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# SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS IN THIN PRISMATIC DOMAINS

Jaiani G.

**Abstract**. The paper is devoted to a dimension reduction method for solving boundary value and initial boundary value problems of systems of partial differential equations in thin non-Lipschitz, in general, prismatic domains.

**Keywords and phrases**: Partial differential equations, order degeneration, dimension reduction method, thin non-Lipschitz prismatic domains.

**AMS subject classification (2000):** 35A25; 35J70; 35J75; 35K65; 35K67; 35L80; 35L81.

The paper deals with the system of $n$ first order linear partial differential equations

$$A_{ijk}u_{j,k} + B_{ij}u_j + C_i(x) = 0, \quad i = 1, 2, \dots, n, \tag{1}$$

where

$$A_{ijk}, B_{ij} = const \ \text{ and functions } \ C_i(x), \quad i, j = 1, 2, \dots, n, \quad k = 1, 2, 3, \text{ are given}$$

(under repeated index $j$ the sum from 1 to $n$ is meant, under repeated $k$ the sum from 1 to 3 is meant, and under repeated Greek indices the sum from 1 to 2 is meant), in $n$ unknown functions $u_i\,(x_1, x_2, x_3)$ of three variables in the following non-Lipschitz, in general, 3D prismatic domain with the Lipschitz 2D projection $\omega$ on $x_3 = 0$:

$$\Omega := \left\{ x := (x_1, x_2, x_3) \in R^3 : (x_1, x_2) \in \omega, \quad \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2) \right\}$$

where $2h := \overset{(+)}{h} - \overset{(-)}{h} > 0$ in $\omega \cup \gamma_1$, $2h = 0$ on $\gamma_0$; $\partial\omega = \bar{\gamma}_0 \cup \bar{\gamma}_1$, $\nu$ is an inward normal to $\partial\omega$. Each of $\gamma_0$ and $\gamma_1$ may be empty but , clearly, not at the same time. When $\dfrac{\partial h}{\partial \nu} = 0$ on $\gamma_0$, the domain $\Omega$ is a non-Lipschitz one.

The boundary value problems for the system (1) in the $3D$ non-Lipschitz, in general, domain $\Omega$ can be reduced to the boundary value problems in the Lipschitz $2D$ domain $\omega$ for the infinite system of singular first order partial differential equations with respect to the s. c. weighted Legendre moments (see [1,2]) of the unknown functions $u_i\,(x_1, x_2, x_3)$:

$$v_{ir}(x_1, x_2) = \frac{u_{ir}(x_1, x_2)}{h^{r+1}}, \quad i = 1, 2, ..., n, \quad r = 0, 1, ..., \tag{2}$$

where

$$u_{ir}\,(x_1, x_2) = \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} u_i\,(x_1, x_2, x_3)\, P_r\,(ax_3 - b)\, dx_3,$$

$$a = \frac{1}{h}, \quad b = \frac{\tilde{h}}{h}, \quad 2\tilde{h} = \overset{(+)}{h} + \overset{(-)}{h}.$$

By this approach difficulties caused by the geometrical singularity of the 3D domain are reduced to the singularity of the equations. In other words, we avoid consideration of 3D non-Lipschitz domains but we get the infinite system of partial differential equations with singular coefficients in 2D Lipschitz domains. In order to present this we apply I.Vekua's dimension reduction method [1,2]. To this end we multiply both the sides of the system (1) by $P_r(ax_3-b)$ and the obtained expressions integrate within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$:

$$
\begin{aligned}
&A_{ij\alpha}\Bigg[u_{jr,\alpha}-\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)+(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\\
&-\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}\left(a_{,\alpha}x_3-b_{,\alpha}\right)P_r'\left(ax_3-b\right)u_j(x_1,x_2,x_3)dx_3\Bigg]\\
&+A_{ij3}\Bigg[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r u_j\left(x_1,x_2,\overset{(-)}{h}\right)\\
&-a\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}P_r'(ax_3-b)\,u_j(x_1,x_2,x_3)\,dx_3\Bigg]\\
&+B_{ij}u_{jr}+C_{ir}\left(x_1,x_2\right)=0,\quad (x_1,x_2)\in\omega,\quad i=\overline{1,n},\ r=0,1,\ldots
\end{aligned}
$$

(under repeated $\alpha$ the sum from 1 to 2 is meant), i.e.,

$$
\begin{aligned}
&A_{ij\alpha}\left(u_{jr,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{\alpha s}u_{js}\right)+A_{ij3}\sum_{s=0}^{r}\overset{r}{a}_{3s}u_{js}+B_{ij}u_{jr}\\
&+A_{ij\alpha}\left[-\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)+(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\
&+A_{ij3}\left[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\
&+C_{ir}\left(x_1,x_2\right)=0,\quad i=1,2,3,\quad r=0,1,\ldots,
\end{aligned}
$$

where

$$
\overset{r}{a}_{\alpha r}:=r\frac{h_{,\alpha}}{h},\quad \overset{r}{a}_{\alpha s}:=(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h},\quad s\neq r,\quad \alpha=1,2,
$$

$$
\overset{r}{a}_{3s}:=-(2s+1)\frac{1-(-1)^{s+r}}{2h}.
$$

The last system is the system of singular partial differential equations which can be easily rewritten in terms of $v_{ir}$. The obtained infinite system of partial differential equations will be a system with the order degeneration for a nonempty $\gamma_0$:

$$A_{ij\alpha}\left[(h^{r+1}v_{jr})_{,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{\alpha s}h^{s+1}v_{js}\right]+A_{ij3}\sum_{s=0}^{r}\overset{r}{a}_{3s}h^{s+1}v_{js}+B_{ij}h^{r+1}v_{jr}=F_{ir}, \tag{3}$$

i.e.,

$$A_{ij\alpha}(h^{r+1}v_{jr})_{,\alpha}+\sum_{s=0}^{r}\overset{r}{E}_{ijs}h^{s+1}v_{js}=F_{ir},\quad i=1,2,...,n,\ \ r=0,1,2,...,$$

where

$$\overset{r}{E}_{ijs}:=A_{ijk}\,\overset{r}{a}_{ks}+B_{ij}\delta_{rs},$$

$$\delta_{rs}=\begin{cases}1, & r=s;\\ 0, & r\neq s,\end{cases}\quad i,j=1,2,...n,\ \ s=0,1,...,r,\ \ r=0,1,2,...,$$

$$\begin{aligned}F_{ir}:=&A_{ij\alpha}\left[\overset{(+)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r\overset{(-)}{h}_{,\alpha}u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]\\ &-A_{ij3}\left[u_j\left(x_1,x_2,\overset{(+)}{h}\right)-(-1)^r u_j\left(x_1,x_2,\overset{(-)}{h}\right)\right]-C_{ir}\left(x_1,x_2\right),\\ &\qquad i=1,2,...,n,\ \ r=0,1,2,\cdots.\end{aligned} \tag{4}$$

Those of $u_j(x_1,x_2,\overset{(+)}{h})$, $u_j(x_1,x_2,\overset{(-)}{h})$ which are given in $3D$ problem on $x_3=\overset{(+)}{h}(x_1,x_2)$ and $x_3=\overset{(-)}{h}(x_1,x_2)$ remain with its given boundary values in the right hand side $F_{ir}$ of the system (3), those of $u_j(x_1,x_2,\overset{(+)}{h})$, $u_j(x_1,x_2,\overset{(-)}{h})$ which are not given on the above surfaces should be replaced by their Legendre-Fourier expansions there, i.e.,

$$u_j(x_1,x_2,\overset{(\pm)}{h})=\sum_{s=0}^{\infty}(\pm1)^s(s+\frac{1}{2})h^s v_{js}(x_1,x_2),$$

containing unknown functions $v_{js}(x_1,x_2)$. The last terms are to be transferred to the left hand side of the system (3), since they contain unknown functions which are sought for.

On the lateral subsurface

$$\Gamma:=\{(x_1,x_2,x_3)\in R^3\ :\ (x_1,x_2)\in\partial\omega,\ \ \overset{(-)}{h}(x_1,x_2)\leq x_3\leq\overset{(+)}{h}(x_1,x_2)\}$$

of $\partial\Omega$ the boundary conditions should be reformulated as follows:

$(i)$ where $\overset{(+)}{h}(x_1,x_2)>\overset{(-)}{h}(x_1,x_2)$, the functions $v_{jr}$ should be calculated by given $u_j(x_1,x_2,x_3)|_{(x_1,x_2)\in\partial\omega}$ by means of the formulas

$$v_{jr}(x_1,x_2)=\frac{1}{h^{r+1}(x_1,x_2)}\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}u_j(x_1,x_2,x_3)\,P_r(ax_3-b)\,dx_3,\quad(x_1,x_2)\in\partial\omega; \tag{5}$$

(*ii*) where $\overset{(+)}{h}(x_1, x_2) = \overset{(-)}{h}(x_1, x_2)$, i.e., on the cusped (in particular, cuspidal) edge, depending on the sharpening geometry of the cusped edge, the unknown functions $v_{jr}$ either should be prescribed or not, but how to calculate them from boundary conditions of $3D$ problem is the subject of special investigation.

The system (1), in particular, contains the governing first order system of the linear theory of elasticity with respect to the stress tensor and displacement vector components. This approach is already successfully applied to the investigation of cusped prismatic shells with cuspidal edges (see [3-7]).

This method can be also applied to the systems of higher order partial differential equations as a method of dimension reduction from $R^m$ to $R^{m-1}$, $m \geq 2$.

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# SOLUTION OF THE THIRD AND FOURTH BVPs OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY FOR THE SPHERE AND FOR SPACE WITH A SPHERICAL CAVE

Basheleishvili M., Bitsadze L.

**Abstract**. The purpose of this paper is to explicitly solve the basic third and the fourth boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with a spherical cavity. The obtained solutions are represented as absolutely and uniformly convergent series.

**Keywords and phrases**: Porous media, double porosity, absolutely and uniformly convergent series, spherical harmonic.

**AMS subject classification (2000):** 74G05; 74G10.

## Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flows and deformation processes occur simultaneously , three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory.They also solved several representative boundary value problems. In part II of these series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity,while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and references cited therein). The basic results and the historical information on the theory of porous media were summarized by de Boer [4].

The main goal of this investigation is to construct explicitly, in the form of absolutely and uniformly convergent series, the solutions of the basic the third and the fourth boundary value problems (BVPs) of the theory of consolidation with double porosity for the sphere and for the whole space with spherical cave.

## 1. Formulation of boundary value problems and uniqueness theorems

The basic Aifantis' equations of statics of the theory of consolidation with double porosity are given in the form [1], [2]

$$\mu\Delta u + (\lambda+\mu) graddivu - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1.1}$$

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0, \tag{1.2}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, $p_1$ is the fluid pressure within the primary pores and $p_2$ is the fluid pressure within the secondary pores. . The constant $\lambda$ is the Lame modulus, $\mu$ is the shear modulus and the constants $\beta_1$ and $\beta_2$ measure the change of porosities due to an applied volumetric strain. $m_j = \dfrac{k_j}{\mu^*}, j = 1, 2.$ The constants $k_1$ and $k_2$ are the permeabilities of the primary and secondary systems of pores, the constant $\mu^*$ denotes the viscosity of the pore fluid and the constant $k$ measures the transfer of fluid from the secondary pores to the primary pores. The quantities $\lambda,\ \ \mu,\ \ k,\ \ \beta_j,\ \ k_j\ \ (j = 1, 2)$ and $\mu^*$ are all positive constants. $\triangle$ is Laplace operator.

Let $D^+ = \{x \in E_3 || x| < a\}$ be an open sphere of radius $a$ centered at point 0 in space $E_3$ and let $S = \{x \in E_3 || x| = a\}$ be a spherical surface of radius $a$. Denote by $D^-$-whole space with a spherical cave.

Introduce the definition of a regular vector-function.

**Definition 1**. A vector-function $U(x) = (u_1, u_2, u_3, p_1, p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in $D^+(D^-)$, and $U$ itself and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U \in C^2(D^+) \bigcap C^1(\overline{D^+})$, $(U \in C^2(D^-) \bigcap C^1(\overline{D^-}))$. Note that for the infinite domain $D^-$ the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U(x) = O(|x|^{-1}), \quad \frac{\partial U_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2 + x_2^3, \ \ j = 1, 2, 3. \tag{1.3}$$

For the equations (1.1)-(1.2) we pose the following boundary value problems:

The third internal and external problem (**Problem** $(III)^{\pm}$). Find in $D^+(D^-)$ a regular solution $U$, of the equations (1.1)-(1.2), by the boundary conditions

$$u^{\pm}(z) = f(z)^{\pm}, \left(\frac{\partial p_1(z)}{\partial n}\right)^{\pm} = f_4^{\pm}, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^{\pm} = f_5^{\pm}(z), \ \ z \in S,$$

where

$$f^{\pm} \in C^{1,\alpha}(S), \quad f_k^{\pm} \in C^{0,\alpha}(S), \ \ 0 < \alpha \leq 1, \ \ k = 4, 5,$$

are given functions.

The fourth internal and external problem (**Problem** $(IV)^{\pm}$).

Find in $D^+(D^-)$ a regular solution $U$, of the equations (1.1)-(1.2), by the boundary conditions

$$(Pu)^{\pm} = f(z)^{\pm}, \qquad p_1^{\pm}(z) = f_4^{\pm}, \ \ p_2^{\pm}(z) = f_5^{\pm}(z), \quad z \in S,$$

where $f^{\pm} \in C^{0,\alpha}(S)$, $f_k^{\pm} \in C^{1,\alpha}(S)$, $0 < \alpha \leq 1$, $k = 4, 5$, are given functions, $Pu$ is a stress vector, which acts on an elements of the $S$ with the normal $n = (n_1, n_2, n_3)$

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.4}$$

here $T(\partial x, n)$ is a stress tensor [7]

$$\begin{aligned} &T(\partial x, n) = \| \, T_{kj}(\partial x, n) \, \|_{3x3}, \\ &T_{kj}(\partial x, n) = \mu \delta_{kj} \frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \quad k, j, = 1, 2, 3. \end{aligned} \tag{1.5}$$

Further we assume that $p_j$ is known, when $x \in D^+$ or $x \in D^-$. Substitute $\beta_1 p_1 + \beta_2 p_2$ in (1.1) and search the particular solution of the following equation

$$\mu \Delta u + (\lambda + \mu) graddivu = grad(\beta_1 p_1 + \beta_2 p_2).$$

It is known, that a particular solution of the equation (1.1) is the following potential [7]

$$u_0(x) = -\frac{1}{4\pi} \iiint\limits_D \Gamma(x - y) grad(\beta_1 p_1 + \beta_2 p_2) dy, \tag{1.6}$$

where

$$\Gamma(x - y) = \frac{1}{4\mu(\lambda + 2\mu)} \left\| \frac{(\lambda + 3\mu)\delta_{kj}}{r} + \frac{(\lambda + \mu)(x_k - y_k)(x_j - y_j)}{r^3} \right\|_{3\times 3},$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2.$$

Substituting the volume potential $u_0$ into (1.1) we obtain (see [7])

$$\mu \Delta u_0 + (\lambda + \mu) graddivu_0 = grad(\beta_1 p_1 + \beta_2 p_2).$$

Thus we have proved that $u_0(x)$ is a particular solution of the equation (1.1). In (1.6) $D$ denotes either $D^+$ or $D^-$, $grad(\beta_1 p_1 + \beta_2 p_2)$ is a continuous vector in $D^+$ along with its first derivatives. When $D = D^-$ the vector $grad(\beta_1 p_1 + \beta_2 p_2)$ has to satisfy the following condition at infinity

$$grad(\beta_1 p_1 + \beta_2 p_2) = O(|x|^{-2-\alpha}), \alpha > 0.$$

Thus the general solution of the equation (1.1) is representable in the form $u = V + u_0$, where

$$A(\partial x)V = \mu \Delta V + (\lambda + \mu) graddivV = 0. \tag{1.7}$$

The latter equation is the equation of an isotropic elastic body. i.e. we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

## 2. Some auxiliary formulas

The spherical coordinates are defined by the equalities

$$\begin{aligned} &x_1=\rho\sin\vartheta\cos\varphi,\quad x_2=\rho\sin\vartheta\sin\varphi,\quad x_3=\rho\cos\vartheta,\quad x\in D^+,\\ &y_1=a\sin\vartheta_0\cos\varphi_0,\quad y_2=a\sin\vartheta_0\sin\varphi_0,\quad y_3=a\cos\vartheta_0,\quad y\in S,\\ &\rho^2=x_1^2+x_2^2+x_3^2,\quad 0\le\vartheta\le\pi,\quad 0\le\varphi\le 2\pi,\end{aligned} \tag{2.1}$$

Let

$$f(z)=\sum_{m=0}^{\infty} f_m(\vartheta,\varphi),$$

where $f_m$ is the sperical function of order $m$ :

$$f_m(\vartheta,\varphi)=\frac{2m+1}{4\pi a^2}\int\limits_S P_m(\cos\gamma)f(y)dS_y,$$

$P_m$ is Legender polynomial of the m-th order, $\gamma$ is an angle formed by the radius-vector $Ox$ and $Oy$,

$$\cos\gamma=\frac{1}{|x||y|}\sum_{m=1}^{3}x_k y_k.$$

The general solutions of the equation $(\Delta-\lambda_0^2)\psi=0$ in the domains $D^+(D^-)$ have the form ([6])

$$\begin{aligned}\psi(x)&=\sum_{n=0}^{\infty}\frac{J_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi),\quad \rho<a,\\ \psi(x)&=\sum_{n=0}^{\infty}\frac{H^{(2)}_{n+\frac12}(i\lambda_0\rho)}{\sqrt{\rho}}Y_n(\vartheta,\varphi),\quad \rho>a,\end{aligned}\tag{2.2}$$

$$\lambda_0^2=\frac{k}{m_1}+\frac{k}{m_2}>0.$$

$Y_n(\vartheta,\varphi)$ is the spherical harmonic.

The general solutions of the equation $\Delta\phi=0$ in the domains $D^+(D^-)$ have the form ([5], p.505)

$$\begin{aligned}\phi(x)&=\sum_{n=0}^{\infty}\frac{\rho^n}{(2n+1)a^{n-1}}Z_n(\vartheta,\varphi),\quad \rho<a,\\ \phi(x)&=\sum_{n=0}^{\infty}\frac{a^{n+2}}{(2n+1)\rho^{n+1}}Z_n(\vartheta,\varphi),\quad \rho>a,\end{aligned}\tag{2.3}$$

$Z_n(\vartheta,\varphi)$ is the spherical harmonic.

It is easy to show that the general solution of the equation (1.2) is representable in the form

$$p_1=-m_2\psi+\phi,\quad p_2=m_1\psi+\phi \tag{2.4}$$

where $\psi$ and $\phi$ are arbitrary solutions of the following equations

$$(\Delta - \lambda_0^2)\psi = 0, \quad \Delta\phi = 0.$$

The following theorems are valid and we cite them without proof.

**Theorem 1.** *The boundary value problems $(III)^-$, $(IV)^-$ have at most one regular solution in the domain $D^-$.*

**Theorem 2.** *Two regular solutions of the boundary value problem $(III)^+$ in the domain $D^+$ may differ by the vector $V(u, p_1, p_2)$, where $u = 0$, and $p_1 = p_2 = c$.*

**Theorem 3.** *Two regular solutions of the boundary value problem $(IY)^+$ may differ by the vector $V(u, p_1, p_2)$, where $u$ vector is a rigid displacement $u_1 = c_1 - \epsilon x_2$, $u_2 = c_2 + \epsilon x_1$, and $p_1 = p_2 = 0$, $x \in D^+$, $\epsilon$ and $c_j$, $j = 1, 2$, are arbitrary real constants.*

### 3. Solution of the third boundary value problem

**Problem** $(III)^+$. First of all we construct a solution for the equations (1.2). A solution of the boundary value problem ($[\frac{\partial p_1}{\partial n}]^+ = f_4^+(z)$, $[\frac{\partial p_2}{\partial n}]^+ = f_5^+(z)$) we seek in the following form

$$\begin{aligned} p_1 &= -m_2 \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{\rho^n}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi), \quad \rho < a, \\ p_2 &= m_1 \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{\rho^n}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi)), \quad \rho < a. \end{aligned} \tag{3.1}$$

Taking into account the fact that $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial \rho}$, from the last equation we obtain

$$\begin{aligned} \frac{\partial p_1}{\partial n} = \frac{\partial p_1}{\partial \rho} &= -m_2 \sum_{n=0}^{\infty} \frac{\partial}{\partial \rho} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{n\rho^{n-1}}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi), \quad \rho < a, \\ \frac{\partial p_2}{\partial n} = \frac{\partial p_2}{\partial \rho} &= m_1 \sum_{n=0}^{\infty} \frac{\partial}{\partial \rho} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}} Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{n\rho^{n-1}}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi), \quad \rho < a. \end{aligned} \tag{3.2}$$

Let us rewrite (3.2) as

$$\begin{aligned} \frac{\partial p_1}{\partial \rho} &= -m_2 \sum_{n=0}^{\infty} H_n(\rho) Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{n\rho^{n-1}}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi)), \quad \rho < a, \\ \frac{\partial p_2}{\partial \rho} &= m_1 \sum_{n=0}^{\infty} H_n(\rho) Y_n(\vartheta, \varphi) + \sum_{n=0}^{\infty} \frac{n\rho^{n-1}}{(2n+1)a^{n-1}} Z_n(\vartheta, \varphi), \quad \rho < a, \end{aligned} \tag{3.3}$$

where $H_n(\rho) = \dfrac{\partial}{\partial \rho} \dfrac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}$.

Let

$$f_k(z) = \sum_{n=0}^{\infty} \widehat{f}_{nk}(\vartheta_0, \varphi_0),$$

where $\widehat{f}_{nk}, \quad k=4,5$ is the sperical function of order $n$ :

$$\widehat{f}_{nk}(\vartheta_0,\varphi_0)=\frac{2n+1}{4\pi a^2}\int\limits_S P_n(\cos\gamma)f_k(y)dS_y, \quad k=4,5.$$

Passing to the limit in (3.3) as $D^+\ni\rho\to a$, we obtain

$$\begin{aligned}
&-m_2\sum_{n=0}^{\infty}H_n(a)Y_n(\vartheta_0,\varphi_0)+\sum_{n=0}^{\infty}\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^{\infty}\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\
&m_1\sum_{n=0}^{\infty}H_n(a)Y_n(\vartheta_0,\varphi_0)+\sum_{n=0}^{\infty}\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^{\infty}\widehat{f}_{5n}(\vartheta_0,\varphi_0).
\end{aligned}\tag{3.4}$$

For the coefficients of $Y_n$ and $Z_n$, (3.4) yields the following equations:

$$\begin{aligned}
&-m_2H_n(a)Y_n(\vartheta_0,\varphi_0)+\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\
&m_1H_n(a)Y_n(\vartheta_0,\varphi_0)+\frac{n}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{5n}(\vartheta_0,\varphi_0), \quad n=1,2,..
\end{aligned}\tag{3.5}$$

By elementary calculation from (3.5) we define $Y_n$ and $Z_n$, for $n\geq 1$

$$\begin{aligned}
&Y_n(\vartheta_0,\varphi_0)=\frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0)-\widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2)H_n(a)},\\
&Z_n(\vartheta_0,\varphi_0)=\frac{(2n+1)[m_1\widehat{f}_{4n}(\vartheta_0,\varphi_0)+m_2\widehat{f}_{5n}(\vartheta_0,\varphi_0)]}{n(m_1+m_2)}, \quad n=1,2,...
\end{aligned}\tag{3.6}$$

Note that $Z_0$ is an arbitrary constant and

$$Y_0=\int\limits_S f_4dS=\int\limits_S f_5dS=0.$$

Substituting (3.6) into (3.1), we obtain a solution of the BVP in the form of series

$$\begin{aligned}
p_1&=\frac{-m_2}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]\\
&+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c, \quad \rho<a,\\
p_2&=\frac{m_1}{(m_1+m_2)\sqrt{\rho}}\sum_{n=1}^{\infty}\frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{H_n(a)}[\widehat{f}_{5n}(\vartheta,\varphi)-\widehat{f}_{4n}(\vartheta,\varphi)]\\
&+\frac{1}{m_1+m_2}\sum_{n=1}^{\infty}\frac{\rho^n}{na^{n-1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi)+m_2\widehat{f}_{5n}(\vartheta,\varphi)]+c, \quad \rho<a.
\end{aligned}\tag{3.7}$$

**Problem** $(III)^{-}$. The boundary value problem $[\frac{\partial p_1}{\partial n}]^{-} = f_4^{-}(z)$, $[\frac{\partial p_2}{\partial n}]^{-} = f_5^{-}(z)$ can be solved analogously and we have

$$\begin{aligned} p_1 &= \frac{-m_2}{(m_1+m_2)\sqrt{\rho}} \sum_{n=1}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)] \\ &- \frac{1}{m_1+m_2} \sum_{n=1}^{\infty} \frac{a^{n+2}}{(n+1)\rho^{n+1}} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a, \\ p_2 &= \frac{m_1}{(m_1+m_2)\sqrt{\rho}} \sum_{n=1}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{h_n(a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)] \\ &- \frac{1}{m_1+m_2} \sum_{n=1}^{\infty} \frac{a^{n+2}}{(n+1)\varrho^{n+1}} [m_1 \widehat{f}_{4n}(\vartheta,\varphi) + m_2 \widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a, \end{aligned} \tag{3.8}$$

where $\quad h_n(\rho) = \dfrac{\partial}{\partial \rho} \dfrac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{\sqrt{\rho}}.$

The functions $\dfrac{\partial p_k}{\partial \rho}$ can be calculated from (3.7)-(3.8).

The solution of the equation

$$\mu \Delta V + (\lambda+\mu) graddiv V = 0,$$

when $V^{\pm} = F^{\pm}$ for a ball is due to Natroshvili D. [8]. (A detailed exposition of the solution can be found in monograph [7]).

$$V(x) = \iint_S \overset{(1)+}{\mathrm{K}}(x,y) F^{+}(y) d_y S, \quad x \in D^{+}, \quad y \in S,$$

$$V(x) = \iint_S \overset{(1)-}{\mathrm{K}}(x,y) F^{-}(y) d_y S, \quad x \in D^{-}, \quad y \in S,$$

where

$$\overset{(1)+}{\mathrm{K}} = \| \overset{(1)+}{\underset{kj}{\mathrm{K}}} \|_{3\times 3},$$

$$\overset{(1)+}{\underset{kj}{\mathrm{K}}} = \frac{1}{4\pi a} \left[ \frac{a^2-\rho^2}{r^3} \delta_{ij} + \beta (a^2-\rho^2) \frac{\partial^2 \Phi(x,y)}{\partial x_i \partial x_j} \right],$$

$$\Phi(x,y) = \int_0^1 \left[ \frac{a^2-\rho^2 t^2}{Q(t)} - \frac{1}{a} - \frac{3t\rho cos\gamma}{a^2} \right] \frac{dt}{t^{1+\alpha}},$$

$$Q(t) = (a^2 - 2a\rho t cos\gamma + \rho^2 t^2)^{\frac{3}{2}},$$

$$\overset{(1)-}{\mathrm{K}} = \| \overset{(1)-}{\underset{kj}{\mathrm{K}}} \|_{3\times 3},$$

$$\overset{(1)-}{\underset{kj}{\mathrm{K}}} = \frac{1}{4\pi a}\left[\frac{\rho^2-a^2}{r^3}\delta_{ij}+\beta(\rho^2-a^2)\frac{\partial^2\Phi^*(x,y)}{\partial x_i\partial x_j}\right],$$

$$\Phi^*(x,y)=\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}t^\alpha dt,\quad Q^*(t)=(\rho^2-2a\rho t cos\gamma+a^2t^2)^{\frac{3}{2}},$$

$$cos\gamma=\frac{x_1y_1+x_2y_2+x_3y_3}{ar}=sin\theta sin\theta' cos(\phi-\phi')+cos\theta cos\theta',$$

$$r^2=a^2-2atcos\gamma+\rho^2,\quad \beta=\frac{\lambda+\mu}{(2\lambda+3\mu)},\quad \alpha=\frac{\lambda+2\mu}{2(\lambda+3\mu)}<1,\quad F^\pm\in C^{1,\alpha}(S).$$

Finally we have proved the following

**Theorem 4.** *The third BVP $(III)^-$is uniquely solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{1,\alpha}(S)$, $\alpha>\frac{1}{2}$. The solution of third BVP $(III)^+$ is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{1,\alpha}(S)$, $\alpha>\frac{1}{2}$ and two regular solutions of the boundary value problem $(III)^+$ in the domain $D^+$ may differ only to within additive constant $c, p_j=c, j=1,2$.*

## 4. Solution of the fourth boundary value problem

**Problem** $(IV)^+$.First of all we will construct a solution for the equations (1.2). A solution of the boundary value problem ($p_1^+(z)=f_4^+,\ p_2^+(z)=f_5^+(z),$) is sought in the form (3.1):

Passing to the limit in (3.1) as $D^+\ni\rho\to a$, we have

$$\begin{aligned}&-m_2\sum_{n=0}^\infty\frac{J_{n+\frac{1}{2}}(i\lambda_0a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0)+a\sum_{n=0}^\infty\frac{1}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^\infty\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\&m_1\sum_{n=0}^\infty\frac{J_{n+\frac{1}{2}}(i\lambda_0a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0)+a\sum_{n=0}^\infty\frac{1}{(2n+1)}Z_n(\vartheta_0,\varphi_0)=\sum_{n=0}^\infty\widehat{f}_{5n}(\vartheta_0,\varphi_0),\end{aligned}\tag{4.1}$$

For the coefficients of $Y_n$ and $Z_n$, (4.1) yields the following equations:

$$\begin{aligned}&-m_2\frac{J_{n+\frac{1}{2}}(i\lambda_0a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0)+\frac{a}{2n+1}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{4n}(\vartheta_0,\varphi_0),\\&m_1\frac{J_{n+\frac{1}{2}}(i\lambda_0a)}{\sqrt{a}}Y_n(\vartheta_0,\varphi_0)+\frac{a}{2n+1}Z_n(\vartheta_0,\varphi_0)=\widehat{f}_{5n}(\vartheta_0,\varphi_0),\end{aligned}\tag{4.2}$$

By elementary calculation from (4.2) we obtain

$$\begin{aligned}&Y_n(\vartheta_0,\varphi_0)=\frac{\widehat{f}_{5n}(\vartheta_0,\varphi_0)-\widehat{f}_{4n}(\vartheta_0,\varphi_0)}{(m_1+m_2)J_{n+\frac{1}{2}}(i\lambda_0a)}\sqrt{a},\\&Z_n(\vartheta_0,\varphi_0)=\frac{(2n+1)[m_1\widehat{f}_{4n}(\vartheta_0,\varphi_0)+m_2\widehat{f}_{5n}(\vartheta_0,\varphi_0))]}{a(m_1+m_2)}.\end{aligned}\tag{4.3}$$

Substituting (4.3) into (3.1), we obtain a solution of the BVP in the form of a series

$$p_1 = \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)} (\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi))$$
$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\rho^n}{a^n}[m_1\widehat{f}_{4n}(\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$
$$p_2 = \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{J_{n+\frac{1}{2}}(i\lambda_0\rho)}{J_{n+\frac{1}{2}}(i\lambda_0 a)} (\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi))$$
$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{\rho^n}{a^n}[m_1\widehat{f}_{4n}(\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho < a,$$

**Problem** $(IV)^-$. Analogously we construct a solution of the BVP $p_1^-(z) = f_4^-, \quad p_2^-(z) = f_5^-(z)$, in the domain $D^-$

$$p_1 = \frac{-m_2\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)} [\widehat{f}_{5n}(\vartheta,\varphi) - \widehat{f}_{4n}(\vartheta,\varphi)]$$
$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{a^{n+1}}{\rho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)],$$
$$p_2 = \frac{m_1\sqrt{a}}{(m_1+m_2)\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0\rho)}{H^{(2)}_{n+\frac{1}{2}}(i\lambda_0 a)} [\widehat{f}_{5n}(\theta,\phi) - \widehat{f}_{4n}(\theta,\phi)]$$
$$+\frac{1}{(m_1+m_2)} \sum_{n=0}^{\infty} \frac{a^{n+1}}{\rho^{n+1}}[m_1\widehat{f}_{4n}(\vartheta,\varphi) + m_2\widehat{f}_{5n}(\vartheta,\varphi)], \quad \rho > a.$$

For these series together with their first derivatives to be absolutely and uniformly convergent it is sufficient that $f_k^{\pm} \in C^{1,\alpha}(S), \ \ 0 < \alpha \le 1, \ \ k = 4,5$. Solutions obtained under such conditions are regular in $D^+$.

