental matrix of the system (1.1_0) under the condition

$$Y(0) = I_n.$$

If the condition

$$\det\big(I_n + (-1)^j G_j(k)\big) \neq 0 \ \text{ for } \ k \in \{1, 2, ...\} \ \ (j = 1, 2) \tag{1.5}$$

is valid, then the fundamental matrix Y of the system (1.1_0) exists and

$$Y(k) = \prod_{l=k}^{0} \big(I_n - G_1(l)\big)^{-1}\big(I_n + G_2(l)\big) \ \text{ for } \ k \in \{1, 2, ...\}. \tag{1.6}$$

We assume

$$D = \sum_{l=0}^{\infty} L(l)Y(l) \ \text{ and } \ D(j) = \sum_{l=0}^{j} L(l)Y(l) \ \ (j = 0, 1, ...). \tag{1.7}$$

If

$$\det D \neq 0, \tag{1.8}$$

then we assume

$$\mathcal{G}(k,j) = \begin{cases} Y(k)D^{-1}D(j-1)Y^{-1}(j)\big(I_n - G_1(j)\big)^{-1} \ \text{ for } \ 0 \le j < k, \\ -Y(k)\big(I_n - D^{-1}D(j-1)\big)Y^{-1}(j)\big(I_n - G_1(j)\big)^{-1} \ \text{ for } \ 0 \le k < j, \\ O_{n\times n} \ \text{ for } \ k = j, \end{cases} \tag{1.9}$$

where $Y(k)$ is the fundamental matrix of the system (1.1_0) defined by (1.6). The matrix function $\mathcal{G}(k,j)$ is called the Green matrix of the problem (1.1_0),(1.2_0).

Theorem 1.1. *Let the condition (1.5) hold and let the system (1.1) be regular. Then the boundary value problem (1.1),(1.2) has a unique solution if and only if the corresponding homogeneous problem* $(1.1_0), (1.2_0)$ *has only the trivial solution. If the letter condition holds, then the solution y of problem (1.1),(1.2) admits the representation*

$$y(k) = Y(k)D^{-1}c_0 + \sum_{l=1}^{\infty} \mathcal{G}(k,l)g(l) \ \textit{ for } \ k \in \mathbb{N}_0, \tag{1.10}$$

where $\mathcal{G}(k,l)$ is the Green matrix of the problem $(1.1_0),(1.2_0)$.

Remark 1.1. We note the homogeneous problem $(1.1_0),(1.2_0)$ has only the trivial solution (as well problem (1.1),(1.2) is uniquely solvable) if and only if the condition (1.8) is valid. Therefore, there exist the Green matrix appearing in Theorem 1.1.

Remark 1.2. If the condition (1.8) is not fulfilled, then for every regular $g \in E(\mathbb{N}_0,\mathbb{R}^n)$ there exists a vector $c_0 \in \mathbb{R}^n$ such that problem (1.1),(1.2) has no solution. In addition, if $\mathcal{L} : E(\mathbb{N}_0,\mathbb{R}^n) \to \mathbb{R}^n)$ is the onto mapping, then for every $c_0 \in \mathbb{R}^n$ there exists a regular function $g \in E(\mathbb{N}_0,\mathbb{R}^n)$ such that the problem (1.1),(1.2) is not solvable.

We give a successive approximation method of construction of the solution of the system (1.1), too, under the Cauchy condition

$$y(k_0) = c_0, \tag{1.11}$$

where $k_0 \in \mathbb{N}$, $c_0 \in \mathbb{R}^n$.

Theorem 1.2 *Let*

$$\det\big(I_n + (-1)^j G_j(k)\big) \neq 0 \ \ \textit{for} \ \ (-1)^j(k-k_0) < 0 \ \ (j=1,2). \tag{1.12}$$

Then the Cauchy problem (1.1),(1.11) has a unique solution $y \in E(\mathbb{N},\mathbb{R}^n)$ and

$$\lim_{m\to\infty} y_m(k) = y(k) \ \ \textit{uniformly for} \ \ k \in \mathbb{N}_0, \tag{1.13}$$

where

$$y_m(k_0) = c_0 \ \ (m = 0,1,...),$$
$$y_0(k) = \big(I_n + (-1)^j G_j(k+j-1)\big)^{-1} c_0 \ \ \textit{for} \ \ (-1)^j(k-k_0) < 0 \ \ (j=1,2)$$

and

$$y_m(k) = \big(I_n + (-1)^j G_j(k+j-1)\big)^{-1} \Bigg[c_0 + (-1)^j G_j(k+j-1) y_{m-1}(k)$$
$$-(-1)^j \sum_{l=k_0+1+(j-1)(k-k_0)}^{k-(j-1)(k-k_0)} \big(G_1(l) y_{m-1}(l) + G_2(l) y_{m-1}(l-1)\big)\Bigg]$$
$$\textit{for} \ \ (-1)^j(k-k_0) < 0 \ \ (j=1,2).$$

2. Generalized differential equations

We give some necessary definition to formulate bases of the theory of the generalized ordinary differential equations.

The interest in the theory of generalized ordinary differential equations has also been stimulated to a considerable extent by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g. [1-10] and the references therein).

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $V_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i = 1,...,n; j = 1,...,m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = V_a^t(x_{ij})$ for $a < t \le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-) = X(a)$, $X(b+) = X(b)$).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the Banach space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $V_a^b(X) < \infty$) with the norm $\|X\|_v = \|X(a)\| + V_a^b(X)$.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$, corresponding to the function $s_0(g)$.

If $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$, and if $a > b$, then we assume $\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t)$.

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0,1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\right)_{i,j=1}^{l,m}.$$

Let $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a,b],\mathbb{R}^n)$.

Under a solution of the system of linear generalized ordinary differential equations

$$dx(t) = dA(t)\cdot x(t) + df(t) \tag{2.1}$$

we understand a vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \ \text{ for } \ a \le s < t \le b.$$

We consider system (2.1) with the boundary value condition

$$\ell(x) = c, \tag{2.2}$$

where $\ell : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n)$ is a linear bounded operator, and $c \in \mathbb{R}^n$ is a constant vector.

The question of the unique solvability of the generalized boundary value problem (2.1),(2.2) is investigated in [1,2,10] (see also the references therein).

3. Proof of the results

We will rewrite problem (1.1),(1.2) in the form of problem (2.1),(2.2) in order to apply the results from [1,2,10] to the last generalized problem.

Let Y be the fundamental matrix of system (1.1) under the condition $Y(0) = I_n$. Then by (1.3) and (1.6) there exists a positive number $r > 0$ such that

$$\|Y(k)\| < r \ \text{ for } \ k \in \mathbb{N}_0. \tag{3.1}$$

We assume

$$G_j(0) = O_{n\times n} \ \ (j = 1,2), \ \ g(0) = 0_n.$$

Let $y \in E(\mathbb{N}_0,\mathbb{R}^n)$ be an arbitrary solution of the problem (1.1),(1.2) and let $z = (z_i)_{i=1}^2$, where $z_i \in E(\mathbb{N}_0,\mathbb{R}^n)(i = 1,2)$ be functions, defined by

$$z_1(k) = z_2(k) = y(k) \ \ (k = 0,1,...).$$

Then by (3.1) we get

$$\|y(k)\| < r\|y(0)\| \ \text{ for } \ k \in \mathbb{N}_0.$$

From this by (1.1),(1.3) and (1.4) we have

$$\sum_{k=0}^{\infty} \|\Delta y(k-1)\| < +\infty$$

and

$$\sum_{k=0}^{\infty} \|\Delta z(k-1)\| < +\infty. \tag{3.2}$$

Moreover, it is evident that

$$\Delta \begin{pmatrix} z_1(k-1) \\ z_2(k-1) \end{pmatrix} = \begin{pmatrix} G_1(k)z_1(k) + G_2(k)z_2(k-1) + g(k) \\ G_1(k)z_1(k) + G_2(k)z_2(k-1) + g(k) \end{pmatrix} \ \text{ for } \ k \in \mathbb{N}_0 \tag{3.3}$$

and

$$\zeta_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \end{pmatrix}, \ \ \zeta_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.4}$$

where

$$\zeta_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{L}(z_1) \\ 0 \end{pmatrix},$$

and $\zeta_2 : E(\mathbb{N}_0, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ is an arbitrary operator such that the condition $\zeta_2(z) = 0$ guarantees the equality $z_1(k_0) = z_2(k_0)$ for some $k_0 \in \mathbb{N}_0$.

We will assume that

$$\zeta_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2(k_0) - z_1(k_0) \\ z_2(k_0) - z_1(k_0) \end{pmatrix},$$

where k_0 is an arbitrary fixed integer from $\mathbb{N}_0$.

The contrary is evident too. If the vector-function $z = (z_i)_{i=1}^2$ is a solution of problem (3.1),(3.2) then $z_1(k) \equiv z_2(k)$ and this discrete vector function is a solution of problem (1.1),(1.2). Therefore, problems (1.1),(1.2) and (3.3),(3.4) are equivalent among themselves.

We note that by (1.3) there exists $k_0 \in \mathbb{N}$ such that $\|G_j(k_0)\| < 1/2$ $(j = 1, 2)$ and, therefore, the inverse matrices $\big(I_n + (-1)^j G_j(k)\big)^{-1}(j = 1, 2)$ exist for $k \geq k_0$. From this, taking into account the condition (1.3) we get that there exists a constant $r_1 > 0$ such that

$$\|\big(I_n + (-1)^j G_j(k)\big)^{-1}\| < r_1 \text{ for } k \geq k_0 \ (j = 1, 2). \tag{3.5}$$

Let now

$$I_{1k} = [t_k, t_{k+1}[\ \ \textit{and} \ \ I_{1k} =]t_k, t_{k+1}] \text{ for } k \in \mathbb{N}_0,$$

where $t_k = k/(k + 1)$ $(k = 0, 1, ...)$.

Let $x = (x_i)_{i=1}^2$ be a vector function defined by

$$x_i(t) = z_i(k) \text{ for } t \in I_{ik} \ (i = 1, 2; k = 0, 1, ...). \tag{3.6}$$

Then by (3.2) we have $x \in \mathrm{BV}([0, 1], \mathbb{R}^{2n})$.

It is not difficult to verify that the vector function x will be a solution of the 2n-dimension problem (2.1),(2.2) with $a = 0, b = 1$,

$$A(t) \equiv (A_{ij}(t))_{i,j=1}^2, \tag{3.7}$$

$$A_{ij}(t) = \sum_{l=0}^{k} G_j(l) \text{ for } t \in I_{ik} \ (i, j = 1, 2; k = 0, 1, ...); \tag{3.8}$$

$$f(t) \equiv (f_i(t))_{i=1}^2, \tag{3.9}$$

$$f_i(t) = \sum_{l=0}^{k} g(l) \text{ for } t \in I_{ik} \ (i, j = 1, 2; k = 0, 1, ...); \tag{3.10}$$

$$\ell(x) = \big(\zeta_i(z)\big)_{i=1}^2 \text{ for } x = (x_i)_{i=1}^2, \ x_i \in \mathrm{BV}([0, 1], \mathbb{R}^n), \ (i = 1, 2) \tag{3.11}$$

and

$$c = \begin{pmatrix} c_0 \\ 0 \end{pmatrix}.$$

It is evident that the inverse proposition is true as well. So that the following lemma is true.

Lemma 1.1 *Let* $y \in E(\mathbb{N}_0, \mathbb{R}^n)$ *be a solution of problem (1.1),(1.2). Then the vector function* $x = (x_i)_{i=1}^n \, \mathrm{BV}([0,1], \mathbb{R}^n)$*, defined by (3.6), will be a solution of the* $2n$*-dimensional generalized boundary value problem (2.1),(3.2), where* $a = 0, b = 1$*, and matrix-and vector functions* A *and* f*, linear operator* ℓ *and constant vector* c *are defined, respectively, by (3.7)-(3.11). On the contrary, if the vector-function* $x = (x)_{i=1}^n \in BV([0,1], \mathbb{R}^{2n})$ *is a solution of the last 2n-dimensional problem (2.1),(3.2), then the vector-function* $y \in E(\mathbb{N}_0, \mathbb{R}^n)$*,* $y(k) \equiv z_1(k)$*, will be a solution of the problem (1.1),(1.2), where*

$$G_i(k) \equiv \Delta A_{1i}(k) \ \ (i = 1, 2), \ \ g(k) \equiv \Delta f_1(k),$$

and $\mathcal{L}(y)$ *and* c_0 *are* n*-vectors whose* i*-th component coincides with* i*-th component of* $\ell(y)$ *and* c*, respectively, for every* $i \in \{1, ..., n\}$*.*

Using the lemma we conclude that the theorems and remarks immediately follow from corresponding results of paper [1,2,10].

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EFFECTIVE SOLUTION OF ONE BOUNDARY VALUE PROBLEM OF THE LINEAR THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR A SPHERICAL RING

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Abstract. In this paper the expansion of regular solution for the equations of the theory of thermoelasticity with microtemperatures is obtained, that we use for explicitly solving one basic boundary value problem (BVP) of the linear equilibrium theory of thermoelasticity with microtemperatures for the spherical ring. The obtained solutions are represented as absolutely and uniformly convergent series.

Keywords and phrases: Thermoelasticity with microtemperatures, absolutely and uniformly convergent series, spherical harmonic.

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Introduction

The linear theory of thermoelasticity for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was constructed by Iesan and Quintanilla [1]. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze [2]. The representations of the Galerkin type and general solutions of the system in this theory were obtained by Scalia, Svanadze and Tracinà [3]. The 3D linear theory of thermoelasticity for microstretch elastic materials with microtemperatures was constructed by Iesan [4] where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved.

The purpose of this paper is to solve explicitly one basic boundary value problem (BVP) of the linear equilibrium theory of thermoelasticity with microtemperatures for the spherical ring. The obtained solutions are represented as absolutely and uniformly convergent series.

Basic equations

Let D be a bounded (respectively, an unbounded) domain of the Euclidean 3D space E_3, bounded by the surface S. Let $\mathbf{x}=(x_1,x_2,x_3)\in E_3$, $\rho=|\mathbf{x}|$, $\partial\mathbf{x}=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$. Assume that the domain D is filled with isotropic elastic materials with the thermoelastic properties possessing microtemperatures.

The basic homogeneous (i.e., body forces are neglected) system of equations of the linear equilibrium theory of thermoelasticity with microtemperatures has the form [1]

$$\mu\Delta\mathbf{u}+(\lambda+\mu)graddiv\mathbf{u}-\beta grad\theta=0, \tag{1}$$

$$k_6\Delta\mathbf{w}+(k_4+k_5)graddiv\mathbf{w}-k_3grad\theta-k_2\mathbf{w}=0, \tag{2}$$

$$k\Delta\theta + k_1 div\mathbf{w} = 0, \tag{3}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ is the displacement vector, $\mathbf{w} = (w_1, w_2, w_3)^T$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads), λ, μ, β, k, k_j, $j = 1, ..., 6$, are constitutive coefficients, Δ is the 3D Laplace operator. The superscript "T" denotes transposition.

Definition 1. A vector-function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$ defined in the domain D is called regular if it has integrable continuous second order derivatives in D , and $\mathbf{U}$ itself and its first order derivatives are continuously extendible at every point of the boundary of D, that is $\mathbf{U} \in C^2(D) \cap C^1(\overline{D})$.

Note that BVPs for the system (2),(3), that contain only $\mathbf{w}$ and θ, can be investigated separately. Then supposing θ, as known, we can study BVPs for the system (1) with respect to $\mathbf{u}$. Combining the results obtained we arrive at explicit solution for BVPs for the system (1)-(3). First we assume that $\theta(\mathbf{x})$ is known, when $\mathbf{x} \in D$, then for $\mathbf{u}$ we get the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu)graddiv\mathbf{u} = \beta grad\theta. \tag{4}$$

It is known that the volume potential $\mathbf{u}_0$ [6]

$$\mathbf{u}_0 = -\frac{\beta}{\pi}\int\limits_D \mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) grad\theta ds, \tag{5}$$

where

$$\mathbf{\Gamma} = \|\Gamma_{kj}\|_{3x3},$$

$$\Gamma_{kj} = \frac{\lambda + 3\mu}{2a\mu}\frac{\delta_{kj}}{r} + \frac{\lambda + \mu}{2a\mu}\frac{x_k x_j}{r^3}, \quad k, j = 1, 2, 3.$$

is a particular solution of (4). In (5) $grad\theta$ is a continuous vector in D along with its first order derivatives.

Thus, the general solution of the equation (4) is representable in the form $\mathbf{u} = \mathbf{V} + \mathbf{u}_0$ where

$$\mu\Delta\mathbf{V} + (\lambda + \mu)graddiv\mathbf{V} = 0. \tag{6}$$

The last equation is the equation of an isotropic elastic body. So we have reduced the solution of basic BVPs under consideration to the solution of the basic BVPs for the equation of an isotropic elastic body.

The solution of the BVPs for the equation (6) is given in [6]. So it remains to solve BVPs for the system (2),(3).

Expansion of regular solutions

In this section the general solution for the equations (2),(3) is obtained that gives possibility to solve the BVP for the spherical ring.

Theorem 1. *The regular solution $\boldsymbol{W} = (\boldsymbol{w}, \theta)$ of equations (2),(3) admits in the domain of regularity a representation*

$$\boldsymbol{W}(\boldsymbol{x}) = (\overset{\mathbf{1}}{\mathbf{w}} + \overset{\mathbf{2}}{\mathbf{w}}, \theta), \tag{7}$$

where

$$\Delta(\Delta - s_1^2)\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad rot\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad (\Delta - s_1^2)div\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad (\Delta - s_2^2)\overset{\mathbf{2}}{\mathbf{w}} = 0,$$
$$div\overset{\mathbf{2}}{\mathbf{w}} = 0, \quad \Delta(\Delta - s_1^2)\theta = 0, \quad s_1^2 = \frac{kk_2 - k_1k_3}{kk_7} > 0, \quad s_2^2 = \frac{k_2}{k_6} > 0. \tag{8}$$

Proof. Let $\mathbf{W}$ be certain solution of the equation (2),(3). Let us prove that $\mathbf{W}$ can be represented in the form (7) and it satisfies the conditions (8). Using the identity

$$\Delta\mathbf{w} = graddiv\mathbf{w} - rotrot\mathbf{w}$$

rewrite equation (2) as follows

$$\mathbf{w} = \frac{k_7}{k_2}graddiv\mathbf{w} - \frac{k_6}{k_2}rotrot\mathbf{w} - \frac{k_3}{k_2}grad\theta.$$

Let

$$\overset{\mathbf{1}}{\mathbf{w}} = \frac{k_7}{k_2}graddiv\mathbf{w} - \frac{k_3}{k_2}grad\theta, \tag{9}$$

$$\overset{\mathbf{2}}{\mathbf{w}} = -\frac{k_6}{k_2}rotrot\mathbf{w}. \tag{10}$$

Clearly, from (9),(10) we obtain

$$rot\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{w}} = 0, \quad (\Delta - s_2^2)\overset{\mathbf{2}}{\mathbf{w}} = 0. \tag{11}$$

(2),(3) yield

$$(k_7\Delta - k_2)div\mathbf{w} - k_3\Delta\theta = 0. \tag{12}$$

Substitution of the value $div\mathbf{w} = -\frac{k}{k_1}\Delta\theta$ from (3) in (12) results in

$$\Delta(\Delta - s_1^2)\theta = 0. \tag{13}$$

From (9) and (10) we have

$$\Delta(\Delta - s_1^2)\overset{\mathbf{1}}{\mathbf{w}} = 0 \quad (\Delta - s_1^2)div\overset{\mathbf{1}}{\mathbf{w}} = 0. \tag{14}$$

Formulas (11),(13),(14) prove the theorem.