The solution of the problem $(TV)^{\pm} = F^{\pm}$, for the equation (1.8) for a ball is given in the work by D. Natroshvili [8] (A detailed exposition of the solution can be found in monograph [7]).

$$V(x) = \iint\limits_S \overset{(2)+}{\mathrm{K}}(x,y)F^+(y)d_ys + a_1 + [\omega, x] + \frac{c(\beta_1+\beta_2)}{3\lambda+2\mu}x, \quad x \in D^+,$$
$$TV = \frac{1}{4\pi\rho}\iint\limits_S \| \ \frac{a^2-\rho^2}{r^3}\delta_{ij} + (a^2-\rho^2)\frac{\partial^2\Phi_4(x,y)}{\partial x_i \partial x_j} \ \|_{3x3} \ F^+(y)ds, \quad x \in D^+,$$
$$V(x) = \iint\limits_S \overset{(2)-}{\mathrm{K}}(x,y)F^-(y)d_ys, \quad x \in D^-,$$
$$TV = \frac{1}{4\pi\rho}\iint\limits_S \| \ \frac{\rho^2-a^2}{r^3}\delta_{ij} + (\rho^2-a^2)\frac{\partial^2\Phi_4^*(x,y)}{\partial x_i \partial x_j} \ \|_{3x3} \ F^-(y)ds, \quad x \in D^-,$$

where

$$\overset{(2)+}{\mathrm{K}} = \| \overset{(2)+}{\underset{kj}{\mathrm{K}}} \|_{3\times 3},$$

$$\overset{(2)+}{\underset{kj}{\mathrm{K}}} = \frac{1}{8\mu\pi}\left[(\Phi_1+\Phi_2)\delta_{ij} + \frac{a^2-3\rho^2}{2}\frac{\partial^2\Phi_3(x,y)}{\partial x_i\partial y_j} + x_j\frac{\partial}{\partial x_i}(\Phi_1-\Phi_2) - 2x_i\frac{\partial\Phi_1}{\partial x_j}\right]$$
$$+\frac{1}{8\mu\pi}\left[x_i\frac{\partial}{\partial x_j}(2\rho\frac{\partial\Phi_3}{\partial\rho}-\Phi_3)+\rho^2\left(\frac{\partial^2\Phi_2(x,y)}{\partial x_i\partial y_j}-\frac{\partial^2\Phi_1(x,y)}{\partial x_i\partial y_j}\right)\right],$$
$$\Phi_1(x,y)=\int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)}-\frac{1}{a}\right]\frac{dt}{t},\quad Q(t)=(a^2-2a\rho t cos\gamma+\rho^2t^2)^{\frac{3}{2}},$$
$$\Phi_2(x,y)=\int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)}-\frac{1}{a}-\frac{3t\rho cos\gamma}{a^2}\right]\frac{dt}{t^2},$$
$$\Phi_0(x,y)=\int_0^1\left[\frac{a^2-\rho^2t^2}{Q(t)}-\frac{1}{a}\right]\frac{dt}{t^{1+\alpha_1}},\quad \Phi_3=\frac{1}{b_1}Im\Phi_0,\quad \Phi_4=Re(b_2\Phi_0),$$
$$\alpha_1=b_0+ib_1=\frac{\mu+i\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}{2(\lambda+\mu)},\quad b_2=\frac{1}{2}+\frac{3\lambda+4\mu}{2\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}},$$

$$\overset{(2)-}{\mathrm{K}} =\| \overset{(2)-}{\underset{kj}{\mathrm{K}}} \|_{3\times3},$$

$$\overset{(2)-}{\underset{kj}{\mathrm{K}}} = \frac{1}{8\mu\pi}\left[-(\Phi_1^*+\Phi_2^*)\delta_{ij} + \frac{a^2-3\rho^2}{2}\frac{\partial^2\Phi_3^*(x,y)}{\partial x_i\partial y_j} - x_j\frac{\partial}{\partial x_i}(\Phi_1-\Phi_2) + 2x_i\frac{\partial\Phi_1^*}{\partial x_j}\right]$$
$$+\frac{1}{8\mu\pi}\left[x_i\frac{\partial}{\partial x_j}(2\rho\frac{\partial\Phi_3^*}{\partial\rho}-\Phi_3^*)-\rho\left(\frac{\partial^2\Phi_2(x,y)}{\partial x_i\partial y_j}-\frac{\partial^1\Phi_1(x,y)}{\partial x_i\partial y_j}\right)\right],$$

$$\Phi_l^*(x,y)=\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}t^{l-1}dt,\quad l=1,2,\quad \Phi_3^*=\frac{2(\lambda+\mu)}{\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}Im\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}\frac{dt}{t^{\alpha_2}}$$
$$\Phi_4^*(x,y)=ReA\int_0^1\frac{\rho^2-a^2t^2}{Q^*(t)}\frac{dt}{t^{\alpha_2}},\quad Q^*(t)=(\rho^2-2a\rho t cos\gamma+a^2t^2)^{\frac{3}{2}},$$
$$\alpha_2=\frac{-\mu+i\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}{2(\lambda+\mu)},\quad A=\frac{1}{2}-i\frac{3\lambda+4\mu}{2\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}.$$

Thus we have proved the following

**Theorem 5.** *For the solvability of the BVP $(IV)^+$ it is necessary that the principal vector and the principal moment of external forces be equal to zero. The BVP $(IV)^+$ is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series if the boundary data are from space $C^{0,\alpha}(S)$, $\alpha>\frac{1}{2}$. Two regular solutions of BVP $(IV)^+$ may differ only to within additive vector $a+[b,x]$, where $a,b$, are arbitrary real constant vectors, $x=x(x_1,x_2,x_3)$. The BVP $(IV)^-$ is solvable in the class of regular functions and the solution is represented in the form of absolutely and uniformly convergent series.*

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# ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF GENERALIZED EMDEN-FOWLER EQUATIONS WITH ADVANCED ARGUMENT

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**Abstract**. The generalized Emden-Fowler Equation

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0$$

is considered, where $p \in L_{\rm loc}(R_+; R_-)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R_+)$ and $\sigma(t) \geq t$ for $t \in R_+$. Oscillatory properties of solutions of the equation are studied. In particular, sufficient conditions are established for the equation to have Property **B**.

**Keywords and phrases**: Functional-differential equation, oscillation, Property B.

**AMS subject classification (2000):** 34K06; 34K11.

## 1. Introduction

In the paper the following differential equation is considered:

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0, \tag{1.1}$$

where

$$\begin{gathered} p \in L_{\rm loc}(R_+; R_-), \quad \mu \in C(R_+; (0, +\infty)), \quad \sigma \in C(R_+; R_+) \\ \text{and} \quad \sigma(t) \geq t \quad \text{for} \quad t \in R_+. \end{gathered} \tag{1.2}$$

New sufficient conditions are established for oscillation of solutions of (1.1). Specifically, sufficient conditions are given for the equation (1.1) to have Property **B** (see below the definition of Property **B**).

A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of (1.1), if it is locally absolutely continuous together with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq t_0$ and satisfies (1.1) almost everywhere on $[t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of the (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution $u$ is said to be nonoscillatory.

**Definition.** We say that the equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1)$$

or

$$\big|u^{(i)}(t)\big| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1), \tag{1.3}$$

when $n$ is even and either is oscillatory or satisfies (1.3) when $n$ is odd.

In the present paper sufficient conditions of new type will be given for the equation (1.1) to have Property **B**. Analogous results for Property **A** are presented in [1]. As to almost linear equations (i.e. when $\lim\limits_{t\to+\infty}\mu(t)=1$), analogous issues for them are substantively studied in [2–4]. The result of the present paper make somewhat more complete those of [5] in case of Property **B**.

Let $t_0\in R_+$ and $\ell\in\{1,\dots,n-1\}$. By $\mathbf{U}_{\ell,t_0}$ we denote the set of all proper solutions $u:[t_0,+\infty)\to R$ of the equation (1.1) satisfying the conditions

$$\begin{aligned} u^{(i)}(t)>0 \quad &\text{for}\quad t\ge t_* \quad (i=0,\dots,\ell-1),\\ (-1)^{i+\ell}u^{(i)}(t)>0 \quad &\text{for}\quad t\ge t_* \quad (i=\ell,\dots,n-1), \end{aligned} \tag{1.4$_\ell$}$$

where $t_*\in[t_0,+\infty)$.

## 2. Sufficient conditions of nonexistence of solutions of the type $(1.4_\ell)$

The assumption of the Theorems presented below contain one of the following two conditions:

$$\mu(t)\le\lambda<1 \quad\text{for}\quad t\in R_+ \tag{2.1}$$

or

$$\mu(t)\ge\lambda \quad\text{for}\quad t\in R_+ \quad\text{and}\quad \lambda\in(0,1). \tag{2.2}$$

The results of this section play an important role in establishing sufficient conditions for the equation (1.1) to have Property **B**.

**Theorem 2.1.** *Let the conditions* (1.2), (2.1) *and*

$$\int_0^{+\infty} t^{n-\ell-1}\big(\sigma(t)\big)^{\ell\mu(t)}|p(t)|dt=+\infty \tag{2.3$_\ell$}$$

*be fulfilled and for some $\gamma\in(0,1)$*

$$\liminf_{t\to+\infty} t^{\gamma}\int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda}\big(\sigma(t)\big)^{(\ell-1)\mu(s)}|p(s)|ds>0, \tag{2.4$_\ell$}$$

*where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ even. If, moreover, for some $\delta\in[0,\lambda]$ and $\sigma_*\in C(R_+)$ such that*

$$t\le\sigma_*(t)\le\sigma(t) \quad\textit{for}\quad t\in R_+, \tag{2.5}$$

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}\big(\sigma_*(t)\big)^{\mu(t)-\lambda+\frac{\delta(1-\gamma)}{1-\lambda}}\big(\sigma(t)\big)^{(\ell-1)\mu(t)}|p(t)|dt=+\infty,$$

*then for any $t_0\in R_+$ we have $\mathbf{U}_{\ell,t_0}=\varnothing$.*

**Theorem 2.2** *Let the conditions* (1.2), (2.1), $(2.3_\ell)$ *and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}|p(s)|ds>0 \tag{2.6$_\ell$}$$

*be fulfilled, where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ even. If, moreover, for some $\delta\in[0,\lambda]$ and $\sigma_*\in C(R_+)$ satisfying the condition* (2.5) *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}\big(\sigma_*(t)\big)^{\mu(t)-\lambda}\big(\sigma(t)\big)^{\mu(t)(\ell-1)} \times\big(\ln(1+\sigma_*(t))\big)^{\frac{\delta}{1-\lambda}}|p(s)|ds=+\infty$$

*holds, then for any* $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

**Theorem 2.3.** *Let the conditions* (1.2), (2.2), (2.3$_\ell$) *and*

$$\liminf_{t\to+\infty} t^\gamma \int_t^{+\infty} s^{n-\ell-1}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}|p(s)|ds > 0 \tag{2.7$_\ell$}$$

*be fulfilled, where* $\gamma \in (0,1)$ *and* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *even. If, moreover, for some* $\delta \in [0,\lambda]$ *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\delta}\big(\sigma(t)\big)^{(\ell-1)\mu(t)+\frac{(\mu(t)-\delta)(1-\gamma)}{1-\lambda}}|p(t)|dt = +\infty$$

*holds, then for any* $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

**Theorem 2.4.** *Let the conditions* (1.2), (2.2) *and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-\ell-1}\big(\sigma(s)\big)^{(\ell-1)\mu(s)}p(s)ds > 0 \tag{2.8$_\ell$}$$

*be fulfilled, where* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *odd. If, moreover, for some* $\delta \in [0,\lambda]$ *the equality*

$$\int_0^{+\infty} t^{n-\ell-1+\lambda+\delta}\big(\sigma(t)\big)^{(\ell-1)\mu(t)}\big(\ln(1+\sigma(t)\big)^{\frac{\mu(t)-\delta}{1-\lambda}}|p(t)|dt = +\infty$$

*holds, then for any* $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

### 3. Differential equations with property B (case $\mu(t) \le \lambda$)

**Theorem 31.** *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.4$_1$) *and*

$$\liminf_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} > 0, \tag{3.1}$$

*be fulfilled. If, moreover,*

$$\int_0^{+\infty} t^{n-2+\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}|p(t)|dt = +\infty,$$

*then the equation* (1.1) *has Property* **B**.

**Theorem 3.2.** *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.4$_1$) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}|p(t)|dt = +\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 3.3.** *Let the conditions* (1.2), (2.1), (2.3$_1$), (2.6$_1$) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2+\mu(t)-\lambda}\big(\ln(1+t)\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt = +\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 3.4.** *Let the conditions* (1.2), (2.1), (3.1), $(2.3_1)$ *and* $(2.6_1)$ *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\mu(t)-\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 3.5.** *Let the conditions* (1.2), (2.1), $(2.3_{n-1})$, $(2.4_{n-2})$ *and*

$$\limsup_{t\to+\infty}\frac{(\sigma(t))^{\mu(t)}}{t}<+\infty \tag{3.2}$$

*be fulfilled. If, moreover,*

$$\int_0^{+\infty} t^{\mu(t)+\frac{\lambda(\lambda-\gamma)}{1-\lambda}}\big(\sigma(t)\big)^{\mu(t)(n-3)}|p(t)|dt=+\infty,$$

*then the equation* (1.1) *has Property* **B**.

**Theorem 3.6.** *Let the conditions* (1.2), (2.1), $(2.3_{n-1})$, $(3.4_{n-2})$ *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-2)\mu(t)+\frac{\lambda(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 3.7** *Let the conditions* (1.2), (2.1), $(2.3_{n-1})$, $(2.6_{n-2})$ *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\mu(t)-\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+t)\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 3.8.** *Let the conditions* (1.2), (2.1), $(2.3_{n-2})$, $(2.6_{n-2})$ *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} \big(\sigma(t)\big)^{(n-1)\mu(t)-\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

## 4. Differential equations with property B (case $\mu(t)\geq\lambda$)

**Theorem 4.1.** *Let the conditions* (1.2), (2.2), $(2.3_1)$, $(2.9_1)$ *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\sigma(t)\big)^{\frac{\mu(t)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.2.** *Let the conditions* (1.1), (1.2), $(2.3_1)$, $(2.9_1)$ *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{\frac{(\mu(t)-\lambda)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.3.** *Let the conditions* (1.2), (2.2), (3.1), ($2.3_1$) *and* ($2.8_1$) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.4.** *Let the conditions* (1.2), (2.2), ($2.3_1$), ($2.8_1$) *and* (3.1) *be fulfilled and*

$$\int_0^{+\infty} t^{n-2+\lambda}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)-\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.5.** *Let the conditions* (1.2), (2.2), ($2.3_{n-1}$), ($2.7_{n-2}$) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-3)\mu(t)+\frac{\mu(t)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.6.** *Let the conditions* (1.2), (2.2), ($2.3_{n-1}$), ($2.7_{n-2}$) *and* (3.2) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)+\frac{(\mu(t)-\lambda)(1-\gamma)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.7.** *Let the conditions* (1.2), (2.2), ($2.3_{n-1}$), (3.2) *and* ($2.8_{n-2}$), *be fulfilled and*

$$\int_0^{+\infty} t\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

**Theorem 4.8.** *Let the conditions* (1.2), (2.2), ($2.3_{n-1}$) *and* ($2.8_{n-2}$) *be fulfilled and*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{(n-3)\mu(t)}\big(\ln(1+\sigma(t))\big)^{\frac{\mu(t)-\lambda}{1-\lambda}}|p(t)|dt=+\infty.$$

*Then the equation* (1.1) *has Property* **B**.

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# UNIQUENESS AND EXISTENCE THEOREMS OF STATICS BVPs OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

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**Abstract**. The purpose of this paper is to consider two-dimensional version of statics of the Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs).

In this work we intend to extend potential method and the theory of integral equation to BVPs of the theory of consolidation with double porosity. The potential method and the theory of integral equation are applied to the investigation of the first and second BVPs of statics of the theory of consolidation with double porosity. For their problems we construct Fredholm type integral equations. Using these equations, the potential method and generalized Green's Formulas, we prove the existence and uniqueness theorems of solutions for the first and second BVPs for the bounded and unbounded domains. For the Aifantis' equation of statics we construct one particular solution and we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic body.

**Keywords and phrases**: Porous media, double porosity, consolidation, fundamental solution.

**AMS subject classification (2000):** 74G25; 74G30.

## 1. Introduction

In a material with two degrees of porosity, there are two pore system, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1, 2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1-3], where analytical solutions of the relevant equations are also given. In part I of a series of paper on the subject, Wilson and Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by Beskos and Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1-3] and references cited therein).The basic results and the historical information on the theory of porous media were summarized by de Boer [4].

In this work we prove the existence and uniqueness theorems of solutions of basic BVPs of the theory of consolidation with double porosity for bounded and unbounded domains. We used the potential method for the proof of all theorems. The basic results on this method are given in [6].

## 2. Basic equations and boundary value problems

Let $\mathbf{x} = (x_1, x_2)$ be the point of the Euclidean two-dimensional space $E^2$. The basic equations of statics of the theory of consolidation with double porosity in the case of plane deformation have the following form [1-2]

$$\begin{aligned} &\mathbf{B}(\partial x)\mathbf{u} = \mu\,\Delta\mathbf{u} + (\lambda+\mu) grad\, div\,\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \\ &(m_1\Delta - k)p_1 + kp_2 = 0, \qquad kp_1 + (m_2\Delta - k)p_2 = 0, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2)$ is the displacement vector, $p_1$ is the fluid pressure within the primary pores and $p_2$ is the fluid pressure within the secondary pores, $m_j = \frac{k_j}{\mu^*}, j = 1, 2$. The constant $\lambda$ is the Lame modulus, $\mu$ is the shear modulus, the constants $\beta_1$ and $\beta_2$ are measure the change of porosities due to an applied volumetric strain. The constant $\mu^*$ denotes the viscosity of the pore fluid, the constant $k$ measures the transfer of fluid from the secondary pores to the primary pores. The quantities $\lambda, \mu, \beta_j, k$ $(j = 1, 2)$ and $\mu^*$ are all positive constants. $\triangle = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is two-dimensional Laplace operator.

The equation (1) can be written in matrix-vector form

$$\mathbf{A}(\partial x)\mathbf{U}(x) = 0, \tag{2}$$

where $\mathbf{U}(x) = (u_1, u_2, p_1, p_2)$,

$$\begin{aligned} &\mathbf{A}(\partial x) = \parallel A_{pq}(\partial x) \parallel_{4x4}, \; A_{jj}(\partial x) = \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_j^2}, \\ &A_{12}(\partial x) = A_{21}(\partial x) = (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2}, \\ &A_{j3}(\partial x) = -\beta_1\frac{\partial}{\partial x_j}, \quad A_{j4}(\partial x) = -\beta_2\frac{\partial}{\partial x_j}, \\ &A_{3j}(\partial x) = 0, \; A_{4j}(\partial x) = 0, \qquad A_{33}(\partial x) = m_1\Delta - k, \\ &A_{34}(\partial x) = A_{43}(\partial x) = k, \qquad A_{44}(\partial x) = m_2\Delta - k, \qquad j = 1, 2. \end{aligned}$$

Let $D^+(D^-)$ be a bounded (an unbounded) two-dimensional domain surrounded by the contour $S$. $\overline{D^+} = D^+ \cup S$, $D^- = E_2 \setminus \overline{D^+}$. Suppose that $S \in C^{1,\alpha}$, $0 < \alpha \leq 1$.

First of all we introduce the definition of a regular vector-function.

**Definition 1**. A vector-function $\mathbf{U} = (u_1, u_2, p_1, p_2)$ defined in $D^+$ (or in $D^-$) is called regular if $U \in C^2(D^+) \bigcap C^1(\overline{D^+})$ (or $\mathbf{U} \in C^2(D^-) \bigcap C^1(\overline{D^-})$) and in the unbounded domain $D^-$ the vector $U$ additionally satisfies the following conditions at infinity:

$$\mathbf{U}(x) = o(1), \quad \frac{\partial \mathbf{U}_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2. \tag{3}$$

The internal and external basic BVPs are formulated as follows:

Find a regular vector $\mathbf{U}$ satisfying in $D^+(D^-)$ the equation (1) and on the boundary $S$ one of the following conditions is given:

**Problem** $(I)^{\pm}_f$. The displacement vector and the fluid pressures are given on $S$ :

$$\mathbf{u}^{\pm} = \mathbf{f}^{\pm}(z), \qquad p_1^{\pm} = f_3^{\pm}(z), \qquad p_2^{\pm} = f_4^{\pm}(z), \qquad z \in S,$$

**Problem** $(II)^{\pm}_f$. The stress vector and the normal derivatives of the preasure functions $\dfrac{\partial p_j}{\partial n}$, $\quad j = 1, 2$, are given on $S$ :

$$[\mathbf{P}(\partial x, n)\mathbf{u}]^{\pm} = \mathbf{f}^{\pm}(z), \quad \left(\frac{\partial p_1}{\partial n}\right)^{\pm} = f_3^{\pm}(z), \qquad \left(\frac{\partial p_2}{\partial n}\right)^{\pm} = f_4^{\pm}(z), \qquad z \in S.$$

**Problem** $(III)^{\pm}_f$. The displacement vector and the normal derivatives of the pressure functions $\dfrac{\partial p_j}{\partial n}$, $\quad j = 1, 2$, are given on $S$ :

$$\mathbf{u}^{\pm} = \mathbf{f}^{\pm}(z), \qquad \left(\frac{\partial p_1}{\partial n}\right)^{\pm} = f_3^{\pm}(z), \qquad \left(\frac{\partial p_2}{\partial n}\right)^{\pm} = f_4^{\pm}(z), \qquad z \in S.$$

**Problem** $(IV)^{\pm}_f$. The stress vector and the fluid pressures are given on $S$ :

$$[\mathbf{P}(\partial x, n)\mathbf{u}]^{\pm} = \mathbf{f}^{\pm}(z), \qquad p_1^{\pm} = f_3^{\pm}(z), \qquad p_2^{\pm} = f_4^{\pm}(z), \qquad z \in S,$$

where $(.)^+$ denotes the limiting value from $D^+$, $(.)^-$ denotes the limiting value from $D^-$ and $\mathbf{f} = (f_1, f_2), f_3, f_4$ are the given functions, $\mathbf{P}(\partial x, n)\mathbf{u}$ is a stress vector which acts on the elements of the arc with the exterior to $D^+$ unit normal vector $\mathbf{n} = (n_1, n_2)$ at the point $x \in S$,

$$\mathbf{P}(\partial x, n)\mathbf{u} = \mathbf{T}(\partial x, n)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{4}$$

and [6]

$$\begin{aligned} &\mathbf{T}(\partial x, n) = \| \, T_{kj} \, \|_{2x2}, \\ &T_{kj}(\partial x, n) = \mu\delta_{kj}\frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \\ &\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \qquad k, j, = 1, 2. \end{aligned} \tag{5}$$

Now we introduce the generalized stress vector $\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u}$, where

$$\overset{\kappa}{\mathbf{P}}(\partial x, n)\mathbf{u} = \overset{\kappa}{\mathbf{T}}(\partial x, n)\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2),$$

$\kappa$ is an arbitrary positive constant and

$$\overset{\kappa}{\mathbf{T}}(\partial x,n)\mathbf{u}=(2\mu-\kappa)\frac{\partial\mathbf{u}}{\partial n}+(\lambda+\kappa)\mathbf{n}div\mathbf{u}+(\kappa-\mu)\mathbf{s}\omega,$$
$$\mathbf{s}=\begin{pmatrix}-n_2\\ n_1\end{pmatrix},\quad \mathbf{n}=\begin{pmatrix}n_1\\ n_2\end{pmatrix},\qquad omega=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}. \tag{6}$$

If $\kappa=0$ from (6) we have $\overset{\kappa}{\mathbf{T}}(\partial x,n)\mathbf{u}=\mathbf{T}(\partial x,n)\mathbf{u}$. We set $\overset{\kappa}{\mathbf{T}}(\partial x,n)\mathbf{u}=\mathbf{N}(\partial x,n)\mathbf{u}$ for $\kappa=\frac{\mu(\lambda+\mu)}{\lambda+3\mu}$.

### 3. Generalized Green's formulas

Let us write the generalized Green's formulas for the domains $D^+$ and $D^-$. Let $\mathbf{u}(x)$ be a regular solution of equation (1) in $D^+$. Multiply the first equation of (1) by $\mathbf{u}(x)$. Integration the result over $D^+$ and apply the integration by parts formula to obtain

$$\int\limits_{D^+}\mathbf{u}\mathbf{B}(\partial x)\mathbf{u}d\sigma=\int\limits_{S}\mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x,n)\mathbf{u}ds-\int\limits_{D^+}\left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})-(\beta_1p_1+\beta_2p_2)div\mathbf{u}\right]d\sigma.$$

If the vector $\mathbf{u}$ is a solution of homogeneous equation $\mathbf{B}(\partial x)\mathbf{u}=0$, then the last equation gives

$$\int\limits_{D^+}\left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})-(\beta_1p_1+\beta_2p_2)div\mathbf{u}\right]d\sigma=\int\limits_{S}\mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x,n)\mathbf{u}\ ds, \tag{7}$$

where

$$2\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})=(2\lambda+2\mu-\kappa)(div\mathbf{u})^2+(2\mu-\kappa)\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2\right]+\frac{\kappa}{2}\omega^2.$$

For the positive definiteness of the potential energy the inequality $0<\kappa\leq 2\mu$ is necessary and sufficient. Obviously the potential energy $E(\mathbf{u},\mathbf{u})$ is obtained from $\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})$ if we set $\kappa=0$.

If the vector $\mathbf{u}(x)$ satisfies the conditions (3) the Green's formula for the region $D^-$ takes the form

$$\int\limits_{D^-}\left[\overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})-(\beta_1p_1+\beta_2p_2)div\ \mathbf{u}\right]d\sigma=-\int\limits_{S}\mathbf{u}\overset{\kappa}{\mathbf{P}}(\partial x,n)\mathbf{u}ds. \tag{8}$$

Analogously we obtain the Green's formula for $p_j,\quad j=1,2,$

$$\int\limits_{D^+}\left[m_1(gradp_1)^2+m_2(gradp_2)^2+k(p_1-p_2)^2\right]d\sigma=\int\limits_{S}\left[m_1p_1\frac{\partial p_1}{\partial n}+m_2p_2\frac{\partial p_2}{\partial n}\right]ds, \tag{9}$$

$$\int\limits_{D^+} \left[m_1(gradp_1)^2 + m_2(gradp_2)^2 + k(p_1 - p_2)^2\right] d\sigma = -\int\limits_{S} \left[m_1 p_1 \frac{\partial p_1}{\partial n} + m_2 p_2 \frac{\partial p_2}{\partial n}\right] ds. \tag{10}$$

**Remark**. Note that if $\beta_1 p_1 + \beta_2 p_2 = const$, in view of the equality $\int\limits_{D^+} div\mathbf{u} = \int\limits_{S} \mathbf{nu}ds$, from (7) we get

$$\int\limits_{D^+} \overset{\kappa}{\mathrm{E}}(\mathbf{u},\mathbf{u})d\sigma = \int\limits_{S} \mathbf{u}\, \overset{\kappa}{\mathrm{T}}(\partial x, n)\mathbf{u}\ ds. \tag{11}$$

**4. The uniqueness theorems**

In this subsection we prove the uniqueness theorems of solutions to the above formulated problems. Let above formulated problems have two regular solutions $\mathbf{U}^{(1)}(x)$ and $\mathbf{U}^{(2)}(x)$, where $\mathbf{U}^{(k)}(x) = (u_1^{(k)}, u_2^{(k)}, p_1^{(k)}, p_2^{(k)}), \quad k = 1, 2$. Let's consider

$$\mathbf{U}(x) = \mathbf{U}^{(1)}(x) - \mathbf{U}^{(2)}(x).$$

Evidently, the vector $\mathbf{U}(x)$ satisfies (1) and the homogeneous boundary conditions $(\mathbf{f} = 0, \quad f_3 = 0, \quad f_4 = 0)$.

Now we prove the following theorems:

**Theorem 1.** *The first internal boundary value problem $(I)_f^+$ admit at most one regular solution in the domain $D^+$.*

**Proof**. Evidently, the vector $\mathbf{U}(x)$ satisfies the system (1) and the boundary condition $\mathbf{U}(x) = 0$ on $S$. From (9) we obtain $p_1 = p_2 = c, \quad x \in D^+$. Since $p_k^+ = 0$, we have $c = 0$, and $p_1 = p_2 = 0, \quad x \in D^+$. Note that if $\mathbf{u}$ is a regular solution of the equation (1), we have Green's formula (7). Using (7), when $\kappa = 0$ and taking into account the fact that the potential energy is positive definite, we conclude that $u_1 = c_1 - \epsilon x_2, \quad u_2 = c_2 + \epsilon x_1 \quad x \in D^+$, where $\epsilon,\ c = const$. Since $\mathbf{U}^+ = 0$, we have $c = 0, \quad \epsilon = 0$ and $\mathbf{u}(x) = 0, \quad x \in D^+$.

**Theorem 2.** *The first external boundary value problem $(I)_f^-$ has at most one regular solution in the domain $D^-$.*

**Proof**. The vectors $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ in the domain $D^-$ must satisfy the condition (3). In this case the formulas (8)-(10) are valid and $\mathbf{U}(x) = \mathbf{C}, \quad x \in D^-$, where $\mathbf{C}$ is again the constant vector. But $\mathbf{U}$ on the boundary satisfies the condition $\mathbf{U}^- = 0$, which implies that $\mathbf{C} = 0$ and $\mathbf{U}(x) = 0, \quad x \in D^-$.

Analogously the following theorems can be proved :

**Theorem 3.** *If the condition $0 < \kappa \leq 2\mu$ is satisfied then any two regular solutions of the second internal boundary value problem $(II)_f^+$ may differ only to within additive vector $\boldsymbol{V} = (\boldsymbol{u}, p_1, p_2)$, where*

$$u_1 = c_1 - \epsilon x_2 + c_1 x_1, \quad u_2 = c_2 + \epsilon x_1 + c_1 x_2 \quad p_k = c, \quad x \in D^+,$$

*$\epsilon$ and $c$ are arbitrary real constants and* $c_1 = \frac{c(\beta_1+\beta_2)}{2(\lambda+\mu)}$.

**Theorem 4.** *The boundary value problems $(II)_f^-$, $(III)_f^-$, $(IV)_f^-$ admit at most one regular solution in the domain $D^-$.*

**Theorem 5.** *Two regular solutions of the $(III)_f^+$ boundary value problem in the domain $D^+$ may differ by the vector $\boldsymbol{V} = (\boldsymbol{u}, p_1, p_2)$, where $\boldsymbol{u} = 0$, and $p_1 = p_2 = c$.*

**Theorem 6.** *Two regular solutions of the $(IY)_f^+$ boundary value problem may differ by the vector $\boldsymbol{V}(\boldsymbol{u}, p_1, p_2)$, where $\boldsymbol{u}$ vector is a rigid displacement and $p_1 = p_2 = 0$.*

## 5. An existence theorems

In this section we establish the existence of regular solutions of the basic BVPs $(I)_f^\pm$ and $(II)_f^\pm$ by means of the potential method and the theory of singular integral equations.

**Problem $(I)_f^+$.** First let us show that the nonhomogeneous system

$$(m_1\Delta - k)p_1 + kp_2 = F_3(x), \quad kp_1 + (m_2\Delta - k)p_2 = F_4(x) \tag{12}$$

always reduces to the homogeneous system by seeking one particular solution. We choose more simple method for constructing particular solution. A solution $p_k$, $k = 1, 2$ is sought in the form

$$p_1 = -\frac{m_2}{2\pi s^2}\int\limits_{D^+}[K_0(sr) + \ln r]\,F_3(y)d\sigma, \quad p_2 = \frac{m_1}{2\pi s^2}\int\limits_{D^+}[K_0(sr) + \ln r]\,F_4(y)d\sigma, \tag{13}$$

where [5]

$$K_0(sr) = -I_0(sr)\left(\ln\frac{sr}{2} + C\right) - 2\sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{sr}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + ... + 1\right),$$
$$I_0(sr) = \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{sr}{2}\right)^{2k}, \quad s^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right), \; r^2 = (x_1 - y_1)^2 + (x_1 - y_1)^2.$$

It is obvious that integrand in (13) contains the terms $r^{2k}\ln r$, $k = 1, 2, ...$ and we can write

$$(\Delta - s^2)p_1 = \frac{m_2}{2\pi}\int\limits_{D^+}\ln r F_3(y)d\sigma, \quad (\Delta - s^2)p_2 = -\frac{m_1}{2\pi}\int\limits_{D^+}\ln r F_4(y)d\sigma.$$

From this we get

$$\Delta(\Delta - s^2)p_1 = m_2F_3(x), \quad \Delta(\Delta - s^2)p_2 = -m_1F_4(x).$$

Thus we obtain that the particular solutions of the equation (12) are

$$\begin{aligned} p_1 &= \frac{1}{2\pi m_2 s^2}\int\limits_{D^+}[K_0(sr) + \ln r]F_3(y)d\sigma, \\ p_2 &= -\frac{1}{2\pi m_1 s^2}\int\limits_{D^+}[K_0(sr) + \ln r]F_4(y)d\sigma. \end{aligned} \tag{14}$$

At first let's search the fundamental solution of the following equation

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0. \tag{15}$$

It is obvious that

$$\begin{pmatrix} m_1\Delta - k & k \\ k & m_2\Delta - k \end{pmatrix}\begin{pmatrix} m_2\Delta - k & -k \\ -k & m_1\Delta - k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m_1 m_2 \Delta(\Delta - s^2), \tag{16}$$

where $s^2 = k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)$. The fundamental solution of the equation $\Delta(\Delta - s^2)\psi = 0$ is

$$\psi = \alpha_0 K_0(sr) + \alpha_1 \ln r.$$

For the unknown coefficient $\alpha_j$ we obtain the following equations

$$-\alpha_0 + \alpha_1 = 0, \ \ \alpha_0 s^2 = 1.$$

from here we obtain $\alpha_0 = \alpha_1 = \dfrac{1}{s^2}$ and $\psi = \dfrac{1}{s^2}(K_0(sr) + \ln r)$. Obviously $\Delta\psi$ contains a logarithmic singularity as $x \to y$.

From the reduced discussion it is evident that the fundamental matrix of the equation (15) must have the form

$$\mathbf{\Gamma^{(1)}}(x-y) = \begin{pmatrix} m_2 K_0(sr) - \dfrac{k}{s^2}[K_0(sr) + \ln r] & -\dfrac{k}{s^2}[K_0(sr) + \ln r] \\ -\dfrac{k}{s^2}[K_0(sr) + \ln r] & m_1 K_0(sr) - \dfrac{k}{s^2}[K_0(sr) + \ln r] \end{pmatrix}. \tag{17}$$

The matrix $\mathbf{\Gamma^{(1)}}(x-y)$ has a logarithmic singularity as $x \to y$. It is evident that every column of the matrix $\mathbf{\Gamma^{(1)}}(x-y)$ is a solution of the system (15) with respect to the point $x$ , if $x \neq y$.

First let us prove the existence of solution of the first BVP ($p_1^+ = f_4^+, \quad p_2^+ = f_5^+,$ ) for the equation (15) in the domain $D^+$. A solution will be sought in the form of the double layer potential

$$\mathbf{p}(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\mathbf{\Gamma^{(1)}}(y-x)\mathbf{g}(y)ds_y, \quad x \in D^+. \tag{18}$$

Passing the limit as $x \to z \in S$ and taking into account the boundary condition, for determining the unknown vector function $\mathbf{g}(y) = (g_3, g_4)$, we obtain the following Fredholm integral equation of the second kind

$$-m_2 g_3(z) + p_1(z) = f_3^+(z), \quad -m_2 g_4(z) + p_2(z) = f_4^+(z), \tag{19}$$

where $f_j^+(z), \quad j = 3, 4$, are given continuous functions and

$$\begin{aligned} p_1(z) &= \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\left[m_2 K_0(sr) g_3(y) - \frac{k}{s^2}(K_0(sr) + \ln r)(g_3(y) + g_4(y))\right] ds_y, \\ p_2(z) &= \frac{1}{\pi}\int_S \frac{\partial}{\partial n}\left[m_2 K_0(sr) g_4(y) - \frac{k}{s^2}(K_0(sr) + \ln r)(g_3(y) + g_4(y))\right] ds_y. \end{aligned} \tag{20}$$

Let us prove that the equation (19) is solvable for any continuous right-hand side. Let us prove that the homogeneous version of (19) ($f_j = 0$) has only the trivial solution. Let the vector $\mathbf{g} \neq 0$ be some solution to it. Obviously $(p_j)^+ = 0, \quad j = 1,2$. Using Green's formula in $D^+$

$$\int_{D^+} \left[m_1(gradp_1)^2 + m_2(gradp_2)^2 + k(p_1 - p_2)^2\right] ds = \int_S \left[m_1 p_1 \frac{\partial p_1}{\partial n} + m_2 p_2 \frac{\partial p_2}{\partial n}\right] ds,$$

we obtain $p_1 = p_2 = c,\ x \in D^+$. ($c$ is an arbitrary constant). It is easy to show that $g_j$ has a continuous derivative, then we have the following formula

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ = \left(\frac{\partial p_1}{\partial n}\right)^-, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ = \left(\frac{\partial p_2}{\partial n}\right)^-.$$

Using Green's formula in $D^-$, we obtain $p_1 = p_2 = c_1$, where $c_1$ is an arbitrary constant, i.e. we have $(p_1)^+ - (p_1)^- = -2m_2 g_3, \quad (p_2)^+ - (p_2)^- = -2m_1 g_4$. If we substitute the last identity in (20), after elementary transformation we obtain $g_3 = \dfrac{c - c_1}{2m_2}, \quad g_4 = \dfrac{c - c_1}{2m_1}$ and (18) takes the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{\pi} \int_S \frac{\partial lnr}{\partial n} \begin{pmatrix} -1 \\ -1 \end{pmatrix} (c - c_1) ds = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (c - c_1).$$

From here we get $c = c_1, \quad g_3 = g_4 = 0$ and hence the homogeneous equation $(19)_0$ corresponding to the equation (19) has only the trivial solution. This implies that the equation (19) is solvable for any continuous right-hand side.

**Remark**. Analogously we prove the existence of solution of external first BVP ($p_1^- = f_3^-, \quad p_2^- = f_4^-,$ ) for the equation (15) in the domain $D^-$. A solution of the first boundary value problem has the form

$$\mathbf{P}(x) = \frac{1}{\pi} \int_S \frac{\partial}{\partial n} \mathbf{\Gamma}^{(1)}(y - x)\mathbf{g}(y) ds_y, \quad x \in D^-, \tag{21}$$

where $g(y)$ is a solution of the following Fredholm integral equation of the second kind

$$m_2 g_1(z) + p_1(z) = f_3^-(z), \quad m_2 g_2(z) + p_2(z) = f_4^-(z), \tag{22}$$

$f_j(z), \quad j = 3,4$, are given continuous functions and $p_j, \quad j = 1,2$, are given by (20).

Further we assume that $\mathbf{P}(x)$ is known, when $x \in D^+$ or $x \in D^+$(see (18) and (21)). Substitute the $\beta_1 p_1 + \beta_2 p_2$ in (1). Let's search the particular solution of the following equation

$$\mu \Delta \mathbf{u} + (\lambda + \mu) graddiv \mathbf{u} = grad(\beta_1 p_1 + \beta_2 p_2). \tag{23}$$

We put

$$\mathbf{u}_0 = \frac{1}{\pi} \int_D \mathbf{\Gamma}(x - y) grad(\beta_1 p_1 + \beta_2 p_2) ds, \tag{24}$$

where [6]

$$\mathbf{\Gamma}(x-y)=\begin{pmatrix} \dfrac{\lambda+3\mu}{2a\mu}\ln r-\dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_1}\right)^2, & -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} \\ -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2}, & \dfrac{\lambda+3\mu}{2a\mu}\ln r-\dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_2}\right)^2 \end{pmatrix}.$$

Substituting the volume potential $\mathbf{u}_0$ into (23), we obtain [6]

$$\mu\Delta\mathbf{u}_0+(\lambda+\mu)graddiv\mathbf{u}_0=grad(\beta_1p_1+\beta_2p_2). \tag{25}$$

Thus we have proved that $\mathbf{u}_0$ is a particular solution of the equation (23). In (24) $D$ denotes either $D^+$ or $D^-$, $grad(\beta_1p_1+\beta_2p_2)$ is a continuous vector in $D^+$ along with its first order derivatives. When $D=D^-$, the vector $grad(\beta_1p_1+\beta_2p_2)$ has to satisfy the following decay condition at infinity

$$grad(\beta_1p_1+\beta_2p_2)=O(|x|^{-2-\alpha}),\quad \alpha>0.$$

Thus the general solution of the equation (23) is representable in the form $\mathbf{u}=\mathbf{V}+\mathbf{u}_0$, where

$$\mu\Delta\mathbf{V}+(\lambda+\mu)graddiv\mathbf{V}=0. \tag{26}$$

This equation is the equation of an isotropic elastic body. Thus we have reduced the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

The solution of the first BVP ( $V^+=F^+$) is given in the form [6]

$$\mathbf{V}(x)=\frac{1}{\pi}\int\limits_S \mathbf{N}(\partial y,n)\mathbf{\Gamma}(x-y)\mathbf{g}(y)ds, \tag{27}$$

where

$$\mathbf{N}(\partial y,n)\mathbf{\Gamma}(x-y)=\begin{pmatrix} 1+\dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta, & \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta \\ \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta, & 1-\dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta \end{pmatrix}\frac{\partial\theta}{\partial s},$$

$$\theta=\arctan\frac{y_2-x_2}{y_1-x_1},\quad \frac{\partial}{\partial s}=n_1\frac{\partial}{\partial x_2}-n_2\frac{\partial}{\partial x_1},$$

$\mathbf{g}$ is a solution of Fredholm integral equation of the second kind

$$\mathbf{g}(z)+\frac{1}{\pi}\int\limits_S \mathbf{N}(\partial y,\mathbf{n})\mathbf{\Gamma}(y-z)\mathbf{g}(y)ds=\mathbf{f}^+(z). \tag{28}$$

To prove the regularity of the double layer potential in the domain $D^+$, it is sufficient to assume that $S\in C^{2,\beta},\ \ 0<\beta<1,\ \ f\in C^{1,\alpha}(S),\ \ 0<\alpha<\beta.$

We have thereby proved the following theorem.

**Theorem 7.** *If $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$, then a regular solution of problem $(I)_f^+$ exists, it is unique and represented by the potential of double-layer (18) and (27), where $\boldsymbol{g}$ is a solution of the Fredholm integral equations (19) and (28) respectively which are always solvable for arbitrary functions $f_3, f_4$, and $\boldsymbol{f}$.*

**Problem $(I)_f^-$.** Now consider the first BVP $(\mathbf{V}^-(z) = \mathbf{f}^-(z))$ in the domain $D^-$. The solution is sought in the form [6]

$$\mathbf{V}(x) = \frac{1}{\pi} \int_S \left[ \mathbf{N}(\partial y, n)\boldsymbol{\Gamma}(x-y) + \frac{1}{2}\mathbf{N}(\partial y, n)\boldsymbol{\Gamma}(y) \right] \mathbf{g}(y)ds. \tag{29}$$

For determining the unknown vector $\mathbf{g}$ we obtain the following Fredholm integral equation of the second kind

$$-\mathbf{g}(z) + \frac{1}{\pi} \int_S \left[ \mathbf{N}(\partial y, \mathbf{n})\boldsymbol{\Gamma}(y-z) + \frac{1}{2}\mathbf{N}(\partial y, n)\boldsymbol{\Gamma}(y) \right] \mathbf{g}(y)ds = \mathbf{f}^-(z). \tag{30}$$

Here we assume that $\int\limits_S \mathbf{g}(y)ds = 0$ which implies the single layer potential vanishing at infinity.

The equation (30) is always solvable if the condition $\int\limits_S \mathbf{g}(y)ds = \int\limits_S \mathbf{f}(y)ds = 0$ is fulfilled [6].

To prove the regularity of the potential defined by (29) in the domain $D^-$, it is sufficient to assume that $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad \mathbf{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$.

**Theorem 8.** $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$, *then a regular solution of problem $(I)_f^-$ exists, it is unique and represented by the potentials of double-layer (21) and (29), where $\boldsymbol{g}$ is a solution of the Fredholm integral equations (22) and (30) respectively which are always solvable for an arbitrary right hand side.*

Thus we have proved the solvability of the first boundary value problem in the domains $D^+$ and $D^-$.

**Problem $(II)_f^+$.** A solution of BVP $\left(\dfrac{\partial p_1}{\partial n}\right)^+ = f_3(z)$, $\quad \left(\dfrac{\partial p_2}{\partial n}\right)^+ = f_4(z)$ of the equation (15) will be sought in the form

$$\mathbf{p}(x) = \frac{1}{\pi} \int_S \boldsymbol{\Gamma}^{(1)}(x-y) \begin{pmatrix} g_3(y) \\ g_4(y) \end{pmatrix} ds_y, \tag{31}$$

where $\boldsymbol{\Gamma}^{(1)}(x-y)$ is given by formula (17), $S \in C^{1,\beta}, 0 < \beta \le 1$ is a closed Lyapunow curve, $g_k, k = 3, 4$, are unknown functions.

Taking into account the boundary conditions for determining the functions $g_k$,we obtain Fredholm integral equations of the second kind

$$m_2 g_3(z) + \frac{\partial p_1(z)}{\partial n} = f_3(z), \quad m_1 g_4(z) + \frac{\partial p_2(z)}{\partial n} = f_4(z), \quad z \in S. \tag{32}$$

The origin is assumed to be in the domain $D^+$. Let us prove that the equation (32) is always solvable. To this end, we consider the homogeneous equation obtained from (32) for $f_j = 0$ and prove that it has only the trivial solution. Let $g_0 \neq 0$ be any solution of this equation. Since $f_j = 0$, we have

$$\left(\frac{\partial p_1}{\partial n}\right)^+ = 0, \quad \left(\frac{\partial p_2}{\partial n}\right)^+ = 0.$$

Using Green's formula (9), we obtain

$$p_k = c = const, \quad k = 1, 2, \quad x \in D^+. \tag{33}$$

But the potential (31) is a continuous function when the point $x$ tends to any point $z$ of the boundary and we get $p_k(x) = c, \quad x \in D^-$. From last conditions it follows that

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ = \left(\frac{\partial p_1}{\partial n}\right)^-, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ = \left(\frac{\partial p_2}{\partial n}\right)^-,$$

$$0 = \left(\frac{\partial p_1}{\partial n}\right)^+ - \left(\frac{\partial p_1}{\partial n}\right)^- = 2m_2 g_3, \quad 0 = \left(\frac{\partial p_2}{\partial n}\right)^+ - \left(\frac{\partial p_2}{\partial n}\right)^- = 2m_1 g_4.$$

Finally we conclude that the homogeneous equation, corresponding to the equation (32) has only the trivial solution. Thus the equation (32) is always solvable for any continuous right-hand side.

As above, the equation

$$\mu\Delta\ \mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0$$

has the particular solution $\mathbf{u}_0(x)$ (see (24)) and the last equation has a solution $u = u_0 + V$, where

$$\mu\Delta\ \mathbf{V} + (\lambda+\mu) graddiv\ \mathbf{V} = 0. \tag{34}$$

As it is already clear here $(\mathbf{TV})^+$ is given. Thus we have the second BVP for the equation of an isotropic elastic body. The solution is sought in the form [6]

$$\mathbf{V}(x) = \frac{1}{\pi}\int_S [\mathbf{M}(x,y) - \mathbf{M}(0,y)\mathbf{g}(y) ds, \tag{35}$$

where $\mathbf{g}$ is an unknown function and $\mathbf{M}(x,y)$ has the form

$$\mathbf{M}(x,y) = \frac{1}{2\mu(\lambda+\mu)} Im \begin{pmatrix} ia\ln\sigma - i(\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} & -\mu\ln\sigma + (\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} \\ \mu\ln\sigma + (\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma}, & ia\ln\sigma + i(\lambda+\mu)\dfrac{\overline{\sigma}}{2\sigma} \end{pmatrix},$$

where

$$\sigma = x_1 - y_1 + i(x_2 - y_2).$$

From (35), after some operations we find that

$$\mathbf{T}(\partial x, n)\mathbf{V}(x) = \frac{1}{\pi}\int_S \mathbf{T}(\partial x, n)\mathbf{M}(x,y)\mathbf{g}(y) ds, \quad x \in D^+, \tag{36}$$

where

$$\begin{gathered}\mathbf{T}(\partial x, n)\mathbf{M}(x,y) = \begin{pmatrix} 1+\cos 2\theta, & \sin 2\theta \\ \sin 2\theta, & 1-\cos 2\theta \end{pmatrix} \frac{\partial \theta}{\partial s}, \\ \theta = \arctan \frac{y_2 - x_2}{y_1 - x_1}, \quad \frac{\partial \ln r}{\partial n} = \frac{\partial \theta}{\partial s}. \end{gathered} \tag{37}$$

When $x \to z \in S$, for determining the vector $\mathbf{g}$ we obtain the following integral equation

$$-\mathbf{g}(z) + \frac{1}{\pi}\int\limits_S \begin{pmatrix} 1+\cos 2\theta, & \sin 2\theta \\ \sin 2\theta, & 1-\cos 2\theta \end{pmatrix} \frac{\partial \theta}{\partial s}\mathbf{g}ds = \mathbf{f}^+(z). \tag{38}$$

The homogeneous equation, corresponding to the equation (38) has nontrivial solution. It is expedient to modify the preceding equation. Therefore we consider the following equation

$$\begin{gathered}-\mathbf{g}(z) + \frac{1}{\pi}\int\limits_S \mathbf{T}_z\mathbf{M}(z,y)g(y)ds + \frac{1}{2\pi}\mathbf{T}_z\mathbf{M}(z)\int\limits_S \mathbf{g}(y)ds - \\ \frac{1}{2\pi}\frac{d}{d\psi}\begin{pmatrix} -\sin\psi\sin 2\psi \\ -2\sin^3\psi \end{pmatrix} M = \mathbf{f}^+(z), \quad z \in S, \end{gathered} \tag{39}$$

$$\psi = \arctan\frac{x_2}{x_1}, \quad M = \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)_{x_1 = x_2 = 0}. \tag{40}$$

Performing elementary calculation, from (39) we get

$$\int\limits_S \mathbf{g}(y)ds = \int\limits_S \mathbf{f}^+ds, \quad M = \int\limits_S [x_1 f_2^+ - x_2 f_1^+]ds. \tag{41}$$

If the principal vector $\int\limits_S \mathbf{f}^+(y)ds$ and the principal moment $\int\limits_S (x_1 f_2^+ - x_2 f_1^+)ds$ are equal to zero, then $\int\limits_S \mathbf{g}ds = 0$ and $M = 0$. Then every solution $\mathbf{g}$ of the equation (39) is, at the same time, a solution of the integral equation (38).

Let us prove that the equation (39) is always solvable if the the principal vector and the principal moment are equal to zero. To this end we consider the homogeneous equation obtained from (39) for $\mathbf{f}^+ = 0$ and prove that it has only trivial solution. Let $\mathbf{g}_0$ be any solution of that equation. Since $\mathbf{f}^+ = 0$ it is obvious that $\int\limits_S \mathbf{f}^+ds = 0$, $M = 0$ are fulfilled for $\mathbf{g}_0$. In this case the obtained homogeneous equation corresponds to the boundary condition $(\mathbf{Tu}_0)^+ = 0$, where $\mathbf{u}_0$ is obtained from (35), if instead of $\mathbf{g}$ we take $\mathbf{g}_0$. Using the uniqueness theorem for the second BVP for $D^+$, we obtain

$$\mathbf{u}_0(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \varepsilon \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D^+,$$

where $c_j$, and $\varepsilon$ are arbitrary constants.

Noting that $M_0 = 0$ and $\mathbf{V}(0) = 0$, therefore $\mathbf{u}_0(x) = 0$, whence [6] ( $\mathbf{u}_0$ and $\mathbf{W}_0$ are the conjugate vectors, $\boldsymbol{\phi} = \boldsymbol{u}_0 + i\boldsymbol{W}_0$)

$$0 = \mathbf{N}(\partial x, n)\mathbf{u}_0(x) = \frac{\lambda + 3\mu}{2a\mu}\frac{\partial \mathbf{W}_0}{\partial S(x)}, \quad x \in D^+. \tag{42}$$

From here $\mathbf{W}_0 = c, \ \ x \in D^+$, where [6]

$$\mathbf{W}_0 = \frac{1}{\pi} Re \int\limits_S \begin{pmatrix} \ln\sigma - \dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} & -\dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} \\ -\dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma}, & \ln\sigma - \dfrac{\lambda+\mu}{\lambda+3\mu}\dfrac{\overline{\sigma}}{2\sigma} \end{pmatrix} \mathbf{g}(y)ds. \tag{43}$$

We can easily establish that if $\mathbf{g}_0$ is a continuous vector, then $(\mathbf{TW}_0)^+ - (\mathbf{TW}_0)^- = 0$. But since $(\mathbf{TW}_0)^+ = 0$, from the last formula we obtain $(\mathbf{TW}_0)^- = 0$. By virtue of $\int\limits_S \mathbf{g}ds = 0$, the vector $\mathbf{W}_0$ is one valued on the entire plane and of order $|x|^{-1}$ at infinity, $\mathbf{W}(\infty) = 0$. Using this fact and uniqueness theorem we obtain

$$\mathbf{W}_0(x) = 0, \quad x \in D^-. \tag{44}$$

The formula $\mathbf{W}_0 = c, \ \ x \in D^+$ and (44) yield $(\mathbf{L}(\partial x, n)\mathbf{W}_0)^+ = 0, \ \ x \in D^+$, $(\mathbf{L}(\partial x, n)\mathbf{W}_0)^- = 0, \ \ x \in D^-$, where the operator $\mathbf{L}(\partial x, n)$ is obtained from $\overset{\boldsymbol{\kappa}}{\mathbf{T}}(\partial x, n)$ for $\kappa = 2\mu$. Further, if we use the formula [6]

$$0 = (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^+ - (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^- = \frac{2\mu}{a}\mathbf{g}_0(z),$$

we obtain $\mathbf{g}_0 = 0$.

Thus the homogeneous equation corresponding to the (39) has only trivial solution. Consequently, the equation (39) has a unique solution $\mathbf{g}$. Substituting $\mathbf{g}$ in (35), we get solution of the second BVP, provided the principal vector and the principal moment of external stresses are equal to zero.

**Theorem 9.** *If $S \in C^{2,\beta}$, $0 < \beta < 1$, $f, f_j \in C^{1,\alpha}(S)$, $0 < \alpha < \beta$, then a regular solution of problem $(II)_f^+$ exists, it is unique and represented by the potentials of singlelayer (31) and (35), where $\mathbf{g}$ is a solution of the Fredholm integral equations (32) and (39) respectively which are always solvable for an arbitrary right hand side.*

**Problem** $(II)_f^-$. Now let us prove the existence of solution of the second BVP $((\mathbf{TV})^- = \mathbf{f}^-)$ in the domain $D^-$. The solution is sought in the form

$$\mathbf{V}(x) = \frac{1}{\pi}\int\limits_S \mathbf{M}(z, y)\mathbf{g}(y)ds + \frac{\mu}{a\rho}\begin{pmatrix} \cos\psi\cos 2\psi \\ \cos\psi \end{pmatrix} M, \tag{45}$$

where

$$\psi = \arctan\frac{x_2}{x_1}, \ \ M = \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)_{x_1 = x_2 = 0}. \tag{46}$$

Here $\mathbf{g}$ is an unknown vector. For the vector $\mathbf{V}$ to be single valued and bounded at infinity, we assume that the condition $\int\limits_S \mathbf{g}ds = 0$, is fulfilled. Later on the principal vector will be assumed to be equal to zero.