Theorem 2. *In the domain of regularity the regular solution of equations (2),(3) can be represented in the form*

$$\boldsymbol{W} = \overset{\mathbf{1}}{\mathbf{V}} + \overset{\mathbf{2}}{\mathbf{V}} + \overset{\mathbf{3}}{\mathbf{V}}, \tag{15}$$

where

$$\overset{\mathbf{1}}{\mathbf{V}} = (\boldsymbol{v}^{(1)}, \varphi_1), \quad \overset{\mathbf{2}}{\mathbf{V}} = (\boldsymbol{v}^{(2)}, \varphi_2), \quad \overset{\mathbf{3}}{\mathbf{V}} = (\boldsymbol{v}^{(3)}, 0) \tag{16}$$

and

$$\Delta\mathbf{v}^{(1)} = 0, \quad (\Delta - s_1^2)\mathbf{v}^{(2)} = 0, \quad (\Delta - s_2^2)\mathbf{v}^{(3)} = 0, \quad rot\mathbf{v}^{(1)} = 0,$$

$$rot\mathbf{v}^{(2)}=0,\quad div\mathbf{v}^{(3)}=0,\qquad \Delta\varphi_1=0,\qquad (\Delta-s_1^2)\varphi_2=0.$$

Proof. Let

$$\overset{1}{\mathbf{v}}=-\frac{(\Delta-s_1^2)\overset{1}{\mathbf{w}}}{s_1^2},\qquad \overset{2}{\mathbf{v}}=\frac{\Delta\overset{1}{\mathbf{w}}}{s_1^2},\qquad \varphi_1=-\frac{(\Delta-s_1^2)\theta}{s_1^2},\qquad \varphi_2=\frac{\Delta\theta}{s_1^2},\tag{17}$$

then, by virtue of (14), it follows

$$\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}}=\overset{1}{\mathbf{w}},\qquad \Delta\overset{1}{\mathbf{v}}=0,\qquad (\Delta-s_1^2)\overset{2}{\mathbf{v}}=0.$$

θ is the solution of a scalar equation of the same type that it satisfied by the vector $\mathbf{w}^{(1)}$; therefore, by analogy we will have $\theta=\varphi_1+\varphi_2$, where

$$\Delta\varphi_1=0,\quad (\Delta-s_1^2)\varphi_2.$$

Now, it is clear that if we take $\mathbf{v}^{(3)}=\overset{2}{\mathbf{w}}$, we obtain representation (15). Hence

$$\begin{aligned}
&\overset{1}{\mathbf{w}}=\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}},\quad \theta=\varphi_1+\varphi_2,\quad rot\overset{1}{\mathbf{w}}=0,\quad div\overset{2}{\mathbf{w}}=0,\\
&\Delta\overset{1}{\mathbf{v}}=0,\qquad \Delta div\overset{1}{\mathbf{v}}=0,\qquad (\Delta-s_1^2)div\overset{2}{\mathbf{v}}=0,\qquad (\Delta-s_1^2)\overset{2}{\mathbf{v}}=0,\\
&\Delta\varphi_1=0,\qquad (\Delta-s_1^2)\varphi_2=0,\qquad (\Delta-s_2^2)\overset{2}{\mathbf{w}}=0.
\end{aligned}\tag{18}$$

Substituting in (2),(3) $\mathbf{w}=\overset{1}{\mathbf{w}}+\overset{2}{\mathbf{w}}$ and replacing $\overset{1}{\mathbf{w}}$ and θ by their values from (17), we have

$$\begin{aligned}
&k_7s_1^2\overset{2}{\mathbf{v}}-k_2(\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}})=k_3grad(\varphi_1+\varphi_2),\\
&k\Delta\varphi_2+k_1div\overset{2}{\mathbf{v}}=0.
\end{aligned}\tag{19}$$

Equation(19) is satisfied by

$$\overset{1}{\mathbf{v}}=-\frac{k_3}{k_2}grad\varphi_1,\qquad \overset{2}{\mathbf{v}}=-\frac{k}{k_1}grad\varphi_2.$$

Finally, if we take

$$\overset{1}{\mathbf{v}}=-\frac{k_3}{k_2}grad\varphi_1,\qquad \overset{2}{\mathbf{v}}=-\frac{k}{k_1}grad\varphi_2$$

and they satisfy the conditions

$$\Delta\overset{1}{\mathbf{v}}=0,\quad (\Delta-s_1^2)\overset{2}{\mathbf{v}}=0,$$

then the general solution of the thermoelasticity equations (2),(3) takes the form

$$\begin{aligned}
&\mathbf{w}(\mathbf{x})=a\ grad\varphi_1(\mathbf{x})+b\ grad\varphi_2(\mathbf{x})+\overset{2}{\mathbf{w}}(\mathbf{x}),\\
&\theta(\mathbf{x})=\varphi_1(\mathbf{x})+\varphi_2(\mathbf{x}),\quad a=-\frac{k_3}{k_2},\qquad b=-\frac{k}{k_1},
\end{aligned}\tag{20}$$

where $\overset{2}{\mathbf{w}}$ satisfies the equations $(\Delta - s_2^2)\overset{2}{\mathbf{w}} = 0, \quad div\overset{2}{\mathbf{w}} = 0.$

Now let us prove that if the vector $\mathbf{W}(\mathbf{w}, \theta) = 0$, then $\varphi_1 = \varphi_2 = 0, \quad \overset{2}{\mathbf{w}} = 0$. It follows from (20) that

$$a\, grad\varphi_1(\mathbf{x}) + b\, grad\varphi_2(\mathbf{x}) + \overset{2}{\mathbf{w}}(\mathbf{x}) = 0,$$

$$\varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) = 0.$$

From here, after simple transformations we find that

$$div[a\, grad\varphi_1(\mathbf{x}) + b\, grad\varphi_2(\mathbf{x}) + \overset{2}{\mathbf{w}}(\mathbf{x})] = bs_1^2\varphi_2 = 0.$$

Thus we have $\varphi_1 = \varphi_2 = 0, \quad \overset{2}{\mathbf{w}} = 0$ and the proof is completed.

Let us assume that D^+ is a ball of radius R_1, centered at point $O(0,0,0)$ in space E_3 and S is a spherical surface of radius R_1.

Let us consider the metaharmonic equation

$$(\Delta + \nu^2)\psi = 0, \quad \nu \neq 0.$$

For this equation the following theorems are valid and we cite them without proof.

Lemma 1. *If regular vector $\boldsymbol{\psi}$ satisfies the conditions*

$$(\Delta + \nu^2)\boldsymbol{\psi} = 0, \quad \nu \neq 0, \quad div\boldsymbol{\psi} = 0,$$

$$(\boldsymbol{x}.\boldsymbol{\psi}) = 0, \quad \boldsymbol{x} \in D^+,$$

then it can be represented in the form

$$\boldsymbol{\psi}(\boldsymbol{x}) = [\boldsymbol{x}.\nabla]h(\boldsymbol{x})),$$

where

$$(\Delta + \nu^2)h(\boldsymbol{x}) = 0,$$

in addition if

$$\int\limits_{S(0,a_1)} h(\boldsymbol{x})ds = 0,$$

where $S(0,a_1) \subset D^+$ is an arbitrary spherical surface of radius a_1, then the function h in D^+ can be defined uniquely by means of vector ψ.

Lemma 2. *If regular vector $\boldsymbol{\psi}$ satisfies the conditions*

$$(\Delta + \nu^2)\boldsymbol{\psi} = 0, \quad \nu \neq 0 \quad div\boldsymbol{\psi} = 0, \quad \boldsymbol{x} \in D^+,$$

then it can be represented in the form

$$\boldsymbol{\psi}(\mathbf{x}) = [\mathbf{x}.\nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}),$$

where

$$(\Delta - s_2^2)\varphi_j = 0, \quad j = 3, 4,$$

in addition if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0, \quad j = 3,4,$$

where $S(0,a_1) \subset D^+$ is an arbitrary spherical surface of radius a_1, then the functions $\varphi_j \quad j = 3,4$ in D^+ can be defined uniquely by means of vector ψ.

Lemma 1 and Lemma 2 are proved in [7].

Now from these theorems it follows that the following theorem is valid.

Theorem 3. *The regular solution $\boldsymbol{W} = (\boldsymbol{w}, \theta)$, where $\boldsymbol{w} = (w_1, w_2, w_3)$, of the homogeneous equations (2),(3), in D^+, can be represented in the form*

$$\begin{aligned} &\boldsymbol{w}(\boldsymbol{x}) = a\, grad\varphi_1(\boldsymbol{x}) + b\, grad\varphi_2(\boldsymbol{x}) + c\, rot\boldsymbol{\varphi^3}(\boldsymbol{x}), \\ &\theta(\boldsymbol{x}) = \varphi_1(\boldsymbol{x}) + \varphi_2(\boldsymbol{x}), \end{aligned} \tag{21}$$

where

$$\Delta\varphi_1 = 0, \qquad (\Delta - s_1^2)\varphi_2 = 0, \qquad (\Delta - s_2^2)\boldsymbol{\varphi^3} = 0, \qquad div\boldsymbol{\varphi^3} = 0,$$

$$s_1^2 = \frac{kk_2 - k_1k_3}{kk_7} > 0, \qquad s_2^2 = \frac{k_2}{k_6} > 0, \qquad a = -\frac{k_3}{k_2}, \qquad b = -\frac{k}{k_1}, \qquad c = -\frac{k_6}{k_2},$$

$$\boldsymbol{\varphi^3}(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}), \qquad (\Delta - s_2^2)\varphi_j = 0, \qquad j = 3,4. \tag{22}$$

In addition if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0,$$

where $S(0,a_1) \subset D^+$ is an arbitrary spherical surface of radius a_1. Between the vector $\boldsymbol{W}(\boldsymbol{x}) = (\boldsymbol{w}, \theta)$ and the functions $\varphi_j, \quad j = 1,..,4$, there exists one-to one correspondence.

Remark. By virtue of the equality

$$rotrot[x.\nabla]\varphi_4 = -\Delta[x.\nabla]\varphi_4,$$

formula (21) can be rewritten in the form

$$\begin{aligned} &\mathbf{w}(\mathbf{x}) = a\, grad\varphi_1(\mathbf{x}) + b\, grad\varphi_2(\mathbf{x}) + [\mathbf{x}.\nabla]\varphi_4(\mathbf{x}) + c\, rot[\mathbf{x}.\nabla]\varphi_3(\mathbf{x}), \\ &\theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}). \end{aligned} \tag{23}$$

Below we shall use solution (23) to solve the BVP for spherical ring.

Some auxiliary formulas

Let us introduce the spherical coordinates

$$x_1=\rho\sin\vartheta\cos\varphi,\quad x_2=\rho\sin\vartheta\sin\varphi,\quad x_3=\rho\cos\vartheta,\quad x\in\Omega,$$
$$y_1=R_1\sin\vartheta_0\cos\varphi_0,\quad y_2=R_1\sin\vartheta_0\sin\varphi_0,\quad y_3=R_1\cos\vartheta_0,\quad y\in S,\tag{24}$$
$$\rho^2=x_1^2+x_2^2+x_3^2,\quad 0\le\vartheta\le\pi,\quad 0\le\varphi\le 2\pi\quad 0\le\rho\le R_1.$$

In the sequel we use the following notations: If $\mathbf{g}(\mathbf{x})=\mathbf{g}(g_1,g_2,g_3)$ and $\mathbf{q}(\mathbf{x})=\mathbf{q}(q_1,q_2,q_3)$ then by symbols $(\mathbf{g}\cdot\mathbf{q})$ and $[\mathbf{g}\cdot\mathbf{q}]$ will be denote scalar product and vector product, respectively

$$(\mathbf{g}\cdot\mathbf{q})=\sum_{k=1}^{3}g_kq_k,\quad [\mathbf{g}\cdot\mathbf{q}]=(g_2q_3-g_3q_2,g_3q_1-g_1q_3,g_1q_2-g_2q_1).$$

The operator $\dfrac{\partial}{\partial S_k(x)}$ is determined as follows

$$[\mathbf{x}\cdot\nabla]_k=\frac{\partial}{\partial S_k(x)},\quad k=1,2,3,\qquad \nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right).$$

Simple calculations give

$$\frac{\partial}{\partial S_1(x)}=x_2\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_2}=-cos\varphi ctg\vartheta\frac{\partial}{\partial\varphi}-sin\varphi\frac{\partial}{\partial\vartheta}$$
$$\frac{\partial}{\partial S_2(x)}=x_3\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_3}=-sin\varphi ctg\vartheta\frac{\partial}{\partial\varphi}+cos\varphi\frac{\partial}{\partial\vartheta}$$
$$\frac{\partial}{\partial S_3(x)}=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}=\frac{\partial}{\partial\varphi}.$$

Below we use the following identities [7]

$$(\mathbf{x}.rotg(\mathbf{x}))=\sum_{k=1}^{3}\frac{\partial g_k(z)}{\partial S_k(z)},\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}(rot[\mathbf{x}.\nabla]h)_k=0,$$
$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}(rot\mathbf{g}(\mathbf{x}))_k=\rho\frac{\partial}{\partial\rho}div\mathbf{g}(\mathbf{x})-\sum_{k=1}^{3}x_k\Delta\mathbf{g}_k(\mathbf{x}),\tag{25}$$
$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}.\mathbf{g}]_k=\rho^2div\mathbf{g}(\mathbf{x})-(\mathbf{x}.\mathbf{g}(\mathbf{x}))-\rho\frac{\partial}{\partial\rho}(\mathbf{x}.\mathbf{g}(\mathbf{x})),$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial S_k(z)}[\mathbf{x}.rot\mathbf{g}(\mathbf{x})]_k = -(\rho\frac{\partial}{\partial\rho}+1)\sum_{k=1}^{3}\frac{\partial g_k(z)}{\partial S_k(z)},$$

$$\sum_{k=1}^{3} x_k \frac{\partial}{\partial S_k(x)} = 0, \quad \frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}\frac{\partial}{\partial S_k(x)},$$

$$\sum_{k=1}^{3} \frac{\partial^2}{\partial S_k^2(x)} = \frac{\partial^2}{\partial\vartheta^2} + ctg\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}, \quad \frac{\partial x_k}{\partial S_k} = 0,$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k} = 0, \quad \frac{\partial g(\rho)Y(\vartheta,\varphi)}{\partial S_k(x)} = g(\rho)\frac{\partial Y(\vartheta,\varphi)}{\partial S_k(x)}.$$

From this formulas it follows that, if g_m is the spherical harmonic, the operator $\frac{\partial}{\partial S_k}$, $k = 1, 2, 3$, does not affect the order of the spherical function:

$$\sum_{k=1}^{3} \frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(x)} = -m(m+1)g_m(\mathbf{x}).$$

We introduce the following notations:

$$(\mathbf{z}.\mathbf{f}^+) = h_1^+(\mathbf{z}), \quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}[\mathbf{z}.\mathbf{f}^+]_k = h_2^+(\mathbf{z}), \quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}f_k^+ = h_3^+(\mathbf{z}), \quad f_4^+ = h_4^+(\mathbf{z}).$$

$$(\mathbf{z}.\mathbf{f}^-) = h_1^-(\mathbf{z}), \quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}[\mathbf{z}.\mathbf{f}^-]_k = h_2^-(\mathbf{z}), \quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(z)}f_k^- = h_3^-(\mathbf{z}), \quad f_4^- = h_4^+(\mathbf{z}).$$

Let us assume that $\mathbf{f}$ and f_4 are sufficiently smooth(differentiable) functions and h_k can be represented in the form

$$h_k^{\pm}(\mathbf{z}) = \sum_{m=0}^{\infty} h_{km}^{\pm}(\vartheta,\varphi),$$

where $h_{km}^{\pm}$ is the spherical harmonic of order m :

$$h_{km}^{\pm} = \frac{2m+1}{4\pi R_1^2}\int\limits_S P_m(\cos\gamma)h_m^{\pm}(y)dS_y,$$

P_m is Legender polynomial of the m-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma = \frac{1}{|x||y|}\sum_{m=1}^{3} x_k y_k.$$

The BVP for the spherical ring

Let us assume that Ω is a spherical ring, $R_1 < |\mathbf{x}| < R_2$, centered at point $O(0,0,0)$ in the Euclidean 3D space E_3, S_1 is a spherical surface of radius R_1 and S_2 is a spherical surface of radius R_2. $S = S_1 \cup S_2$.

The boundary value problem for the spherical ring is formulated as follows:

Find in the domain Ω a regular solution $\mathbf{U}(\mathbf{u},\mathbf{w},\theta)$ of equations (1),(2),(3) by the boundary conditions

$$(\mathbf{u})^- = \mathbf{F}^-(\mathbf{y}), \quad (\mathbf{w})^- = \mathbf{f}^-(\mathbf{y}), \quad \left(k\frac{\partial\theta}{\partial\mathbf{n}} + k_1\mathbf{n}\mathbf{w}\right)^- = f_4^-(\mathbf{y}), \quad \rho = R_1,$$

$$(\mathbf{u})^+ = \mathbf{F}^+(\mathbf{y}), \quad (\mathbf{w})^+ = \mathbf{f}^+(\mathbf{y}), \quad \left(k\frac{\partial\theta}{\partial\mathbf{n}} + k_1\mathbf{n}\mathbf{w}\right)^+ = f_4^+(\mathbf{y}), \quad \rho = R_2,$$

where $\mathbf{F}^\pm, \mathbf{f}^\pm, f_4^\pm$ are the given functions on S.

Theorem 4. *Two regular solutions of the considered BVP problem may differ by the vector* $\boldsymbol{V}(\boldsymbol{u},\boldsymbol{w},\theta)$, *where* $\mathbf{u}=0$, $\mathbf{w}=0$, $\theta = const.$

The general solution of the equations $(\Delta - s_k^2)\psi = 0$, $k=1,2$, in the domain Ω has the form ([7])

$$\psi(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\phi_m^{(2)}(is_k\rho)Y_m(\vartheta,\varphi) + \Psi_m^{(2)}(is_k\rho)Z_m(\vartheta,\varphi)\right], \quad R_1<\rho<R_2,$$

The general solution of the equation $\Delta\phi = 0$ in the domains Ω has the form

$$\phi(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\frac{\rho^m}{(2m+1)R_2^{m-1}}Y_m(\vartheta,\varphi) + \frac{R_1^{m+2}}{(2m+1)\rho^{m+1}}Z_m(\vartheta,\varphi)\right], \quad R_1<\rho<R_2,$$

where $Y_m(\theta,\varphi), Z_m(\theta,\varphi)$ are the spherical harmonics,

$$\phi_m^{(2)}(is_k\rho) = \frac{\sqrt{R_2}J_{m+\frac{1}{2}}(is_k\rho)}{\sqrt{\rho}J_{m+\frac{1}{2}}(is_kR_2)}, \quad \Psi_m^{(2)}(is_k\rho) = \frac{\sqrt{R_1}H^{(1)}_{m+\frac{1}{2}}(is_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac{1}{2}}(is_kR_1)}.$$

Using (23), we have

$$(\mathbf{x}\cdot\mathbf{w}) = a\rho\frac{\partial\varphi_1(\mathbf{x})}{\partial\rho} + b\rho\frac{\partial\varphi_2(\mathbf{x})}{\partial\rho} + c\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)},$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{w}]_k = a\sum_{k=1}^{3}\frac{\partial^2\varphi_1(\mathbf{x})}{\partial S_k^2(x)} + b\sum_{k=1}^{3}\frac{\partial^2\varphi_2(\mathbf{x})}{\partial S_k^2(x)} - c\left(\rho\frac{\partial}{\partial\rho}+1\right)\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)}, \quad (26)$$

$$\sum_{k=1}^{3}\frac{\partial w_k}{\partial S_k(x)} = \sum_{k=1}^{3}\frac{\partial^2\varphi_4(\mathbf{x})}{\partial S_k^2(x)}, \quad \theta(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}).$$

Let the functions $\varphi_m(\mathbf{x})$, $m=1,2,3,4$, be sought in the form

$$\varphi_1(\mathbf{x}) = \sum_{m=0}^{\infty}\left[\frac{\rho^m}{(2m+1)R_2^{m-1}}Y_{1m}(\vartheta,\varphi) + \frac{R_1^{m+2}}{(2m+1)\rho^{m+1}}Z_{1m}(\vartheta,\varphi)\right],$$

$$\varphi_2(\mathbf{x}) = \sum_{m=0}^{\infty} \left[\phi_m^{(2)}(is_1\rho)Y_{2m}(\vartheta,\varphi) + \Psi_m^{(2)}(is_1\rho)Z_{2m}(\vartheta,\varphi)\right],$$

$$\varphi_j(\mathbf{x}) = \sum_{m=0}^{\infty} \left[\phi_m^{(2)}(is_2\rho)Y_{jm}(\vartheta,\varphi) + \Psi_m^{(2)}(is_2\rho)Z_{jm}(\vartheta,\varphi)\right], j = 3,4,$$