For determining of vector $\mathbf{g}$ we obtain the following Fredholm integral equation of the second kind

$$\mathbf{g}(z) + \frac{1}{\pi}\int\limits_S \begin{pmatrix} 1+\cos 2\theta, & \sin 2\theta \\ \sin 2\theta, & 1 - \cos 2\theta \end{pmatrix}\frac{\partial\theta}{\partial s}\mathbf{g}(y)ds + \frac{\mu}{a}\frac{\partial}{\partial\psi}\begin{pmatrix} \cos\psi\cos 2\psi \\ \cos\psi \end{pmatrix} M = \mathbf{f}^-(z). \tag{47}$$

By integration, from (47) we obtain

$$\int_S \mathbf{g}ds = \int_S \mathbf{f}^- ds. \tag{48}$$

Now we will establish that the equation (47) is always solvable. To this end, we consider the homogeneous equation obtained from (47) for $\mathbf{f}^- = 0$. Let's prove that this equation has only trivial solution. Let's assume the contrary and denote by $\mathbf{g}_0$ any solution of the homogeneous equation. Since $\mathbf{f}^- = 0$, from (48) we have $\int_S \mathbf{g}_0 ds = 0$. Note that the homogeneous equation corresponds now to the boundary condition $(\mathbf{TV})^- = 0$. Taking into account the uniqueness theorem for the second BVP in the domain $D^-$, we obtain $\mathbf{V}_0(x) = 0, x \in D^-$. In this case $(\mathbf{LV}_0)^- = (\mathbf{LV}_0)^+ = 0$. Therefore

$$0 = \int_S [x_1(\mathbf{LV}_0)_1^+ + x_2(\mathbf{LV}_0)_2^-]ds = M_0, \ \ x \in D^-$$

and (45) takes the form

$$\mathbf{V}_0 = u_0(x) = \frac{1}{\pi}\int_S \mathbf{M}(x,y)\mathbf{g}ds = 0, x \in D^-.$$

From here

$$0 = \mathbf{N}(\partial x, n)\mathbf{u}_0 = \frac{\lambda + 3\mu}{2a\mu}\frac{\partial \mathbf{W}_0}{\partial s(x)}.$$

The last equation gives $\mathbf{W}_0 = c, \ \ x \in D^-$. As since $\mathbf{W}_0(\infty) = 0$, we obtain $c = 0$ and $\mathbf{W}_0 = 0, \ \ x \in D^-$. From here it follows that $(\mathbf{TW}_0)^- = 0$. But $(\mathbf{TW}_0)^- = (\mathbf{TW}_0)^+$. Therefore $(\mathbf{TW}_0)^+ = 0$ and

$$\mathbf{W}_0(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \varepsilon \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \ \ x \in D^+.$$

By appling (46) we obtain $M_0 = \varepsilon = 0$ and $\mathbf{W}_0 = c, \ \ x \in D^+$.

Later having used the formula

$$0 = (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^+ - (\mathbf{L}(\partial x, n)\mathbf{W}_0(x))^- = \frac{2\mu}{a}\mathbf{g}_0(z).$$

we obtain $\mathbf{g}_0 = 0$.

Consequently (47) has a unique solution, provided the principal vector is equal to zero.

**Remark**. As above the solution of BVP $\left[\dfrac{\partial p_1}{\partial n}\right]^- = f_3^-(z), \left[\dfrac{\partial p_2}{\partial n}\right]^- = f_4^-(z)$, will be represented by the singlelayer potential (31), where $g_3$ and $g_4$ are the solutions of Fredholm integral equations of the second kind

$$-m_2 g_3(z) + \frac{\partial p_1(z)}{\partial n} = f_3(z), \quad -m_1 g_4(z) + \frac{\partial p_2(z)}{\partial n} = f_4(z), \quad z \in S. \tag{49}$$

Thus the existence of the solution of the second boundary value problem in the domain $D^-$ is proved.

**Theorem 10.** *If* $S \in C^{2,\beta}$, $\quad 0 < \beta < 1$, $\quad f_3, f_4, \boldsymbol{f} \in C^{1,\alpha}(S)$, $\quad 0 < \alpha < \beta$, *then a regular solution of problem* $(II)_f^-$ *exists, it is unique and represented by the potentials of singlelayer (45) and (31), where* $\boldsymbol{g}$ *is a solution of the Fredholm integral equations (47) and (49) respectively which are always solvable for an arbitrary right hand side.*

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# EXISTENCE OF OPTIMAL INITIAL DATA AND CONTINUITY OF THE INTEGRAL FUNCTIONAL MINIMUM WITH RESPECT TO PERTURBATIONS FOR A CLASS OF NEUTRAL DIFFERENTIAL EQUATION

Tadumadze T.

**Abstract**. For the system of differential equations, linear with respect to prehistory of velocity, sufficient conditions of existence of optimal initial data are obtained. Under initial data we imply the collection of constant delays, initial moment and vector, initial functions. The question of the continuity of the integral functional minimum with respect to perturbations of the right-hand side of equation and integrand is investigated.

**Keywords and phrases**: Neutral differential equation; existence of initial data; continuity of the integral functional minimum.

**AMS subject classification (2000):** 49J25; 34K35.

Let $0<\tau_{1i}<\tau_{2i}, i=\overline{1,s}, 0<\eta_{1j}<\eta_{2j}, j=\overline{1,m}, t_1<t_2<t_3$ be given numbers, with $t_3-t_2>\tau=\max(\tau_{21},...,\tau_{2s},\eta_{21},...,\eta_{2m})$; *let* $R^n$ be the $n$-dimensional vector space of points

$$x=(x^1,...,x^n)^T, |x|^2=\sum_{i=1}^{n}(x^i)^2,$$

where $T$ means transpose; the functions

$$F_i(t,x,y)=(f_i^0(t,x,y),f_i(t,x,y))^T\in R^{1+n}, i=\overline{1,s}$$

are continuous on the set $I\times R^n\times R^n$, where $I=[t_1,t_3]$, and continuously differentiable with respect to $x,y\in R^n$; suppose that $\Phi\subset R^n, X_0\subset R^n$ are compact sets, $V\subset R^n$ is a compact and convex set. By $\Delta_1$ and $\Delta_2$ we denote sets of measurable $\varphi(t)\in\Phi, t\in I_1=[\hat{\tau},t_2], \hat{\tau}=a-\tau$, and $v(t)\in V, t\in I_1$ initial functions, respectively. Further, $R^{n\times n}$ is the space of matrices

$$A=(a_{ij})_{i,j=1}^n, |A|^2=\sum_{i,j=1}^{n}(a_{ij})^2;$$

the functions $A_j(t)\in R^{n\times n}, a_j(t)=(a_j^1(t),...,a_j^n(t)), j=\overline{1,m}$ are continuous on the interval $I$.

The collection of initial data $\tau_i, i=\overline{1,s}, \eta_j, j=\overline{1,m}, t_0, x_0, \varphi(t), v(t)$ is said to be initial element and we denote it by $w$.

To each initial element

$$w=(\tau_1,...,\tau_s,\eta_1,...,\eta_m,t_0,x_0,\varphi,v)\in W=[\tau_{11},\tau_{21}]\times\cdots\times[\tau_{1s},\tau_{2s}]$$

$$\times[\eta_{11},\eta_{21}]\times...\times[\eta_{1m},\eta_{2m}]\times[t_1,t_2]\times X_0\times\Delta_1\times\Delta_2$$

we assign the neutral differential equation

$$\dot{x}(t)=\sum_{i=1}^{s} f_i(t,x(t),x(t-\tau_i))+\sum_{j=1}^{m} A_j(t)\dot{x}(t-\eta_j), t\in[t_0,t_3] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t), \dot{x}(t)=v(t), t\in[\hat{\tau},t_0), x(t_0)=x_0. \tag{2}$$

**Remark.** The symbol $\dot{x}(t)$ on the interval $[\hat{\tau},t_0)$ is not connected with derivative of the function $\varphi(t)$.

**Definition 1.** Let $w\in W$. A function $x(t)=x(t;w)\in R^n, t\in[\hat{\tau},t_3]$ is called a solution, corresponding to the element $w$, if it satisfies condition (2), is absolutely continuous on the interval $[t_0,t_3]$ and satisfies Eq.(1) almost everywhere on $[t_0,t_3]$.

By $W_0$ we denote the set of such initial elements $w\in W$ for which there exists the corresponding solution $x(t;w)$. In what follows we will assume that $W_0\neq\emptyset$.

We note that, if the following condition

$$|f_x(t,x,y)|+|f_y(t,x,y)|\leq L, \forall(t,x,y)\in I\times R^n\times R^n$$

is fulfilled, where $L>0$ is a given number, then $W_0=W$.

Let us consider the following functional

$$J(w)=\sum_{i=1}^{s}\int_{t_0}^{t_3}[f_i^0(t,x(t),x(t-\tau_i))$$
$$+\sum_{j=1}^{m} a_j(t)\dot{x}(t-\eta_j)]dt, w\in W_0,$$

where $x(t)=x(t;w)$.

**Definition 2.** An initial element $w_0=(\tau_{10},...,\tau_{s0},\eta_{10},...,\eta_{m0},t_{00},x_{00},\varphi_0,v_0)\in W_0$ is said to be optimal for the differential equation (1) if

$$J(w_0)\leq J(w)$$

for any $w\in W_0$.

**Theorem 1.** *Let the following conditions hold:*

1) *there exists a compact $K_0\subset R^n$ such that*

$$x(t;w)\in K_0, t\in[t_0,t_3], \forall w\in W_0;$$

2) *for any $(\xi_i,x_i)\in I\times K_0, i=\overline{1,s}$ the set*

$$\big\{(F_1(\xi_1,x_1,y),...,F_s(\xi_s,x_s,y)):y\in\Phi\big\}\subset R^{(1+n)s}$$

*is convex. Then there exists an optimal initial element $w_0$.*

**Theorem 2.** *Let $f_i(t,x,y)=B_i(t,x)y, B(t,x)\in R^{n\times n}$ and function $f_i^0(t,x,y)$ is convex with respect to $y$. Moreover, let the set $\Phi$ be convex and let the condition 1) of Theorem 1 be fulfilled. Then there exists an optimal initial element $w_0$.*

Theorems 1,2 are proved by a scheme given in [1,2].

Let us consider the perturbed differential equation

$$\dot{x}(t) = \sum_{i=1}^{s} \big[f_i(t, x(t), x(t-\tau_i)) + g_{i\delta}(t, x(t), x(t-\tau_i))\big]$$

$$+\sum_{j=1}^{m}[A_j(t) + A_{j\delta}(t)]\dot{x}(t-\eta_j), t \in [t_0, t_3] \tag{3}$$

with the initial condition (2) and the perturbed functional

$$J(w;\delta) = \int_{t_0}^{t_3} \Big\{ \sum_{i=1}^{s} \big[f_i^0(t, x(t), x(t-\tau_i)) + g_{i\delta}^0(t, x(t), x(t-\tau_i))$$

$$+\sum_{j=1}^{m} \big[a_j(t) + a_{j\delta}(t)\big]\dot{x}(t-\eta_j)\Big\}dt,$$

where the functions $G_{i\delta}(t,x,y) = (g_{i\delta}^0(t,x,y), g_{i\delta}(t,x,y))^T, i = \overline{1,s}$ are continuous on the set $I\times R^n\times R^n$ and continuously differentiable with respect to $x, y \in R^n$; $A_{j\delta}(t), a_{j\delta}(t)$, $j = \overline{1,m}, t \in I$ are continuous functions.

**Definition 3.** An initial element $w_{0\delta} = (\tau_{1\delta}, ..., \tau_{s\delta}, \eta_{1\delta}, ..., \eta_{m\delta}, t_{0\delta}, x_{0\delta}, \varphi_\delta, v_\delta) \in W_0$ is said to be optimal for the differential equation (3) if

$$J(w_{0\delta};\delta) \le J(w;\delta)$$

for any $w \in W_0$.

**Theorem 3.** *Let the conditions of the Theorem 1 hold . Then for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for arbitrary functions $G_{i\delta}(t,x,y), i = \overline{1,s}$; $A_{j\delta}(t), a_{j\delta}(t)$, $j = \overline{1,m}$ satisfying the conditions:*

$$\sum_{i=1}^{s}\int_{t_1}^{t_3} \sup\Big\{\Big|\frac{\partial G_{i\delta}(t,x,y)}{\partial x}\Big| + \Big|\frac{\partial G_{i\delta}(t,x,y)}{\partial y}\Big| : x, y \in K_1\Big\}dt \le C, \tag{4}$$

$$\sum_{i=1}^{s}\int_{t_1}^{t_3} \sup\Big\{\Big|G_{i\delta}(t,x,y)\Big| : x, y \in K_1\Big\}dt + \sum_{j=1}^{m}\Big[\|A_{j\delta}\| + \|a_{j\delta}\|\Big] \le \delta \tag{5}$$

*and the set*

$$\Big\{(F_1(\xi_1, x_1, y) + G_{1\delta}(\xi_1, x_1, y), ..., F_s(\xi_s, x_s, y) + G_{s\delta}(\xi_s, x_s, y)) : y \in \Phi\Big\}$$

*is convex, where $C > 0$ is a fixed number,*

$$\|A_{j\delta}\| = \sup\{|A_{j\delta}(t)| : t \in I\}$$

*and $K_1 \subset R^n$ is a compact set containing some neighborhood of set $K_0 \cup \Phi$; there exists an optimal initial element $w_{0\delta}$ and the following inequality*

$$|J(w_{0\delta};\delta) - J(w_0)| \le \varepsilon \tag{6}$$

*is fulfilled.*

**Theorem 4.** *Let the conditions of Theorem 2 hold. Then for any* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for arbitrary functions*

$$G_{i\delta}(t,x,y) = (g^0_{i\delta}(t,x,y), B_{i\delta}(t,x)y)^T, i = \overline{1,s}$$

*and* $A_{j\delta}(t), a^0_{j\delta}(t), j = \overline{1,m}$ *satisfying the conditions (4),(5) and the functions* $g^0_{i\delta}(t,x,y)$, $i = \overline{1,s}$ *are convex with respect to* $y$; *there exists an optimal element* $w_{0\delta}$ *and the inequality (6) is fulfilled.*

Theorems 3,4 are proved by a scheme given in [3]. The similar questions are considered for delay differential equations in [4].

Finally, we note that Theorems 1-4 play an important role in solving inverse problems for neutral differential equations [5].

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# NECESSARY OPTIMALITY CONDITIONS OF SINGULAR CONTROLS IN CONTROL PROBLEM FOR VOLTERRA TYPE TWO-DIMENSIONAL DIFFERENCE EQUATION

Amirova R., Mansimov K.

**Abstract**. Necessary optimality condition is obtained in the form of discrete maximum principle in an optimal control problem described by a system of two-dimensional difference equations of Volterra type. Moreover, the case of degeneration of discrete maximum condition is considered.

**Keywords and phrases**: Necessary optimality condition, system of Volterra difference equations, discrete principle of the maximum, singular controls.

**AMS subject classification (2000):** 49K15; 49K22; 49K99; 34H05; 49K25.

## 1. Introduction

Optimization problems for Volterra integral equations occupy an important place in the theory of optimal control. The Volterra integral equations are widely used in modeling some phenomena of continuum mechanics and biomechanics [1-8]. The optimal control problems described by Volterra integral equations have been studied in [8-11]. The present paper deals with investigation of an optimal control problem described by system of Volterra two-dimensional difference equations. The necessary optimality condition is proved in the form of discrete maximum condition. Moreover, necessary optimality conditions are proved for controls which are singular optimal controls in the sense of Pontryagin's maximum principle.

## 2. Statement of the problem

Consider a problem on minimum of the functional

$$S(u) = \varphi(z(t_1, x_1)), \tag{2.1}$$

under restrictions

$$\begin{aligned} u(t,x) \in U \subset R^r,\ (t,x) \in T \times X = \{(t,x):\ t = t_0, t_0+1, ..., t_1 \\ ;\ x = x_0, x_0+1, ..., x_1\}, \end{aligned} \tag{2.2}$$

$$z(t,x) = \sum_{\tau=t_0}^{t} \sum_{s=x_0}^{x} f(t,x,\tau,s,z(\tau,s),u(\tau,s)), \quad (t,x) \in T \times X \tag{2.3}$$

Here $\varphi(z)$ is a given twice differentiable scalar function, $t_0,\ t_1,\ x_0,\ x_1$ are given numbers, $f(t,x,\tau,s,z,u)$ is a given $n$ -dimensional vector-function continuous by the aggregate of variables together with partial derivatives with respect to $z$ up to the second order inclusive, $u(t,x)$ is a control function, $U$ is a given non-empty and bounded set.

A control function $u(t,x)$ satisfying the restriction (2.2) and the pair $(u(t,x),\ z(t,x))$ will be called an admissible control and an admissible process, respectively.

Equation (2.3) is a difference analogue of Volterra two-dimensional integral equation. It is assumed that to each admissible control $u(t,x)$ corresponds unique solution of discrete equation (2.2). The existence, uniqueness and boundedness problems of solutions of Volterra one-dimensional difference equations have been investigated in [5, 12-14].

We note that different aspects of multi parameter, in particular two-parameter discrete control systems have been studied in [15-22].

The admissible control $u(t,x)$ minimizing the functional (2.1) under restrictions (2.2), (2.3) is said to be an optimal control, the corresponding process $(u(t,x),\ z(t,x))$ an optimal process.

### 3. The second order increment formula

In this section we derive representation for the increments of cost functional $S(u)$. Let the set

$$f(t,x,\tau,s,z,U)=\{\alpha:\ \alpha=f(t,x,\tau,s,z,v),\ v\in U\}\,, \tag{3.1}$$

be convex for all $(t,x,\tau,s,z)$ .

Let $(u(t,x),\ z(t,x))$ be a fixed admissible process, by $u(t,x;\,\varepsilon)$ we denote an arbitrary admissible control such that its appropriate state of the process $z(t,x;\,\varepsilon)$ satisfies the relation

$$z(t,x;\varepsilon)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}f(t,x,\tau,s,z(\tau,s:\varepsilon),u(\tau,s:\varepsilon))\equiv\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f(t,x,\tau,s,z(\tau,s:\varepsilon)$$
$$,u(\tau,s))+\varepsilon\left[f(t,x,\tau,s,z(\tau,s:\varepsilon),v(\tau,s))-f(t,x,\tau,s,z(\tau,s:\varepsilon),u(\tau,s))\right]\Big], \tag{3.2}$$

where $\varepsilon\in[0,1]$ is an arbitrary number, $v(t,x)\in U,\ (t,x)\in T\times X$ is an arbitrary admissible control.

Such an admissible control $u(t,x;\varepsilon)$ exists by the convexity of set (3.1).

Introduce the functions

$$y(t,x)=\left.\frac{\partial z(t,x:\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0};\quad Y(t,x)=\left.\frac{\partial^2 z(t,x:\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0}. \tag{3.3}$$

Using (3.2), and taking into account the smoothness of the function $f(t,x,\tau,s,z,u)$, it is proved that $y(t,x)$ and $Y(t,x)$ are the solutions of Volterra type linear inhomogeneous difference equations

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\big], \tag{3.4}$$

$$Y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)+2\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s)$$
$$,u(\tau,s))y(\tau,s)+y^{'}(\tau,s)f_{zz}(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\big]. \qquad (3.5)$$

Here and in the sequel, we use the denotation

$$\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\equiv f(t,x,\tau,s,z(\tau,s),v(\tau,s))$$
$$-f(t,x,\tau,s,z(\tau,s),u(\tau,s)),$$

($'$) prime means a scalar product for the vectors, the transpose operation for the matrices. Moreover, special increment of functional (2.1) responding to admissible controls $u(t,x;\varepsilon)$ and $u(t,x)$ will be written in the form

$$\Delta S_\varepsilon(u)=S(u(t,x;\varepsilon))-S(u(t,x))=\varepsilon\frac{\partial\varphi^{'}(z(t_1,x_1))}{\partial z}y(t_1,x_1)$$
$$+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)+\frac{\varepsilon^2}{2}\frac{\partial\varphi(z(t_1,x_1))}{\partial z}Y(t_1,x_1)+0(\varepsilon^2). \qquad (3.6)$$

Now we Introduce the Hamilton-Pontryagins function

$$H(t,x,z(t,x),u(t,x),\psi(t,x))=\sum_{\tau=t}^{t_1}\sum_{s=x}^{x_1}\psi^{'}(\tau,s)f(\tau,s,t,x,z(t,x),u(t,x))$$
$$-\varphi_z^{'}(z(t_1,x_1))f(t_1,x_1,t,x,z(t,x),u(t,x)),$$

where $\psi=\psi(t,x)$ is $n$-dimensional vector-function of conjugated variables being a solution of the equation

$$\psi(t,x)=H_z(t,x,z(t,x),u(t,x),\psi(t,x)). \qquad (3.7)$$

Equation (3.7) is an analogy of the conjugated system [23-25] for control problem (2.1)-(2.3) and is a Volterra linear nonhomogeneous equation with respect to $\psi(t,x)$.

**Theorem 3.1** *The second order increment of functional (2.1) can be represented by the following formula*

$$\Delta S_\varepsilon(u)=-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H(t,x,z(t,x),u(t,x),\psi(t,x))+\frac{\varepsilon^2}{2}\Big\{y^{'}(t_1,x_1)\frac{\partial^2\varphi(z(t_1,x_1))}{\partial z^2}$$
$$\times y(t_1,x_1)-\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$-2\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Big\}+0(\varepsilon^2). \qquad (3.8)$$

**Proof.** Multiplying scalarly the both sides of relations (3.4), (3.5) from the left by $\psi(t,x)$, and summing the both sides of the obtained relations over $t$ ($x$) from $t_0$ ($x_0$)

to $t_1$ $(x_1)$, we get

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\Bigg[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+\Delta_{v(\tau,s)}f(t,x,\tau,s,z(\tau,s),u(\tau,s))\Big]\Bigg]. \tag{3.9}$$

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)Y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\Bigg[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)$$
$$+2\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+y^{'}(\tau,s)f_{zz}(t,x,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\Big]\Bigg]. \tag{3.10}$$

The following statement is true.

**Lemma 3.1** *Let $L(t,x,\tau,s)$ and $K(t,x,\tau,s)$ be given $(n\times n)$ matrix functions. Then the identity*

$$\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}L(m,\ell,t,x)K(t,x,\tau,s)\right]=\sum_{t=t_0}^{m}\sum_{x=x_0}^{\ell}\left[\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}L(m,\ell,\tau,s)K(\tau,s,t,x)\right]$$

*is valid.*

The lemma is a two-dimensional discrete analogue of Fubini formula [1, 7]. Using this lemma and assuming

$$M(t,x,z(t,x),u(t,x),\psi(t,x))=\sum_{\tau=t}^{t_1}\sum_{s=x}^{x_1}\psi^{'}(\tau,s)f(\tau,s,t,x,z(t,x),u(t,x)),$$

identities (3.9), (3.10) can be transformed into the form

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+\Delta_{v(t,x)}M(t,x,z(t,x),u(t,x),\psi(t,x))\Big], \tag{3.11}$$

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)Y(t,x)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)$$
$$+2\Delta_{v(t,x)}M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+y^{'}(t,x)M_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Big]. \tag{3.12}$$

Further, it is clear that from (3.4), (3.5) follows

$$y(t_1,x_1)=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$

$$+\Delta_{v(t,x)} f(t_1,x_1,t,x,z(t,x),u(t,x))\Big].$$

$$Y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\Big[f_z(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))Y(\tau,s)$$
$$+2\Delta_{v(\tau,s)}f_z(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)$$
$$+y^{'}(\tau,s)f_{zz}(t_1,x_1,\tau,s,z(\tau,s),u(\tau,s))y(\tau,s)\Big].$$

Taking into account identities (3.11)-(3.13) in (3.6), we get

$$\Delta S_\varepsilon(u)=\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\frac{\partial\varphi^{'}(z(t_1,x_1))}{\partial z}\Big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$
$$+\Delta_{v(t,x)}f(t_1,x_1,t,x,z(t,x),u(t,x))\Big]+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\frac{\partial\varphi(z(t_1,x_1))}{\partial z}\big[f_z(t_1,x_1,t,x,z(t,x),u(t,x))Y(t,x)$$
$$+2\Delta_{v(\tau,s)}f_z(t_1,x_1,t,x,z(t,x),u(t,x))y(t,x)$$
$$+y^{'}(t,x)f_{zz}(t_1,x_1,t,x,z(t,x),u(t,x))\big]+\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,y(t,x)$$
$$-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)+\Delta_{v(t,x)}M(t,x,z(t,x),u(t,x),\psi(t,x))\Big]$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,Y(t,x)-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)$$
$$+2\Delta_{v(t,x)}M_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+y^{'}(t,x)M_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\big]+0(\varepsilon^2).$$

Hence, grouping the similar terms and taking into consideration the expressions of Hamilton-Pontryagins function, we have

$$\Delta S_\varepsilon(u)=-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Y(t,x)+\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,y(t,x)$$
$$+\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\psi^{'}(t,x)\,Y(t,x)-\varepsilon\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H^{'}(t,x,z(t,x),u(t,x),\psi(t,x))$$
$$+\frac{\varepsilon^2}{2}y^{'}(t_1,x_1)\frac{\partial^2\varphi^{'}(z(t_1,x_1))}{\partial z^2}y(t_1,x_1)-\varepsilon^2\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))$$
$$\times y(t,x)-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)+0(\varepsilon^2).$$

Hence, with regard to the fact that $\psi(t,x)$ is a solution of equation (3.7), we obtain formula (3.8).

## 4. The second order increment formula

By arbitrariness of $\varepsilon \in [0,1]$ the following theorem immediately follows from expression (3.8)

**Theorem 4.1.** *If the set (3.1) is convex, then for optimality of the admissible control $u(t,x)$ the inequality*

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1} \Delta_{v(t,x)} H(t,x,z(t,x),u(t,x),\psi(t,x)) \le 0 \tag{4.1}$$

*should be fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$.*

Theorem 4.1 is an analogue of Pontryagins discrete maximum principle [22-25] for the considered problem and is a first order necessary optimality condition. Therefore, the number of non-optimal controls satisfying the maximum condition (4.1) may be sufficiently great. Besides, possibility of degeneration of optimality condition (4.1) (see [26]) is not excluded.

Now we investigate the case of degeneration of necessary optimality condition (4.1).

**Definition 4.1.** The admissible control $u(t,x)$ is called singular control in the sense of Pontryagins maximum principle, if the relation

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1} \Delta_{v(t,x)} H(t,x,z(t,x),u(t,x),\psi(t,x)) = 0 \tag{4.2}$$

is fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$. By definition, the singular controls satisfy first order necessary optimality conditions and consequently to analyze them from the optimality point of view we need second order and sometimes higher order optimality conditions [26].

Allowing for (4.2), the following statement follows from expression (3.8).

**Theorem 4.2.** *If the set (3.1) is convex, then for optimality of the singular control $u(t,x)$ the inequality*

$$y'(t_1,x_1)\varphi_{zz}(z(t_1,x_1))y(t_1,x_1) - \sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[y'(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$+2\Delta_{v(t,x)}H'_z(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)\Big] \ge 0 \tag{4.3}$$

*should be fulfilled for all $v(t,x) \in U$, $(t,x) \in T \times X$.*

Here, $y(t,x)$ is a solution of the equation in variations (3.4). Inequality (4.3) is a sufficiently general necessary optimality condition of singular controls. Based on this inequality, in some cases we can get constructively verifiable necessary optimality conditions of singular controls that are expressed obviously by the parameters of the problem (2.1)-(2.3).

The equations in variations (3.4) is a Volterra type linear, nonhomogeneous two-dimensional difference equation.

Using the scheme of the papers [1, 3-7], it is proved that the solution of the equation in variations (3.4) $y(t,x)$ allows the representation

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[\Delta_{v(\tau,s)}f_z(t,x,\tau,s,z(\tau,s),u(\tau,s))$$
$$+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)\Delta_{v(\tau,s)}f(\alpha,\beta,\tau,s,z(\tau,s),u(\tau,s))\Big]. \tag{4.4}$$

Here, $R(t,x,\tau,s)$ is a solution of the Volterra type linear nonhomogeneous matrix difference equation

$$R(m,\ell,t,x)=\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}R(m,\ell,\tau,s)f_z(\tau,s,t,x,z(t,x),u(t,x))$$
$$-f_z(m,\ell,t,x,z(t,x),u(t,x)). \tag{4.5}$$

Equation (4.5) is a discrete analogue of the resolvent of Volterra type integral equation. By means of the scheme, for example of the paper [1], it is proved that $R(m,\ell,t,x)$ is also a solution of the equation

$$R(m,\ell,t,x)=\sum_{\tau=t}^{m}\sum_{s=x}^{\ell}f_z(m,\ell,\tau,s,z(\tau,s),u(\tau,s))R(\tau,s,t,x)$$
$$-f_z(m,\ell,t,x,z(t,x),u(t,x)). \tag{4.6}$$

By analogy with the papers [1, 3-7], we call the matrix function $R(m,\ell,t,x)$ a resolvent of the equation in variations (3.4) and equations (4.5), (4.6) the equations of the resolvent. Assume that the right-hand side of system (2.3) has the form:

$$f(t,x,\tau,s,z,u)=A(t,x,\tau,s)\,g(\tau,s,z,u). \tag{4.7}$$

Then representation (4.4) takes the form

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Big[A(t,x,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))$$
$$+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s)\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\Big]$$
$$=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\left\{\left[A(t,x,\tau,s)+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s)\right]\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\right\}.$$

Assuming

$$Q(t,x,\tau,s)=A(t,x,\tau,s)+\sum_{\alpha=\tau}^{t}\sum_{\beta=s}^{x}R(t,x,\alpha,\beta)A(\alpha,\beta,\tau,s),$$

this formula can be written in the form

$$y(t,x)=\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}Q(t,x,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s)). \tag{4.8}$$

It is clear that from representation (4.8) we have

$$y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{\alpha=t_0}^{t_1}Q(t_1,x_1,\tau,s)\,\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s)).$$

Therefore we get

$$y^{'}(t_1,x_1)\varphi_{zz}(z(t_1,x_1))y(t_1,x_1)=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))^{'}$$
$$\times\varphi_{zz}(z(t_1,x_1))\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta)). \tag{4.9}$$

Thus,

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$=\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}\Big[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\tau,s)$$
$$\times\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\Big]. \tag{4.10}$$

Finally, using the scheme of the papers [20, 21], we prove the identity

$$\sum_{t=t_0}^{t_1}\sum_{x=x_0}^{x_1}y^{'}(t,x)H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))y(t,x)$$
$$=\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g^{'}(\tau,s,z(\tau,s),u(\tau,s))$$
$$\times\left\{\sum_{t=max(\tau,\alpha)}^{t_1}\;\sum_{x=max(s,\beta)}^{x_1}Q(^{'}t,x,\tau,s)\,H_{zz}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\alpha,\beta)\right\}$$
$$\times\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta)). \tag{4.11}$$

Taking into account identities (4.9)-(4.11) in inequality (4.3), we get the relation

$$\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\sum_{\alpha=t_0}^{t_1}\sum_{\beta=x_0}^{x_1}\Delta_{v(\tau,s)}g^{'}(\tau,s,z(\tau,s),u(\tau,s))M(\tau,s,\alpha,\beta)\Delta_{v(\alpha,\beta)}g(\alpha,\beta,z(\alpha,\beta),u(\alpha,\beta))$$
$$+2\sum_{\tau=t_0}^{t_1}\sum_{s=x_0}^{x_1}\left[\sum_{\tau=t_0}^{t}\sum_{s=x_0}^{x}\Delta_{v(t,x)}H_z^{'}(t,x,z(t,x),u(t,x),\psi(t,x))Q(t,x,\tau,s)\right.$$
$$\left.\times\Delta_{v(\tau,s)}g(\tau,s,z(\tau,s),u(\tau,s))\right]\leq 0 \tag{4.12}$$

where

$$M(\tau, s, \alpha, \beta) = -Q'(t_1, x_1, \tau, s)\, \varphi_{zz}(z(t_1, x_1))\, Q(t_1, x_1, \alpha, \beta)$$

$$+ \sum_{t=max(\tau,\alpha)}^{t_1} \sum_{x=max(s,\beta)}^{x_1} Q('t, x, \tau, s)\, H_{zz}(t, x, z(t, x), u(t, x), \psi(t, x)) Q(t, x, \alpha, \beta). \quad (4.13)$$

Now we formulate the obtained result.

**Theorem 4.3.** *If the function $f(t, x, \tau, s, z, u)$ has the form (4.7) and the set (3.1) is convex, then for optimality of the singular control $u(t, x)$ the inequality (4.12) should be fulfilled for all $v(t, x) \in U$, $(t, x) \in T \times X$.*

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# NECESSARY OPTIMALITY CONDITIONS OF SECOND ORDER IN DISCRETE TWO-PARAMETER STEPWISE CONTROL PROBLEMS

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**Abstract**. A stepwise optimal control problem described by two-dimensional discrete systems is considered. Under openness of a control domain, necessary optimality conditions of first and second order are obtained.

**Keywords and phrases**: Discrete two-parameter system, stepwise control problem, necessary optimality conditions, singular in the classic sense controls, classical extremals.

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## 1. Introduction

Discrete dynamical models of controlled systems are an important class among of mathematical models. Such models arise in modeling of real processes and discretization of continuous models [1-10]. Optimization problems of stepwise or variable structure systems occupy an important place in the theory of optimal control [11-21]. The present paper is devoted to derivation of necessary optimality conditions for one class of control problem described by two-dimensional stepwise discrete system. Finally, we note that various necessary and sufficient optimality conditions for discrete two-dimensional controlled systems are obtained in [8, 22-27].

## 2. Statement of the problem

Let the controlled system be described by the following discrete two-parametric system of equations

$$\begin{cases} z_i(t+1,x+1) = f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x)), \\ (t,x) \in D_i, i=\overline{1,3}, \end{cases} \tag{2.1}$$

with boundary conditions

$$\begin{cases} z_1(t_0,x) = a_1(x),\ x=x_0,x_0+1,...,X,\ z_1(t,x_0)=\beta_1(t),\ t=t_0,t_0+1,...,t_1, \\ z_2(t_1,x) = z_1(t_1,x),\ x=x_0,x_0+1,...,X,\ z_2(t,x_0)=\beta_2(t),\ t=t_1,t_1+1,...,t_2, \\ z_3(t_2,x) = z_2(t_2,x),\ x=x_0,x_0+1,...,X,\ z_3(t,x_0)=\beta_3(t),\ t=t_2,t_2+1,...,t_3, \\ a_1(x_0)=\beta_1(t_0),\ z_1(t_1,x_0)=\beta_2(t_1),\ z_2(t_2,x_0)=\beta_3(t_2). \end{cases} \tag{2.2}$$

Here, $D_i = \{(t,x) : t = t_{i-1}, t_{i-1}+1, ..., t_i - 1; x = x_0, x_0+1, ..., X-1\}, i=\overline{1,3}$, where $x_0, X, t_i, i=\overline{1,3}$ are fixed numbers; $f_i(t,x,a_i,b_i,u_i), i=\overline{1,3}$ are $n$-dimensional vector-functions continuous in the aggregate of variables together with partial derivatives with respect to $(z_i,a_i,b_i,u_i), i=\overline{1,3}$ up to the second order inclusive, $\alpha_1(x), \beta_i(t), i=\overline{1,3}$

are given $n$-dimensional vector-functions, and $u_i(t,x), i=\overline{1,3}$ are $r$-dimensional control functions with values from the given non-empty, bounded and open sets $U_i \subset R^r, i=\overline{1,3}$, i.e.

$$u_i(t,x) \in U_i \subset R^r, \quad (t,x) \in D_i, \quad i=\overline{1,3}. \tag{2.3}$$

The triple $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))'$ with the above mentioned properties and its corresponding solution $z(t,x) = (z_1(t,x), z_2(t,x), z_3(t,x))'$ of boundary value problem (2.1)-(2.2) will be called an admissible control and admissible state of the process, respectively. The pair $(u(t,x), z(t,x))$ is said to be an admissible process.

The problem is to minimize the cost functional

$$S(u) = \sum_{i=1}^{3} \varphi_i(z_i(t_i, X)) \tag{2.4}$$

determined on the solutions of boundary value problem (2.1)-(2.2) generated by all possible admissible controls.
Here, $\varphi_i(z_i), i=\overline{1,3}$ are the given twice continuously differentiable scalar functions.
In the sequel, the problem on the minimum of the functional (2.4) under restrictions (2.1)-(2.3) will be called problem (2.1)-(2.4). The admissible process $(u(t,x), z(t,x))$ being a solution of problem (2.1)-(2.4) will be called an optimal process.

## 3. Auxiliary facts and variations of cost functional

Let $(u(t,x), z(t,x))$ be a fixed admissible process. In the sequel, the following denotations will be used:

$$H_i(t,x,z_i,a_i,b_i,u_i,\psi_i) = \psi_i' f_i(t,x,z_i,a_i,b_i,u_i),$$

$$\frac{\partial f_i[t,x]}{\partial a_i} = \frac{\partial f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x))}{\partial a_i},$$

$$\frac{\partial H_i[t,x]}{\partial z_i} = \frac{\partial H_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i},$$

$$\frac{\partial^2 H_i[t,x]}{\partial z_i^2} = \frac{\partial^2 H_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i^2},$$

where $\psi_i = \psi_i(t,x),\ i=\overline{1,3}$ are $n$-dimensional vector-functions of conjugated being the solutions of the problem

$$\psi_i(t-1,x-1) = \frac{\partial H_i[t,x]}{\partial z_i} + \frac{\partial H_i[t-1,x]}{\partial a_i} + \frac{\partial H_i[t,x-1]}{\partial b_i}, \quad i=\overline{1,3}, \tag{3.1}$$

$$\psi_1(t_1-1,X-1) = \psi_2(t_1-1,X-1) - \frac{\partial \varphi_1(z_1(t_1,X))}{\partial z_1},$$

$$\psi_1(t_1-1,x-1) = \psi_2(t_1-1,x-1) + \frac{\partial H_1[t_1-1,x]}{\partial a_1} - \frac{\partial H_2[t_1-1,x]}{\partial a_2},$$

$$\psi_1(t-1,X-1) = \frac{\partial H_1[t-1,X-1]}{\partial b_1},$$

$$\psi_2(t_2-1,X-1) = \psi_3(t_2-1,X-1) - \frac{\partial \varphi_2(z_2(t_2,X))}{\partial z_2},$$

$$\psi_2(t_2-1,x-1)=\psi_3(t_2-1,x-1)+\frac{\partial H_2[t_2-1,x]}{\partial a_2}-\frac{\partial H_3[t_2-1,x]}{\partial a_3},$$
$$\psi_2(t-1,X-1)=\frac{\partial H_2[t,X-1]}{\partial b_2},\quad \psi_3(t_3-1,X-1)=-\frac{\partial\varphi_3(z_3(t_3,X))}{\partial z_3},$$
$$\psi_3(t_3-1,x-1)=\frac{\partial H_3[t_3,x]}{\partial a_3},\quad \psi_3(t-1,X-1)=\frac{\partial H_3[t,X-1]}{\partial b_3}. \tag{3.2}$$

Using a scheme for example from [23, 28, 29] we can show that the first and second variations (in the classical sense) of functional (2.4) have the form

$$\delta^1 S(u;\,\delta u)=-\sum_{i=1}^{3}\left[\sum_{t=t_{i-1}}^{t_1-1}\sum_{x=x_0}^{X-1}\frac{\partial H_i'[t,x]}{\partial u_i}\delta u_i(t,x)\right], \tag{3.3}$$

$$\delta^2 S(u;\,\delta u)=\sum_{i=1}^{3}\delta z_i'(t_i,X)\frac{\partial^2\varphi_i(z_i(t_i,X)}{\partial z_i^2}\delta z_i(t_i,X))$$
$$-\sum_{i=1}^{3}\left[\sum_{t=t_{i-1}}^{t_1-1}\sum_{x=x_0}^{X-1}\left[\delta z_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i^2}\delta z_i(t,x)+\delta z_i'(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial z_i}\delta z_i(t,x)+\delta z_i'(t,x)\right.\right.$$
$$\times\frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial a_i}\delta z_i(t+1,x)+\delta z_i'(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i^2}\delta z_i(t+1,x)+\delta z_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial b_i}$$
$$\times\delta z_i(t,x+1)+\delta z_i'(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i\,\partial z_i}\delta z_i(t,x)+\delta z_i'(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i^2}\delta z_i(t,x+1)$$
$$+\delta z_i'(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial b_i}\delta z_i(t,x+1)+\delta z_i'(t,x+1)\frac{\partial^2 H_i[t,x]}{\partial b_i\,\partial a_i}\delta z_i(t+1,x)$$
$$+2\delta u_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial z_i}\delta z_i(t,x)+2\delta u_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial a_i}\delta z_i(t+1,x)+2\delta u_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i\,\partial b_i}$$
$$\left.\left.\times\delta z_i(t,x+1)+\delta u_i'(t,x)\frac{\partial^2 H_i[t,x]}{\partial u_i^2}\delta u_i(t,x)\right]\right] \tag{3.4}$$

respectively, where $\delta u_i(t,x)\in R^r$, $(t,x)\in D_i$, $i=\overline{1,3}$ is an arbitrary bounded vector-function called an admissible variation of the control $u_i(t,x)$, $i=\overline{1,3}$, and $\delta z_i(t,x)$ is a variation of the trajectory $z_i(t,x)$ being a solution of the equation in variations

$$\delta z_i(t+1,x+1)=\frac{\partial f_i[t,x]}{\partial z_i}\delta z_i(t,x)+\frac{\partial f_i[t,x]}{\partial a_i}\delta z_i(t+1,x)+\frac{\partial f_i[t,x]}{\partial b_i}\delta z_i(t,x+1)$$
$$+\frac{\partial f_i[t,x]}{\partial u_i}\delta u_i(t,x),\quad i=\overline{1,3}, \tag{3.5}$$

with boundary conditions

$$\begin{cases}\delta z_1(t_0,x)=0,\ x=x_0,x_0+1,...,X,\ \delta z_1(t,x_0)=0,\ t=t_0,t_0+1,...,t_1,\\ \delta z_2(t_1,x)=\delta z_1(t_1,x),\ x=x_0,x_0+1,...,X,\ \delta z_2(t,x)=0,\ t=t_1,t_1+1,...,t_2,\\ \delta z_3(t_2,x)=\delta z_2(t_2,x),\ x=x_0,x_0+1,...,X,\ \delta z_3(t,x_0)=0,\ t=t_2,t_2+1,...,t_3.\end{cases} \tag{3.6}$$

The system of difference equations (3.5) is linear and nonhomogeneous. Therefore, we can represent (see [23, 26, 27]) the solution of problem (3.5)-(3.6) in the form

$$\delta z_1(t,x)=\sum_{\tau=t_0}^{t-1}\sum_{s=x_0}^{x-1}R_1(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s), \tag{3.7}$$

$$\delta z_2(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x-1}Q_1(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s)$$
$$+\sum_{\tau=t_1}^{t-1}\sum_{s=x_0}^{x-1}R_2(t,x;\tau,s)\frac{\partial f_2[\tau,s]}{\partial u_2}\delta u_2(\tau,s), \tag{3.8}$$

$$\delta z_3(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x-1}Q_2(t,x;\tau,s)\frac{\partial f_1[\tau,s]}{\partial u_1}\delta u_1(\tau,s)+\sum_{\tau=t_1}^{t_2-1}\sum_{s=x_0}^{x-1}Q_3(t,x;\tau,s)$$
$$\times\frac{\partial f_2[\tau,s]}{\partial u_2}\delta u_2(\tau,s)+\sum_{\tau=t_2}^{t-1}\sum_{s=x_0}^{x-1}R_3(t,x;\tau,s)\frac{\partial f_3[\tau,s]}{\partial u_3}\delta u_3(\tau,s), \tag{3.9}$$

where by definition

$$Q_1(t,x;\tau,s)=R_2(t,x;t_1-1,x-1)\,R_1(t,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[R_2(t,x;t_1-1,\beta-1)-R_2(t,x;t_1-1,\beta)\frac{\partial f_2[t_1-1,\beta]}{\partial a_2}\right]R_1(t_1,\beta;\,\tau,s),$$

$$Q_2(t,x;\tau,s)=R_3(t,x;t_2-1,x-1)\,Q_1(t_2,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[[R_3(t,x;t_2-1,\beta-1)-R_3(t,x;t_2-1,\beta)]\,\frac{\partial f_3[t_2-1,\beta]}{\partial a_3}\right]Q_1(t_2,\beta;\,\tau,s),$$

$$Q_3(t,x;\tau,s)=R_3(t,x;t_2-1,x-1)\,R_2(t_2,x;\tau,s)$$
$$+\sum_{\beta=s+1}^{x-1}\left[[R_3(t,x;t_2-1,\beta-1)-R_3(t,x;t_2-1,\beta)]\,\frac{\partial f_3[t_2-1,\beta]}{\partial a_3}\right]R_2(t_2,\beta;\,\tau,s),$$

Here, $R_i(t,x;\tau,s)$, $i=\overline{1,3}$ are $(n\times n)$ dimensional matrix functions being the solutions of the following problems:

$$R_i(t,x;\tau-1,s-1)=R_i(t,x;\tau,s)\frac{\partial f_i[\tau,s]}{\partial z_i}+R_i(t,x;\tau-1,s)\frac{\partial f_i[\tau-1,s]}{\partial a_i}$$
$$+R_i(t,x;\tau,s-1)\frac{\partial f_i[\tau,s-1]}{\partial b_i},$$
$$R_i(t,x;t-1,s-1)=R_i(t,x;t_1-1,s)\frac{\partial f_i[t-1,s]}{\partial a_i},$$
$$R_i(t,x;\tau-1,x-1)=R_i(t,x;\tau,x-1)\frac{\partial f_i[\tau,x-1]}{\partial b_i},$$
$$R_i(t,x;t-1,x-1)=E,\ (E-(n\times n)\text{ is a unit function}).$$

Let $(u(t,x), z(t,x))$ be an optimal process. Then, along this process, for all the admissible variations $\delta u(t,x)$ of the control $u(t,x)$, the first variation (3.3) of functional (2.4) should equal zero, the second variation (3.4) of functional (2.4) should be non-negative, i.e.

$$\delta^1 S(u;\ \delta u) = 0, \tag{3.10}$$

$$\delta^2 S(u;\ \delta u) \geq 0. \tag{3.11}$$

The relations (3.10) and (3.11) are implicit necessary conditions of first and second orders, respectively.

In the next section, using these relations we obtain the explicit necessary optimality conditions expressed directly by the parameter of the stated problem.

## 4. Necessary optimality conditions

Allowing for representation (3.10), by independence of the admissible variations $\delta u_i(t,x)$, $i = \overline{1,3}$ of the control it follows from (3.3) that along the optimal process

$$\frac{\partial H_i[\theta,\xi]}{\partial u_i} = 0, \quad \text{for all } \ (\theta,\xi) \in D_i,\ i = \overline{1,3}. \tag{4.1}$$

The relation (4.1) representing a first order necessary optimality conditions is an analogy of Euler equation for problem (2.1)-(2.4).

Each admissible control $u(t,x)$ satisfying Euler equation (4.1) is said to be classic extremal in problem (2.1)-(2.4).

Using inequality (3.11), in many cases we can get explicit necessary optimality condition of second order.

To this end, assume that in system (2.1)

$$f_i(t,x,z_i,a_i,b_i,u_i) = B_i(t,x)\, b_i + F(t,x,z_i,a_i,u_i). \tag{4.2}$$

Assume

$$K_1(\tau,s) = -R_1^{'}(t_1,X;\,\theta,\tau)\,\frac{\partial^2\varphi_1(z_1(t_1,X))}{\partial z_1^2}\,R_1(t_1,X;\,\theta,s) - Q_1^{'}(t_2,X;\,\theta,\tau)$$

$$\times\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}\,Q_2(t_2,X;\,\theta,s) - Q_3^{'}(t_3,X;\,\theta,\tau)\,\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,Q_3(t_3,X;\,\theta,s)$$

$$+\sum_{t=\theta+1}^{t_1-1}\ \sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_1^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial z_1^2}\,R_1(t,x;\,\theta,s) + R_1^{'}(t,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial z_1\,\partial a_1}$$

$$\times R_1(t+1,x;\,\theta,s) + R_1^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial a_1\,\partial z_1}\,R_1(t,x;\,\theta,s)\Bigg]$$

$$+\sum_{t=\theta}^{t_1-1}\ \sum_{x=max(\tau,s)+1}^{X-1} R_1^{'}(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_1[t,x]}{\partial a_1^2}\,R_1(t+1,x;\,\theta,s)$$

$$+\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_1'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\,Q_1(t,x;\,\theta,s)+Q_1'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}$$

$$\times Q_1(t+1,x;\,\theta,s)+Q_1'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}\,Q_1(t,x;\,\theta,s)+Q_1'(t+1,x;\,\theta,\tau)$$

$$\times\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\,Q_1(t+1,x;\,\theta,s)\Bigg]+\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_2'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,Q_2(t,x;\,\theta,s)$$

$$+Q_2'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\,Q_2(t+1,x;\,\theta,s)+Q_2'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\,Q_2(t,x;\,\theta,s)$$

$$+Q_2'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,Q_2(t+1,x;\,\theta,s)\Bigg], \tag{4.3}$$

$$K_2(\tau,s)=-R_2'(t_2,X;\,\theta,\tau)\,\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}\,R_2(t_2,X;\,\theta,s)-Q_3'(t_3,X;\,\theta,\tau)$$

$$\times\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,Q_3(t_3,X;\,\theta,s)+\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_2'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2^2}$$

$$\times R_2(t,x;\,\theta,s)+R_2'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\,R_2(t+1,x;\,\theta,s)+R_2'(t+1,x;\,\theta,\tau)$$

$$\times\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\,R_2(t,x;\,\theta,s)\Bigg]+\sum_{t=\theta}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_2[t,x]}{\partial a_2^2}$$

$$\times R_2(t+1,x;\,\theta,s)+\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[Q_3'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,Q_3(t,x;\,\theta,s)$$

$$+Q_3'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}Q_3(t+1,x;\,\theta,s)+Q_3'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}\,Q_3(t,x;\,\theta,s)$$

$$+Q_3'(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,Q_3(t+1,x;\,\theta,s)\Bigg], \tag{4.4}$$

$$K_3(\tau,s)=-R_3'(t_3,X;\,\theta,\tau)\,\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}\,R_3(t_3,X;\,\theta,s)$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}\Bigg[R_3'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\,R_3(t,x;\,\theta,s)$$

$$+R_3'(t,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\,R_3(t+1,x;\,\theta,s)+R_3'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\,R_3(t,x;\,\theta,s)\Bigg]$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}R_3'(t+1,x;\,\theta,\tau)\,\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\,R_3(t+1,x;\,\theta,s). \tag{4.5}$$

Using the discrete variants of line variations [30], we prove the following

**Theorem 4.1** *If the sets $U_i$, $i=\overline{1,3}$ are open, then under the assumptions made for optimality of the classical extremal $u(t,x)$ in problem (2.1)-(2.4), (4.2) the following relations*

$$1) \qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_1^{'}(\tau)\,\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}\,K_1(\tau,s)\,\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)+\sum_{x=x_0}^{x_1-1} v_1^{'}(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1^2}\,v_1(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_1^{'}(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1\,\partial a_1}\,R_1(\theta+1,x;\,\theta,s)\,\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)\right]\le 0 \qquad (4.6)$$

*should be fulfilled for all $v_1(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_1=\{t_0,\,t_0+1,\,\dots,\,t_1-1\}$,*

$$2) \qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_2^{'}(\tau)\,\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}\,K_2(\tau,s)\,\frac{\partial f_2[\theta,s]}{\partial u_2}\,v_2(s)+\sum_{x=x_0}^{x_1-1} v_2^{'}(x)\,\frac{\partial^2 H_2[\theta,x]}{\partial u_2^2}\,v_2(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_2^{'}(x)\,\frac{\partial^2 H_2[\theta,x]}{\partial u_2\,\partial a_2}\,R_2(\theta+1,x;\,\theta,s)\,\frac{\partial f_2[\theta,s]}{\partial u_2}\,v_2(s)\right]\le 0 \qquad (4.7)$$

*for all $v_2(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_2=\{t_1,\,t_1+1,\,\dots,\,t_2-1\}$,*

$$3) \qquad \sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1} v_3^{'}(\tau)\,\frac{\partial f_3^{'}[\theta,\tau]}{\partial u_3}\,K_3(\tau,s)\,\frac{\partial f_3[\theta,s]}{\partial u_3}\,v_3(s)+\sum_{x=x_0}^{x_1-1} v_3^{'}(x)\,\frac{\partial^2 H_3[\theta,x]}{\partial u_3^2}\,v_3(x)$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1} v_3^{'}(x)\,\frac{\partial^2 H_3[\theta,x]}{\partial u_3\,\partial a_3}\,R_3(\theta+1,x;\,\theta,s)\,\frac{\partial f_3[\theta,s]}{\partial u_3}\,v_3(s)\right]\le 0 \qquad (4.8)$$

*for all $v_3(x)\in R^r$, $x=x_0,\,x_0+1,\dots,X-1$, $\theta\in T_3=\{t_2,\,t_2+1,\,\dots,\,t_3-1\}$.*

**Proof.** Using arbitrariness of admissible variations of the control $u(t,x)=(u_1(t,x),\,u_2(t,x),\,u_3(t,x))$, we assume

$$\delta u_1^*(t,x)=\begin{cases} v_1(x), \quad t=\theta\in T_1;\;\; x=x_0,\,x_0+1,\dots,\,X-1,\\ 0, \quad t\neq\theta;\;\; x=x_0,\,x_0+1,\dots,\,X-1,\end{cases} \qquad (4.9)$$

$$\delta u_1^*(t,x)=0, \quad (t,x)\in D_i, \quad i=1,2.$$

Here, $v_1(x)\in R^r$, $x=x_0,\,x_0+1,\dots,\,X-1$ is an arbitrary bounded vector-function, $\theta\in T_1=\{t_0,\,t_0+1,\,\dots,\,t_1-1\}$ is an arbitrary point.

By $\delta z^*(t,x)=(\delta z_1^*(t,x),\,\delta z_2^*(t,x),\,\delta z_3^*(t,x))$ we denote the solution of problems (3.5)-(3.6) that corresponds to special variation (4.9) of control. It follows from representations (3.7)-(3.9) that

$$\delta z_1^*(t,x)=\begin{cases} 0, \quad t=t_0,\,t_0+1,\dots,\,\theta;\;\; x=x_0,\,x_0+1,\dots,\,X,\\ \sum\limits_{s=x_0}^{x-1} R_1(t,x;\theta,s)\,\dfrac{\partial f_1[\theta,s]}{\partial u_1}v_1(s), \quad t\ge\theta+1;\;\; x=x_0,x_0+1,...,X,\end{cases} \qquad (4.10)$$

$$\begin{cases} \delta z_2^*(t,x) = \sum_{s=x_0}^{x-1} Q_1(t,x;\theta,s)\, \dfrac{\partial f_1[\theta,s]}{\partial u_1} v_1(s), \\ t = t_1, t_1+1, ..., t_2; \quad x = x_0, ..., X, \end{cases} \tag{4.11}$$

$$\begin{cases} \delta z_3^*(t,x) = \sum_{s=x_0}^{x-1} Q_2(t,x;\theta,s)\, \dfrac{\partial f_1[\theta,s]}{\partial u_1} v_1(s), \\ t = t_2, t_2+1, ..., t_3; \quad x = x_0, ..., X, \end{cases} \tag{4.12}$$

Allowing for (3.4), (4.5), (4.9) from (3.11) we get that for the optimality of classic singular control $u(t,x) = (u_1(t,x),\, u_2(t,x),\, u_3(t,x))$ in problem (2.1)-(2.4), (4.2) the inequality

$$\sum_{i=1}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) - \frac{1}{2} \sum_{i=1}^{3} \Bigg[ \sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \Bigg[ \delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i^2}$$
$$\times \delta z_i^*(t,x) + \delta z_i^{*'}(t,x) \frac{\partial^2 H_i[t,x]}{\partial z_i\, \partial a_i} \delta z_i^*(t+1,x) + \delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i\, \partial z_i} \delta z_i^*(t,x)$$
$$+ \delta z_i^{*'}(t+1,x) \frac{\partial^2 H_i[t,x]}{\partial a_i^2} \delta z_i^*(t+1,x) \Bigg]\Bigg] - 2 \sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{x_1-1} \Bigg[ \delta u_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial u_1\, \partial z_1} \delta z_1^*(t,x)$$
$$+ \delta u_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial u_1\, \partial a_1} \delta z_1^*(t+1,x) \Bigg] - \sum_{x=x_0}^{X-1} v_1'(x) \frac{\partial^2 H_1[t,x]}{\partial u_1^2} v_1(x) \geq 0, \tag{4.13}$$

should be fulfilled for all $v_1(x) \in R^r$, $x = x_0, x_0+1, ..., X-1$.

Further, using representations (4.10)-(4.12), we get

$$\sum_{i=1}^{3} \delta z_i^{*'}(t_i,X) \frac{\partial^2 \varphi_i(z_i(t_i,X))}{\partial z_i^2} \delta z_i^*(t_i,X) = \sum_{\tau=x_0}^{X-1} \sum_{s=x_0}^{X-1} v_1'(\tau) \frac{\partial f_1'[\theta,\tau]}{\partial u_1} \Bigg[ R_1'(t_1,X;\,\theta,\tau)$$
$$\times \frac{\partial^2 \varphi_1(z_1(t_1,X))}{\partial z_1^2} R_1(t_1,X;\,\theta,s) + Q_1'(t_2,X;\,\theta,\tau)\, \frac{\partial^2 \varphi_2(z_2(t_2,X))}{\partial z_2^2} Q_1(t_2,X;\,\theta,s)$$
$$+ Q_2'(t_3,X;\,\theta,\tau)\, \frac{\partial^2 \varphi_3(z_3(t_3,X))}{\partial z_3^2} Q_2(t_3,X;\,\theta,s) \Bigg]\, \frac{\partial f_1[\theta,s]}{\partial u_1} v_1(s). \tag{4.14}$$

By the scheme given in [25, 26], we have

$$\sum_{t=t_0}^{t_1-1} \sum_{x=x_0}^{X-1} \delta z_1^{*'}(t,x) \frac{\partial^2 H_1[t,x]}{\partial z_1^2} \delta z_1^*(t,x) = \sum_{\tau=x_0}^{X-1} \sum_{s=x_0}^{X-1} v_1'(\tau) \frac{\partial f_1'[\theta,\tau]}{\partial u_1}$$
$$\times \left[ \sum_{t=\theta+1}^{t_1-1} \sum_{x=max(\tau,s)+1}^{X-1} R_1'(t,x;\,\theta,\tau) \frac{\partial^2 H_1[t,x]}{\partial z_1^2} R_1(t,x;\,\theta,s) \right] \frac{\partial f_1[\theta,s]}{\partial u_1} v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\delta z_2^*(t,x)=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}Q_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\delta z_3^*(t,x)=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_2(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),\tag{4.15}$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t,x)\frac{\partial^2 H_1[t,x]}{\partial z_1\,\partial a_1}\delta z_1^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=\theta+1}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial z_1\partial a_1}R_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t+1,x)\frac{\partial^2 H_1[t,x]}{\partial a_1\,\partial z_1}\delta z_1^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=\theta+1}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1\partial z_1}R_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\partial a_2}Q_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\delta z_2^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}Q_1(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\partial a_3}Q_2(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}Q_2(t,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta z_1^{*'}(t+1,x)\frac{\partial^2 H_1[t,x]}{\partial a_1^2}\delta z_1^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=\theta}^{t_1-1}\sum_{x=max(\tau,s)+1}^{X-1}R_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1^2}R_1(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_1^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}Q_2(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s),$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_1^{'}(\tau)\frac{\partial f_1^{'}[\theta,\tau]}{\partial u_1}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_3(t+1,x;\,\theta,s)\right]\frac{\partial f_1[\theta,s]}{\partial u_1}v_1(s).$$

Further, on the basis of discrete analogy of Fubini theorem (see [20, 28, 29]), we get

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{X-1}\delta u_1^{*'}(t,x)\,\frac{\partial^2 H_1[t,x]}{\partial u_1\partial a_1}\,\delta z_1^*(t+1,x)=\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{X-1}v_1^{'}(x)\,\frac{\partial^2 H_1[\theta,x]}{\partial u_1\,\partial a_1}\,R_1(t+1,x;\,\theta,s)\right.$$
$$\left.\times\frac{\partial f_1[\theta,s]}{\partial u_1}\,v_1(s)\right]=\sum_{x=x_0}^{X-1}\left[\sum_{s=x+1}^{X-1}v_1^{'}(s)\,\frac{\partial^2 H_1[\theta,s]}{\partial u_1\,\partial a_1}\,R_1(t+1,s;\,\theta,x)\right]$$
$$\times\frac{\partial f_1[\theta,x]}{\partial u_1}\,v_1(x),\tag{4.16}$$

Taking into account relations (4.14)-(4.16) and denotation (3.10) in relation (4.13) we arrive at inequality (4.6).