The conditions $\int\limits_{S(0,a_1)} \varphi_j ds = 0 \quad j = 3,4$ in fact mean that

$$Y_{30} = Y_{40} = 0, \qquad Z_{30} = Z_{40} = 0.$$

Substitute in (26) the functions $\varphi_j(\mathbf{x})$, passing to the limit as $\rho \to R_1$, $\rho \to R_2$ and taking into account boundary conditions , for determining the unknown values Y_{jm} and Z_{jm}, we obtain the following system of algebraic equations

$$\frac{maR_1^m}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{(m+1)aR_1}{2m+1}Z_{1m} + b\left[\rho\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{2m}$$
$$+b\left[\rho\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{2m} - cm(m+1)\left\{\left[\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{3m} + Z_{3m}\right\} = h_{1m}^-,$$
$$\frac{maR_2}{(2m+1)}Y_{1m} - \frac{(m+1)aR_1^{m+2}}{(2m+1)R_2^{m+1}}Z_{1m} + b\left[\rho\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{2m}$$
$$+b\left[\rho\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{2m} - cm(m+1)\left\{Y_{3m} + \left[\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Z_{3m}\right\} = h_{1m}^+,$$
$$-\frac{m(m+1)aR_1^m}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{am(m+1)R_1}{2m+1}Z_{1m} - bm(m+1)\left\{\phi_m^{(2)}(is_1R_1)Y_{2m} + Z_{2m}\right\}$$
$$+cm(m+1)\left\{\left[(\rho\frac{\partial}{\partial\rho}+1)\phi_m^{(2)}(is_2\rho)\right]_{\rho=R_1} Y_{3m} + \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_1} Z_{3m}\right\} = h_{2m}^-,$$
$$-\frac{m(m+1)aR_2}{2m+1}Y_{1m} - \frac{am(m+1)R_1^{m+2}}{(2m+1)R_2^{m+1}}Z_{1m} - bm(m+1)\left\{Y_{2m} + \Psi_m^{(2)}(is_1R_2)Z_{2m}\right\}$$
$$+cm(m+1)\left\{\left[(\rho\frac{\partial}{\partial\rho}+1)\phi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Y_{3m} + \left[(\rho\frac{\partial}{\partial\rho}+1)\Psi_m^{(2)}(is_2\rho)\right]_{\rho=R_2} Z_{3m}\right\} = h_{2m}^+,$$

$$-m(m+1)\{\Phi_m^{(2)}(is_2R_1)Y_{4m} + Z_{4m}\} = h_{3m}^-,$$

$$-m(m+1)\{Y_{4m} + \Psi_m^{(2)}(is_2R_2)Z_{4m}\} = h_{3m}^+, \tag{27}$$

$$\frac{mR_1^{m-1}}{(2m+1)R_2^{m-1}}Y_{1m} - \frac{m+1}{2m+1}Z_{1m} + \left[\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{2m}$$

$$+\left[\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{2m}=\frac{h_{4m}^-}{k}+\frac{1}{R_1 b}h_{1m}^-,$$

$$\frac{m}{2m+1}Y_{1m}-\frac{(m+1)R_1^{m+2}}{(2m+1)R_2^{m+2}}Z_{1m}+\left[\frac{\partial}{\partial\rho}\phi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{2m}$$

$$+\left[\frac{\partial}{\partial\rho}\Psi_m^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{2m}=\frac{h_{4m}^+}{k}+\frac{1}{R_2 b}h_{1m}^+.$$

Note that for $m=0$, (27) is transformed to the system

$$\begin{gathered}
Z_{10}=\frac{b}{k(a-b)}h_{40}^-,\quad 0=h_{20}^+,\quad 0=h_{20}^-,\quad 0=h_{30}^+,\quad 0=h_{30}^-,\\
0\cdot Y_{10}+\left[\frac{\partial}{\partial\rho}\phi_0^{(2)}(is_1\rho)\right]_{\rho=R_1} Y_{20}+\left[\frac{\partial}{\partial\rho}\Psi_0^{(2)}(is_1\rho)\right]_{\rho=R_1} Z_{20}=\frac{h_{10}^-}{bR_1}+\frac{a}{b}Z_{10},\\
0\cdot Y_{10}+\left[\frac{\partial}{\partial\rho}\phi_0^{(2)}(is_1\rho)\right]_{\rho=R_2} Y_{20}+\left[\frac{\partial}{\partial\rho}\Psi_0^{(2)}(is_1\rho)\right]_{\rho=R_2} Z_{20}=\frac{h_{10}^+}{bR_2}+\frac{aR_1^2}{bR_2^2}Z_{10}.
\end{gathered}\tag{28}$$

Taking into account the identities $J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z,\quad H_{\frac{1}{2}}^{(1)}(z)=-i\sqrt{\frac{2}{\pi z}}\exp(iz)$, after certain calculations, the determinant of system (28) takes the form

$$\delta=\frac{\exp R_1 s_1}{R_1 R_2\sinh s_1 R_2}\{(s_1^2 R_1 R_2-1)\sinh s_1(R_2-R_1)$$

$$+s_1(R_2-R_1)\cosh s_1(R_2-R_1)\}\neq 0.$$

Thus we have shown that Y_{10} is an arbitrary constant and for the solution to exist it is necessary that the conditions $h_{20}^+=0,\quad h_{20}^-=0,\quad R_2^2h_{40}^+=R_1^2h_{40}^-$ be fulfilled. By virtue of the uniqueness theorems of solutions of the BVP, we conclude that the determinant of system (26) for $m\geq 1$ is different from zero and we obtain the required solution of problem in the form of series.

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ON THE EXISTENCE OF UNBOUNDED OSCILLATORY SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. The statements on the existence of unbounded oscillatory solutions are proved. It is also shown that non-oscillatory solutions vanish at infinity for linear ordinary differential equations of third order.

Keywords and phrases: Linear differential equations of third order, vanishing at infinity, non-oscillatory solution.

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Let us consider the linear ordinary differential equation of third order

$$u''' + p_1(t)u'' + p_2(t)u' + p_3(t)u = 0, \tag{1}$$

where $p_k : R_+ \to R$ $(k = 1, 2, 3)$ are continuous functions.

A nontrivial solution of equation (1) is called oscillatory if it has an infinite number of zeros, and non-oscillatory otherwise. In the present paper, when p_3 is non-negative, we prove the statements on the existence of unbounded oscillatory solutions, and also show that non-oscillatory solutions vanish at infinity.

We will first prove some auxiliary propositions.

Lemma 1. *Let $\alpha \leq 1$, let the conditions*

$$\limsup_{t\to+\infty} t^{k\alpha}|p_k(t)| < +\infty \quad (k = 1, 2, 3) \tag{2}$$

be fulfilled and let equation (1) *have a solution, satisfying for some $\mu \geq 0$ the condition*

$$\limsup_{t\to+\infty} t^{-\mu}|u(t)| < +\infty. \tag{3}$$

Then

$$\limsup_{t\to+\infty} t^{-\mu+j\alpha}|u^{(j)}(t)| < +\infty \quad (j = 1, 2). \tag{4}$$

Proof. By (2) and (3) we can choose numbers $t_0 \geq 1$ and $c > 1$ such that

$$t^{k\alpha}|p_k(t)| < c \quad (k = 1, 2, 3) \ \text{ for } \ t \geq t_0, \tag{5}$$

$$t^{-\mu}|u(t)| < c \ \text{ for } \ t \geq t_0. \tag{6}$$

Therefore

$$|u'''(t)| \leq c\sum_{j=0}^{2} t^{(j-3)\alpha}|u^{(j)}(t)| \ \text{ for } \ t \geq t_0. \tag{7}$$

Assume that the lemma is not true, i.e.

$$\limsup_{t\to+\infty}\sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| = +\infty.$$

Then there exist increasing sequences $(t_i)_{i=1}^{+\infty}$, $(M_i)_{i=1}^{+\infty}$ such that $t_1 > t_0$, $t_i \to +\infty$, $M_i \to +\infty$ as $i \to +\infty$ and

$$M_i = \sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| = \max\Big\{\sum_{j=1}^{2} t^{-\mu+j\alpha}|u^{(j)}(t)| :\ t_0 \le t \le t_i\Big\}.$$

Thus we can assume that there exists $l \in \{1,2\}$ such that for any $i \in N$

$$t_i^{-\mu+l\alpha}|u^{(l)}(t_i)| \ge \frac{M_i}{2}\,.$$

Suppose first that $l = 2$ and $h > 0$ satisfies the inequalities

$$hc < \frac{1}{4}\,, \quad hc(1-h)^{\mu-3\alpha} < \frac{1}{4}\,.$$

Then by virtue of (7)

$$|u''(t)| \ge |u''(t_i)| - \int_t^{t_i} |u'''(s)|\,ds \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - \int_t^{t_i} cM_i s^{\mu-3\alpha}\,ds$$

and therefore if $\mu - 3\alpha \ge 0$, then

$$|u''(t)| \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - cM_i t_i^{\mu-3\alpha} h t_i^{\alpha} \ge \frac{M_i}{4}\,t_i^{\mu-2\alpha} \ \text{ for } \ t \in [t_i - ht_i^{\alpha}; t_i],$$

and if $\mu - 3\alpha < 0$, then

$$|u''(t)| \ge \frac{M_i}{2}\,t_i^{\mu-2\alpha} - cM_i t_i^{\mu-3\alpha}(1-h)^{\mu-3\alpha} h t_i^{\alpha} \ge \frac{M_i}{4}\,t_i^{\mu-2\alpha} \quad \text{for } \ t \in [t_i - ht_i^{\alpha}; t_i].$$

Let $s_0 = t_i - ht_i^{\alpha}$, $s_1 = t_i - \frac{ht_i^{\alpha}}{2}$, $s_2 = t_i$. Then there exists $\xi \in [s_0, s_2]$ such that

$$\frac{u(\xi)}{2} = \frac{u(s_0)}{(s_1-s_0)(s_2-s_0)} - \frac{u(s_1)}{(s_1-s_0)(s_2-s_1)} + \frac{u(s_2)}{(s_2-s_0)(s_2-s_1)}\,.$$

Hence by virtue of (6) we obtain

$$\frac{M_i}{4}\,t_i^{\mu-2\alpha} \le |u''(\xi)| \le 2\sum_{j=0}^{2} \frac{|u(s_j)|}{(\frac{ht_i^{\alpha}}{2})^2} \le \frac{8cc_\mu}{h^2}\,t_i^{\mu-2\alpha},$$

where $c_\mu = 1$ if $\mu \geq 0$, and $c_\mu = (1-h)^\mu$ if $\mu < 0$. Therefore

$$M_i \leq \frac{32cc_\mu}{h^2}\,.$$

For any $i \in N$, which is a contradiction. In an analogous manner we obtain a contradiction when $l = 1$. The lemma is proved.

Remark 1. For $\alpha = \mu = 0$, Lemma 1 is proved in [1]. For second order equations see [2].

Lemma 2. *Let $\beta > 0$, $\alpha \geq 0$, let the conditions*

$$\limsup_{t\to+\infty} |p_k(t)| \exp(-\alpha k t^\beta) < +\infty \quad (k = 1, 2, 3)$$

be fulfilled and for some $\mu > 0$ let equations (1) *have a solution, satisfying the condition*

$$\limsup_{t\to+\infty} |u(t)| \exp(-\mu t^\beta) < +\infty.$$

Then

$$\limsup_{t\to+\infty} |u^{(j)}(t)| \exp(-(\mu + j\alpha)t^\beta) < +\infty \quad (j = 1, 2). \tag{8}$$

Proof. By transformation of the variable

$$u(t) = \exp(\mu t^\beta) v(s), \quad s = \int\limits_0^t \exp(\alpha \tau^\beta)\, d\tau, \tag{9}$$

equation (1) takes the form

$$v'''(s) + \widetilde{p}_1(s)v''(s) + \widetilde{p}_2(s)v'(s) + \widetilde{p}_3(s)v(s) = 0, \tag{10}$$

where

$$\begin{aligned}
\widetilde{p}_1(s) &= \Big(p_1(t) + \mu\beta t^{\beta-1} + (\mu+\alpha)\beta t^{\beta-1} + \beta(\mu+2\alpha)t^{\beta-1}\Big)\exp(-\alpha t^\beta),\\
\widetilde{p}_2(s) &= \Big[p_2(t) + p_1(t)\big(\mu\beta t^{\beta-1} + (\mu+\alpha)\beta t^{\beta-1}\big) + \mu\beta(\beta-1)t^{\beta-2} +\\
&\quad + \mu^2\beta^2 t^{2\beta-2} + \mu\beta(\beta-1)t^{\beta-2} + \mu(\mu+\alpha)\beta t^{2\beta-2} +\\
&\quad + (\mu+\alpha)\beta(\beta-1)t^{\beta-2} + (\mu+\alpha)^2\beta^2 t^{2\beta-2}\Big]\exp(-2\alpha t^\beta),\\
\widetilde{p}_3(s) &= \Big[p_3(t) + p_2(t)\mu\beta t^{\beta-1} +\\
&\quad + p_1(t)\big(\mu\beta(\beta-1)t^{\beta-2} + \mu^2\beta^2 t^{2\beta-2}\big) + \mu^3\beta^3 t^{3\beta-3}\Big]\exp(-3\alpha t^\beta).
\end{aligned}$$

It is obvious that for equation (10) the conditions of Lemma 1 are fulfilled if it is assumed that $\mu = 0$ and $\alpha = 0$. Therefore

$$\limsup_{t\to+\infty} |v^{(j)}(s)| < +\infty \quad (j = 1, 2).$$

This, by virtue of (9), implies inequality (8). The Lemma is proved.

Theorem 1. *If the inequalities*

$$p_2(t) \le 0, \quad p_3(t) \ge 0 \ \ \textit{for} \ \ t \in R_+, \tag{11}$$

$$\int\limits_0^{+\infty} [p_1(t)]_+ \, dt < +\infty \tag{12}$$

are fulfilled, then there exists a solution of equation (1) *such that*

$$\limsup_{t\to+\infty} t^{-\frac{3}{2}+j} |u^{(j)}(t)| > 0 \ \ (j = 1, 2). \tag{13}$$

If, besides, condition (2) *is fulfilled for some* $\alpha \le 1$, *then equation* (1) *has a solution which, in addition to* (13), *also satisfies the condition*

$$\limsup_{t\to+\infty} t^{-1-\frac{\alpha}{2}} |u(t)| > 0. \tag{14}$$

Proof. Let u_1 and u_2 be solutions of equation (1) which satisfy the initial conditions

$$\begin{aligned} &u_1(0) = 0, \quad u_1'(0) = 1, \quad u_1(0) = 0, \\ &u_2(0) = 0, \quad u_2'(0) = 0, \quad u_2''(0) = 1. \end{aligned}$$

Let us introduce the notation

$$\begin{aligned} v_{01}(t) &= u_1(t)u_2'(t) - u_1'(t)u_2(t), \\ v_{02}(t) &= \exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg)\big(u_1(t)u_2''(t) - u_1''(t)u_2(t)\big), \\ v_{12}(t) &= \exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg)\big(u_1'(t)u_2''(t) - u_1''(t)u_2'(t)\big). \end{aligned}$$

The vector-function $x = colon(v_{01}, v_{02}, v_{12})$ is a solution of the problem

$$x' = A(t)x, \quad x(0) = colon(0, 0, 1),$$

where

$$A(t) = \begin{pmatrix} 0 & \exp\bigg(-\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & 0 \\ -p_2(t)\exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & [p_1(t)]_- & 1 \\ p_3(t)\exp\bigg(\int\limits_0^t [p_1(s)]_+ \, ds\bigg) & 0 & [p_1(t)]_- \end{pmatrix}.$$

Let
$$y(t) = colon\bigg(\int\limits_0^t s\exp\bigg(-\int\limits_0^s [p_1(\tau)]_+\,d\tau\bigg)\,ds, t, 1\bigg).$$
Then y satisfies the system
$$y' = B(t)y,$$
where
$$B(t) = \begin{pmatrix} 0 & \exp\bigg(-\int\limits_0^t [p_1(s)]_+\,ds\bigg) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
Since $x(0) \geq y(0) \geq 0$ and
$$A(t) \geq B(t) \geq 0 \ \text{ for } \ t \geq 0$$
it is easy to show that
$$x(t) \geq y(t) \ \text{ for } \ t \geq 0.$$
Therefore
$$v_{01}(t) \geq \int\limits_0^t s\exp\bigg(-\int\limits_0^s [p_1(\tau)]_+\,d\tau\bigg)\,ds \ \text{ for } \ t \geq 0.$$
With (12) taken into account, we obtain
$$\limsup_{t\to+\infty} \frac{v_{01}(t)}{t^2} > 0. \tag{15}$$
Let us show that u_1 or u_2 satisfies condition (13). Indeed, assuming the contrary, we have
$$\lim_{t\to+\infty} t^{-\frac{1}{2}} u_i'(t) = \lim_{t\to+\infty} t^{-\frac{3}{2}} u_i(t) = 0 \ \ (i = 1, 2),$$
which contradicts condition (15).

Now assume that conditions (2) are fulfilled, then
$$\lim_{t\to+\infty} t^{-1-\frac{\alpha}{2}} u_i(t) = 0 \ \ (i = 1, 2).$$
In that case, by virtue of Lemma 1
$$\limsup_{t\to+\infty} t^{-1+\frac{\alpha}{2}} u_i'(t) < +\infty \ \ (i = 1, 2)$$
and therefore
$$\lim_{t\to+\infty} t^{-2} v_{01}(t) = 0,$$
which contradicts inequality (15). The theorem is proved.

Corollaries 1.2.1, 1.3.1 (see [3], pp. 453, 455]) and Theorem 1 immediately give rise to the following propositions.

Corollary 1.1. Let $\alpha < 1$, conditions (11),(12) and

$$\lim_{t\to+\infty} t^{k\alpha} p_k(t) = 0 \ \ (k = 1, 2),$$
$$0 < \liminf_{t\to+\infty} t^{3\alpha} p_3(t) \le \limsup_{t\to+\infty} t^{3\alpha} p_3(t) < +\infty$$

be fulfilled. Then equation (1) has an oscillatory solution which satisfies conditions (13) and (14).

Theorem 2. *Let* (11),(12) *and let one of the following two conditions*

$$\lim_{t\to+\infty} t^3 p_3(t) = +\infty \tag{16}$$

or

$$\lim_{t\to+\infty} t^2 p_2(t) = +\infty \tag{17}$$

be fulfilled. Then equation (1) *has a solution such that*

$$\limsup_{t\to+\infty} t^{-\mu} |u^{(j)}(t)| = +\infty \tag{18}$$

for any $\mu > 0$ *and* $j \in \{1, 2\}$. *If, besides, conditions* (2) *hold for some* $\alpha < 1$, *then there exists a solution of equation* (1) *which satisfies condition* (18) *for any* $\mu > 0$ *and* $j \in \{0, 1, 2\}$.

Proof. It is analogous to the proof of Theorem 1, now for $t \ge t_0 > 0$ we put

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int\limits_0^t [p_1(s)]_+ ds\Big) & 0 \\ 0 & 0 & 1 \\ \dfrac{\nu(\nu-1)t^{\nu-2}}{\int\limits_0^t s^\nu \exp\big(-\int\limits_0^s [p_1(\tau)]_+ d\tau\big)\, ds} & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\Big(\int\limits_0^t s^\nu \exp\Big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\Big)\, ds, t^\nu, \nu t^{\nu-1}\Big)$$

if conditions (16) are fulfilled, and

$$B(t) = \begin{pmatrix} 0 & \exp\Big(-\int\limits_0^t [p_1(s)]_+ \, ds\Big) & 0 \\ \dfrac{\nu t^{\nu 1}}{\int\limits_0^t s^\nu \exp\big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\big)\, ds} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\Big(\int\limits_0^t s^\nu \exp\Big(-\int\limits_0^s [p_1(\tau)]_+ \, d\tau\Big)\, ds, t^\nu, 1\Big)$$

if (17) is fulfilled.

Remark 2. In Theorems 1 and 2, the requirement that $p_2(t) \le 0$ for $t \ge 0$ is an essential one.

Indeed, let us consider the differential equation

$$u''' - u'' + \frac{1}{4}\,u' + \frac{9}{4}\,u = 0, \tag{19}$$

which has a fundamental system of solutions

$$e^{-t}, \quad e^{-t}\sin\frac{\sqrt{5}}{2}\,t, \quad e^{-t}\cos\frac{\sqrt{5}}{2}\,t.$$

Thus equation (19) has no unbounded solution though all the conditions of Theorems 1 and 2 are fulfilled except the condition that the function p_2 is non-positive.

According to Theorem 3.2 [5], Theorem 2 immediately implies

Corollary 2.1. Let $\alpha < 1$, let conditions (11), (12), (16) and

$$\limsup_{t\to+\infty} t^k |p_k(t)| < +\infty \ \ (k = 1, 2), \quad \limsup_{t\to+\infty} t^{3\alpha} p_3(t) < +\infty$$

be fulfilled. Then equation (1) has an oscillatory solution, satisfying conditions (18) for any $\mu > 0$ and $j \in \{0, 1, 2\}$.