Now, we introduce the special variation of the control $u(t,x)$ by the formula

$$\begin{cases} \delta u_1^*(t,x)=0, \quad (t,x)\in D_i, \quad i=1,3, \\ \delta u_2^*(t,x)=\begin{cases} v_2(x), \quad t=\theta\in T_2; \;\; x=x_0,\, x_0+1,\ldots,\, X-1, \\ 0, \quad t\neq\theta; \;\; x=x_0,\, x_0+1,\ldots,\, X-1. \end{cases} \end{cases} \tag{4.17}$$

Here, $v_2(x)\in R^r$, $x=x_0,\, x_0+1,\ldots,\, X-1$ is an arbitrary $r$-dimensional bounded vector-function, $\theta\in T_2=\{t_1,\, t_1+1,\, \ldots,\, t_2-1\}$ is an arbitrary point.

Denote by $\delta z^*(t,x)=(\delta z_1^*(t,x),\, \delta z_2^*(t,x),\, \delta z_3^*(t,x))$ the solution of problems (3.5)-(3.6) that corresponds to the special variation (35) of the control.

It follows from representations (3.7)-(3.9) that

$$\delta z_1^*(t,x)=0,$$

$$\delta z_2^*(t,x)=\begin{cases} 0, \quad t=t_1,\, t_1+1,\ldots,\, \theta; \;\; x=x_0,\, x_0+1,\ldots,\, X, \\ \sum\limits_{s=x_0}^{x-1} R_2(t,x;\theta,s)\,\dfrac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad t\geq\theta+1, \end{cases} \tag{4.18}$$

$$\delta z_3^*(t,x)=\sum_{s=x_0}^{x-1} Q_3(t,x;\theta,s)\,\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \quad t=t_2,t_2+1,...,t_3; \;\; x=x_0,...,X.$$

Allowing for (3.4), (4.17), from (3.11) we obtain that for optimality of the classic extremal $u(t,x)=(u_1(t,x),\, u_2(t,x),\, u_3(t,x))$ in problem (1)-(4), (17) the inequality

$$\sum_{i=2}^{3}\delta z_i^{*'}(t_i,X)\frac{\partial^2\varphi_i(z_i(t_i,X))}{\partial z_i^2}\delta z_i^*(t_i,X)-\frac{1}{2}\sum_{i=2}^{3}\left[\sum_{t=t_{i-1}}^{t_i-1}\sum_{x=x_0}^{X-1}\left[\delta z_i^{*'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i^2}\right.\right.$$
$$\times\delta z_i^*(t,x)+\delta z_i^{*'}(t,x)\frac{\partial^2 H_i[t,x]}{\partial z_i\,\partial a_i}\delta z_i^*(t+1,x)+\delta z_i^{*'}(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i\,\partial z_i}\delta z_i^*(t,x)$$
$$\left.\left.+\delta z_i^{*'}(t+1,x)\frac{\partial^2 H_i[t,x]}{\partial a_i^2}\delta z_i^*(t+1,x)\right]\right]-2\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{x_1-1}\left[\delta u_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial u_2\,\partial z_2}\delta z_2^*(t,x)\right.$$
$$\left.+\delta u_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial u_2\,\partial a_2}\delta z_2^*(t+1,x)\right]-\sum_{x=x_0}^{X-1}v_2'(x)\frac{\partial^2 H_2[t,x]}{\partial u_2^2}v_2(x)\geq 0, \tag{4.19}$$

should be fulfilled for all $v_2(x)\in R^r$, $x=x_0,x_0+1,...,X-1$.

Further, using representations (4.18), we get

$$\sum_{i=1}^{3}\delta z_i^{*'}(t_i,X)\frac{\partial^2\varphi_i(z_i(t_i,X))}{\partial z_i^2}\delta z_i^*(t_i,X)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2'(\tau)\frac{\partial f_2'[\theta,\tau]}{\partial u_2}\left[R_2'(t_2,X;\,\theta,\tau)\right.$$
$$\left.\times\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}R_2(t_2,X;\,\theta,s)+Q_3'(t_3,X;\,\theta,\tau)\,\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}Q_3(t_3,X;\,\theta,s)\right]$$
$$\times\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s). \tag{4.20}$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}\delta z_2^*(t,x)=\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}R_2(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.21}$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_3(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.22}$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\partial a_2}R_2(t+1,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.23}$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}\delta z_2^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=\theta+1}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\partial z_2}R_2(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.24}$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t,x)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3^{'}(t,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\partial a_3}Q_3(t+1,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.25}$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}\delta z_3^*(t,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\partial z_3}Q_3(t,x;\,\theta,s)\right]\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s),\tag{4.26}$$

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta z_2^{*'}(t+1,x)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}\delta z_2^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=\theta}^{t_2-1}\sum_{x=max(\tau,s)+1}^{X-1}R_2^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}R_2(t+1,x;\,\theta,s)\right]$$
$$\times\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s), \tag{4.27}$$

$$\sum_{t=t_2}^{t_3-1}\sum_{x=x_0}^{X-1}\delta z_3^{*'}(t+1,x)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}\delta z_3^*(t+1,x)=\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_2^{'}(\tau)\frac{\partial f_2^{'}[\theta,\tau]}{\partial u_2}$$
$$\times\left[\sum_{t=t_2}^{t_3-1}\sum_{x=max(\tau,s)+1}^{X-1}Q_3^{'}(t+1,x;\,\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_3(t+1,x;\,\theta,s)\right]$$
$$\times\frac{\partial f_2[\theta,s]}{\partial u_2}v_2(s). \tag{4.28}$$

Using the discrete analogy of Foubini theorem [23], we have

$$\sum_{t=t_1}^{t_2-1}\sum_{x=x_0}^{X-1}\delta u_2^{*'}(t,x)\frac{\partial^2 H_2[t,x]}{\partial u_2\,\partial a_2}\delta z_2^*(t+1,x)$$
$$=\sum_{x=x_0}^{X-1}\left[\sum_{s=x+1}^{X-1}v_2^{'}(s)\frac{\partial^2 H_2[\theta,s]}{\partial u_2\,\partial a_2}R_2(t+1,s;\,\theta,x)\right]\frac{\partial f_2[\theta,x]}{\partial u_2}v_2(x). \tag{4.29}$$

Taking into account identities (4.20)-(4.29), and also denotation (4.4) in inequality (4.19), we arrive at relation (4.7). Inequality (4.8) is also proved by the appropriate arguments. This completes the proof of the theorem.

**Remark.** Similar symmetric results are obtained in the case when the right-hand side of system (2.1) has the form

$$f_i(t,x,z_i,a_i,b_i,u_i)=A_i(t,x)\,a_i+Q_i(t,x,z_i,b_i,u_i).$$

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# A REMARK CONCERNING PECULIARITIES OF TWO MODELS OF CUSPED PRISMATIC SHELLS

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**Abstract**. Comparative analysis of peculiarities of setting of boundary value problems are carried out for cusped prismatic shells within the framework of the zero approximation of hierarchical models when on the face surfaces either stress or displacement vectors are assumed to be known.

**Keywords and phrases**: Cusped plates, cusped prismatic shells, mathematical modeling, linear elasticity, degenerate and singular elliptic and hyperbolic equations and systems.

**AMS subject classification (2000):** 74K20; 74K25; 74B05; 35Q74; 35J70; 35J75; 35L80; 35L81.

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin $O$. We conditionally assume the $x_3$-axis vertical. The elastic body is called a prismatic shell if it is bounded above and below by, respectively, the surfaces (so called face surfaces)

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2),$$

laterally by a cylindrical surface $\Gamma$ of generatrix parallel to the $x_3$-axis and its vertical dimension is sufficiently small compared with other dimensions of the body.

In other words, the 3D elastic prismatic shell-like body occupies a bounded region $\overline{\Omega}$ with boundary $\partial\Omega$, which is defined as:

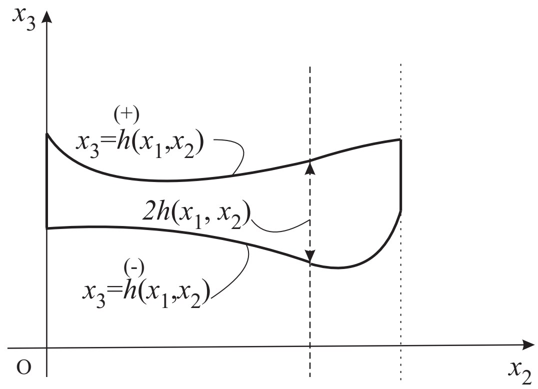


Fig.1. A cross-section of a typical non-cusped prismatic shell

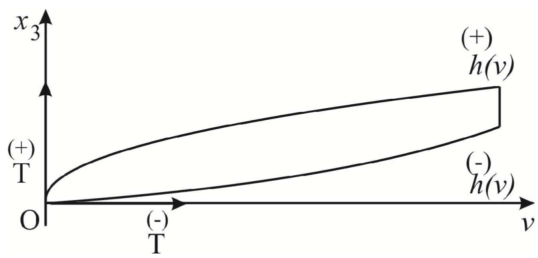


Fig.2. A cross-section of a blunt cusped prismatic shell

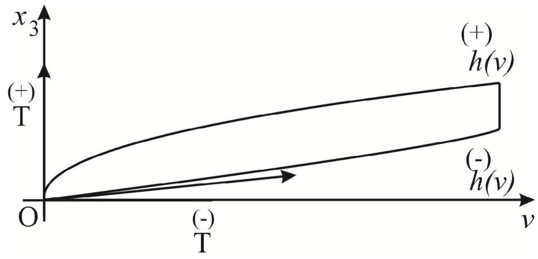


Fig.3. A cross-section of a blunt cusped prismatic shell ($\varphi \in ]0, \frac{\pi}{2}[$)

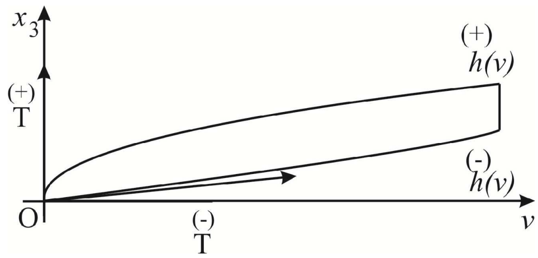


Fig.4. A cross-section of a blunt cusped prismatic shell ($\varphi = 0$)

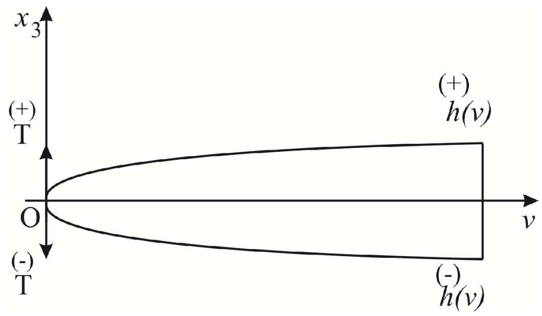


Fig.5. A cross-section of a blunt cusped plate ($\varphi = \pi$)

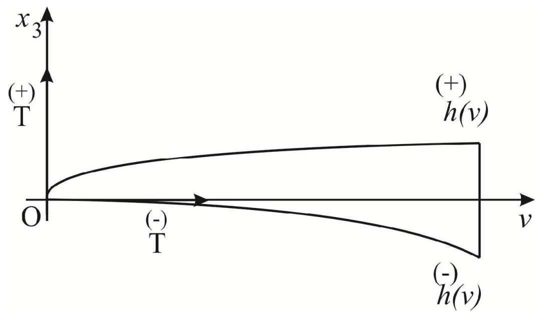


Fig.6. A cross-section of a blunt cusped prismatic shell ($\varphi = \frac{\pi}{2}$)

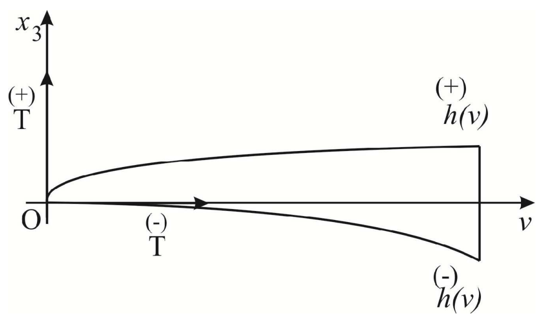


Fig.7. A cross-section of a blunt cusped prismatic shell ($\varphi \in ]\frac{\pi}{2}, \pi[$)

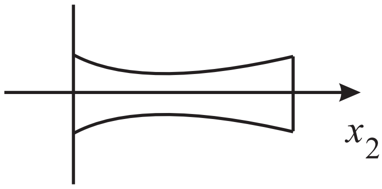


Fig.8. Non-cusped edges

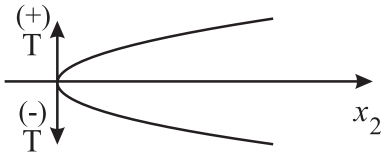


Fig.9. $\varphi = \pi$

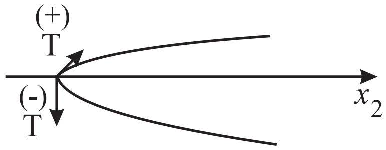


Fig.10. $\frac{\pi}{2} < \varphi < \pi$

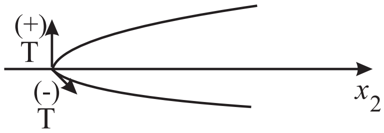


Fig.11. $\frac{\pi}{2} < \varphi < \pi$

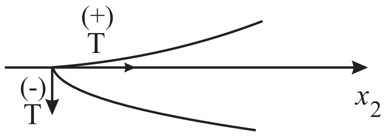


Fig.12. $\varphi = \frac{\pi}{2}$

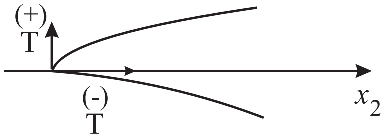


Fig.13. $\varphi = \frac{\pi}{2}$

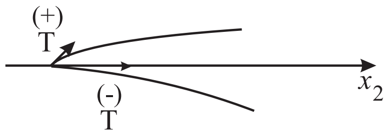


Fig.14. $0 < \varphi < \frac{\pi}{2}$

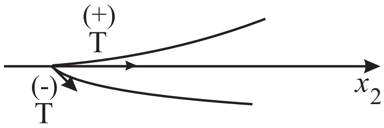


Fig.15. $0 < \varphi < \frac{\pi}{2}$

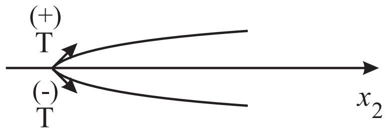


Fig.16. $0 < \varphi < \pi$

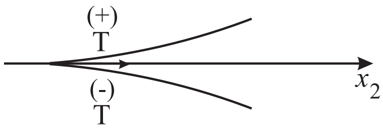


Fig.17. $\varphi = 0$

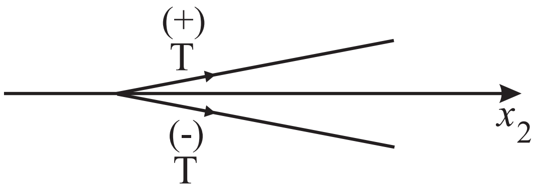


Fig.18. Wedge

Typical cross-sections of prismatic shells

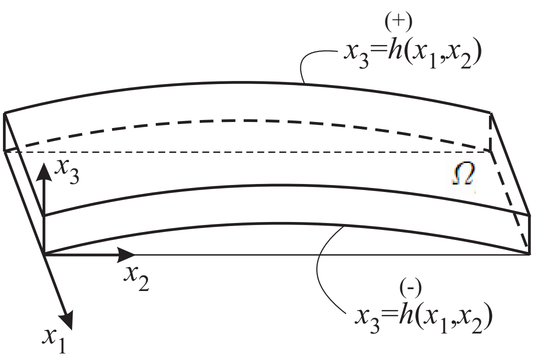


Fig.19. Prismatic shell of constant thickness

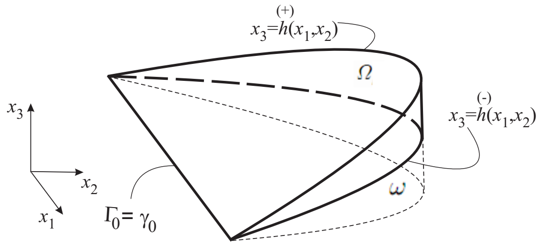


Fig.20. A sharp cusped prismatic shell with a semicircle projection

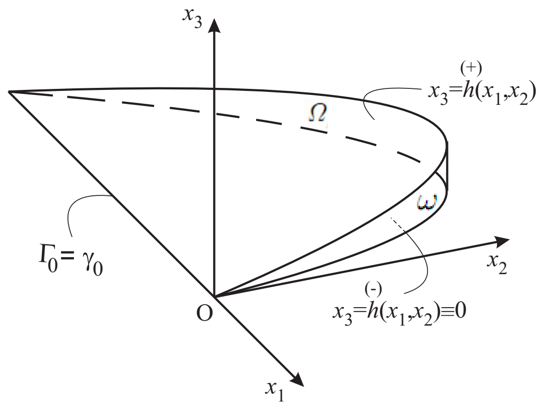


Fig.21. A sharp cusped prismatic shell with a semicircle projection

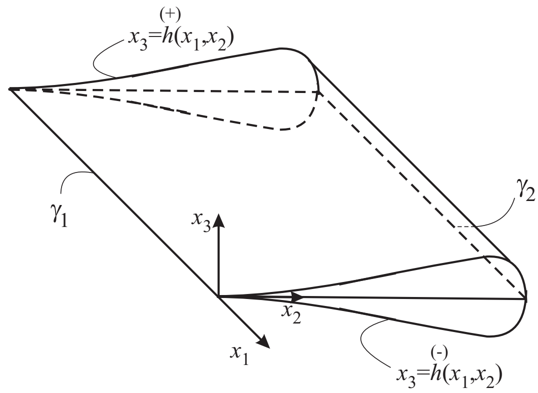


Fig.22. A cusped plate with sharp $\gamma_1$ and blunt $\gamma_2$ edges, $\gamma_0 = \gamma_1 \cup \gamma_2$

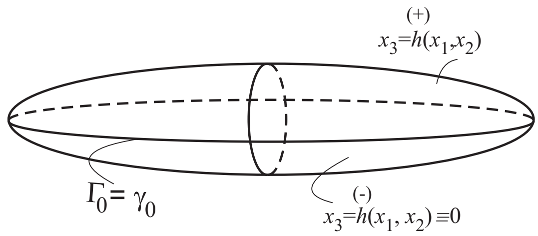


Fig.23. A blunt cusped plate with the edge $\gamma_0$

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \ (x_1, x_2) \in \omega, \ \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2) \right\},$$

where $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell $\overline{\Omega} := \Omega \cup \partial\Omega$ (see Figures 1-18, where typical cross-sections of prismatic shells with an angle $\varphi$ between tangents $\overset{(+)}{T}$ and $\overset{(-)}{T}$ are given and Figures 19-23); $\gamma = \partial\omega$ and $\partial\Omega$ denote boundaries of $\omega$ and $\Omega$, respectively; $\mathbb{R}^n$ is an $n$-dimensional Euclidian space.

In what follows we assume that

$$\overset{(\pm)}{h}(x_1, x_2) \in C^2(\omega) \cap C(\bar{\omega}),^1$$

and

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \begin{cases} > 0 & \text{for} \quad (x_1, x_2) \in \omega, \\ \geq 0 & \text{for} \quad (x_1, x_2) \in \partial\omega \end{cases}$$

is the thickness of the prismatic shell $\overline{\Omega}$ at the points $(x_1, x_2) \in \bar{\omega} = \omega \cup \partial\omega$. $\max\{2h\}$ is essentially less than characteristic dimensions of $\omega$. Let

$$2\overline{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2).$$

In the symmetric case of the prismatic shells, i.e., when

$$\overset{(-)}{h}(x_1, x_2) = -\overset{(+)}{h}(x_1, x_2), \quad \text{i.e.,} \quad 2\overline{h}(x_1, x_2) = 0,$$

we have to do with plates of variable thickness $2h(x_1, x_2)$ and a middle-plane $\omega$ (see Figures 22, 23). Prismatic shells are called cusped ones if a set $\gamma_0$, consisting of $(x_1, x_2) \in \partial\omega$ for which $2h(x_1, x_2) = 0$, is not empty. For such prismatic shells $\partial\Omega$ may be non-Lipschitz boundary (see Fig. 22)

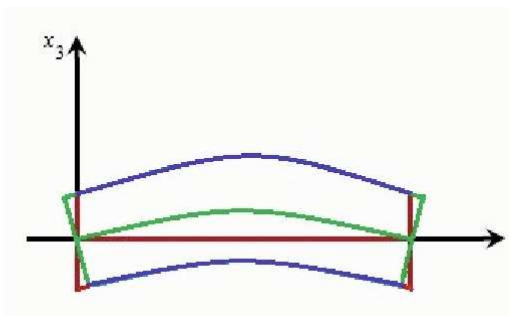


Fig.24. Comparison of cross-sections of prismatic and standard shells

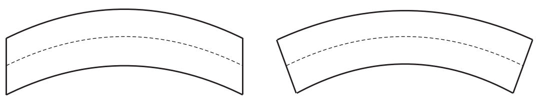

Fig.25. Cross-sections of a prismatic (left) and a standard shell with the same mid-surface

Distinctions between the prismatic shell of constant thickness and the standard shell of constant thickness are shown on Figures 24 and 25. The lateral boundary of the standard shell is orthogonal to the middle surface of the shell, while the lateral

[1] $C(\bar{\omega})$ denotes a class of continuous on $\bar{\omega}$ functions; $C^2(\omega)$ denotes a class of twice continuously dofferentiable functions with respect to $x_1$, $x_2$, $(x_1, x_2) \in \omega$.

boundary of the prismatic shell is orthogonal to the projection of the prismatic shell on $x_3 = 0$.

In what follows $X_{ij}$ and $e_{ij}$ are the stress and strain tensors, respectively, $u_i$ are the displacements, $\Phi_i$ are the volume force components, $\rho$ is the density, $\lambda$ and $\mu$ are the Lamé constants, $\delta_{ij}$ is the Kroneker delta, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Moreover, repeated indices imply summation (Greek letters run from 1 to 2, and Latin letters run from 1 to 3, unless stated otherwise).

I.Vekua's hierarchical models for elastic prismatic shells are the mathematical models, which were introduced by I. Vekua [1, 2], and which were constructed by the multiplication of the basic equations of linear elasticity

**Motion Equations**

$$X_{ij,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad t > t_0, \quad i = 1, 2, 3;$$

**Generalized Hooke's law (isotropic case)**

$$X_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}, \quad i, j = 1, 2, 3, \quad \theta := e_{ii};$$

**Kinematic Relations**

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3,$$

by Legendre polynomials $P_l(ax_3 - b)$, $l = 0, 1, 2, \ldots$, where

$$a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \quad b(x_1, x_2) := \frac{\overline{h}(x_1, x_2)}{h(x_1, x_2)},$$

and then integration with respect to $x_3$ within the limits $\overset{(-)}{h}(x_1, x_2)$ and $\overset{(+)}{h}(x_1, x_2)$. By these calculations in Vekua's first version on upper and lower face surfaces stress-vectors are assumed as prescribed, while values of the displacements are calculated there from their (displacements') Fourier-Legendre series expansions on the segment $x_3 \in \left[\overset{(-)}{h}(x_1, x_2), \overset{(+)}{h}(x_1, x_2)\right]$ and vice versa in his second version. So, we get the equivalent infinite system of relations with respect to the so called $l$-th order moments

$$\begin{aligned}\Big(X_{ijl},\, e_{ijl},\, u_{il}\Big)(x_1, x_2, t) \quad := \quad & \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Big(X_{ij},\, e_{ij},\, u_i\Big)(x_1, x_2, x_3, t) \\ \times \quad & P_l(ax_3 - b)\; dx_3. \end{aligned} \qquad (1)$$

Then, having followed the usual procedure used in the theory of elasticity, we get an equivalent infinite system with respect to the $l$-th order moments $u_{il}$. After this if we assume that the moments whose subscripts, indicating order of moments are greater than $N$ equal zero and consider only the first $N + 1$ equations (for every $i = 1, 2, 3$)

in the obtained infinite system of equations with respect to the $l$-th order moments $u_{il}$ we obtain the $N-$th order approximation (hierarchical model) governing system with respect to $\overset{N}{u}_{il}$ (roughly speaking $\overset{N}{u}_{il}$ is an "approximate value" of $u_{il}$).

In the zero approximation of I.Vekua's hierarchical models of shallow prismatic shells the governing system has the form

$$\mu\big[(hv_{\alpha 0,\beta})_{,\alpha} + (hv_{\beta 0,\alpha})_{,\alpha}\big] + \lambda(hv_{\gamma 0,\gamma})_{,\beta} = -\overset{0}{X}_{\beta} + \rho h\ddot{v}_{\beta 0}, \quad \beta = 1,2, \tag{2}$$

$$\mu(hv_{30,\alpha})_{,\alpha} = -\overset{0}{X}_3 + \rho h\ddot{v}_{30}, \tag{3}$$

where $v_{k0} := \frac{u_{k0}}{h}$, $k = 1,2,3$, are unknown so called weighted "moments" of displacements,

$$\begin{aligned}
\overset{0}{X}_j &:= \overset{(+)}{\sigma}_{3j} - \overset{(+)}{\sigma}_{\alpha j}\overset{(+)}{h}_{,\alpha} + (-1)^r\Big[-\overset{(-)}{\sigma}_{3j} + \overset{(-)}{\sigma}_{\alpha j}\overset{(-)}{h}_{,\alpha}\Big] + \Phi_{j0} \\
&= Q_{\overset{(+)}{n}j}\sqrt{1 + \Big(\overset{(+)}{h}_{,1}\Big)^2 + \Big(\overset{(+)}{h}_{,2}\Big)^2} \\
&+(-1)^r Q_{\overset{(-)}{n}j}\sqrt{1 + \Big(\overset{(-)}{h}_{,1}\Big)^2 + \Big(\overset{(-)}{h}_{,2}\Big)^2} + \Phi_{j0}, \quad j = 1,2,3, \quad r = \overline{0,N}.
\end{aligned}$$

By $Q_{\overset{(+)}{n}j}$ and $Q_{\overset{(-)}{n}j}$ components of the stress vectors acting on the upper and lower surfaces, respectively, are denoted. By $\Phi_{j0}$ we denote the zero order moments of the components of the volume forces.