Theorem 3. *Let* $\sigma > 0$,

$$\limsup_{t\to+\infty} t^{-\sigma} \int_0^t [p_1(s)]_+\, ds < +\infty, \tag{20}$$

let inequality (11) *and one of the following two conditions*

$$\lim_{t\to+\infty} t^{3-3\sigma} p_3(t) = +\infty \tag{21}$$

or

$$\lim_{t\to+\infty} t^{2-2\sigma} |p_2(t)| = +\infty \tag{22}$$

be fulfilled. Then (1) *has a solution such that*

$$\limsup_{t\to+\infty} |u^{(j)}(t)| \exp(-\mu t^\sigma) = +\infty \tag{23}$$

for any $\mu > 0$ *and* $j \in \{1, 2\}$*. If, besides, for some* $\alpha \ge 0$

$$\limsup_{t\to+\infty} |p_k(t)| \exp(-\alpha k t^\sigma) < +\infty \ \ (k = 1, 2, 3), \tag{24}$$

Then there exists a solution of equation (1) *which satisfies condition* (23) *for any* $\mu > 0$ *and* $j \in \{0, 1, 2\}$.

Proof. We begin by assuming that condition (21) is fulfilled. Let $\mu > 0$ and u_1, u_2, v_{01}, v_{02}, v_{12}, x, A be defined as they were in proving Theorem 1, and let ν be chosen so that

$$\int_0^t [p_1(s)]_+\, ds \le (\nu - 2\mu) t^\sigma \ \text{ for } \ t \ge t_0. \tag{25}$$

Put

$$y(t) =$$

$$= colon\bigg(\int_0^t \exp\bigg(-\int_0^s [p_1(\tau)]_+\,d\tau\bigg)\exp(\nu s^\sigma)\,ds, \exp(\nu t^\sigma), \nu\sigma t^{\sigma-1}\exp(\nu t^\sigma)\bigg).$$

Then y on the interval $]0,+\infty[$ satisfies the system

$$y' = B(t)y,$$

where

$$B(t) = \begin{pmatrix} 0 & \exp\bigg(-\int_0^t [p_1(s)]_+\,ds\bigg) & 0 \\ 0 & 0 & 1 \\ b_{\sigma\nu}(t) & 0 & 0 \end{pmatrix},$$

$$b_{\sigma\nu}(t) = \frac{\big(\nu(\sigma-1)\sigma t^{\sigma-2} + \nu^2\sigma^2 t^{2\sigma-2}\big)\exp(\nu t^\sigma)}{\int_0^t \exp\big(-\int_0^s [p_1(\tau)]_+\,d\tau\big)\exp(\nu s^\sigma)\,ds}.$$

By (21) it is easy to verify that

$$\limsup_{t\to+\infty} \frac{p_3(t)\exp\big(\int_0^t [p_1(s)]_+\,ds\big)}{b_{\sigma\nu}(t)} = +\infty.$$

If $\varepsilon > 0$ is such that

$$x(t_0) \geq \varepsilon y(t_0) \geq 0,$$
$$A(t) \geq B(t) \geq 0 \ \text{ for } \ t \geq t_0,$$

then it can be easily shown that

$$x(t) \geq \varepsilon y(t) \ \text{ for } \ t \geq t_0.$$

Therefore

$$v_{01}(t) \geq \varepsilon \int_0^t \exp\bigg(-\int_0^s [p_1(\tau)]_+\,d\tau\bigg)\exp(\nu s^\sigma)\,ds \ \text{ for } \ t \geq t_0.$$

Hence by virtue of (25) we obtain

$$\limsup_{t\to+\infty} \frac{v_{01}(t)}{t^{1-\sigma}\exp(2\mu t^\sigma)} = +\infty. \tag{26}$$

Let us show that u_1 or u_2 satisfies condition (23). Indeed, if we assume the contrary, we have

$$\limsup_{t\to+\infty} |u_i'(t)| \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2),$$
$$\limsup_{t\to+\infty} |u_i(t)| t^{\sigma-1} \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2).$$

Then

$$\limsup_{t\to+\infty} v_{01}(t) t^{\sigma-1} \exp(-2\mu t^\sigma) < +\infty,$$

which contradicts (26). Thus u_1 or u_2 satisfies condition (23).

If, besides, (24) holds and

$$\limsup_{t\to+\infty} |u_i(t)| \exp(-\mu t^\sigma) < +\infty \quad (i = 1, 2),$$

then by virtue of Lemma 2 we obtain

$$\limsup_{t\to+\infty} |u_i'(t)| \exp(-(\mu+\alpha) t^\sigma) < +\infty \quad (i = 1, 2)$$

and

$$\limsup_{t\to+\infty} v_{01}(t) \exp(-(2\mu+\alpha) t^\sigma) < +\infty.$$

But, as above, this is a contradiction.

Now assume that condition (22) is fulfilled. Then the proof is carried out as above, only in this case

$$B(t) = \begin{pmatrix} 0 & \exp\left(-\int\limits_0^t [p_1(s)]_+\, ds\right) & 0 \\ \dfrac{\nu\sigma t^{\sigma-1}\exp(\nu t^\sigma)}{\int\limits_0^t \exp\big(-\int\limits_0^s [p_1(\tau)]_+\, d\tau\big)\exp(\nu s^\sigma)\, ds} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y(t) = colon\left(\int\limits_0^t \exp\left(-\int\limits_0^s [p_1(\tau)]_+\, d\tau\right)\exp(\nu s^\sigma) ds, \exp(\nu t^\sigma), 1\right).$$

The theorem is proved.

According to Theorem 3.2 [5], Theorem 3 immediately implies

Corollary 3.1. Let conditions (11), (20) be fulfilled and

$$\limsup_{t\to+\infty} |p_k(t)| < +\infty \quad (k = 1, 2), \qquad \lim_{t\to+\infty} p_3(t) = +\infty,$$
$$\limsup_{t\to+\infty} p_3(t) \exp(-3\alpha t^\sigma) < +\infty.$$

Then equation (1) has an oscillatory solution, satisfying conditions (23) for any $j \in \{0, 1, 2\}$.

In conclusion, we present a theorem on an asymptotic oscillatory solution of equation (1) when p_3 is a non-negative function.

Theorem 4. *If equation* (1) *is oscillatory,*

$$p_1(t) \geq 0, \quad p_2(t) \leq 0, \quad p_3(t) \geq 0 \ \textit{for} \ t \geq 0 \tag{27}$$

and

$$\int_0^{+\infty} p_1(t)\,dt < +\infty,$$

then equation (1) *has a non-oscillatory solution and any of such solutions satisfies the condition*

$$u(t)u'(t) \leq 0 \ \textit{for} \ t \geq 0, \quad \lim_{t\to+\infty} u(t) = 0. \tag{28}$$

To prove this theorem we need lemmas on the asymptotic properties of solutions of the differential equation

$$\Big(\frac{1}{a_2(t)}\Big(\frac{x'}{a_1(t)}\Big)'\Big)' + p(t)x = 0, \tag{29}$$

where $a_i(t) : R_+ \to]0, +\infty[\ (i = 1, 2)$, $p : R_+ \to R_+$ are continuous functions.

Lemma 3. *Let*

$$\int_0^{+\infty} a_2(t)\,dt = +\infty, \quad \int_0^{+\infty} a_1(t) \int_0^t a_2(s)\,ds\,dt = +\infty \tag{30}$$

and equation (1) *have the solution x which for some $t_0 \geq 0$ satisfies the conditions*

$$x(t) > 0, \quad x'(t) > 0, \quad \Big(\frac{1}{a_1(t)} x'(t)\Big)' > 0 \ \textit{for} \ t \geq t_0.$$

Then equation (29) *is non-oscillatory.*

For the proof of this lemma see ([6], Lemma 4.2).

Lemma 4. *If p is not identically zero in the neighborhood of $+\infty$, conditions* (30) *are fulfilled and x is a solution of equation* (29) *that satisfies the inequality*

$$x(t) > 0 \ \textit{for} \ t \geq t_0. \tag{31}$$

Then there exists $t_1 \geq t_0$ such that either

$$x'(t) > 0, \quad \Big(\frac{1}{a_1(t)}\, x'(t)\Big)' > 0 \ \textit{for} \ t \geq t_1$$

or

$$x'(t) < 0, \quad \Big(\frac{1}{a_1(t)}\, x'(t)\Big)' > 0 \ \textit{for} \ t \geq t_0.$$

Proof. To prove the lemma it suffices to show that

$$\frac{1}{a_2(t)}\Big(\frac{1}{a_1(t)}\,x'(t)\Big)' > 0 \ \text{ for } \ t \ge t_0. \tag{32}$$

Since $p(t) \ge 0$, the function

$$\frac{1}{a_2}\Big(\frac{1}{a_1}\,x'\Big)'$$

does not increase. If (32) does not hold, then since p is not identically zero in the neighborhood of ∞, there are $t_1 \ge t_0$ and $c_0 < 0$ such that

$$\frac{1}{a_2(t)}\Big(\frac{1}{a_1(t)}\,x'(t)\Big)' \le c_0 \ \text{ for } \ t \ge t_1.$$

This inequality readily implies that

$$x(t) \le c_0 \int\limits_{t_1}^{t} a_1(s_1) \int\limits_{t_1}^{s_1} a_2(s_2)\, ds_2\, ds_1 + \frac{x'(t_1)}{a_1(t_1)} \int\limits_{t_1}^{t} a_1(s)\, ds + x(t_1) \ \text{ for } \ t \ge t_1.$$

If in the latter inequality we pass to the limit as $t \to +\infty$, then, taking (30) into account, we have

$$\lim_{t\to+\infty} x(t) = -\infty.$$

The obtained contradiction proves (32). The lemma is proved.

Lemma 5. *Let condition* (30) *be fulfilled. Then for the existence of a solution x of equation* (29) *that satisfies the condition*

$$\lim_{t\to+\infty} x(t) = 1, \tag{33}$$

it is necessary and sufficient that

$$\int\limits_{0}^{+\infty} \int\limits_{0}^{s_3} a_2(s_2) \int\limits_{0}^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 < +\infty. \tag{34}$$

Proof. *Sufficiency.* Choose such a large t_0 that

$$\int\limits_{t_0}^{+\infty} \int\limits_{t_0}^{s_3} a_2(s_2) \int\limits_{t_0}^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 = \Theta < 1.$$

Let

$$S = \Big\{x \in C([t_0,+\infty[) : \ 0 \le x(t) \le 2 \text{ for } t \ge t_0\Big\}.$$

Consider the integral operator $F : S \to S$ defined by the equality

$$F(x)(t) = 1 + \int\limits_{t}^{+\infty} \int\limits_{t}^{s_3} a_2(s_2) \int\limits_{t}^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3) x(s_3)\, ds_3.$$

If $u, v \in S$, then

$$\begin{aligned}&\big|F(u)(t) - F(v)(t)\big| \\ &\leq \Big| \int\limits_t^{+\infty}\int\limits_t^{s_3} a_2(s_2) \int\limits_t^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\big(u(s_3) - v(s_3)\big)\, ds_3\Big| \\ &\leq \|u - v\| \cdot \Theta \ \text{ for } \ t \geq t_0.\end{aligned}$$

This means that F is a contracting operator and by virtue of the well-known Banach theorem, F has a fixed point, i.e. there exists $x \in S$ such that

$$x(t) = 1 + \int\limits_t^{+\infty}\int\limits_t^{s_3} a_2(s_2) \int\limits_t^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 \ \text{ for } \ t \geq t_0.$$

It is easy to verify that x is a solution of equation (29) that satisfies (33).

Necessity. Assume that x is a solution of equation (29) that satisfies condition (33). Then by virtue of Lemma 4 there exists $t_0 > 0$ such that

$$x(t) > 0, \quad x'(t) < 0, \quad \Big(\frac{1}{a_1(t)}\, x'(t)\Big)' > 0 \ \text{ for } \ t \geq t_0.$$

The equality

$$\begin{aligned}&\int\limits_{t_0}^{t}\int\limits_{t_0}^{s} a_2(s_2) \int\limits_{t_0}^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s) x(s)\, ds \\ &= -\int\limits_{t_0}^{t} a_2(s_2) \int\limits_{t_0}^{s_2} a_1(s_1)\, ds_1\, ds_2\, \frac{1}{a_2(t)} \Big(\frac{x'(t)}{a_1(t)}\Big)' \\ &\qquad + \int\limits_{t_0}^{t} a_1(s_1)\, ds_1\, \frac{x'(t)}{a_1(t)} - x(t) + x(t_0) \ \text{ for } \ t \geq t_0\end{aligned}$$

implies (34). The lemma is proved.

Lemma 6. *Let condition* (30) *be fulfilled. Then for the existence of a solution x of equation* (29) *that satisfies the condition*

$$\lim_{t \to +\infty} \frac{x(t)}{\int\limits_0^t a_1(s) \int\limits_0^s a(\tau)\, d\tau\, ds} = 1. \tag{35}$$

It is necessary and sufficient that

$$\int\limits_0^{+\infty} p(s_3) \int\limits_0^{s_3} a_1(s_1) \int\limits_0^{s_1} a_2(s_2)\, ds_2\, ds_1\, ds_3 < +\infty. \tag{36}$$

Proof. The *sufficiency* is proved as in Lemma 5, but in this case the set S and the operator $F : S \to S$are defined as follows

$$S = \Big\{u \in C([t_0, +\infty[) :\ 0 \le u(t) \le \int\limits_0^t a_1(s) \int\limits_0^s a_2(\tau)\, d\tau\, ds \text{ for } t \ge t_0\Big\},$$

$$F(u)(t) = \int\limits_{t_0}^t a_1(s_1) \int\limits_{t_0}^{s_1} a_2(s_2)\, ds_2\, ds_1 + \int\limits_{t_0}^t a_1(s_1) \int\limits_{t_0}^{s_1} a_2(s_2) \int\limits_{s_2}^{+\infty} p(s_3)u(s_3)\, ds_3\, ds_2\, ds_1.$$

Necessity. If x is a solution of equation (29) that satisfies condition (35), then, taking into account Lemma 4, we obtain

$$x(t) > 0, \quad x'(t) > 0, \quad \Big(\frac{1}{a_1(t)}\, x'(t)\Big)' > 0 \ \text{ for } \ t \ge t_0.$$

Then by virtue of (35) from the equality

$$\int\limits_{t_0}^t p(s)x(s)\, ds = -\frac{1}{a_2(t)}\Big(\frac{x'(t)}{a_1(t)}\Big)' + \frac{1}{a_2(t)}\Big(\frac{x'(t)}{a_1(t)}\Big)'\Big|_{t=t_0}$$

we have (36). The lemma is proved.

Lemma 7. *Let equation* (29) *be oscillatory and let condition* (30) *be fulfilled. In addition to this, assume that there is a number $c > 0$ such that the inequality*

$$\frac{a_1(s_1)}{a_2(s_1)} \ge \frac{a_1(s_2)}{a_2(s_2)} \cdot c$$

holds for for any $s_1 > 0$ and $s_2 > 0$, where $s_1 \le s_2$. Then equation (29) *has a non-oscillatory solution and any such solution tends to zero at infinity.*

Proof. The existence of a non-oscillatory solution follows from Theorem 14.2.1 in [7]. Since equation (29) is oscillatory, by virtue of Lemmas 3, 4, 6

$$\int\limits_0^{+\infty} p(s_3) \int\limits_0^{s_3} a_1(s_1) \int\limits_0^{s_1} a_2(s_2)\, ds_2\, ds_1\, ds_3 = +\infty.$$

Then, since

$$\int_0^{+\infty}\int_0^{s_3} a_2(s_2)\int_0^{s_2} a_1(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3$$

$$= \int_0^{+\infty}\int_0^{s_3} a_2(s_2)\int_0^{s_2} \frac{a_1(s_1)}{a_2(s_1)}\,a_2(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3$$

$$\geq c\int_0^{+\infty}\int_0^{s_3} a_1(s_2)\int_0^{s_2} a_2(s_1)\,ds_1\,ds_2 p(s_3)\,ds_3,$$

we have

$$\int_0^{+\infty} p(s_3)\int_0^{s_3} a_2(s_2)\int_0^{s_2} a_1(s_1)\,ds_1\,ds_2\,ds_3 = +\infty.$$

Therefore, if x is a non-oscillatory solution of equation (29), by virtue of Lemmas 3, 4, 5

$$\lim_{t\to+\infty} x(t) = 0.$$

The lemma is proved.

Proof. [Proof of Theorem 4] Equation (1) on the interval $[0,+\infty[$ can be written in the form (29), where

$$p(t) = p_3(t)v(t)\exp\bigg(\int_0^t p_1(s)\,ds\bigg),$$

a_1, a_2 are defined by the equalities

$$a_1(t) = v(t), \quad a_2(t) = v^{-2}(t)\exp\bigg(-\int_0^t p_1(\tau)\,d\tau\bigg),$$

and v is a solution of the equation

$$(g(t)v')' + q(t) = 0,$$

where

$$g(t) = \exp\bigg(\int_0^t p_1(\tau)\,d\tau\bigg), \quad q(t) = g(t)p_2(t),$$

which satisfies the condition

$$v(t) > 0, \quad v'(t) \leq 0 \ \text{ for } \ t \geq 0.$$

Then, as is known (see [7, pp. 419–422]), condition (30) is fulfilled.

Moreover,

$$\frac{a_1(s_1)}{a_2(s_1)} = \frac{a_1(s_2)}{a_2(s_2)} \cdot \frac{v_1^3(s_1)}{v_1^3(s_2)} \exp\Big(- \int\limits_{s_1}^{s_2} p_1(\tau)\, d\tau \Big)$$
$$\geq \frac{a_1(s_2)}{a_2(s_2)} \cdot c \ \text{ for } \ s_2 \geq s_1 \geq 0,$$

where

$$c = \exp\Big(- \int\limits_0^{+\infty} p_1(\tau)\, d\tau \Big).$$

Thus all the conditions of Lemma 7 are fulfilled. This lemma immediately implies the validity of the theorem.

Remark 3. In Theorem 4 the condition $p_2(t) \leq 0$ for $t \geq 0$ is an essential one.

Indeed, let us consider the equation

$$u''' + \frac{1}{4t^2}\, u' + \frac{c}{t^3\, \ln^{3/2} t} u = 0 \ \ (t \geq a > 1), \tag{37}$$

where $c > 0$. By Theorem 5 [8] this equation is oscillatory. Equation (37) can be written in the form (29), where

$$a_1(t) = t^{\frac{1}{2}}, \quad a_2(t) = \frac{1}{t}, \quad p(t) = \frac{c}{t^{5/2}\, \ln^{3/2} t}.$$

Since

$$\int\limits_a^{+\infty} \int\limits_a^{s_3} a_2(s_2) \int\limits_a^{s_2} a_1(s_1)\, ds_1\, ds_2 p(s_3)\, ds_3 < +\infty.$$

By virtue of Lemma 5, equation (37) has a solution, satisfying condition (33).

Corollaries 1.1, 2.1 and Theorem 4 immediately give rise to the following propositions.

Corollary 4.1. Let $\alpha < 1$, conditions (27) be fulfilled and

$$\int\limits_0^{+\infty} p_1(t)\, dt < +\infty, \quad \lim_{t \to +\infty} t^{k\alpha} p_k(t) = 0 \ \ (k = 1, 2),$$
$$0 < \liminf_{t \to +\infty} t^{3\alpha} p_3(t) \leq \limsup_{t \to +\infty} t^{3\alpha} p_3(t) < +\infty.$$

Then equation (1) has both non-oscillatory solutions, satisfying condition (28) and oscillatory solutions, satisfying conditions (13), (14).

Corollary 4.2. Let conditions (27) be fulfilled and

$$\int\limits_0^{+\infty} p_1(t)\, dt < +\infty, \quad \lim_{t \to +\infty} t^{k} p_k(t) = 0 \ \ (k = 1, 2),$$
$$\frac{2\sqrt{3}}{9} < \liminf_{t \to +\infty} t^{3} p_3(t) \leq \limsup_{t \to +\infty} t^{3} p_3(t) < +\infty.$$

Then equation (1) has both oscillatory solutions satisfying both condition (28) and conditions (13), (14).

Remark 4. From the results of [9] (see also [10], [11]) it follows that under the conditions of Theorem 4, the solution of equation (1), satisfying condition (28), is unique to within a constant multiplier.

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COMPARISON THEOREMS AND SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Some two-point singular boundary value problems for second order linear differential equations are investigated.

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1. Introduction

Consider the differential equations

$$u'' + p(t)\,u = 0 \tag{1.1}$$

and

$$v'' + q(t)\,v = 0, \tag{1.2}$$

where $p, q \in C((a,b); R)$. For these equations Sturm [1] proved a comparison theorem, which later was widely used in studying the boundary value problems and asymptotic behavior of solutions. For generalizations of Sturm's theorems see [2], and for singular case see [3-5].

2. Some auxiliary lemmas

Lemma 2.1. *Let $a < t_0 < b$,*

$$p, q \in C((a, t_0]; R_+) \tag{2.1}$$

and

$$\int_t^{t_0} (p(s) - q(s))ds \geq 0 \quad \text{for} \quad t \in (a, t_0]. \tag{2.2}$$

Let $v \in C^{(2)}((a, t_0]; [0, +\infty))$ be a solution of equation (1.2) under conditions

$$\lim_{t \to a_0+} v(t) = v(a_0+) = 0, \quad v'(t_0) = 0$$

and

$$\int_{a_0}^{t_0} \frac{ds}{v^2(s)} = +\infty,$$

where $a \leq a_0$, $v(t) > 0$ for $t \in (a_0, t_0]$ and $v(a_0) = 0$. If $u \in C^{(2)}((a_0, t_0]; R)$ is a solution of equation (1.1), then at least one of the conditions

1) *there exist $t_* \in (a, t_0)$ such that $u(t_*) = 0$*

or

2) $u'(t_0) \leq 0$

is fulfilled.

Lemma 2.2. *Let* $a < t_0 < b$

$$p, q \in C([t_0, b); R_+) \tag{2.3}$$

and

$$\int_{t_0}^{t} (p(s) - q(s))ds \geq 0 \quad \textit{for} \quad t \in [t_0, b).$$

Let $v \in C^{(2)}([t_0 b); [0, +\infty))$ *be a solution of equation* (1.2) *under conditions*

$$v(b_0-) = \lim_{t \to b_0-} v(t) = 0, \quad v'(t_0) = 0$$

and

$$\int_{t_0}^{b_0} \frac{ds}{v^2(s)} = +\infty,$$

where $b_0 \leq b$, $v(t) > 0$ *for* $t \in [t_0, b_0)$ *and* $v(b_0) = 0$. *If* $u \in C^{(2)}[t_0, b_0); R)$ *is a solution of equation* (1.1), *then at least one of the conditions*

1) *there exist* $t_* \in (t_0, b_0)$ *such that* $u(t_*) = 0$

or

2) $u'(t_0) \geq 0$

is fulfilled.

Remark 2.1. If in Lemmas 2.1 and 2.2 $p(t) \geq q(t)$ for $t \in (a, b)$, then conditions (2.1) and (2.3) are unnecessary.

3. Two-point boundary value problems

Consider the problems on the existence of solution of the equation

$$u'' + q(t)\, u = f(t), \tag{3.1}$$

where $q, f \in C((a, b); R)$, under conditions

$$u(a+) = 0, \quad u'(b-) = 0, \tag{3.2}$$

$$u'(a+) = 0, \quad u(b-) = 0 \tag{3.3}$$

and

$$u(a+) = u(b-) = 0. \tag{3.4}$$

Theorem 3.1. *Let* $q \in C((a, b]; R_+)$, $(t-a)q \in L([a, b])$ *and there exists a function* $p \in C((a, b]; R_+)$ *such that*

$$\int_{t}^{b} (p(s) - q(s))ds \geq 0 \quad \textit{for} \quad t \in (a, b),$$

and equation (1.1) *has a solution* $u : (a, b) \to (0, +\infty)$ *such that* $u'(b) > 0$. *Then problem* (3.1), (3.2) *has only one solution.*

Corollary 3.1. Let $q \in C((a,b];R_+)$, $(t-a)q \in L([a,b])$ and

$$\int_t^b p(s)ds \le \frac{b-t}{4(b-a)(t-a)} \quad \text{for} \quad t \in (a,b].$$

Then problem (3.1), (3.2) has only one solution.

Corollary 3.2. Let $q \in C((a,b];R_+)$, $(t-a)q \in L([a,b])$,

$$q(t) \le \frac{1}{4(t-a)^2} \quad \text{for} \quad t \in [a,b).$$

Then problem (3.1), (3.3) has only one solution.

Theorem 3.2. *Let $q \in C([a,b);R_+)$, $(b-t)q \in L([a,b])$ and there exists a function $p \in C([a,b];R_+)$ such that*

$$\int_a^t (p(s)-q(s))ds \ge 0 \quad \textit{for} \quad t \in [a,b),$$

and equation (1.1) *has a solution $u : (a,b) \to (0,+\infty)$ such that $u'(a) < 0$. Then problem* (3.1), (3.3) *has only one solution.*

Corollary 3.3. Let $q \in C([a,b);R_+)$, $(b-t)q \in L([a,b]$ and

$$\int_a^t q(s)ds \le \frac{t-a}{4(b-a)(b-t)} \quad \text{for} \quad t \in [a,b).$$

Then problem (3.1), (3.3) has only one solution.

Corollary 3.4. Let $q \in C([a,b);R)$, $(b-t)q \in L([a,b])$ and

$$q(t) \le \frac{1}{4(b-t)^2} \quad \text{for} \quad a \le t < b.$$

Then problem (3.1), (3.2) has only one solution.

Theorem 3.3 *Let $p;q \in C((a,b);R)$, $(t-a)(b-t)q \in L([a,b])$ and*

$$q(t) \le p(t) \quad \textit{for} \quad t \in (a,b). \tag{3.5}$$

If there exist $t_* \in (a,b)$ and solution $u \in C^{(2)}((a,b);(0,+\infty))$ of equation (1.1) *such that $u'(t) > 0$ for $t \in (a,t_*]$ or $u'(t) < 0$ for $t \in [t_*,b)$, then problem* (3.1), (3.4) *has only one solution.*

Corollary 3.5. Let $q \in C((a,b);R)$, $(t-a)(b-t)q \in L([a,b])$ and let (3.5) be fulfilled, where

$$p(t) = \begin{cases} \dfrac{1}{4(t-a)^2} & \text{for} \quad t \in \left(a, \frac{a+b}{2}\right], \\ \dfrac{1}{4(b-t)^2} & \text{for} \quad t \in \left[\frac{a+b}{2}, b\right). \end{cases}$$

Then problem (3.1), (3.4) has only one solution.

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ON ONE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURES FOR A SQUARE WHICH IS WEAKENED BY A HOLE AND BY CUTTINGS AT VERTICES

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Abstract. In the present work we consider the problem of statics of the linear theory of elastic mixtures for a square which is weakened by a hole and by cuttings at vertices about of finding an equally strong contour. The hole and cutting boundaries are assumed to be free from external forces, and to the remaing part of the square boundary are applied the same absolutely rigid punches, subjected to the action of external normal contractive forces with the given principal vectors.

Relying on the analogous to Kolosov-Muskhelishvilis formulas, in the linear theory of elastic mixtures, the problem reduces to a mixed problem of the theory of analytic functions (the Keldysh-Sedov problem), and the solution of the latter allows us to construct complex potentials and equations of an unknown contour efficiently (in analytical form). The analysis of the obtained results is carried out and the formula of tangential normal stress vector is derived.

Keywords and phrases: Equally strong contour, elastic mixture, generalized Kolosov-Muskhelishvili representation, Keldysh-Sedov problem.

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Introduction

The problems of the plane theory of elasticity for infinite domains weakened by equally strong holes have been studied in [1], [8] and also by many other authors. The same problem for simple and doubly-connected domains with partially unknown boundaries are investigated in [2], [9] etc. The mixed boundary value problems of the plane theory of elasticity for domains with partially unknown boundaries have been studied by R. Bantsuri [3]. Analogous problem in the case of the plane theory of elastic mixtures is considered in [16].

In [14], using the method, suggested by R. Banstsuri in [4], the author gives a solution of the mixed problem of the plane theory of elasticity for a finite multiply connected domain with a partially unknown boundary having the axis of symmetry. Analogous problem in the case of the plane theory of elastic mixtures has been studied in [17]. In the work of R. Bantsuri and G. Kapanadze [5] the problem, of statics of the plane theory of elasticity, of finding an equally strong contour for a square which is weakened by a hole and by cuttings at vertices are considered.

In the present work, in the case of the plane theory of elastic mixtures we study the problem, analogous to that solved in [5]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili's formula [17] and the method, developed in [5].

1. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [7]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + \mathcal{K} \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{1.1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements

$$\mathcal{K} = -\frac{1}{2} e m^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0} \begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$
$$\Delta_0 = m_1 m_3 - m_2^2, \quad m_k = e_k + \frac{1}{2} e_{3+k}, \quad e_1 = a_2/d_2,$$
$$e_2 = -c/d_2, \quad e_3 = a_1/d_2, \quad d_2 = a_1 a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5,$$
$$a_3 = \mu_3 + \lambda_5, \quad e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1, \quad e_3 + e_6 = a/d_1,$$
$$a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2,$$
$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4,$$
$$\rho = \rho_1 + \rho_2, \quad d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2\rho_1/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\rho_2/\rho.$$

Here μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$ are elasticity modules, characterizing mechanical properties of a mixture, ρ_1 and ρ_2 are its particular densities. The elastic constants μ_1, μ_2, μ_3, λ_p $p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of inequality [13].

In [6] M. Bashaleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2} e \overline{\varphi'(z)} + \overline{\psi(z)} \tag{1.2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix}$$
$$= \frac{\partial}{\partial s(x)} [(A - 2E)\varphi(z) + Bz\overline{\varphi(z)'} + 2\mu\overline{\psi(z)}], \tag{1.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions:

$$A = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad B = \mu e, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\frac{\partial}{\partial s(x)} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial n(x)} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad n = (n_1, n_2)^T$$

are the unit vectors of the other normal, $(Tu)_p$, $p = \overline{1,4}$, the stress components [6]

$$(Tu)_1 = r'_{11} n_1 + r'_{21} n_2, \quad (Tu)_2 = r'_{12} n_1 + r'_{22} n_2,$$
$$(Tu)_3 = r''_{11} n_1 + r''_{21} n_2, \quad (Tu)_4 = r''_{12} n_1 + r''_{22} n_2,$$

Consider the following vectors [16] or [17]

$$\overset{(1)}{\tau}=\begin{pmatrix} r'_{11}\\ r''_{11}\end{pmatrix}=\begin{bmatrix} a & c_0\\ c_0 & b\end{bmatrix}\begin{pmatrix}\theta'\\ \theta''\end{pmatrix}-2\frac{\partial}{\partial x_2}\mu\begin{pmatrix}u_2\\ u_4\end{pmatrix},$$
$$\overset{(2)}{\tau}=\begin{pmatrix} r'_{22}\\ r''_{22}\end{pmatrix}=\begin{bmatrix} a & c_0\\ c_0 & b\end{bmatrix}\begin{pmatrix}\theta'\\ \theta''\end{pmatrix}-2\frac{\partial}{\partial x_1}\mu\begin{pmatrix}u_1\\ u_3\end{pmatrix}, \tag{1.4}$$

$$\overset{(1)}{\eta}=\begin{pmatrix} r'_{21}\\ r''_{21}\end{pmatrix}=-\begin{bmatrix} a_1 & c\\ c & a_2\end{bmatrix}\begin{pmatrix}\omega'\\ \omega''\end{pmatrix}+2\frac{\partial}{\partial x_1}\mu\begin{pmatrix}u_2\\ u_4\end{pmatrix},$$
$$\overset{(2)}{\eta}=\begin{pmatrix} r'_{12}\\ r''_{12}\end{pmatrix}=\begin{bmatrix} a_1 & c\\ c & a_2\end{bmatrix}\begin{pmatrix}\omega'\\ \omega''\end{pmatrix}+2\frac{\partial}{\partial x_2}\mu\begin{pmatrix}u_1\\ u_3\end{pmatrix}, \tag{1.5}$$
$$\theta'=\operatorname{div}u',\quad \theta''=\operatorname{div}u'',\quad \omega'=\operatorname{rot}u',\quad \omega\operatorname{rot}u''.$$

Let ($\mathbf{n}$,S) be the right rectangular system, where S and n are respectively, the tangent and the normal of the curve L at the point $t=t_1+it_2$. Assume that $n=(n_1,n_2)^T=(\cos\alpha,\sin\alpha)^T$ and $S^0=(-n_2,n_1)^T=(-sin\alpha,\cos\alpha)^T$, where α is the angle of inclination of the normal n to the ox_1-axis.

Introduce the vectors

$$U_n=(u_1n_1+u_2n_2;u_3n_1+u_4n_2)^T,\ U_s=(u_2n_1-u_1n_2;u_4n_1-u_3n_2)^T; \tag{1.6}$$

$$\sigma_n=\begin{pmatrix}(Tu)_1n_1+(Tu)_2n_2\\ (Tu)_3n_1+(Tu)_4n_2\end{pmatrix},\quad \sigma_s=\begin{pmatrix}(Tu)_2n_1-(Tu)_1n_2\\ (Tu)_4n_1-(Tu)_3n_2\end{pmatrix}, \tag{1.7}$$

$$\sigma_t=\begin{pmatrix}[r'_{21}n_1-r'_{11}n_2,\quad r'_{22}n_1-r'_{12}n_2]^TS^0\\ [r''_{21}n_1-r''_{11}n_2,\quad r''_{22}n_1-r''_{12}n_2]^TS^0\end{pmatrix}. \tag{1.8}$$

Let us call (1.8) vector the tangential normal stress vector in the linear theory of elastic mixture.

After elementary calculations we obtain

$$\sigma_n=\overset{(1)}{\tau}\cos^2\alpha+\overset{(2)}{\tau}\sin^2\alpha+\eta\cos\alpha\sin\alpha,$$
$$\sigma_t=\overset{(1)}{\tau}\sin^2\alpha+\overset{(2)}{\tau}\cos^2\alpha-\eta\cos\alpha\sin\alpha,$$
$$\sigma_s=\frac{1}{2}(\overset{(2)}{\tau}-\overset{(1)}{\tau})\sin 2\alpha+\frac{1}{2}\eta\cos 2\alpha-\frac{1}{2}\varepsilon^*$$

where $\eta=\overset{(1)}{\eta}+\overset{(2)}{\eta}$, $\varepsilon^*=\overset{(1)}{\eta}-\overset{(2)}{\eta}$.

Direct calculations allow us to check on L [16]

$$\sigma_n+\sigma_t=\tau=\overset{(1)}{\tau}+\overset{(2)}{\tau}=2(2E-A-B)\operatorname{Re}\varphi'(t); \tag{1.9}$$

$$\sigma_n+2\mu\Big(\frac{\partial Us}{\partial s}+\frac{U_n}{\rho_0}\Big)+i\Big[\sigma_s-2\mu\Big(\frac{\partial U_n}{\partial s}-\frac{U_s}{\rho_0}\Big)\Big]=2\varphi'(t), \tag{1.10}$$

$$[(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}]_L=-i\int_L e^{i\alpha}(\sigma_n+i\sigma_s)ds; \tag{1.11}$$

where $\det(2E - A - B) > 0$, $\frac{1}{\rho_0}$ is the curvature of L at the point t. Everywhere in the sequel it will be assumed that the components U_n and U_s are bounded [7].

Formulas (1.2), (1.3) and (1.9), (1.10) are analogous to those of Kolosov-Muskhelishvili in the linear theory of elastic mixture [12].

2. Statement of the problem and the method of its solving

Let an isotropic elastic mixture occupy on the plane $z = x_1 + ix_2$ a doubly-connected domain G, a square. The side lenght of square will be denoted by a^0.

Let to the boundary of the square which is weakened by an interior hole and cuttings at vertices be applied the same absolutely smooth rigid punches, subjected to the action of external normal contractive forces with the known principal vectors. The hole and cutting boundary is free from external forces.

We formulate the following problem: Find an elastic equilibrium of the square and analytic form of the hole and cutting contours under the condition that tangential normal stress vector, i. e. (1.8) vector, will take one and the same constant value $\sigma_t = K^0$, $K^0 = (K_1^0, K_2^0) = \text{const}$ on them.

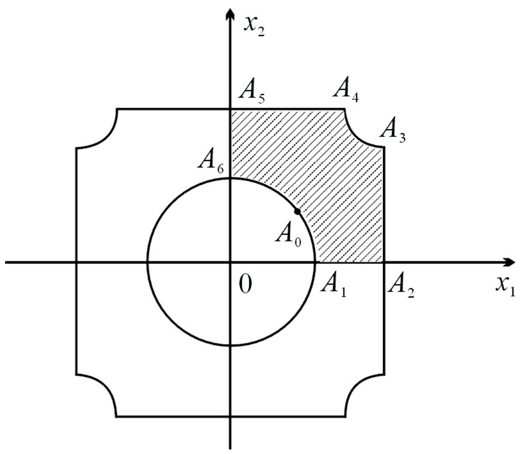

Figure 1:

In these conditions, we call the assemblage of hole and cutting boundaries an equally strong contour. Owing to the symmetry of the problem, we consider the shaded part of the square, i. e. the curvilinear polygon $A_0A_1A_2A_3A_4A_5A_6$ and denote it by D_0, where A_0 is the mid point of the arc A_6A_1 (a shaded in fig.1).

The boundary of the domain D_0 consists of rectilinear segments $L_1 = \cup L_1^{(j)}$, $L_1^{(j)} = A_jA_{j+1}$ $(j = 1, 2, 4, 5)$ and unknown arcs $L_0 = L_0^{(1)} \cup L_0^{(2)}$, $L_0^{(1)} = A_3A_4$, $L_0^{(2)} = A_6A_1$.

The boundary conditions of the problem are of the form $U_n = U^0 = \text{const}$ on $L_1^{(2)} \cup L_1^{(4)}$, and $U_n = 0$ on $L_1^{(1)} \cup L_1^{(5)}$, vector (1.7), is equal to zero on the entire boundary of the domain D_0, i.e. $\sigma_s = 0$ on $L = L_1 \cup L_0$.

Relying on the analogous Kolosov-Muskhelishvilis formulas (1.9)-(1.11), the above posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in D^0

by the boundary conditions on L

$$\operatorname{Re}\varphi'(t) = H, \quad t \in L_0, \quad H = \frac{1}{2}(2E - A - B)^{-1}K^0, \tag{2.1}$$

$$\operatorname{Im}\varphi'(t) = 0, \quad t \in L_1, \tag{2.2}$$

$$\operatorname{Re} e^{-i\alpha(t)}[(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}] = C(t), \quad t \in L_1, \tag{2.3}$$

$$(A-2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)} = B^{(j)}(t), \quad t \in L_0^{(j)}, \quad j = 1, 2; \tag{2.4}$$

where $\alpha(t)$ is the angle, made by the outer normal to the contour L_1 and the $0x_1$-axis,

$$C(t) = \operatorname{Re}\left[-i\int_{A_1}^{t}\sigma_n(t_0)\exp i[\alpha(t_0)-\alpha(t)]dS_0 + \nu\exp(-i\alpha(t))\right], \ t \in L_1 \tag{2.5}$$

$$B^{(j)}(t) = -i\int_{A_1}^{t}\sigma_n(t_0)\exp(i\alpha(t_0))dS_0 + \nu, \ \ t \in L_0^{(j)}, \ \ j = 1, 2, \tag{2.6}$$

$\nu = (\nu_1, \nu_2)^T$ is an arbitrary complex constant vector. It is easy to notice that $C(t)$ is a piecewise constant and $B^{(j)}$ is a constant vector-function.

Moreover, if $t \in L_1$, then we can write

$$\operatorname{Re} e^{-i\alpha(t)}t = \operatorname{Re} e^{-\alpha(t)}A(t) \tag{2.7}$$

where $A(t) = A_k$ for $t \in A_kA_{k+1}$.

In the sequel, the vector-function $\varphi(z)$ will be assumed to be continuous in a closed domain D_0, and $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the boundary of the body D_0 except possibly of the points A_1, A_3, A_4, A_6 in the neighborhood of which they admit the estimate of the type

$$|\varphi_j'(z)|, \quad |\psi_j(z)| < M|z - A_k|^{-\delta_k}, \quad j = 1, 2, \tag{2.8}$$

where $0 < \delta_k < \frac{1}{2}$, $k = 1, 3, 4, 6$, $M = \text{const} > 0$.

The equalities (2.1)–(2.2) are in fact the Keldysh-Sedov problem for the domain D_0.

By virtue of the condition (2.8), the (2.1)–(2.2) problem has a unique solution [10] or [11], $\varphi'(z) = H$.

Consequently, leaving out of account the constant summand we get

$$\varphi(z) = Hz = \frac{1}{2}(2E - A - B)^{-1}K^0 z. \tag{2.9}$$

Here K^0 is to be defined in the course of solving the problem.