When on the face surfaces displacements are prescribed for $N = 0$ approximation the governing system has the following form

$$\begin{aligned}
&\mu\left[(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}\right]_{,\beta} + \lambda\left[(hv_{\gamma 0})_{,\gamma}\right]_{,\alpha} \\
&-(\ln h)_{,\beta}\left\{\lambda\delta_{\alpha\beta}(hv_{\gamma 0})_{,\gamma} + \mu\left[(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha}\right]\right\} \\
&+2\mu\,\Psi_{\alpha\beta,\beta}(x_1,x_2,t) + \lambda\,\Psi_{kk,\alpha}(x_1,x_2,t) \\
&-(\ln h)_{,\beta}\left[\lambda\delta_{\alpha\beta}\Psi_{kk}(x_1,x_2,t) + 2\mu\,\Psi_{\alpha\beta}(x_1,x_2,t)\right] \\
&+\Phi_{\alpha 0}(x_1,x_2,t) = \rho h\ddot{v}_{\alpha 0}, \quad \alpha = 1,2;
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\mu(hv_{30})_{,\beta\beta} - (\ln h)_{,\beta}\,\mu(hv_{30})_{,\beta} + 2\mu\,\Psi_{3\beta,\beta}(x_1,x_2,t) \\
&-2\mu(\ln h)_{,\beta}\,\Psi_{3\beta}(x_1,x_2,t) + \Phi_{30}(x_1,x_2,t) = \rho h\ddot{v}_{30},
\end{aligned} \tag{5}$$

where

$$\Psi_{33}(x_1,x_2,t) := u_3(x_1,x_2,\overset{(+)}{h},t) - u_3(x_1,x_2,\overset{(-)}{h},t),$$

$$2\Psi_{i\beta}(x_1,x_2,t) := u_i\left(x_1,x_2,\overset{(-)}{h},t\right)\overset{(-)}{h}_{,\beta} - u_i\left(x_1,x_2,\overset{(+)}{h},t\right)\overset{(+)}{h}_{,\beta}$$

$$+\begin{cases} -u_\beta\left(x_1,x_2,\overset{(+)}{h},t\right)\overset{(+)}{h}_{,\alpha} + u_\beta\left(x_1,x_2,\overset{(-)}{h},t\right)\overset{(-)}{h}_{,\alpha} & \text{for } i = \alpha, \quad \alpha = 1,2; \\ u_\beta\left(x_1,x_2,\overset{(+)}{h},t\right) - u_\beta\left(x_1,x_2,\overset{(-)}{h},t\right) & \text{for } i = 3. \end{cases}$$

Let now

$$2h = h_0 x_2^\kappa, \quad h_0,\, \kappa = const > 0, \quad x_2 \geq 0. \tag{6}$$

In the static case, for deflections from (3) we get

$$\mu(h v_{30,\alpha})_{,\alpha} = -\overset{0}{X}_3, \quad x_2 \geq 0.$$

Assuming that $u_{30}$ depends only on $x_2$ (i.e., we consider cylindrical deformation)

$$(x_2^\kappa v_{30,\alpha})_{,\alpha} = -2\mu^{-1} h_0^{-1} \overset{0}{X}_3,$$

whence,

$$v_{30,22} + \frac{\kappa}{x_2} v_{30,2} = -2\mu^{-1} h_0^{-1} x_2^{-\kappa} \overset{0}{X}_3, \tag{7}$$

The general solution of the latter has the form

$$v_{30} = 2(\kappa-1)^{-1} \mu^{-1} h_0^{-1} \int\limits_{x_2^0}^{x_2} \left( x_2^{1-\kappa} - \xi^{1-\kappa} \right) \overset{0}{X}_3(\xi) d\xi \tag{8}$$

$$+ c_1 x_2^{1-\kappa} + c_2, \quad \kappa \neq 1, \quad c_1,\, c_2 = const;$$

$$v_{30} = 2\mu^{-1} h_0^{-1} \int\limits_{x_2^0}^{x_2} (\ln \xi - \ln x_2) \overset{0}{X}_3(\xi) d\xi + c_1 \ln x_2 + c_2, \tag{9}$$

$$\kappa = 1, \quad x_2^0 \in ]0, l[, \quad c_1,\, c_2 = const.$$

Hence, under the evident assumption on $\overset{0}{X}_3$, it is easy to conclude that on the boundary $x_2 = 0$ in the class of bounded functions displacement $\frac{v_{30}}{2}$ can be prescribed when $0 \leq \kappa < 1$, while for $\kappa \geq 1$ the boundary $x_2 = 0$ should be freed from the boundary condition (BC). Boundary value problems (BVPs) and initial boundary value problems (IBVPs) for the system (2), (3) and in the general $N$-th approximation are studied sufficiently well in the case of cusped prismatic shells (see [3-18]). For prismatic cusped shells the system (4), (5) is not studied at all. If we consider the case (6) for equation (5), it is easy to see that the systems (2), (3) and (4), (5) qualitatively differ from each other.

In the static case, from (5) we get

$$\begin{gathered} \mu(h v_{30})_{,\beta\beta} - (\ln h)_{,\beta}\, \mu(h v_{30})_{,\beta} + 2\mu\, \Psi_{3\beta,\beta}(x_1, x_2) \\ -2\mu(\ln h)_{,\beta}\, \Psi_{3\beta}(x_1, x_2) + \Phi_{30}(x_1, x_2) = 0, \end{gathered} \tag{10}$$

i.e.,

$$\begin{gathered} h v_{30,\beta\beta} + 2h_{,\beta} v_{30,\beta} + h_{,\beta\beta} v_{30} - (\ln h)_{,\beta} (h v_{30,\beta} + h_{,\beta} v_{30}) \\ = -2\Psi_{3\beta,\beta} + 2(\ln h)_{,\beta}\, \Psi_{3\beta} - \mu^{-1} \Phi_{30}. \end{gathered}$$

Therefore,

$$\begin{gathered} h v_{30,\beta\beta} + h_{,\beta} v_{30,\beta} + [h_{,\beta\beta} - (\ln h)_{,\beta} h_{,\beta}] v_{30} \\ = -2\Psi_{3\beta,\beta} + 2(\ln h)_{,\beta}\, \Psi_{3\beta} - \mu^{-1} \Phi_{30}. \end{gathered} \tag{11}$$

Assuming that $\Phi_{30} \in C(\bar{\omega})$, $u_\alpha \equiv 0$, $\alpha = 1, 2$, and $v_{30}$ depends only on $x_2$, taking into account (6) and dividing the equality (11) on $\frac{h_0}{2}x_2^{\kappa-2}$, from (11) we get

$$x_2^2 v_{30,22} + \kappa x_2 \nu_{30,2} - \kappa v_{30} = 2h_0^{-1}[-2x_2^{2-\kappa}\Psi_{32,2} + 2\kappa x_2^{1-\kappa}\Psi_{32} - \mu^{-1}x_2^{2-\kappa}\Phi_{30}]. \tag{12}$$

The last equation is well-known Euler equation and, since $\kappa+1 > 0$, its general solution has the form

$$\begin{aligned} v_{30} = \frac{u_{30}}{\frac{h_0}{2}x_2^\kappa} &= -2(\kappa+1)^{-1}h_0^{-1}\int\limits_{x_2^0}^{x_2}(x_2\xi^{-\kappa} - x_2^{-\kappa}\xi) \\ &\times\Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}\Big]d\xi \\ &+2h_0^{-1}c_1 x_2 + 2h_0^{-1}c_2 x_2^{-\kappa}, \quad 0 < x_2^0 < L, \end{aligned} \tag{13}$$

where $c_1$ and $c_2$ are arbitrary constants.

The last results can also be achieved as follows: if we rewrite (5) with respect to $u_{30}$

$$\mu(u_{30}),_{\beta\beta} - (\ln h),_\beta\, \mu(u_{30}),_\beta = -2\mu\Psi_{3\beta,\beta} + 2\mu(\ln h),_\beta\, \Psi_{3\beta} - \Phi_{30}$$

and take into account (6) we get

$$u_{30,22} - \frac{\kappa}{x_2}u_{30,2} = -2\Psi_{32,2} + 2\frac{\kappa}{x_2}\Psi_{32} - \mu^{-1}\Phi_{30}. \tag{14}$$

Its general solution has the form

$$u_{30} = -(\kappa+1)^{-1}\int\limits_{x_2^0}^{x_2}(x_2^{1+\kappa}\xi^{-\kappa} - \xi)\Psi(\xi)d\xi + c_1 x_2^{1+\kappa} + c_2, \tag{15}$$

where

$$\Psi(\xi) := 2\Psi_{32,2}(\xi) - \frac{2\kappa}{\xi}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi).$$

Hence, since in the zero approximation it is assumed that

$$u_i(x_1, x_2, x_3, t) = \frac{1}{2h}u_{i0}(x_1, x_2, t) =: \frac{1}{2}v_{i0}(x_1, x_2, t),$$

we obtain (13).

Note that, in view of (15),

$$\begin{aligned} X_{320}(x_2) &= \mu(hv_{30}),_2 + 2\mu\Psi_{32}(x_2) = \mu u_{30,2} + 2\mu\Psi_{32}(x_2) \\ &= \mu c_1(\kappa+1)x_2^\kappa - \mu x_2^\kappa\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi + 2\mu\Psi_{32}(x_2). \end{aligned}$$

Clearly, if $\overset{(+)}{h}(x_2) = h_1 x_2^\kappa$, $\overset{(-)}{h}(x_2) = h_2 x_2^\kappa$, $h_1, h_2 = const$, $h_1 > h_2$ $(h_0 := h_1 - h_2)$,

$$\lim_{x_2\to 0} X_{320}(x_2) = \frac{\mu}{\kappa}\lim_{x_2\to 0}\Big(2x_2\Psi_{32,2} - 2\kappa\Psi_{32} + \mu^{-1}x_2\Phi_{30}\Big) + 2\mu\lim_{x_2\to 0}\Psi_{32}$$

$$= \frac{2\mu}{\kappa}\lim_{x_2\to 0} x_2\Psi_{32,2}$$

$$= \frac{2\mu}{\kappa}\begin{cases} 0 \ \text{ if } \ \kappa > 1 \ \text{ and } \ u_3;\ u_{3,2} = O(1), \ \ x_2 \to 0; \\ \kappa(\kappa-1)(\overset{(-)}{d}_1 h_2 - \overset{(+)}{d}_1 h_1) \ \text{ if } \ 0 < \kappa \le 1 \ \text{ and } \ u_{3,2} = O(1), \ \ x_2 \to 0, \\ \quad u_3(x_1, x_2, \overset{(\pm)}{h}(x_2)) = \overset{(\pm)}{\psi}(x_1,x_2)x_2^{1-\kappa}, \ \ \lim\limits_{x_2\to 0}\overset{(\pm)}{\psi}(x_1,x_2) = \overset{(\pm)}{d}_1; \\ O^*(x_2^{\kappa-1}) = d_0\kappa(\kappa-1)x_2^{\kappa-1}, \ \ x \to 0, \ \text{ if } 0 < \kappa < 1 \ \text{ and } \ u_{3,2} = O(1), \\ \quad \lim\limits_{x_2\to 0} u_3(x_1, x_2, \overset{(\pm)}{h}(x_2)) = d_0 \ne 0. \end{cases}$$

Since under assumption of boundedness of 3D $u_3$, all its moments (because of boundedness of the integrand in (1) and tending of integration limits to 0 as $x_2 \to 0$) vanish at cusped edge, in particular

$$u_{30}(0) = 0$$

should be fulfilled. It will be achieved if in (15) we take

$$c_2 = -(\kappa+1)^{-1}\int\limits_{x_2^0}^{0} \xi\Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi)\Big]d\xi, \tag{16}$$

This is easily seen because of

$$\lim_{x_2\to 0} x_2^{\kappa+1}\int\limits_{x_2^0}^{x_2} \xi^{-\kappa}\Big[2\Psi_{32,2}(\xi) - 2\kappa\xi^{-1}\Psi_{32}(\xi) + \mu^{-1}\Phi_{30}(\xi)\Big]d\xi = 0.$$

If (16) is violated, then, by virtue of (15), taking into account the last limit, $u_{30}(0) \ne 0$ and from (13) it follows that $v_{30}$ is unbounded as $x_2 \to 0$, which contradicts the boundedness of $u_3$.

Applying the general representation (13) of $v_{30}$, let us analyze the setting of bending BVPs on $[0, L]$.

If $c_2$ has the form (16), then, by virtue of (13), (15),

$$\lim_{x_2\to 0} v_{30} = \lim_{x_2\to 0}\frac{u_{30}}{\frac{h_0}{2}x_2^\kappa} = \lim_{x_2\to 0}\frac{2\Big\{c_2 - (\kappa+1)^{-1}\int\limits_{x_2^0}^{x_2}(x_2^{\kappa+1}\xi^{-\kappa} - \xi)\Psi(\xi)d\xi\Big\}}{h_0 x_2^\kappa}$$

$$= \lim_{x_2\to 0}\frac{-2(\kappa+1)^{-1}(x_2^{\kappa+1}x_2^{-\kappa} - x_2)\Psi(x_2) - x_2^\kappa\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi}{\kappa h_0 x_2^{\kappa-1}}$$

$$= \lim_{x_2\to 0}\left[\frac{0}{\kappa h_0 x_2^{\kappa-1}} - \frac{x_2}{\kappa h_0}\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi\right].$$

Therefore,

$$\lim_{x_2\to 0} v_{30}(x_2) = 0 - \frac{1}{\kappa h_0}\lim_{x_2\to 0} x_2\int\limits_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi \tag{17}$$

if $\Psi$ is such a function that there exists the last limit.

Thus,

$$\begin{aligned} v_{30}(x_2) &= 2h_0^{-1}c_1 x_2 + 2h_0^{-1}(\kappa+1)^{-1}x_2^{-\kappa} \\ &\times\Big\{\int\limits_0^{x_2}\xi\Psi(\xi)d\xi - x_2^{\kappa+1}\int_{x_2^0}^{x_2}\xi^{-\kappa}\Psi(\xi)d\xi\Big\} \end{aligned} \tag{18}$$

is bounded near $x_2 = 0$ under some restrictions on $\Psi$ and choosing appropriately $c_1$ we can satisfy either BC

$$v_{30}(L) = v_{30}^L \tag{19}$$

or BC

$$X_{320}(L) = \mu(hv_{30})_{,2}\,|_{x_2=L} + 2\mu\Psi_{32}(L) = \mu u_{30,2}|_{x_2=L} + 2\mu\Psi_{32}(L) = X_{320}^L. \tag{20}$$

Namely, correspondingly,

$$c_1 = 2^{-1}h_0 L^{-1}v_{30}^L - (\kappa+1)^{-1}\Big\{L^{-\kappa-1}\int\limits_0^L\xi\Psi(\xi)d\xi - \int\limits_{x_2^0}^L\xi^{-\kappa}\Psi(\xi)d\xi\Big\} \tag{21}$$

and

$$c_1 = (1+\kappa)^{-1}\mu^{-1}L^{-\kappa}X_{320}^L + (1+\kappa)^{-1}\int\limits_{x_2^0}^L\xi^{-\kappa}\Psi(\xi)d\xi - 2(1+\kappa)^{-1}L^{-\kappa}\Psi_{32}(L). \tag{22}$$

Under some restrictions on $\Psi$ from boundedness of $u_3$ there follows boundedness of $v_{30} \in C^2(]0,L[)\cap C(]0,L])$, which given by (18) with (21) is a unique solution of the BVP (12), (19), when $\kappa > 0$. Thus, actually we have solved the Keldysh type BVP.

If volume forces and the displacement on the face surfaces are equal to zero, i.e., $\Phi_{30}\equiv 0$, $\Psi_{32}\equiv 0$, it is natural to set BC on the edge $x_2 = 0$ as

$$v_{30}(0) = 0 \tag{23}$$

since the last follows from (17).

(18) with (21) gives a unique solution of BVP $(12)_0$[2], (23), (19), of the form

$$v_{30}(x_2) = \frac{\nu_{30}^L}{L}x_2.$$

[2] $(12)_0$ means homogeneous equation (12).

This BVP is not correct since by inhomogeneous BC (23) it will not be solvable. In order to get correct BVP, BC (23) should be replaced by boundedness of the solution, so, we again arrive at the correct Keldysh type BVP.

As it follows from the general representation (8), (9) of the solution $v_{30}$ of equation (7) analogous BVP for equation (7) (the model, when stress vectors on the face surfaces are prescribed) is uniquely solvable only if $0 \leq \kappa < 1$, moreover, the non-homogenous BC (23) is admissible in contrast to the previous model (see (12)). When $\kappa \geq 1$ under condition of boundedness of $v_{30}$ it is possible to satisfy only one BC.

**Remark.** In the case under consideration under assumption of boundedness of 3D displacements it follows from (14), (15) that

$$u_{30,22} - \frac{\kappa}{x_2} u_{30} = 0, \tag{24}$$

$$u_{30} = c_1 x_2^{1+\kappa} + c_2.$$

Evidently, BVP (24),

$$u_{30}(0) = u_{30}^0, \quad u_{30}(L) = u_{30}^L,$$

is uniquely solvable provided that $u_{30}^0$ and $u_{30}^L$ are assumed to be known. From 3D BVP in displacements $u_{30}^L$ is known, while $u_{30}^0 = 0$ and cannot be arbitrarily prescribed. If nevertheless we find $u_{30}^0$ to be assigned, displacement $v_{30}$ will become unbounded as $x_2 \to 0$, which will be nonsense since $\infty$ cannot be approximate value of 0. While zero can be considered as approximate boundary value since we consider small deflections. In such sense we could consider (23) as BC when $\Psi_{32} \not\equiv 0$.

Now, let us analyze the possibility of prescribing the stress vectors on the prismatic shell edges.

Since

$$X_{320}(x_2) = \mu u_{30,2} = \frac{1}{2}\mu h_0 (x_2^{\kappa} v_{30})_{,2}\,,$$

by virtue of (15),

$$X_{320}(x_2) = \mu(1+\kappa) c_1 x_2^{\kappa}.$$

The last means that

$$X_{320}(0) = 0.$$

Hence, $X_{320}$ can be arbitrarily prescribed only at non-cusped edge $x_2 = L$.

For the homogeneous equation $(12)_0$ besides the BC (23) we can set the BC (20), i.e., on the edge $x_2 = L$ the stress vector is given.

(18) with (22) gives a unique solution of BVP $(12)_0$, (23), (20) of the form

$$v_{30} = \frac{2X_{320}^L}{\mu h_0(\kappa+1)L^{\kappa}} x_2.$$

Considering (8) we easily conclude that analogous BVP $(7)_0$,(23),(20), is uniquely solvable for the model (7), provided that $0 \leq \kappa < 1$ (in this case also the non-homogenous BC (19) is admissible). For $\kappa \geq 1$ from (8), (9) it is easily seen that only bounded solution is a constant and if $X_{320}^L \neq 0$, BVP $(7)_0$, (23), (20), is not solvable. If $X^L_{320} = 0$, then a solution of BVP $(7)_0$, nonhomogeneous (23), $(20)_0$ is a constant given at $x_2 = 0$.

**Conclusion.** In the case of the first model [see (7)] the Dirichlet problem is correct for $0 < \kappa < 1$ and the Keldysh problem is correct for $\kappa \geq 1$, while in the case of the second model [see (12)] the Keldysh problem is correct for $\kappa > 0$.

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# ON EFFECTS OF CONSTANT DELAY PERTURBATION AND THE DISCONTINUOUS INITIAL CONDITION IN VARIATION FORMULAS OF SOLUTION OF DELAY CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATION

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**Abstract**. Variation formulas of solution (variation formulas) are proved for a controlled nonlinear delay functional-differential equation with the discontinuous initial condition, under perturbations of initial moment, delay parameter, initial vector, initial and control functions. The effects of delay perturbation and the discontinuous initial condition are discovered in the variation formulas. The discontinuity of the initial condition means that the values of the initial function and the trajectory, generally, do not coincide at the initial moment.

**Keywords and phrases**: Controlled delay functional-differential equation; variation formula of solution; effect of delay perturbation; effect of the discontinuous initial condition.

**AMS subject classification (2000):** 34K99.

## 1. Introduction

Linear representation of the main part of the increment of a solution of an equation with respect to perturbations is called the variation formula. The variation formula allows one to construct an approximate solution of the perturbed equation in an analytical form on the one hand, and in the theory of optimal control plays the basic role in proving the necessary conditions of optimality [1-11], on the other. Variation formulas for various classes of functional-differential equations without perturbation of delay are given in [6,10,12-14].Variation formulas for delay functional-differential equations with the continuous and discontinuous initial condition taking into consideration constant delay perturbation are proved in [15] and [16], respectively. Variation formulas for controlled delay functional-differential equations with the continuous initial condition taking into consideration constant delay perturbation are proved in [17]. In this paper the variation formulas are proved for the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_0), u_0(t))$$

with the discontinuous initial condition

$$x(t) = \varphi_0(t), t \in [t_{00} - \tau_0, t_{00}), x(t_{00}) = x_{00}$$

under perturbations of initial moment $t_{00}$, delay parameter $\tau_0$, initial vector $x_{00}$, initial function $\varphi_0(t)$ and control function $u_0(t)$.

## 2. Notation and auxiliary assertions

Let $R^n_x$ be the $n$-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where $T$ means transpose; suppose that $O \subset R^n_x$ and $V \subset R^r_u$ are open sets. Let the $n$-dimensional function $f(t,x,y,u)$ satisfy the following conditions: for almost all $t \in I = [a,b]$, the function $f(t,\cdot) : O^2 \times V \to R^n_x$ is continuously differentiable; for any $(x,y,u) \in O^2 \times V$, the functions $f(t,x,y,u), f_x(\cdot), f_y(\cdot), f_u(\cdot)$ are measurable on $I$; for arbitrary compacts $K \subset O, U \subset V$ there exists a function $m_{K,U}(\cdot) \in L(I,[0,\infty))$, such that for any $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t,x,y,u) | + | f_x(\cdot) | + | f_y(\cdot) | + | f_u(\cdot) | \leq m_{K,U}(t).$$

Further, let $0 < \tau_1 < \tau_2$ be given numbers; Let $E_\varphi$ be the space of continuous functions $\varphi : I_1 \to R^n_x$, where $I_1 = [\hat{\tau}, b], \hat{\tau} = a - \tau_2; \Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ is a set of initial functions; let $E_u$ be the space of bounded measurable functions $u : I \to R^r_u$ and let $\Omega = \{u \in E_u : clu(I) \subset V\}$ be a set of control functions, where $u(I) = \{u(t) : t \in I\}$ and $clu(I)$ is the closer of the set $u(I)$.

To each element $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda = (a,b) \times (\tau_1, \tau_2) \times O \times \Phi \times \Omega$ we assign the controlled delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t)) \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0. \tag{2.2}$$

The condition (2.2) is said to be the discontinuous initial condition since generally $x(t_0) \neq \varphi(t_0)$.

**Definition 2.1.** Let $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to $\mu$ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times R^n_x \times E_\varphi \times E_u$ we introduce the set of variations:

$$V = \{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta u) \in E_\mu - \mu_0 : | \delta t_0 | \leq \alpha, | \delta\tau | \leq \alpha, | \delta x_0 | \leq \alpha,$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, | \lambda_i | \leq \alpha, i = \overline{1,k}\}, \tag{2.3}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta u_i \in E_u - u_0, i = \overline{1,k}$ are fixed functions ; $\alpha > 0$ is a fixed number.

**Lemma 2.1.** *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, x_0, \varphi_0, u_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$ and let $K_0 \subset O$ and $U_0 \subset V$ be compact sets containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10,}])$ and $clu_0(I)$, respectively. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover,*

$$\begin{cases} x(t; \mu_0 + \varepsilon\delta\mu) \in K_0, t \in [\hat{\tau}, t_{10} + \delta_1], \\ u_0(t) + \varepsilon\delta u(t) \in U_0, t \in I. \end{cases} \tag{2.4}$$

This lemma is a result of Theorem 5.3 in [18, p.111].

**Remark 2.1.** Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10}+\delta_1]$. Therefore, in the sequel the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10}+\delta_1]$.

Lemma 2.1 allows one to define the increment of the solution $x_0(t)=x(t;\mu_0)$ :

$$\begin{cases}\Delta x(t)=\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\\ (t,\varepsilon,\delta\mu)\in[\hat{\tau},t_{10}+\delta_1]\times[0,\varepsilon_1]\times V.\end{cases} \tag{2.5}$$

**Lemma 2.2.** *Let the following conditions hold:*

2.1. $t_{00}+\tau_0<t_{10}$;

2.2. *the function* $\varphi_0(t), t\in I_1$ *is absolutely continuous and the function* $\dot{\varphi}_0(t)$ *is bounded;*

2.3. *there exist compact sets* $K_0\subset O$ *and* $U_0\subset V$ *containing neighborhoods of sets* $\varphi_0(J_1)\cup x_0([t_{00},t_{10}])$ *and* $clu_0(I)$, *respectively, such that the function* $f(t,x,y,u)$ *is bounded on the set* $I\times K_0^2\times U_0$;

2.4. *there exists the limit*

$$\lim_{w\to w_0} f(w,u_0(t))=f_0^-, w=(t,x,y)\in(a,t_{00}]\times O^2,$$

*where* $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_0))$. *Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that*

$$\max_{t\in[t_{00},t_{10}+\delta_2]}|\Delta x(t)|\le O(\varepsilon\delta\mu)^3 \tag{2.6}$$

*for arbitrary* $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$, *where* $V^-=\{\delta\mu\in V:\delta t_0\le 0,\delta\tau\le 0\}$. *Moreover,*

$$\Delta x(t_{00})=\varepsilon\Big[\delta x_0-f_0^-\delta t_0\Big]+o(\varepsilon\delta\mu). \tag{2.7}$$

**Lemma 2.3.** *Let the conditions 2.1-2.3 of Lemma 2.2 hold, and there exists the limit*

$$\lim_{w\to w_0} f(w,u_0(t))=f_0^+, w=(t,x,y)\in[t_{00},b)\times O^2.$$

*Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that the inequality*

$$\max_{t\in[t_0,t_{10}+\delta_2]}|\Delta x(t)|\le O(\varepsilon\delta\mu), \tag{2.8}$$

*is valid for arbitrary* $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$, *where* $t_0=t_{00}+\varepsilon\delta t_0, V^+=\{\delta\mu\in V:\delta t_0\ge 0,\delta\tau\ge 0\}$. *Moreover,*

$$\Delta x(t_0)=\varepsilon\Big[\delta x_0-f_0^+\delta t_0\Big]+o(\varepsilon\delta\mu). \tag{2.9}$$

Lemmas 2.2 and 2.3 can be proved in analogy to Lemma 2.3 (see [15]).

[3]Here and throughout the following, the symbols $O(t;\varepsilon\delta\mu), o(t;\varepsilon\delta\mu)$ stand for quantities (scalar or vector) that have the corresponding order of smallness with respect to $\varepsilon$ uniformly with respect to $t$ and $\delta\mu$.

**Lemma 2.4.** *Let the conditions of Lemma 2.2 hold. Then*

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)=\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[|\Delta x(t-\tau)-\Delta x(t-\tau_0)|\Big]dt$$

$$\le o(\varepsilon\delta\mu), \tag{2.10}$$

*for arbitrary* $(\varepsilon,\delta\mu)\in(0,\varepsilon_2]\times V^-$, *where* $\tau=\tau_0+\varepsilon\delta\tau,\zeta(\cdot)\in L(J,[0,\infty))$, *about* $\varepsilon_2$ *and* $\delta_2$ *see Lemma 2.2.*

**Proof.** It is obvious that $t-\tau\ge t_{00}$ and $t-\tau_0\ge t_{00}$ for $t\in[t_{00}+\tau_0,t_{10}+\delta_2]$. Therefore,

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\le\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[\int_{t-\tau_0}^{t-\tau}|\dot{\Delta}x(\xi)|d\xi\Big]dt$$

$$=\int_{t_{00}+\tau_0}^{t_{10}+\delta_2}\zeta(t)\Big[\int_{t-\tau_0}^{t-\tau}\theta(\xi;\varepsilon\delta\mu)d\xi\Big]dt,$$

where

$$\theta(\xi;\varepsilon\delta\mu)=|f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))$$

$$-f[\xi]|,f[\xi]=f(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))$$

see (2.5).

a) Let $t_{00}+2\tau_0\le t_{10}$ and $\varepsilon_2\in(0,\varepsilon_1]$ be so small that $t_0+2\tau>t_{00}+\tau_0,\forall(\varepsilon,\delta\mu)\in(0,\varepsilon_2]\times V^-$, then we have

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)=\alpha(t_{00}+\tau_0,t_0+2\tau;\varepsilon\delta\mu)+\alpha(t_0+2\tau,t_{00}+2\tau_0;\varepsilon\delta\mu)$$

$$+\alpha(t_{00}+2\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu).$$

The function $\theta(\xi;\varepsilon\delta\mu)$ is bounded (see the condition 2.3 of Lemma 2.2), therefore

$$\alpha(t_0+2\tau,t_{00}+2\tau_0;\varepsilon\delta\mu)\le o(\varepsilon\delta\mu).$$

We note that there exists $L(\cdot)\in L(I,[0,\infty))$ such that

$$|f(t,x_1,y_1,u_1)-f(t,x_2,y_2,u_2)|\le L(t)\Big(|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\Big),$$

$$t\in I,(x_i,y_i,u_i)\in K_0^2\times U_0,i=1,2,3.$$

It is not difficult to see that

$$\alpha(t_{00}+\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu)\le\alpha_1(t_{00}+\tau_0,t_0+2\tau;\varepsilon\delta\mu)+o(\varepsilon\delta\mu)$$

$$+\alpha_1(t_{00}+2\tau_0,t_{10}+\delta_2;\varepsilon\delta\mu), \tag{2.11}$$

where

$$\alpha_1(t',t'';\varepsilon\delta\mu)=\int_{t'}^{t''}\zeta(t)\alpha_2(t;\varepsilon\delta\mu)dt,\alpha_2(t;\varepsilon\delta\mu)$$

$$= \int_{t-\tau_0}^{t-\tau} L(\xi)\Big\{|\Delta x(\xi)| + |x_0(\xi-\tau) - x_0(\xi-\tau_0)| + |\Delta x(\xi-\tau)| + \varepsilon|\delta u(\xi)|\Big\}d\xi.$$

If $t \in [t_{00}+\tau_0, t_0+2\tau]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}, \xi-\tau \leq t_0, \xi-\tau_0 \leq t_0$. Therefore,

$$|\Delta x(\xi)| \leq O(\varepsilon\delta\mu), |x_0(\xi-\tau) - x_0(\xi-\tau_0)| = |\varphi_0(\xi-\tau) - \varphi_0(\xi-\tau_0)|$$

$$= \int_{t-\tau_0}^{t-\tau} |\dot{\varphi}_0(\xi)|d\xi = O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)| = \varepsilon|\delta\varphi(\xi-\tau)|. \tag{2.12}$$

Thus,

$$\alpha_1(t_{00}+\tau_0, t_0+2\tau; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu). \tag{2.13}$$

Further, if $t \in [t_{00}+2\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}+\tau_0, \xi-\tau \geq t_{00}, \xi-\tau_0 \geq t_{00}$. Therefore,

$$|\Delta x(\xi)| \leq O(\varepsilon\delta\mu), |x_0(\xi-\tau) - x_0(\xi-\tau_0)| = \int_{t-\tau_0}^{t-\tau} |\dot{x}_0(\xi)|d\xi$$

$$= \int_{t-\tau_0}^{t-\tau} |f[\xi]|d\xi = O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)| = O(\varepsilon\delta\mu).$$

Consequently,

$$\alpha_1(t_{00}+2\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu). \tag{2.14}$$

From (2.11) by virtue (2.13) and (2.14) we obtain (2.10).

b) Let $t_{00}+2\tau_0 > t_{10}$ and, $\varepsilon_2$ and $\delta_2$ be so small that $t_{00}+2\tau > t_{10}+\delta_2$. It is clear that

$$\alpha(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq \alpha_1(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu).$$

If $t \in [t_{00}+\tau_0, t_{10}+\delta_2]$ and $\xi \in [t-\tau_0, t-\tau]$ then $\xi \geq t_{00}, \xi-\tau \leq t_0, \xi-\tau_0 \leq t_0$. Therefore,

$$\alpha_1(t_{00}+\tau_0, t_{10}+\delta_2; \varepsilon\delta\mu) \leq o(\varepsilon\delta\mu)$$

(see (2.12)). Lemma 2.4 is proved.