On the basis of formulas (1.11), (2.5), (2.6), (2.9) and putting $\nu = 0$, the boundary conditions (2.3), (2.4) and (2.7) yield

$$\begin{gathered}
\operatorname{Im}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=0;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_1^{(1)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Re}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=a^0K^0-P,\ t\in L_1^{(2)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=P,\ t\in L_0^{(1)};\\
\operatorname{Im}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=a^0K^0-P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=P,\ t\in L_1^{(4)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=0;\ \operatorname{Re}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_1^{(5)};\\
\operatorname{Re}\left(\frac{1}{2}K^0t-2\mu\psi(t)\right)=P;\ \operatorname{Im}\left(\frac{1}{2}K^0t+2\mu\psi(t)\right)=0,\ t\in L_0^{(2)};
\end{gathered}\tag{2.10}$$

where

$$P=\int_{L_1^{(j)}}\sigma_n dS,\quad j=1,2,4,5.$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi_1+i\xi_2$ map conformally the upper half-plane ($\operatorname{Im}\zeta>0$) onto the domain D_0. By β_k we denote the preimages of the points A_k ($k=\overline{0,6}$) and assume that $\beta_3=-1$; $\beta_4=1$; $\beta_0=-\infty$. Moreover, owing to the symmetry, we may assume that $\beta_5=-\beta_2$; $\beta_6=-\beta_1$. Note that

$$-\infty<\beta_1<\beta_2<-1<1<-\beta_2<-\beta_1<+\infty.$$

Consider the vector-functions

$$\phi(\zeta)=-i\Big(\frac{1}{2}K^0\omega(\zeta)-2\mu\psi(\omega(\zeta))\Big);\tag{2.11}$$

$$\Psi(\zeta)=\frac{1}{2}K^0\omega(\zeta)+2\mu\psi(w(\zeta)).\tag{2.12}$$

Taking into (2.11) and (2.12), boundary conditions (2.10) take the form

$$\begin{gathered}
\operatorname{Im}\phi(\xi_1)=0,\quad \xi_1\in(-\infty;\beta_1)\cup(-\beta_2;\infty);\\
\operatorname{Re}\phi(\xi_1)=0;\quad \xi_1\in(\beta_1;\beta_2);\\
\operatorname{Im}\phi(\xi_1)=-P;\quad \xi_1\in(\beta_2;1),\\
\operatorname{Re}\phi(\xi_1)=a^0K^0-P,\quad \xi_1\in(1;-\beta_2);
\end{gathered}\tag{2.13}$$

$$\begin{gathered}
\operatorname{Im}\Psi(\xi_1)=0,\quad \xi_1\in(-\infty;\beta_2)\cup(-\beta_1;\infty);\\
\operatorname{Re}\Psi(\xi_1)=a^0K^0-P,\quad \xi_1\in(\beta_2;-1),\\
\operatorname{Im}\Psi(\xi_1)=P,\quad \xi_1\in(-1,-\beta_2),\\
\operatorname{Re}\Psi(\xi_1)=0,\quad \xi_1\in(-\beta_2,-\beta_1).
\end{gathered}\tag{2.14}$$

The above problems are the vector form of the Keldysh-Sedov problems [10], [11] for a half-plane $\operatorname{Im}\zeta>0$.

A solution of problems (2.13) and (2.14) can be represented as follows [10], [5]

$$\phi(\zeta)=\frac{\chi_1(\zeta)}{\pi i}\left[(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{\chi_1(\xi_1)(\xi_1-\zeta)}-iP\int_{\beta_2}^{1}\frac{d\xi_1}{\chi(\xi_1)(\xi_1-\zeta)}\right], \tag{2.15}$$

$$\Psi(\zeta)=\frac{\chi_2(\zeta)}{\pi i}\left[(a^0K^0-P)\int_{\beta_2}^{-1}\frac{d\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}+iP\int_{-1}^{\beta_2}\frac{d\xi_1}{\chi_2(\xi_1)(\xi_1-\zeta)}\right], \tag{2.16}$$

where

$$\chi_1(\zeta)=\sqrt{(\zeta-\beta_1)(\zeta-\beta_2)(\zeta-1)(\zeta+\beta_2)},$$
$$\chi_2(\zeta)=\sqrt{(\zeta+\beta_1)(\zeta+\beta_2)(\zeta+1)(\zeta-\beta_2)}$$

Note that, under the $\chi_j(\zeta)$ sign we mean a branch whose decomposition near the point at infinity has the form

$$\chi_j(\zeta)=\zeta^2+\alpha_1^j\zeta+\alpha_2^{(j)}+\cdots,\quad j=1,2.$$

It is easy to show that

$$\chi_1(\xi_1)=\begin{cases}|\chi_1(\xi_1)|, & \xi_1\in(-\infty,\beta_1)\cup(\beta_2,1)\cup(-\beta_2,\infty)\\ -i|\chi_1(\xi_1)|, & \xi_1\in(\beta_1,\beta_2)\cup(1,-\beta_2);\end{cases} \tag{2.17}$$

$$\chi_2(\xi_1)=\begin{cases}|\chi_2(\xi_1)|, & \xi_1\in(-\infty,\beta_2)\cup(-1,-\beta_2)\cup(-\beta_1,\infty)\\ i|(\xi_1)|, & \xi_1\in(\beta_2,-1)\cup(-\beta_2,-\beta_1);\end{cases} \tag{2.18}$$

$$|\chi_1(\xi_1)|=|\chi_2(-\xi_1)|. \tag{2.19}$$

By virtue of (2.17)–(2.19) formulas (2.15) and (2.16) can be written as

$$\phi(\zeta)=g(\zeta),\quad \Psi(\zeta)=g(-\zeta),\quad \operatorname{Im}\zeta>0, \tag{2.20}$$

where $g=(g_1,g_2)^T$.

$$g(\zeta)=\frac{\chi_1(\zeta)}{\pi i}\left[(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1-\zeta)}-P\int_{\beta_2}^{1}\frac{d\xi_1}{|\chi_1(\zeta)|(\xi_1-\zeta)}\right]. \tag{2.21}$$

Now note that we will seek for a bounded at infinity solution of the problems (2.13) and (2.14). On the other hand, from (2.20) and (2.21) we conclude that, the necessary and sufficient condition for the existence of such a solution is of the form

$$(a^0K^0-P)\int_1^{-\beta_2}\frac{d\xi_1}{|\chi_1(\xi_1)|}-P\int_{\beta_2}^{1}\frac{d\xi_1}{|\chi_1(\xi_1)|}=0. \tag{2.22}$$

Having found the vector-function $\phi(\zeta)$ and $\Psi(\zeta)$, by virtue of (2.12) and (2.20) we can define the vector-functions $K^0\omega(\zeta)$ and $\psi(\omega(\zeta))$ by the formulas

$$K^0\omega(\zeta)=g(-\zeta)+ig(\zeta),\quad \psi(\omega(\zeta))=\frac{1}{4}\mu^{-1}[g(-\zeta)-ig(\zeta)]. \tag{2.23}$$

Let us now pass to finding analytical form of the unknown equally strong contour. Equations for the parts $L_0^{(1)}$ and $L_0^{(2)}$ of the unknown contour can be obtained from the image of the function $\omega(\zeta)$ for $\zeta = \xi_1^0 \in (-1, 1)$ and $\zeta = \xi_1^0 \in (-\infty, \beta_1) \cup (-\beta_1, \infty)$ respectively.

By the Sokhotskii-Plemelj formulas [11] and owing to (2.21) and (2.23), we find that the equations for the arcs $L_0^{(1)}$ and $L_0^{(2)}$ are given by the formulas respectively

$$\omega(\xi_1^0) = \frac{g_1(-\xi_1^0) + P_1 + i(g_1(\xi_1^0) + P_1)}{K_1^0} = \frac{g_2(-\xi_1^0) + i(g_2(\xi_1^0) + P_2)}{K_2^0}, \tag{2.24}$$

$$\omega(\xi_1^0) = \frac{g_1(-\xi_1^0) + i(g_1(\xi_1^0))}{K_1^0} = \frac{g_2(-\xi_1^0) + i(g_2(\xi_1^0))}{K_2^0}, \tag{2.25}$$

where

$$g(\xi_1^0) = (g_1, g_2)^T = \frac{\chi_1(\xi_1^0)}{\pi i}\left[(a^0 K^0 - P)\int_1^{-\beta_2} \frac{d\xi_1}{|\chi_1(\xi_1)|(\xi_1 - \xi_1^0)} - P\int_{\beta_2}^1 \frac{d\xi_1}{|\chi_1(\xi_1^0)(\xi_1 - \xi_1^0)|}\right].$$

Revert now to the condition (2.22). Equality (2.22) yelds

$$K^0 = \frac{1}{a^0} P\Big(1 + \frac{F_1}{F_2}\Big), \tag{2.26}$$

where

$$F_1 = \int_{\beta_2}^1 \frac{d\xi_1}{|\chi_1(\xi_1)|}, \quad F_2 = \int_1^{-\beta_2} \frac{d\xi_1}{|\chi_1(\xi_1)|}. \tag{2.27}$$

It should be noted the integrals appearing in (2.27) and (2.21) are expressed in terms of elliptic integrals of the first and third kind [15].

Of special importance is the definition of parameters K_1^0, K_2^0, β_1 and β_2, involved in the above formulas. For defining above parameters we use the way and results, described in [5].

Refer now to formulas (2.26) and (2.27). the values F_1 and F_2 are the complete elliptic integrals of the first kind [15], namely [5]

$$F_1 = M^{-1} F\left(\frac{\pi}{2}/m_1^0\right); \quad F_2 = M^{-1} F\left(\frac{\pi}{2}/m_2^0\right),$$

where

$$M = \sqrt{2}[\beta_2(\beta_1 - 1)]^{-\frac{1}{2}}, \quad F(\frac{\pi}{2}/m^0) = \int_0^{\frac{\pi}{2}} (1 - m^0 \sin^2\theta)^{-\frac{1}{2}} d\theta,$$

$$m_1^0 = \frac{2\beta_2(\beta_1 - 1)(\beta_1 + \beta_2)}{2\beta_2(\beta_1 - 1)}, \quad m_2^0 = \frac{(\beta_2 + 1)(\beta_1 - \beta_2)}{2\beta_2(\beta_1 - 1)}.$$

(of interest is the fact that $m_1^0 + m_2^0 = 1$ and $m_1^0 > m_2^0$).

Fixing the value of the parameter m_1^0 (and hence of parameter $m_2^0 = 1 - m_1^0$) for finding β_1 and β_2 we obtain the equality

$$\beta_2^2 + (1 - 2m_1^0)(\beta_1 - 1)\beta_2 - \beta_1 = 0 \tag{2.28}$$

the discriminant of the above equation(with respect to β_2) is of the form $D = (1 - 2m_1^0)^2(\beta_1 - 1)^2 + 4\beta_1$.

Introducing the notation $\sqrt{-\beta_1} = x$, from the condition $D \geq 0$, $x > 1$ we get

$$x \geq \frac{1 + 2\sqrt{m_1^0(1 - m_1^0)}}{2m_1^0 - 1} = l.$$

If we assume that $D > 0$, then to every value $x > l$, and hence $\beta_1 < -l^2$, according to (2.28), there correspond two values β_2, both satisfying the condition $\beta_2 < -1$, but this contradicts the condition of the uniqueness of the conformally mapping function $z = \omega(\zeta)$, and hence we should have $D = 0$ from which it follows that

$$\beta_1 = -\Big[\frac{1 + 2\sqrt{m_1^0(1 - m_1^0)}}{2m_1^0 - 1}\Big]^2; \quad \beta_2 = \frac{(2m_1^0 - 1)(\beta_1 - 1)}{2}. \tag{2.29}$$

Summing the obtained results, we conclude that for the fixed m_1^0 in the domain $(\frac{1}{2}, 1)$, from the table of complete elliptic integrals we can find F_1 and F_2, and using formulas (2.26) and (2.27) we define parameters $\mathcal{K}^0$, β_1, β_2 and the conformally mapping function $z = \omega(\zeta)$ formulas (2.24) and (2.25) which establishes analytical form of the unknown equally strong contour.

Direct calculations show that as m_1^0 increases, the length of the contour $L_0^{(1)}$ decreases, $L_0^{(2)}$ increases, and K_j^0 $j = 1, 2$, increases (see [5]).

In a particular case, for $m_1^0 = 0,75$ we have approximately [18]

$$F_1 = 2,156; \quad F_2 = 1,686; \quad K_j^0 = \frac{2,28}{a^0} P_j, \quad j = 1, 2;$$
$$\beta_1 = -13,7; \quad \beta_2 = -3,7; \quad g_j(0) = 0,743P_j, \quad j = 1, 2;$$
$$\omega(0) = (0,764a^0; 0,764a^0);$$
$$g_j(-1) = 0,386P_j, \quad j = 1, 2; \quad \omega(-1) = (a^0; 0,608a^0);$$
$$g_j(\infty) = g_j(-\infty) = 1,08P_j, \quad j = 1, 2;$$
$$\omega(\infty) = \omega(-\infty) = (0,474a^0; 0,474a^0);$$
$$g_j(-\beta_1) = 1,451P_j; \quad \omega(\beta_1) = (0,636a^0; 0).$$

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ON A CONNECTION BETWEEN CONTROLLABILITY OF THE INITIAL AND PERTURBED TWO-STAGE SYSTEMS

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Abstract. Sufficient conditions are established, guaranteeing controllability of the initial two-stage system of ordinary differential equations if a sequence of the perturbed two-stage systems is controllable, when the perturbations of right-hand side of system are small in the integral sense.

Keywords and phrases: Two-stage system, controllability, perturbations, sufficient conditions.

AMS subject classification (2010): 93B05, 93B12, 93C15, 93C73.

1. Formulation of main results

Let $t_{01} < t_{02} < \theta_1 < \theta_2 < t_{11} < t_{12}$ be given numbers and R^n_x be the n-dimensional vector space of points

$$x = (x^1, ..., x^n)^T, |x|^2 = \sum_{i=1}^{n} (x^i)^2,$$

where T means transpose; suppose that $O \subset R^n_x$ and $Y \subset R^m_y$ are open sets, $U \subset R^p_u$ and $V \subset R^q_v$ are compact sets. Further, let $E_f = E_f(I_1 \times O, R^n_x)$, be the space of functions $f(t,x) \in R^n_x$ defined on $I_1 \times O$ and satisfying the following conditions:

1.1. for any $x \in O$ the function $f(t,x)$ is measurable on $I_1 = [t_{01}, \theta_2]$;

1.2. for any function $f \in E_f$ and any compact set $K \subset O$ there exist functions $m_{f,K}(\cdot), L_{f,K}(\cdot) \in L_1(I_1, R_+), R_+ = [0, \infty)$ such that for almost all $t \in I_1$,

$$|f(t,x)| \le m_{f,K}(t), \forall x \in K$$

and

$$|f(t,x_1) - f(t,x_2)| \le L_{f,K}(t)|x_1 - x_2|, \forall (x_1,x_2) \in K^2.$$

Let $E^u_f = E^u_f(I_1 \times O, R^n_x)$ be the space of functions $f(t,x,u) \in R^n_x$ defined on $I_1 \times O \times U$ and satisfying the following conditions:

1.3. for any $(x,u) \in O \times U$ the function $f(t,x,u)$ is measurable on I_1;

1.4. for any function $f \in E^u_f$ and any compact set $K \subset O$ there exist functions $m_{f,K}(\cdot), L_{f,K}(\cdot) \in L_1(I_1, R_+)$ such that for almost all $t \in I_1$,

$$|f(t,x,u)| \le m_{f,K}(t), \forall (x,u) \in K \times U$$

and

$$|f(t,x_1,u) - f(t,x_2,u)| \le L_{f,K}(t)|x_1 - x_2|, \forall (x_1,x_2,u) \in K^2 \times U.$$

Analogously are defined the following spaces $E_g = E_g(I_2 \times Y, R^m_y)$ and $E^v_g = E^v_g(I_2 \times Y \times V, R^m_y)$, where $I_2 = [\theta_1, t_{12}]$.

Let $f_0 \in E_f^u$ and $g_0 \in E_g^v$ be given functions and $x_0 \in O$ and $y_1 \in Y$ be given points. By Ω and Δ we denote sets of measurable functions $u : I_1 \to U$ and $v : I_2 \to V$, respectively.

To each element

$$w = (t_0, \theta, t_1, u(\cdot), v(\cdot)) \in W = [t_{01}, t_{02}] \times [\theta_1, \theta_2] \times [t_{11}, t_{12}] \times \Omega \times \Delta$$

we assign the two-stage system of differential equations

$$\begin{cases} \dot{x} = f_0(t, x, u(t)), t \in [t_0, \theta], \\ \dot{y} = g_0(t, y, v(t)), t \in [\theta, t_1] \end{cases} \tag{1.1}$$

with the initial condition

$$x(t_0) = x_0 \tag{1.2}$$

and the intermediate condition at the switching moment θ

$$y(\theta) = Q(\theta, x(\theta)). \tag{1.3}$$

Here the function $Q(t, x) \in R_y^m$ is continuous on $[\theta_1, \theta_2] \times O$ and continuously differentiable with respect to $x \in 0$.

Definition 1.1. Let $w = (t_0, \theta, t_1, u(\cdot), v(\cdot)) \in W$. The pair of functions $\Phi(w) = \{x(t) = x(t; w) \in O, t \in [t_0, \theta]; y(t) = y(t; w) \in Y, t \in [\theta, t_1]\}$ is called solution corresponding to the element w, if the conditions (1.2) and (1.3) are fulfilled. Moreover, the function $x(t)$ is absolutely continuous and satisfies the first equation of (1.1) almost everywhere (a.e.) on $[t_0, \theta]$; the function $y(t)$ is absolutely continuous and satisfies the second equation of (1.1) a.e. on $[\theta, t_1]$.

Definition 1.2. The element $w \in W$ is admissible if for corresponding solution $\Phi(w)$ the condition

$$y(t_1) = y_1 \tag{1.4}$$

holds.

The set of admissible elements is denoted by W_0.

Definition 1.3. The system (1.1) is called controllable with the conditions (1.2)-(1.4), if $W_0 \neq \emptyset$.

To formulate the main results we introduce the following notation: let $C > 0, N > 0$ and $K \subset O, M \subset Y$ be given numbers and compact sets,

$$F_{K,C} = \Big\{ f \in E_f : \int_{I_1} [m_{f,K}(t) + L_{f,K}(t)]dt \leq C \Big\},$$

$$V_{K,\delta} = \Big\{ f \in F_{K,C} : \Big| \int_{s_1}^{s_2} f(t, x)dt \Big| \leq \delta, \forall s_1, s_2 \in I_1, x \in K \Big\}, \delta > 0;$$

$$G_{M,N} = \Big\{ g \in E_g : \int_{I_2} [m_{g,M}(t) + L_{g,M}(t)]dt \leq N \Big\},$$

$$H_{M,\delta}=\Big\{g\in G_{M,N}:\Big|\int_{s_1}^{s_2}g(t,y)dt\Big|\le\delta,\forall s_1,s_2\in I_2,y\in M\Big\};$$

$$F^u_{K,C}=\Big\{f\in E^u_f:\int_{I_1}[m_{f,K}(t)+L_{f,K}(t)]dt\le C\Big\},$$

$$V^u_{K,\delta}=\Big\{f\in F^u_{K,C}:\int_{I_1}\sup_{(x,u)\in K\times U}|f(t,x,u)|dt\le\delta\Big\},$$

$$G^v_{M,N}=\Big\{g\in E^v_g:\int_{I_2}[m_{g,M}(t)+L_{g,M}(t)]dt\le N\Big\},$$

$$H^v_{M,\delta}=\Big\{g\in G^v_{M,N}:\int_{I_2}\sup_{(y,v)\in M\times V}|g(t,y,v)|dt\le\delta\Big\};$$

$$B_{y_1,\varepsilon}=\Big\{y\in Y:\parallel y_1-y\parallel\le\varepsilon\Big\},\varepsilon>0.$$

Theorem 1.1. *Let the system (1.1) be controllable i.e. there exists $w_0=(t_{00},\theta_0,t_{10},u_0(\cdot),v_0(\cdot))\in W_0$. Then for arbitrary $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for any $f\in V_{K_{01},\delta}$ and $g\in H_{M_{01},\delta}$ the perturbed two-stage system*

$$\begin{cases}\dot x(t)=f_0(t,x,u(t))+f(t,x),t\in[t_0,\theta],\\ \dot y(t)=g_0(t,y,v(t))+g(t,y),t\in[\theta,t_1]\end{cases}\tag{1.5}$$

with the conditions

$$x(t_0)=x_0,y(\theta)=Q(\theta,x(\theta)),y(t_1)\in B_{y_1,\varepsilon}\tag{1.6}$$

is controllable. Here $K_{01}\subset O$ and $M_{01}\subset Y$ are compact sets, containing some neighborhoods of $K_0=\{x(t;w_0):t\in[t_{00},\theta_0]\}$ and $M_0=\{y(t;w_0):t\in[\theta_0,t_{10}]\}$, respectively.

Theorem 1.2. *Let the system (1.1) be controllable. Then for arbitrary $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for any $f\in V^u_{K_{01},\delta}$ and $g\in H^v_{M_{01},\delta}$ the perturbed two-stage system*

$$\begin{cases}\dot x(t)=f_0(t,x,u(t))+f(t,x,u(t)),t\in[t_0,\theta],\\ \dot y(t)=g_0(t,y,v(t))+g(t,y,v(t)),t\in[\theta,t_1]\end{cases}$$

with the conditions (1.6) is controllable.

Definition 1.4. The pair of functions $\hat\Phi(w)=\{\hat x(t)=\hat x(t;w)\in O,t\in I_1;\hat y(t)=\hat y(t;w)\in Y,t\in I_2\}$ is called a continuation of the solution $\Phi(w)$, if $\hat x(t)$ on the interval I_1 is a continuation of the solution $x(t),t\in[t_0,\theta]$ and $\hat y(t)$ on the interval I_2 is a continuation of the solution $y(t),t\in[\theta,t_1]$ (see Definition 1.1).

Theorem 1.3. *Let the following conditions hold:*

1.5. *for any $w\in W$ there exists the continuation solution $\hat\Phi(w)$; moreover, there exist compact sets $K_1\subset O$ and $M_1\subset Y$ such that, for any $w\in W$*

$$\hat x(t;w)\in K_1,t\in I_1\ \ \textit{and}\ \ \hat y(t;w)\in M_1,t\in I_2;$$

1.6. *the sets*

$$f_0(t,x,U)=\{f_0(t,x,u):u\in U\} \text{ for any fixed } (t,x)\in I_1\times O$$

and

$$g_0(s,y,V)=\{g_0(t,x,v):v\in V\} \text{ for any fixed } (s,y)\in I_2\times Y$$

are convex;

1.7. *there exist sequences* $\{\varepsilon_i\}\to 0, \{\delta_i\}\to 0, \{f_i\in V_{K_{11},\delta_i}\}$ *and* $\{g_i\in H_{M_{11},\delta_i}\}$ *such that for any* $i=1,2,...$ *the perturbed system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f_i(t,x), t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g_i(t,y), t\in[\theta,t_1]\end{cases}$$

with the conditions

$$x(t_0)=x_0, y(\theta)=Q(\theta,x(\theta)), y(t_1)\in B_{y_1,\varepsilon_i} \tag{1.7}$$

is controllable i.e. there exists admissible element $w_i=(t_{0i},\theta_i,t_{1i},u_i,v_i)$.
Then the system (1.1) is controllable with the conditions (1.2)-(1.4). Here $K_{11}\subset O$ *and* $M_{11}\subset Y$ *are compact sets, containing some neighborhoods of* K_1 *and* M_1*, respectively.*

Theorem 1.4. *Let the conditions 1.5, 1.6 hold and let there exist sequences* $\{\varepsilon_i\}\to 0, \{\delta_i\}\to 0, \{f_i\in V^u_{K_{11},\delta_i}\}$ *and* $\{g_i\in H^v_{M_{11},\delta_i}\}$ *such that for any* $i=1,2,...$ *the perturbed system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f_i(t,x,u(t)), t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g_i(t,y,v(t)), t\in[\theta,t_1]\end{cases}$$

with the conditions (1.7) is controllable. Then system (1.1) is controllable with conditions (1.2)-(1.4).

Finally, we note that Theorems, analogous to Theorems 1.1-1.4 are given in [1] for ordinary and delay differential equations. Optimal control problems for various classes of the two-stage and multi-stage systems are investigated in [2-17].

2. Auxiliary assertions

Theorem 2.1([1], p.101; [18], p.108). *Let* $\tilde{w}=(\tilde{t}_0,\tilde{\theta},\tilde{t}_1,\tilde{u}(\cdot),\tilde{v}(\cdot))\in W$ *be a given element and let* $\Phi(\tilde{w})$ *be the corresponding solution. For arbitrary* $\varepsilon>0$ *there exists a number* $\delta=\delta(\varepsilon)>0$ *such that for any* $f\in V_{\tilde{K}_1,\delta}$ *and* $g\in H_{\tilde{M}_1,\delta}$ *the perturbed two-stage system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,\tilde{u}(t))+f(t,x), t\in[\tilde{t}_0,\tilde{\theta}],\\ \dot{y}(t)=g_0(t,y,\tilde{v}(t))+g(t,y), t\in[\tilde{\theta},\tilde{t}_1]\end{cases}$$

with the conditions

$$x(\tilde{t}_0)=x_0, y(\tilde{\theta})=Q(\tilde{\theta},x(\tilde{\theta}))$$

has the solution

$$\Phi(\tilde{w};f,g)=\{x(t;\tilde{w},f,g)\in\tilde{K}_1, t\in[\tilde{t}_0,\tilde{\theta}]; y(t;\tilde{w},f,g)\in\tilde{M}_1, t\in[\tilde{\theta},\tilde{t}_1]\}$$

and the following inequalities

$$|x(t;\tilde{w})-x(t;\tilde{w},f,g)|\le\varepsilon, t\in[\tilde{t}_0,\tilde{\theta}]; |y(t;\tilde{w})-y(t;\tilde{w},f,g)|\le\varepsilon, t\in[\tilde{\theta},\tilde{t}_1]$$

hold.Here $\tilde{K}_1\subset O$ *and* $\tilde{M}_1\subset Y$ *are compact sets, containing some neighborhoods of* $\{x(t;\tilde{w}):t\in[\tilde{t}_0,\tilde{\theta}]\}$ *and* $\{y(t;\tilde{w}):t\in[\tilde{\theta},\tilde{t}_1]\}$, *respectively.*

Theorem 2.2([1], p.101; [18], p.108). *Let the condition 1.5 hold. Then for arbitrary* $\varepsilon>0$ *there exists a number* $\delta=\delta(\varepsilon)>0$ *such that for any* $w\in W, f\in V_{K_{11},\delta}$ *and* $g\in H_{M_{11},\delta}$ *the perturbed two-stage system*

$$\begin{cases}\dot{x}(t)=f_0(t,x,u(t))+f(t,x), t\in[t_0,\theta],\\ \dot{y}(t)=g_0(t,y,v(t))+g(t,y), t\in[\theta,t_1]\end{cases}$$

with the conditions

$$x(t_0)=x_0, y(\theta)=Q(\theta,x(\theta))$$

has the solution

$$\hat{\Phi}(w;f,g)=\{\hat{x}(t;w,f,g)\in K_{11}, t\in I_1; \hat{y}(t;w,f,g)\in M_{11}, t\in I_2\}$$

and the following inequalities

$$|\hat{x}(t;w)-\hat{x}(t;w,f,g)|\le\varepsilon, t\in I_1; |\hat{y}(t;w)-\hat{y}(t;w,f,g)|\le\varepsilon, t\in I_2$$

hold.

Lemma 2.1 ([19], p.86). *Let* $x(t)\in O, t\in I_1$ *be a continuous function and let a sequence* $\{f_i\in V_{K,C}\}$ *satisfy the condition*

$$\lim_{i\to\infty}\sup\Big\{\Big|\int_{s_1}^{s_2}f_i(t,x)dt\Big|:s_1,s_2\in I_1, x\in K\Big\}=0.$$

Then

$$\lim_{i\to\infty}\sup\Big\{\Big|\int_{s_1}^{s_2}f_i(t,x(t))dt\Big|:s_1,s_2\in I_1\Big\}=0.$$

Here $K\subset O$ *is a compact set containing some neighborhood of* K.

Let $p(t,u)\in R_x^n$ be a given function, defined on $I_1\times U$ and satisfying the following conditions: for almost all $t\in I_1$ the function $p(t,\cdot)\to R_x^n$ is continuous; for each $u\in U$ the function $p(\cdot,u):I_1\to R_x^n$ is measurable.

Theorem 2.3([20], p.257). *Let the set*

$$P(t)=\{p(t,u):u\in U\}$$

be convex and

$$p_i(\cdot)\in L_1(I_1,R_x^n); p_i(t)\in P(t) \ \ a.e. \ on \ \ I_1, i=1,2,....$$

moreover,

$$\lim_{i\to\infty}p_i(t)=p(t) \quad weakly \ on \ \ I_1.$$

Then

$$p(t) \in P(t) \;\; a.e. \;\; on \;\; I_1$$

and there exists a measurable function $u(t) \in U, t \in I_1$ *such that*

$$p(t, u(t)) = p(t) \;\; a.e. \;\; on \;\; I_1.$$

3. Proof of Theorem 1.1

Let $\varepsilon_0 > 0$ be so small that

$$K_{\varepsilon_0} = \{x \in R^n_x : \exists \hat{x} \in K_0, |x - \hat{x}| \leq \varepsilon_0\} \subset int K_{01}$$

and

$$M_{\varepsilon_0} = \{y \in R^m_y : \exists \hat{y} \in M_0, |y - \hat{y}| \leq \varepsilon_0\} \subset int M_{01}.$$

According to Theorem 2.1 for any $\varepsilon \in (0, \varepsilon_0]$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for any $f \in V_{K_{01},\delta}$ and $g \in H_{M_{01},\delta}$ the perturbed two-stage system

$$\begin{cases} \dot{x}(t) = f_0(t, x, u_0(t)) + f(t, x), t \in [t_{00}, \theta_0], \\ \dot{y}(t) = g_0(t, y, v_0(t)) + g(t, y), t \in [\theta_0, t_{10}] \end{cases}$$

with the conditions

$$x(t_{00}) = x_0, y(\theta_0) = Q(\theta_0, x(\theta_0))$$

has the solution

$$\Phi(w_0; f, g) = \{x(t; w_0, f, g), t \in [t_{00}, \theta_0]; y(t; w_0, f, g), t \in [\theta_0, t_{10}]\}$$

and the following inequalities

$$|x(t; w_0) - x(t; w_0, f, g)| \leq \varepsilon, t \in [t_{00}, \theta_0]; |y(t; w_0) - y(t; w_0, f, g)| \leq \varepsilon, t \in [\theta_0, t_{10}]$$

hold.

Consequently, the element w_0 is admissible for system (1.5) with conditions (1.6) for any $f \in V_{K_{01},\delta}$ and $g \in H_{M_{01},\delta}$.

Remark 2.1. Theorem 1.2 is a simply corollary of Theorem 1.1, since for any $u(\cdot) \in \Omega$ and $v(\cdot) \in \Delta$ we have

$$\sup\left\{\left|\int_{s_1}^{s_2} f(t, x, u(t))dt\right| : s_1, s_2 \in I_1, x \in K\right\} \leq \int_{I_1} \sup_{(x,u) \in K \times U} |f(t, x, u)|dt,$$

$$\sup\left\{\left|\int_{s_1}^{s_2} g(t, y, v(t))dt\right| : s_1, s_2 \in I_2, y \in M\right\} \leq \int_{I_2} \sup_{(y,v) \in M \times V} |g(t, y, v)|dt.$$

4. Proof of Theorem 1.3

Let $\varepsilon_0 > 0$ be so small that

$$K_{1,\varepsilon_0} = \{x \in R^n_x : \exists \hat{x} \in K_1, |x - \hat{x}| \leq \varepsilon_0\} \subset int K_{11}$$

and

$$M_{1,\varepsilon_0} = \{y \in R_y^m : \exists \hat{y} \in M_1, |y - \hat{y}| \le \varepsilon_0\} \subset int M_{11}.$$

It is clear that there exists a subsequence $\{\varepsilon_{i_1}\} \subset \{\varepsilon_1, \varepsilon_2, ...\}$ such that $\varepsilon_{i_1} \in (0, \varepsilon_0], i = 1, 2, ...$ On the basis of Theorem 2.2 for each ε_{i_1} there exists $\delta_{i_1} \in \{\delta_1, \delta_2, ...\}$ such that for $w_{i_1} = (t_{0i_1}, \theta_{i_1}, t_{1,i_1}, u_{i_1}, v_{i_1}), f_{i_1}$ and g_{i_1} we have

$$|x(t; w_{i_1}) - x(t; w_{i_1}, f_{i_1}, g_{i_1})| \le \varepsilon_{i_1}, t \in I_1 \tag{4.1}$$

and

$$|y(t; w_{i_1}) - y(t; w_{i_1}, f_{i_1}, g_{i_1})| \le \varepsilon_{i_1}, t \in I_2. \tag{4.2}$$

Thus,

$$x(t; w_{i_1}, f_{i_1}, g_{i_1}) \in K_{1,\varepsilon_0}, t \in I_1$$

and

$$y(t; w_{i_1}, f_{i_1}, g_{i_1}) \in M_{1,\varepsilon_0}, t \in I_2.$$

The sequences $\{x(t; w_{i_1})\}$ and $\{y(t; w_{i_1})\}$ are uniformly bounded and equicontinuous, since

$$x(t; w_{i_1}) \in K_1, t \in I_1; y(t; w_{i_1}) \in M_1, t \in I_2$$

and

$$|\dot{x}(t; w_{i_1})| \le |f_0(t, x(t; w_{i_1}), u_{i_1}(t))| \le m_{K_1}(t) = m_{f_0,K_1}(t), t \in I_1,$$
$$|\dot{y}(t; w_{i_1})| \le |g_0(t, y(t; w_{i_1}), v_{i_1}(t))| \le m_{M_1}(t) = m_{g_0,M_1}(t), t \in I_2.$$

By the Arzela-Ascoli lemma from sequences $\{x(t; w_{i_1})\}$ and $\{y(t; w_{i_1})\}$ we can extract uniformly convergent subsequences. Without loss of generality, we assume that

$$\lim_{i\to\infty} x(t; w_{i_1}) = x_0(t) \text{ uniformly in } I_1, \tag{4.3}$$

$$\lim_{i\to\infty} y(t; w_{i_1}) = y_0(t) \text{ uniformly in } I_2; \tag{4.4}$$

$$\lim_{i\to\infty} t_{0i_1} = t_{00}, \lim_{i\to\infty} \theta_{i_1} = \theta_0, \lim_{i\to\infty} t_{1i_1} = t_{10}.$$

On the basis of (4.1)-(4.4) we obtain

$$\lim_{i\to\infty} x_{i_1}(t) = x_0(t) \text{ uniformly in } I_1, \lim_{i\to\infty} y_{i_1}(t) = y_0(t) \text{ uniformly in } I_2,$$

where

$$x_{i_1}(t) = x(t; w_{i_1}, f_{i_1}, g_{i_1}), y_{i_1}(t) = y(t; w_{i_1}, g_{i_1}), y_{i_1}).$$

Obviously,

$$x_{i_1}(t_{0i_1}) = x_0, y_{i_1}(\theta_{i_1}) = Q(\theta_{i_1}, x_{i_1}(\theta_{i_1})), y_{i_1}(t_{i_1}) \in B_{y_1,\varepsilon_{i_1}},$$

therefore

$$x_0(t_{00}) = x_0, y_0(\theta_0) = Q(\theta_0, x_0(\theta_0)), y_0(t_{10}) = y_1. \tag{4.5}$$

Further,

$$x_{i_1}(t) = x_0 + \int_{t_{0i_1}}^t [f_0(s, x_{i_1}(s), u_{i_1}(s)) + f_{i_1}(s, x_{i_1}(s))]ds = x_0 + \int_{t_{0i_1}}^t p_i(s)ds + \alpha_i(t)$$

$$+\beta_i(t) + \gamma_i(t), \tag{4.6}$$

where

$$p_i(s) = f_0(s, x_0(s), u_{i_1}(s)), \alpha_i(t) = \int_{t_{0i_1}}^{t} f_{i_1}(s, x_0(s))ds,$$

$$\beta_i(t) = \int_{t_{0i_1}}^{t} [f_0(s, x_{i_1}(s), u_{i_1}(s)) - p_i(s)]ds, \gamma_i(t) = \int_{t_{0i_1}}^{t} [f_{i_1}(s, x_{i_1}(s)) - f_{i_1}(s, x_0(s))]ds.$$

It is not difficult to see that

$$|p_i(s)| \le m_{K_{i_1}}(t), i = 1, 2, ..., |\alpha_i(t)| \le \sup\Big\{\Big|\int_{s_1}^{s_2} f_{i_1}(t, x_0(t))dt\Big| : s_1, s_2 \in I_1\Big\},$$

$$|\beta_i(t)| \le \max_{t\in I_1}\Big|x_{i_1}(t) - x_0(t)\Big|\int_{I_1} L_{K_{11}}(s)ds,$$

$$|\gamma_i(t)| \le \max_{t\in I_1}\Big|x_{i_1}(t) - x_0(t)\Big|\int_{I_1} L_{f_{i_1},K_{11}}(s)ds \le C\Big|x_{i_1}(t) - x_0(t)\Big|.$$

Without loss of generality,we assume that

$$\lim_{i\to\infty} p_i(s) = p(s) \quad \text{weakly on} \quad I_1$$

([20], p.239). Moreover, we have

$$\lim_{i\to\infty} \alpha_i(t) = 0, \lim_{i\to\infty} \beta_i(t) = 0, \lim_{i\to\infty} \gamma_i(t) = 0$$

(see Lemma 2.1, 4.3 and 4.4). From (4.6) it follows

$$x_0(t) = x_0 + \int_{t_{00}}^{t} p(s)ds, t \in [t_{00}, \theta_0].$$

Obviously,

$$p_i(s) \in P(s) = f_{i_1}(s, x_0(s), U), s \in I_1.$$

From Theorem 2.3 follow the inclusion $p(s) \in P(s)$ and existence of such a function $u_0(\cdot) \in \Omega$ that

$$p(s) = f_0(s, x_0(s), u_0(s)), \ \textit{a.e. on } I_1.$$

Thus,

$$x_0(t) = x_0 + \int_{t_{00}}^{t} f_0(s, x_0(s), u_0(s))ds, t \in [t_{00}, \theta_0].$$

In a similar way, taking into account convexity of the set $g_0(t, y, V)$, one can prove

$$y_0(t) = Q(\theta_0, x_0(\theta_0)) + \int_{\theta_0}^{t} g_0(s, y_0(s), v_0(s))ds, t \in [\theta_0, t_{10}], v_0(\cdot) \in \Delta.$$

Consequently, the element $w_0 = (t_{00}, \theta_0, t_{10}, u_0(\cdot), v_0(\cdot))$ is admissible (see (4.5)).

Remark 4.1. Theorem 1.4 is proved analogously to Theorem 1.3.

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THE SOLUTION OF THE STRESS PROBLEM OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR A CIRCULAR RING

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Abstract. The solution of statics of the stress boundary value problem of the theory of thermoelasticity with microtemperatures for the circular ring is presented. The representation of regular solutions for the system of equations of the linear theory of thermoelasticity with microtemperatures by harmonic, biharmonic and metaharmonic functions is obtained. The solution is obtained by means of absolutely and uniformly convergent series. The question on the uniqueness of the solution of the problem is studied.

Keywords and phrases: Thermoelasticity, microtemperature, sress problem, uniqueness theorem, explicit solutions.

AMS subject classification (2010): 74F05, 74G10, 74G30.

1. Basic equations

The basic system of equations of the theory of thermoelasticity with microtemperatures can be written in the form [1,2]:

$$\begin{aligned} &\mu\Delta u(x) + (\lambda+\mu) graddivu(x) = \beta gradu_3(x),\\ &k\Delta u_3(x) + k_1 divw(x) = 0, \\ &k_6\Delta w(x) + (k_4+k_5) graddivw(x) - k_3 gradu_3(x) - k_2 w(x) = 0, \end{aligned} \tag{1}$$

where $\lambda, \mu, \beta, k, k_1, k_2, k_3, k_4, k_5, k_6$ are constitutive coefficients [1]; $u(x)$ is the displacement vector of the point $x = (x_1, x_2)$; $u = (u_1, u_2)$; $w = (w_1, w_2)$ is the microtemperatures vector; u_3 is temperature measured from the constant absolute temperature T_0; Δ is the Laplace operator.

Problem. Find a regular vector $U = (u_1, u_2, u_3, w_1, w_2)$, $(U \in C^1(\overline{D}) \cap C^2(D), \overline{D} = D \cup S_0 \cup S_1)$ satisfying in the ring D a system of equations (1) and on the circumferences S_0 and S_1 the boundary conditions:

$$\begin{aligned} &[T'(\partial_z, n)u(z) - \beta u_3(z) n(z)]^i = f^i(z), \quad \left[k\frac{\partial u_3(z)}{\partial n(z)} + k_1 w(z) n(z)\right]^i = f_3^i(z),\\ &[T''(\partial_z, n)w(z)]^i = p^i(z), \quad i = 0, 1, \end{aligned} \tag{2}$$

where $f = (f_1, f_2)$, $p = (p_1, p_2)$, f_1, f_2, f_3 are the given functions on S_0 and S_1; $T'u$ is the stress vector in the classical theory of elasticity; $T''w$ is stress vector for microtemperatures [1,2]:

$$T'(\partial_x, n)u(x) = \mu\frac{\partial u(x)}{\partial n} + \lambda n(x)divu(x) + \mu\sum_{i=1}^{2} n_i(x)gradu_i(x),$$
$$T''(\partial_x, n)w(x) = (k_5 + k_6)\frac{\partial w(x)}{\partial n} + k_4 n(x)divw(x) + k_5\sum_{i=1}^{2} n_i(x)gradw_i(x). \qquad (3)$$

The above-formulated problem of thermoelasticity with microtemperatures can be considered as a union of two problems A and B, where:

Problem A. find in a ring D the solution $u(x)$ of equation $(1)_1$, if on the circumferences S_0 and S_1 there are given the values of the vector $T'(\partial_z, n)u(z) - \beta u_3(z)n(z)$;

Problem B. find in the ring D the solutions $u_3(x)$ and $w(x)$ of the system of equations $(1)_2$ and $(1)_3$,if on the circumferences S_0 and S_1 there are given the values of the function $k\dfrac{\partial u_3(z)}{\partial n(z)} + k_1 w(z)n(z)$ and of the vector $T''(\partial_z, n)w(z)$.

Let (u', u_3', w') and (u'', u_3'', w'') be two different solutions of any of the problems. Then the differences $u = u' - u''$, $u_3 = u_3' - u_3''$ and $w = w' - w''$ of these solutions, obviously, satisfies the homogeneous system $(1)_0$ and zero boundary conditions $(2)_0$. For a regular solutions of equation $(1)_1$ and equations $(1)_2$ and $(1)_3$ the Green's formulas [2,3]:

$$\int_D [E_1(u(x), u(x)) - \beta u_3(x)divu(x)]dx = -\int_S u^0(y)[T'(\partial_y, n)u(y) - \beta u_3(y)n(y)]^0 d_y S_0$$
$$+\int_S u^1(y)[T'(\partial_y, n)u(y) - \beta u_3(y)n(y)]^1 d_y S_1,$$

$$\int_D [T_0 E_2(w(x), w(x)) + k \mid gradu_3 \mid^2 + (k_1 + k_3T_0)wgradu_3 + k_2T_0 \mid w(x) \mid^2]dx$$
$$= -\int_S u_3^0(y)[k\partial_n u_3(y) + k_1 w(y)n(y)]^0 + T_0 w^0(y)[T''(\partial_y, n)w(y)]^0 d_y S_0 \qquad (4)$$
$$+\int_S u_3^1(y)[k\partial_n u_3(y) + k_1 w(y)n(y)]^1 + T_0 w^1(y)[T''(\partial_y, n)w(y)]^1 d_y S_1$$

is valid, where

$$E_1(u, u) = (\lambda + \mu)(\partial_1 u_1 + \partial_2 u_2)^2 + \mu(\partial_1 u_1 - \partial_2 u_2)^2 + \mu(\partial_2 u_1 + \partial_1 u_2)^2;$$

$$E_2(w, w) = \frac{1}{2}(2k_4 + k_5 + k_6)(\partial_1 w_1 + \partial_2 w_2)^2 + (k_6 + k_5)(\partial_1 w_1 - \partial_2 w_2)^2$$
$$+(k_6 + k_5)(\partial_2 w_1 + \partial_1 w_2)^2 + (k_6 - k_5)(\partial_1 w2 - \partial_2 w_1)^2,$$

under the conditions that: $\lambda+\mu$, $\mu > 0$ and, accordingly, $2k_4+k_5+k_6 > 0, k_6\pm k_5 > 0$ are nonnegative quadratic forms.

Taking into account formula $(4)_2$ under the homogeneous boundary conditions for the problem B, we obtain $E_2(w, w) = 0$, $gradu_3 = 0$, $w = 0$. The solution of the above equations has the form: $u_3(x) = const, w = 0$.

The following uniqueness theorem is valid.

Theorem. *The difference of two arbitrary solutions of the BVP (1), (2) is the vector* $U = (u_1, u_2, u_3, w_1, w_2)$, *where* $u_1(x) = -c_1x_2 + clx_1 + q_1, u_2(x) = -c_1x_1 + clx_1 + q_2, u_3 = c, w_1 = w_2 = 0; c, c_1, q_1, q_2$ *are arbitrary constants,* $l = \frac{\beta}{2(\lambda+\mu)}$.

2. Solution of the problem B

Taking into account formulas: $\dfrac{\partial}{\partial x_2} = n_2\dfrac{\partial}{\partial r} + \dfrac{n_1}{r}\dfrac{\partial}{\partial \psi}$, $\dfrac{\partial}{\partial x_1} = n_1\dfrac{\partial}{\partial r} - \dfrac{n_2}{r}\dfrac{\partial}{\partial \psi}$, we rewrite the representation solutions of the system $[(1)_2, (1)_3]$ and the boundary conditions of the problem B in the tangent and normal components [3]:

$$\begin{aligned} &u_3(x) = \varphi_1(x) + \varphi_2(x),\\ &w_n(x) = a_1\frac{\partial}{\partial r}\varphi_1(x) + a_2\frac{\partial}{\partial r}\varphi_2(x) - a_3\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_3(x),\\ &w_s(x) = a_1\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_1(x) + a_2\frac{1}{r}\frac{\partial}{\partial \psi}\varphi_2(x) + a_3\frac{\partial}{\partial r}\varphi_3(x), \end{aligned} \tag{5}$$

$$\begin{aligned} &k\left[\frac{\partial u_3}{\partial r}\right]^i + k_1\left[w_n\right]^i = f_3^i(z), \quad k_7\left[\frac{\partial w_n}{\partial r}\right]^i + \frac{k_4}{R_i}\left[\frac{\partial w_s}{\partial \psi}\right]^i = p_n^i(z),\\ &k_6\left[\frac{\partial w_s}{\partial r}\right]^i + \frac{k_5}{R_i}\left[\frac{\partial w_n}{\partial \psi}\right]^i = p_s^i(z), \end{aligned} \tag{6}$$

where $w_n = (w \cdot n), w_s = (w \cdot s), p_n = (p \cdot n), p_s = (p \cdot s), n = (n_1, n_2), s = (-n_2, n_1)$, $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$, $i = 0, 1$; $\triangle\varphi_1 = 0, (\triangle + s_1^2)\varphi_2 = 0, (\triangle + s_2^2)\varphi_3 = 0, s_1^2 = -\dfrac{kk_2 - k_1k_3}{kk_7}$, $s_2^2 = -\dfrac{k_2}{k_6}, a_1 = -\dfrac{k_3}{k_2}, a_2 = -\dfrac{k}{k_1}, a_3 = \dfrac{k_6}{k_7}$; $k_7 = k_4 + k_5 + k_6$; $k, k_2, k_6, k_7 > 0$; $w_n = (w \cdot n)$, $w_s = (w \cdot s)$, $p_n = (p \cdot n)$, $p_s = (p \cdot s)$, $n = (n_1, n_2)$, $s = (-n_2, n_1)$; $x = (r, \psi)$, $r^2 = x_1^2 + x_2^2$. R_0 is radius of the boundary S_0; R_1 is radius of the boundary S_1.

The harmonic function φ_1 and metaharmonic functions φ_2 and φ_3 are represented in the form of series in the ring ([4], p.417; [5]):

$$\begin{aligned} &\varphi_1(x) = X_{10}\ln r + Y_{10} + \sum_{m=1}^{\infty}[r^m(X_{1m}\cdot\nu_m(\psi)) + r^{-m}(X_{1m}\cdot\nu_m(\psi))],\\ &\varphi_2(x) = \sum_{m=0}^{\infty}[I_m(s_2r)(X_{2m}\cdot\nu_m(\psi)) + K_m(s_2r)(Y_{2m}\cdot\nu_m(\psi))],\\ &\varphi_3(x) = \sum_{m=0}^{\infty}[I_m(s_3r)(X_{3m}\cdot\ s_m(\psi)) + K_m(s_3r)(Y_{3m}\cdot\ s_m(\psi))], \end{aligned} \tag{7}$$

where $I_m(s_jr)$ and $K_m(s_jr)$ are the Bessel's and modified Hankel's functions of an imaginary argument, respectively; X_{km} and Y_{km} are the unknown two-component constants vectors, $\nu_m(\psi) = (\cos m\psi, \sin m\psi), s_m(\psi) = (-\sin m\psi, \cos m\psi), j = 2, 3; k = 1, 2$.

We substitute (7) into (5) and then the obtained expressions substitute into (6). Passing to the limit, as $r \to R_0$ and $r \to R_1$ for the unknowns X_{mk} and Y_{mk} we obtain a system of algebraic equations:

$$-a_1\frac{1}{R_i^2}X_{10}+a_2s_2^2[I_0''(s_2R_i)X_{20}+K_0''(s_2R_i)Y_{20}]=\frac{A_{10}^i}{2k_7},$$
$$I_0''(s_3R_i)X_{30}+K_0''(s_3R_i)Y_{30}=\frac{A_{20}^i}{2k_6a_3s_3},$$
$$\frac{1}{R_i}(1+k_1a_1)X_{10}+s_2(1+a_2)[I_0'(s_2R_i)X_{20}+K_0'(s_2R_i)Y_{20}]=\frac{A_{30}^i}{2},$$

$$a_1mR_i^{m-2}[k_7(m-1)-k_4m]X_{1m}+a_2\left[k_7s_2^2I_m''(s_2R_i)-k_4\frac{m^2}{R_i^2}I_m(s_2R_i)\right]X_{2m}$$
$$+k_7a_3\frac{m}{R_i}\left[\frac{1}{R_i}I_m(s_3R_i)+s_3I_m'(s_3R_i)\right]X_{3m}+a_1mR_i^{-(m+2)}\left[k_7(m+1)-k_4m\right]Y_{1m}$$
$$+a_2\left[k_7s_2K_m''(s_2R_i)-k_4\frac{m^2}{R_i^2}K_m(s_2R_i)\right]Y_{2m}$$
$$+k_7a_3\frac{m}{R_i}\left[\frac{1}{R_i}K_m(s_3R_i)+s_3K_m'(s_3R_i)\right]Y_{3m}=A_{1m}^i,$$

$$a_1mR_i^{m-2}[k_5m+k_6(m-1)]X_{1m}+a_2\frac{m}{R_i}\left[-k_6\frac{1}{R_i}I_m(s_2R_i)+s_2(k_5+k_6)I_m'(s_2R_i)\right]X_{2m}$$
$$+a_3[k_6s_3^2I_m''(s_3R_i)-k_5\frac{m^2}{R_i^2}I_m(s_3R_i)]X_{3m}-a_1mR_i^{-(m+2)}[k_6(m+1)+k_5m]Y_{1m}$$
$$+a_2\frac{m}{R_i}\left[-k_6\frac{1}{R_i}K_m(s_2R_i)+(k_5+k_6)s_2K_m'(s_2R_i)\right]Y_{2m}$$
$$+a_3\left[-k_5\frac{m^2}{R_i^2}K_m(s_3R_i)+k_6s_3^2K_m''(s_3R_i)\right]Y_{3m}=A_{2m}^i,$$

$$k_1mR_i^{m-1}X_{1m}+s_2I_m'(s_2R_i)(k+k_1a_2)X_{2m}-k_1a_3\frac{m}{R_i}I_m(s_3R_i)X_{3m}$$
$$-k_1mR_i^{-(m+1)}Y_{1m}+s_2(k+k_1a_2)K_m'(s_2R_i)Y_{2m}-k_1a_3\frac{m}{R_i}K_m(s_3R_i)Y_{3m}=A_{3m}^i,$$

where A_{1m}^i, A_{2m}^i and A_{3m}^i are the Fourier coefficients of the functions $p_n(z)$, $p_s(z)$ and $f_3(z)$, respectively; i=0,1; m=1,2,....

3. Solution of the problem A

The solution of the first equation of the system (1) with the boundary condition (2) is represented by the sum

$$u(x)=v_0(x)+v(x), \tag{8}$$

where v_0 is a particular solution of equation $(1)_1$:

$$v_0(x)=\frac{\beta}{\lambda+2\mu}grad[-\frac{1}{s_1^2}\varphi_2(x)+\varphi_0(x)];$$

φ_0 is a biharmonic function: $\triangle\varphi_0 = \varphi_1$; $v(x) = (v_1(x), v_2(x))$ is the solution of the homogeneous equation $\mu\triangle v(x) + (\lambda+\mu) graddivv(x) = 0$ which can be found by means of the formulae [6]

$$v_1(x) = \frac{\partial}{\partial x_1}[\Phi_1(x) + \Phi_2(x)] - \frac{\partial}{\partial x_2}\Phi_3(x), \quad v_2(x) = \frac{\partial}{\partial x_2}[\Phi_1(x) + \Phi_2(x)] + \frac{\partial}{\partial x_1}\Phi_3(x),$$

where $\Delta\Phi_1(x) = 0, \quad \Delta\Delta\Phi_2(x) = 0, \quad \Delta\Delta\Phi_3(x) = 0$;

$$\begin{aligned}
&\Phi_1(x) = \sum_{m=1}^{\infty}\left[\left(\frac{r}{R_1}\right)^m (Z_{1m}\cdot\nu_m(\psi)) + \left(\frac{R_0}{r}\right)^m (Z_{2m}\cdot\nu_m(\psi))\right] + Z_{10}\ln r \\
&\Phi_2(x) = \sum_{m=0}^{\infty}\left(\frac{r}{R_1}\right)^{m+2} (Z_{3m}\cdot\nu_m(\psi)) \\
&+\sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot\nu_m(\psi)) + r\ln r(Z_{41}\cdot\nu_1(\psi)) + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{20} \\
&\Phi_3(x) = -\frac{(\lambda+2\mu)}{\mu}\sum_{m=1}^{\infty}\left(\frac{r}{R_1}\right)^{m+2} (Z_{3m}\cdot s_m(\psi)) \\
&+\frac{\lambda+2\mu}{\mu}\sum_{m=2}^{\infty}\left(\frac{R_0}{r}\right)^{m-2}(Z_{4m}\cdot s_m(\psi)) \\
&+\frac{(\lambda+2\mu)}{\mu} r\ln r(Z_{11}\cdot s_1(\psi)) + Z_{40}\ln r + \frac{1}{2}\left(\frac{r}{R_1}\right)^2 Z_{30},
\end{aligned} \tag{9}$$

where Z_{km} are the unknown two-component vectors, $k = 1,2,3,4$. Taking into account (8) and relying on the condition $(2)_I$, we can write

$$[T'(\partial_z, n)v(z)]^i = \Psi^i(z), \quad z \in S_i, \quad i = 0, 2,$$

where $\Psi^i(z) = f^i(z) + \beta u_3^i(z)n(z) - [T'(\partial_z, n)v_0(z)]^i$ is the known vector, $\Psi^i = (\Psi_1^i, \Psi_2^i)$. We rewrite this conditions in the tangent and normal components:

$$[T'(\partial_z, n)v(z)]_n^i = \Psi_n^i(z), \quad [T'(\partial_z, n)v(z)]_s^i = \Psi_s^i(z), \tag{10}$$

where

$$[T'(\partial_z, n)v(z)]_n^i = (\lambda+2\mu)\frac{\partial v_n^i(z)}{\partial r} + \lambda\frac{1}{R_i}\frac{\partial v_s^i(z)}{\partial\psi},$$

$$[T'(\partial_z, n)v(z)]_s^i = \mu\frac{\partial v_s^i(z)}{\partial r} + \mu\frac{1}{R_i}\frac{\partial v_n^i(z)}{\partial\psi},$$

$$v_n^i(z) = \frac{\partial}{\partial r}(\Phi_1^i(z) + \Phi_2^i(z)) - \frac{1}{r}\frac{\partial}{\partial\psi}\Phi_3^i(z),$$

$$v_s^i(z) = \frac{1}{r}\frac{\partial}{\partial\psi}(\Phi_1^i(z) + \Phi_2^i(z)) + \frac{\partial}{\partial r}(\Phi_3^i(z)).$$

We substitute (9) into (10). Passing to the limit, as $r \to R_0$ and $r \to R_1$ for the unknowns Z_{mk} we obtain a system of algebraic equations:

$$A(m)t^{m-2}Z_{1m} + B(m)Z_{2m} + C(m)t^m Z_{3m} + E_1(m)Z_{4m} = \eta_m^0,$$
$$A(m)Z_{1m} + B(m)t^{m+2}Z_{2m} + C(m)Z_{3m} + E_2(m)Z_{4m} = \eta_m^1,$$
$$A(m)t^{m-2}Z_{1m} + B(m)Z_{2m} + D(m)t^m Z_{3m} + E_3(m)Z_{4m} = \zeta_m^0,$$
$$A(m)Z_{1m} + B(m)t^{m+2}Z_{2m} + D(m)Z_{3m} + E_4(m)Z_{4m} = \zeta_m^1,$$

where

$$t = \frac{R_0}{R_1}, \quad e_1(m) = 2(\lambda+\mu)(m+1), \quad e_2(m) = 2(\lambda+\mu)(m-1),$$

$$A(m) = \frac{2\mu(m-1)m}{R_1^2}, \quad B(m) = \frac{2\mu(m+1)m}{R_0^2}, \quad C(m) = -\frac{e_{1m}(m-2)}{R_1^2},$$
$$D(m) = -\tfrac{e_1(m)m}{R_1^2}, \quad E_1(1) = \frac{2(2\lambda+3\mu)}{R_0}, \quad E_1(m) = -\frac{e_2(m)(m+2)}{R_0^2},$$
$$E_2(1) = \frac{2(2\lambda+3\mu)}{R_1}, \quad E_2(m) = -\frac{e_2(m)(m+2)}{\mu R_0}t^m, \quad E_3(1) = \frac{2\mu}{R_0}, \quad E_3(m) = \frac{e_2(m)m}{R_0^2},$$
$$E_4(1) = \frac{2\mu}{R_1}\ln R_1, \quad E_4(m) = \frac{e_2(m)t^m}{\mu R_0}, \quad m = 2, 3, ...$$

If the principal vector and principal moment of external stresses is equal to zero, then we obtain

$$R_1^2 \int_0^{2\pi} \Psi_s^1(\theta)d\theta - R_0^2 \int_0^{2\pi} \Psi_s^0(\theta)d\theta = 0.$$

From here when $m = 0$, we get: $R_1^2\zeta_0^1 = R_0^2\zeta_0^0$. When $m = 0$ for the unknowns Z_{10}, Z_{20} and Z_{40} we obtain the system

$$-\frac{2\mu}{R_i^2}Z_{10} + \frac{2(\lambda+\mu)}{R_1^2}Z_{20} = \frac{\zeta_0^i}{2}, \quad -\frac{2\mu}{R_0^2}Z_{40} = \frac{\zeta_0^0}{2},$$

Z_{30} is an arbitrary constant, $i = 0, 1$.

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ქართული რეზიუმეები

წრფივი სასაზღვრო ამოცანები მრავალგანზომილებიანი რეგულარული სხვაობიანი სისტემებისთვის

მ. აშორდია, ნ. კეკელია

დამტკიცებულია ზოგადი სახის წრფივი სასაზღვრო ამოცანის ამოხსნადობის გრინის ტიპის თეორემა. მოცემულია კოშის ამოცანის ამონახსნის აგების მიმდევრობითი მიახლოების მეთოდი.

თერმოდრეკადობის წრფივი თეორიის ერთი სასაზღვრო ამოცანა სფერული რგოლისათვის მიკროტემპერატურის გათვალისწინებით

ლ. ბიწაძე

ნაშრომში მიღებულია ამონახსნის ზოგადი წარმოდგენის ფორმულები თერმოდრეკადობის განტოლებებისათვის მიკროტემპერატურის გათვალისწინებით, რომლის საშუალებით ამოხსნილია ერთი ძირითადი სასაზღვრო ამოცანა სფერული რგოლისათვის. ამონახსნი წარმოდგენილია აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით.

მესამე რიგის წრფივი ჩვეულებრივი დიფერენციალური განტოლების შემოუსაზღვრელი რხევადი ამონახსნის არსებობის შესახებ

ნ. ხვედელიძე

ნაშრომში დამტკიცებულია მესამე რიგის წრფივი ჩვეულებრივი დიფერენციალური განტოლების შემოუსაზღვრელი რხევადი ამონახსნის არსებობა, აგრეთვე ისეთი არარხევადი ამონახსნის არსებობა, რომელიც უსასრულობაში კრებადია ნულისაკენ.

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რ. კოპლატაძე

მიღებულია ახალი ტიპის შედარების თეორემები სინგულარული განტოლების შემთხვევაში, მათი გამოყენებით შესწავლილია მეორე რიგის წრფივი დიფერენციალური განტოლებებისათვის ზოგიერთი ორწერტილოვანი სასაზღვრო ამოცანა.

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კ. სვანაძე

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ი. ცაგარელი, მ. სვანაძე

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