**Lemma 2.5.** *Let the conditions of Lemma 2.3 hold. Then*

$$\int_{t_0+\tau}^{t_{10}+\delta_2} \zeta(t)\Big[|\Delta x(t-\tau) - \Delta x(t-\tau_0)|\Big]dt \leq o(\varepsilon\delta\mu).$$

*for arbitrary* $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^+$.

This Lemma can be proved in analogy to Lemma 2.4.

### 3. Formulation of main results

**Theorem 3.1.** *Let the conditions of Lemma 2.2 hold.Moreover, there exits the limit*

$$\lim_{(w_1,w_2)\to(w_{01},w_{02})} [f(w_1, u_0(t)) - f(w_2, u_0(t))] = f_1^-, w_i \in (a, t_{00}+\tau_0] \times O^2, i = 1, 2,$$

*where*

$$w_{01} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), x_{00}), w_{02} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00})).$$

*Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3.1}$$

*for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-$ *and*

$$\delta x(t; \delta\mu) = -\Big\{Y(t_{00}; t)f_0^- + Y(t_{00} + \tau_0; t)f_1^-\Big\}\delta t_0$$
$$-Y(t_{00} + \tau_0; t)f_1^-\delta\tau + \beta(t; \delta\mu), \tag{3.2}$$

where

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta x_0 + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t)f_y[\xi + \tau_0]\delta\varphi(\xi)d\xi$$
$$-\Big\{\int_{t_{00}}^{t} Y(\xi; t)f_y[\xi]\dot{x}_0(\xi - \tau_0)d\xi\Big\}\delta\tau + \int_{t_{00}}^{t} Y(\xi; t)f_u[\xi]\delta u(\xi)d\xi. \tag{3.3}$$

*Here* $Y(\xi; t)$ *is the* $n \times n$*-matrix function satisfying the linear functional-differential equation with advanced argument*

$$Y_\xi(\xi; t) = -Y(\xi; t)f_x[\xi] - Y(\xi + \tau_0; t)f_y[\xi + \tau_0], \xi \in [t_{00}, t], \tag{3.4}$$

*and the condition*

$$Y(\xi; t) = \begin{cases} H \; for \; \xi = t, \\ \Theta \; for \; \xi > t, \end{cases} \tag{3.5}$$

$$f_x = \frac{\partial}{\partial x}f, f_x[\xi] = f_x(\xi, x_0(\xi), x_0(\xi - \tau_0), u_0(\xi));$$

$H$ *is the identity matrix and* $\Theta$ *is the zero matrix.*

**Some comments.** The expression (3.2) is called the variation formula.

**c1.** Theorem 3.1 corresponds to the case when the variations at the points $t_{00}$ and $\tau_0$ are performed simultaneously on the left.

**c2.** The summand

$$-\Big\{Y(t_{00} + \tau_0; t)f_1^- + \int_{t_{00}}^{t} Y(\xi; t)f_y[\xi]\dot{x}_0(\xi - \tau_0)d\xi\Big\}\delta\tau$$

in formula (3.2) (see also (3.3)) is the effect of perturbation of the delay $\tau_0$.

**c3.** The expression

$$-\Big\{Y(t_{00}; t)f_0^- + Y(t_{00} + \tau_0; t)f_1^-\Big\}\delta t_0$$

is the effect of discontinuous initial condition (2.2) and perturbation of the initial moment $t_{00}$.

**c4.** The expression

$$Y(t_{00}; t)\delta x_0 + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi + \tau_0; t)f_y[\xi + \tau_0]\delta\varphi(\xi)d\xi + \int_{t_{00}}^{t} Y(\xi; t)f_u[\xi]\delta u(\xi)d\xi$$

in formula (3.3) is the effect of perturbations of the initial vector $x_0$, initial $\varphi_0(t)$ and control $u_0(t)$ functions.
**c5.** The variation formula allows one to obtain an approximate solution of the perturbed functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_0-\varepsilon\delta\tau), u_0(t)+\varepsilon\delta u(t))$$

with the perturbed initial condition

$$x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), t \in [\hat{\tau}, t_{00}+\varepsilon\delta t_0), x(t_{00}) = x_{00} + \varepsilon\delta x_0.$$

In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ from (3.1) it follows

$$x(t; \mu_0 + \varepsilon\delta\mu) \approx x_0(t) + \varepsilon\delta x(t; \delta\mu)$$

(see (2.5)).
**c6.** Finally we note that the variation formula which is proved in the present work doesn't follows from the formula proved in [15].

**Theorem 3.2.** *Let the conditions of Lemma 2.3 hold.Moreover, there exits the limit*

$$\lim_{(w_1,w_2)\to(w_{01},w_{02})}[f(w_1,u_0(t)) - f(w_2,u_0(t))] = f_1^+, w_i \in [t_{00}+\tau_0, b) \times O^2, i = 1, 2.$$

*Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times [0, \varepsilon_2] \times V^+$, formula (3.1) holds and*

$$\delta x(t; \delta\mu) = -\Big\{Y(t_{00}; t)f_0^+ + Y(t_{00}+\tau_0; t)f_1^+\Big\}\delta t_0$$
$$-Y(t_{00}+\tau_0; t)f_1^+\delta\tau + \beta(t; \delta\mu). \tag{3.6}$$

Theorem 3.2 corresponds to the case when the variations at the points $t_{00}$ and $\tau_0$ are performed simultaneously on the right. Theorems 3.1 and 3.2 are proved by a scheme given in [10].

## 4. Proof of Theorem 3.1

Here and in what follows we shall assume that $t_0 = t_{00} + \varepsilon\delta t_0, \tau = \tau_0 + \varepsilon\delta\tau, \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), u(t) = u_0(t) + \varepsilon\delta u(t)$. Let $\varepsilon_2 \in (0, \varepsilon_1]$ be so small (see Lemma 2.2) that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2] \times V^-$ the following inequalities hold

$$t_{00} - \tau \le t_0, t_0 + \tau \ge t_{00}.$$

The function $\Delta x(t)$ (see (2.5)) satisfies the equation

$$\begin{aligned}\dot{\Delta}x(t) &= f(t, x_0(t)+\Delta x(t), x_0(t-\tau)+\Delta x(t-\tau), u(t)) - f[t]\\ &= f_x[t]\Delta x(t) + f_y[t]\Delta x(t-\tau_0) + \varepsilon f_u[t]\delta u(t) + r(t; \varepsilon\delta\mu)\end{aligned} \tag{4.1}$$

on the interval $[t_{00}, t_{10} + \delta_2]$, where

$$r(t;\varepsilon\delta\mu) = f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u(t)) - f[t]$$
$$-f_x[t]\Delta x(t) - f_y[t]\Delta x(t-\tau_0) - \varepsilon f_u[t]\delta u(t), \tag{4.2}$$

By using the Cauchy formula ([10], p.21), one can represent the solution of equation (4.1) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_u[\xi]\delta u(\xi)d\xi$$
$$+\sum_{i=0}^{1} R_i(t;t_{00},\varepsilon\delta\mu), t \in [t_{00}, t_{10}+\delta_2], \tag{4.3}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\Delta x(\xi)d\xi, \\ \\ R_1(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t) r(\xi;\varepsilon\delta\mu)d\xi \end{cases} \tag{4.4}$$

and $Y(\xi;t)$ is the matrix function satisfying equation (3.4) and condition (3.5).

Let a number $\delta_2 \in (0, \delta_1]$ be so small that $t_{00} + \tau_0 < t_{10} - \delta_2$. The function $Y(\xi;t)$ is continuous on the set

$$\Pi = \{(\xi,t):\ \xi \in [t_{00}, t_{00}+\tau_0], t \in [t_{10}-\delta_2, t_{10}+\delta_2]\}$$

([10], Lemma 2.1.7). Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big[\delta x_0 - f_0^- \delta t_0\Big] + o(t;\varepsilon\delta\mu) \tag{4.5}$$

(see (2.7)). One can readily see that

$$R_0(t;t_{00},\varepsilon\delta\mu) = \varepsilon \int_{t_{00}-\tau_0}^{t_0} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$
$$+\int_{t_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\Delta x(\xi)d\xi = \varepsilon \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$
$$+\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t) f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu), \tag{4.6}$$

where

$$o(t;\varepsilon\delta\mu) = -\varepsilon \int_{t_0}^{t_{00}} Y(\xi+\tau_0;t) f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi.$$

For $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$ we have

$$R_1(t;t_{00},\varepsilon\delta\mu) = \sum_{i=1}^{3} \alpha_i(t;\varepsilon\delta\mu) \tag{4.7}$$

$$\alpha_1(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}r_1(\xi;t,\varepsilon\delta\mu)d\xi,\alpha_2(t;\varepsilon\delta\mu)=\int_{t_0+\tau}^{t_{00}+\tau_0}r_1(\xi;t,\varepsilon\delta\mu)d\xi,$$

$$\alpha_3(t;\varepsilon\delta\mu)=\int_{t_{00}+\tau_0}^{t}r_1(\xi;t,\varepsilon\delta\mu)d\xi,r_1(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)r(\xi;\varepsilon\delta\mu).$$

We introduce the notations:

$$f[t;s,\varepsilon\delta\mu]=f(t,x_0(t)+s\Delta x(t),x_0(t-\tau_0)+s\{x_0(t-\tau)-x_0(t-\tau_0)$$

$$+\Delta x(t-\tau)\},u_0(t)+s\varepsilon\delta u(t)),\sigma(t;s,\varepsilon\delta\mu)=f_x[t;s,\varepsilon\delta\mu]-f_x[t],$$

$$\rho(t;s,\varepsilon\delta\mu)=f_y[t;s,\varepsilon\delta\mu]-f_y[t],\vartheta(t;s,\varepsilon\delta\mu)=f_u[t;s,\varepsilon\delta\mu]-f_u[t].$$

It is easy to see that

$$f(t,x_0(t)+\Delta x(t),x_0(t-\tau)+\Delta x(t-\tau),u_0(t)+\varepsilon\delta u(t))-f[t]$$

$$=\int_0^1\frac{d}{ds}f[t;s,\varepsilon\delta\mu]ds=\int_0^1\Big\{f_x[t;s,\varepsilon\delta\mu]\Delta x(t)+f_y[t;s,\varepsilon\delta\mu]\{x_0(t-\tau)$$

$$-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon f_u[t;s,\varepsilon\delta\mu]\delta u(t)\Big\}ds$$

$$=\Big[\int_0^1\sigma(t;s,\varepsilon\delta\mu)ds\Big]\Delta x(t)+\Big[\int_0^1\rho(t;s,\varepsilon\delta\mu)ds\Big]\{x_0(t-\tau)$$

$$-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon\Big[\int_0^1\vartheta(t;s,\varepsilon\delta\mu)ds\Big]\delta u(t)$$

$$+f_x[t]\Delta x(t)+f_y[t]\{x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)\}+\varepsilon f_u[t]\delta u(t).$$

On account of the last relation we have

$$\alpha_1(t;\varepsilon\delta\mu)=\sum_{i=1}^{5}\alpha_{1i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{11}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi,$$

$$\sigma_1(\xi;\varepsilon\delta\mu)=\int_0^1\sigma(\xi;s,\varepsilon\delta\mu)ds,\ \alpha_{12}(t;\varepsilon\delta\mu)$$

$$=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)\{x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\}d\xi,$$

$$\rho_1(\xi;\varepsilon\delta\mu)=\int_0^1\rho(\xi;s,\varepsilon\delta\mu)ds,\alpha_{13}(t;\varepsilon\delta\mu)$$
$$=\varepsilon\int_{t_{00}}^{t_0+\tau}Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)d\xi,\ \vartheta_1(\xi;\varepsilon\delta\mu)$$
$$=\int_0^1\vartheta(\xi;s,\varepsilon\delta\mu)ds,\ \alpha_{14}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\{\Delta x(\xi-\tau)$$
$$-\Delta x(\xi-\tau_0)\}d\xi,\ \alpha_{15}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\{x_0(\xi-\tau)$$
$$-x_0(\xi-\tau_0)\}d\xi$$

For $\xi\in[t_{00},t_0+\tau]$ we have

$$\begin{cases}|\Delta x(\xi)|\le O(\varepsilon\delta\mu),\Delta x(\xi-\tau)=\varepsilon\delta\varphi(\xi-\tau),\\ \Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)=\varepsilon[\delta\varphi(\xi-\tau)-\delta\varphi(\xi-\tau_0)]\\ x_0(\xi-\tau)-x_0(\xi-\tau_0)=\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)\end{cases}\tag{4.8}$$

(see (4.2)). The function $\varphi_0(t)$ is absolutely continuous, therefore for each fixed Lebesgue point $\xi\in(t_{00},t_{00}+\tau_0)$ of function $\dot\varphi_0(\xi-\tau_0)$ we get

$$\varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)=\int_\xi^{\xi-\varepsilon\delta\tau}\dot\varphi_0(s-\tau_0)ds$$
$$=-\varepsilon\dot\varphi_0(\xi-\tau_0)\delta\tau+\gamma(\xi;\varepsilon\delta\mu),\tag{4.9}$$

with

$$\lim_{\varepsilon\to0}\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}=0\ \text{ uniformly for } \delta\mu\in V^-.\tag{4.10}$$

Thus, (4.9) and (4.10) are valid for almost all points of the interval $(t_{00},t_{00}+\tau_0)$. From (4.9) taking into account boundedness of the function $\dot\varphi_0(t)$ it follows

$$|\ \varphi_0(\xi-\tau)-\varphi_0(\xi-\tau_0)\ |\le O(\varepsilon\delta\mu)\text{ and }\left|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\right|\le const.\tag{4.11}$$

Consequently, for $\alpha_{1i}(t;\varepsilon\delta\mu)),i=\overline{1,4}$ we have

$$\begin{cases}|\ \alpha_{11}(t;\varepsilon\delta\mu)|\le\|\ Y\ \|\ O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),\\ |\ \alpha_{12}(t;\varepsilon\delta\mu)|\le\|\ Y\ \|\ O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu),\\ |\ \alpha_{13}(t;\varepsilon\delta\mu)|\le\varepsilon\ \|\ Y\ \|\ \vartheta_2(\varepsilon\delta\mu),\\ |\ \alpha_{14}(t;\varepsilon\delta\mu)|\le o(\varepsilon\delta\mu),\end{cases}$$

$$\alpha_{15}(t;\varepsilon\delta\mu)=\gamma_1(t;\varepsilon\delta\mu)-\varepsilon\Big[\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\dot\varphi_0(\xi-\tau_0)d\xi\Big]dt,$$

(see (4.8),(4.9),(4.11)). Here

$$\sigma_2(\varepsilon\delta\mu)=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_x(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_x(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt,\rho_2(\varepsilon\delta\mu)$$

$$=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_y(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_y(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt,$$

$$\vartheta_2(\varepsilon\delta\mu)=\int_{t_{00}}^{t_{00}+\tau_0}\Big[\int_0^1\Big|f_u(t,x_0(t)+s\Delta x(t),\varphi_0(t-\tau_0)+s(\varphi_0(t-\tau)-\varphi_0(t-\tau_0))$$

$$+s\delta\varphi(t-\tau),u_0(t)+s\varepsilon\delta u(s))-f_u(t,x_0(t),\varphi_0(t-\tau_0),u_0(t))\Big|ds\Big]dt$$

$$\|Y\|=\sup\Big\{|Y(\xi;t)|:(\xi,t)\in\Pi\Big\},\hat{\gamma}(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\gamma(\xi;\varepsilon\delta\mu)d\xi.$$

Obviously,

$$\Big|\frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\leq\|Y\|\int_{t_{00}}^{t_{00}+\tau_0}|f_y[\xi]|\,\Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|d\xi.$$

By the Lebesguer theorem on passing to the limit under the integral sign, we have

$$\lim_{\varepsilon\to0}\sigma_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to0}\rho_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to0}\vartheta_2(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to0}\Big|\frac{\hat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},t_{00}+\tau_0]\times V^-$ (see (4.10)). Thus,

$$\alpha_{1i}(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),i=\overline{1,4}; \tag{4.12}$$

and

$$\alpha_{15}(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_0+\tau}Y(\xi;t)f_y[\xi]\dot{\varphi}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu).$$

Further,

$$\varepsilon\Big[\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{\varphi}_0(\xi-\tau_0)d\xi\Big]\delta\tau=o(t;\varepsilon\delta\mu),$$

$$\dot{x}_0(\xi-\tau_0)=\dot{\varphi}_0(\xi-\tau_0),\xi\in[t_{00},t_{00}+\tau_0],$$

therefore,

$$\alpha_{15}(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu). \tag{4.13}$$

On the basis of (4.12) and (4.13) we obtain

$$\alpha_1(t;\varepsilon\delta\mu)=-\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\varepsilon\delta\mu). \tag{4.14}$$

Now let us transform $\alpha_2(t;\varepsilon\delta\mu)$. We have

$$\alpha_2(t;\varepsilon\delta\mu)=\sum_{i=1}^{4}\alpha_{2i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{21}(\varepsilon\delta\mu)=\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)\Big[f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))$$

$$-f[\xi]\Big]d\xi,\alpha_{22}(t;\varepsilon\delta\mu)=-\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_x[\xi]\Delta x(\xi)d\xi,\alpha_{23}(t;\varepsilon\delta\mu)$$

$$=-\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi,\alpha_{24}(t;\varepsilon\delta\mu)=-\varepsilon\int_{t_0+\tau}^{t_{00}+\tau_0}Y(\xi;t)f_u[\xi]\Delta\delta u(\xi)d\xi.$$

If $\xi\in[t_0+\tau,t_{00}+\tau_0]$ then

$$|\Delta x(\xi)|\le O(\varepsilon\delta\mu),x_0(\xi-\tau)+\Delta x(\xi-\tau)=x(\xi-\tau;\mu_0+\varepsilon\delta\mu)$$

$$=x_{00}+\varepsilon\delta x_0+\int_{t_0}^{\xi-\tau}f(s,x(s;\mu_0+\varepsilon\delta\mu),x(s-\tau;\mu_0+\varepsilon\delta\mu),u_0(s)+\varepsilon\delta u(s))ds$$

therefore

$$\lim_{\varepsilon\to 0}(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau))=(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00})=w_{02}.$$

Moreover,

$$\lim_{\varepsilon\to 0}(\xi,x_0(\xi),x_0(\xi-\tau_0))=(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}))=w_{01}.$$

Thus,

$$\lim_{\varepsilon\to 0}\Big[f(\xi,x_0(\xi)+\Delta x(\xi),x_0(\xi-\tau)+\Delta x(\xi-\tau),u_0(\xi)+\varepsilon\delta u(\xi))-f[\xi]\Big]$$

$$=\lim_{(w_1,w_2)\to(w_{01},w_{02})}[f(w_1,u_0(t))-f(w_2,u_0(t))]=f_1^-,w_i\in(a,t_{00}+\tau_0]\times O^2,i=1,2,$$

Since the function $Y(\xi;t)$ is continuous on the set $\Pi$, therefore

$$\alpha_{21}(t;\varepsilon\delta\mu)=-\varepsilon Y(t_{00}+\tau_0;t)f_1^-(\delta t_0+\delta\tau)+o(t;\varepsilon\delta\mu).$$

Further, for $\xi\in[t_0+\tau,t_0+\tau_0]$ we have

$$\Delta x(\xi-\tau_0)=\varepsilon\delta\varphi(\xi-\tau_0),$$

therefore

$$\alpha_{23}(t;\varepsilon\delta\mu)=-\varepsilon\int_{t_0+\tau}^{t_0+\tau_0}Y(\xi;t)f_y[\xi]\delta\varphi(\xi-\tau_0)d\xi$$

$$-\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi = -\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi$$
$$+o(t;\varepsilon\delta\mu).$$

Obviously,

$$\alpha_{22}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), \alpha_{24}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu).$$

Finally, for $\alpha_2(t;\varepsilon\delta\mu)$ we get

$$\alpha_2(t;\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^-(\delta t_0+\delta\tau) - \int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi$$
$$+o(t;\varepsilon\delta\mu). \tag{4.15}$$

It remains to estimate $\alpha_3(t;\varepsilon\delta\mu)$. We have

$$\alpha_3(t;\varepsilon\delta\mu) = \sum_{i=1}^{5} \alpha_{3i}(t;\varepsilon\delta\mu),\ \alpha_{31}(t;\varepsilon\delta\mu)$$

where

$$\alpha_{31}(t;\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi, \alpha_{32}(t;\varepsilon\delta\mu)$$
$$= \int_{t_{00}+\tau_0}^{t} Y(\xi;t)\rho_1(\xi;\varepsilon\delta\mu)\{x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\}d\xi,$$
$$\alpha_{33}(t;\varepsilon\delta\mu) = \varepsilon\int_{t_{00}+\tau_0}^{t} Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)d\xi,\ \alpha_{34}(t;\varepsilon\delta\mu)$$
$$= \int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\{\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)\}d\xi,\ \alpha_{35}(t;\varepsilon\delta\mu)$$
$$= \int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\{x_0(\xi-\tau)-x_0(\xi-\tau_0)\}d\xi.$$

For $\xi\in[t_{00}+\tau_0, t_{10}+\delta_2]$ we have

$$|\Delta x(\xi)| \le O(\varepsilon\delta\mu), |\Delta x(\xi-\tau)| \le O(\varepsilon\delta\mu), \tag{4.16}$$

(see (4.2)). For each fixed Lebesgue point $\xi\in(t_{00}+\tau_0, t_{10}+\delta_2)$ of function $\dot x_0(\xi-\tau_0)$ we get

$$x_0(\xi-\tau)-x_0(\xi-\tau_0) = \int_{\xi}^{\xi-\varepsilon\delta\tau} \dot x_0(s-\tau_0)ds$$
$$= -\varepsilon\dot x_0(\xi-\tau_0)\delta\tau + \gamma_1(\xi;\varepsilon\delta\mu), \tag{4.17}$$

with

$$\lim_{\varepsilon\to 0}\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \ \text{ uniformly for } \delta\mu\in V^-. \tag{4.18}$$

Thus, (4.17) and (4.18) are valid for almost all points of the interval $(t_{00}+\tau_0, t_{10}+\delta_2)$. From (4.17) taking into account boundedness of the function $f(t,x,y,u)$ it follows

$$| x_0(\xi-\tau) - x_0(\xi-\tau_0) | \leq O(\varepsilon\delta\mu) \text{ and } \left|\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| \leq const. \tag{4.19}$$

For $\alpha_{3i}(t;\varepsilon\delta\mu)), i=\overline{1,4}$ we have

$$\begin{cases} | \alpha_{31}(t;\varepsilon\delta\mu)| \leq \| Y \| \, O(\varepsilon\delta\mu)\sigma_3(\varepsilon\delta\mu), \\ | \alpha_{32}(t;\varepsilon\delta\mu)| \leq \| Y \| \, O(\varepsilon\delta\mu)\rho_3(\varepsilon\delta\mu), \\ | \alpha_{33}(t;\varepsilon\delta\mu)| \leq \varepsilon \, \| Y \| \, \vartheta_3(\varepsilon\delta\mu), \\ | \alpha_{34}(t;\varepsilon\delta\mu)| \leq o(\varepsilon\delta\mu), \end{cases}$$

(see (4.17),(4.19) and Lemma 2.4). Here

$$\sigma_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \sigma_1(\xi;\varepsilon\delta\mu)d\xi, \rho_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \rho_1(\xi;\varepsilon\delta\mu)d\xi,$$

$$\vartheta_3(\varepsilon\delta\mu) = \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} \vartheta_1(\xi;\varepsilon\delta\mu)d\xi.$$

Obviously,

$$\left|\frac{\hat{\gamma}_1(t;\varepsilon\delta\mu)}{\varepsilon} \leq \| Y \| \int_{t_{00}+\tau_0}^{t_{10}+\delta_2} |f_y[\xi]| \left|\frac{\gamma_1(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| d\xi.$$

By the Lebesguer theorem on passing to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0}\sigma_3(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\rho_3(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\vartheta_3(\varepsilon\delta\mu)=0,\ \lim_{\varepsilon\to 0}\left|\frac{\hat{\gamma}_1(t;\varepsilon\delta\mu)}{\varepsilon}\right|=0$$

uniformly for $(t,\delta\mu) \in [t_{00}, t_{10}+\delta_2] \times V^-$ (see (4.18)).
Thus,

$$\alpha_{3i}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), i=\overline{1,4}$$

and

$$\alpha_{35}(t;\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}+\tau_0}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu).$$

On the basis of last relations we get

$$\alpha_3(t;\varepsilon\delta\mu) = -\varepsilon\Big[\int_{t_{00}}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau + o(t;\varepsilon\delta\mu). \tag{4.20}$$

Taking into account (4.14),(4.15) and (4.20) the expression (4.7) can be represented in the form

$$R_1(t;t_{00},\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^-\delta t_0 - \varepsilon\Big[f_1^- + \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau$$

$$-\int_{t_0+\tau_0}^{t_{00}+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \tag{4.21}$$

Finally, from (4.3) by virtue of (4.6) and (4.21) we obtain (3.1), where $\delta x(t;\delta\mu)$ has the form (3.2).

## 5. Proof of Theorem 3.2

The function $\Delta x(t)$ satisfies equation (4.1) on the interval $[t_0, t_{10}+\delta_2]$. By using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0) + \varepsilon\int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi + \sum_{i=0}^{1} R_i(t;t_0,\varepsilon\delta\mu), \tag{5.1}$$

(see (4.4)). Let a number $\delta_2\in(0,\delta_1]$ be so small that $t_{00}+\tau_0 < t_{10}-\delta_2$. The matrix function $Y(\xi;t)$ is continuous on $\Pi$, therefore

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big[\delta x_0 - f^{+}\delta t_0\Big] + o(t;\varepsilon\delta\mu) \tag{5.2}$$

(see (2.8)).
Now let us transform $R_0(t;t_0,\varepsilon\delta\mu)$. It is not difficult to see that

$$R_0(t;t_0,\varepsilon\delta\mu) = \varepsilon\int_{t_0-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\Delta x(\xi)d\xi = \varepsilon\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}+\tau_0}^{t_0+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \tag{5.3}$$

In a similar way, for $t\in[t_{10}-\delta_2, t_{10}+\delta_2]$ one can prove

$$R_1(t;t_0,\varepsilon\delta\mu) = -\varepsilon Y(t_{00}+\tau_0;t)f_1^{+}\delta t_0 - \varepsilon\Big[f_1^{+} + \int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau$$

$$-\int_{t_{00}+\tau_0}^{t_0+\tau_0} Y(\xi;t)f_y[\xi]\Delta x(\xi-\tau_0)d\xi + o(t;\varepsilon\delta\mu). \tag{5.4}$$

Finally, we note that

$$\varepsilon\int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi = \varepsilon\int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi + o(t;\varepsilon\delta\mu). \tag{5.5}$$

for $t\in[t_{10}-\delta_2, t_{10}+\delta_2]$.

Taking into account (5.2)-(5.5), from (5.1), we obtain (3.1), where $\delta x(t;\varepsilon\delta\mu)$ has form (3.6).

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# ON THE UNIQUENESS OF THE SOLUTION OF AN INVERSE PROBLEM OF THE POTENTIALLY THEORY IN A THREE-DIMENSIONAL SPACE

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**Abstract**. In the present paper we consider the inverse problem for a volume potential. First we consider piecewise-smooth simply-connected domains and after that smooth simply-connected domains in a three-dimensional space.

**Keywords and phrases**: Inverse problem, potential, Keldish theorem, strictly locally convex.

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The solution of an inverse problem of the potential theory is of great theoretical and practical importance. The practical application of inverse problems is so significant that they are regarded as topical problems of modern mathematical analysis.

The uniqueness of the solution of an inverse problem in the class of star domains of constant density was for the first time proved P.S. Novikov [1].

In the present paper we consider the inverse problem for a volume potential.First we consider piecewise-smooth simply-connected domains and after that smooth simply-connected domains in a three-dimensional space.

Let us define volume potentials and simple-layer potentials.

$$V^f(x) = \int_{\Omega} \Gamma(x,y) f(y) dS_y, \quad U^{\psi}(x) = \int_{\partial\Omega} \Gamma(x,y)\psi(y) dS_y,$$

where $\Omega$ is a bounded piecewise-smooth domain, $f \in C(\partial\Omega)$, $\psi \in C(\partial\Omega)$, $\Gamma(x,y) = |x-y|^{-1}$. We denote by $\Omega_\infty$ the simply-connected component of $R^3 - \overline{\Omega}$ which contains a point at infinity, and by $\emptyset$ an empty set. $C_k$, $k = 1,2,3,\dots$ are positive constants.

**Definition 1.** Let $Q$ be a simply-connected bounded piecewise-smooth domain from $R^3$. We will set the domain $Q$ is strictly convex if for any points $z_1 \in \overline{Q}$, $z_2 \in \overline{Q}$ an interval point of a segment $\overline{z_1 z_2}$ is an interval point for the domain $Q$.

**Definition 2.** Let $\Omega$ be a simply-connected bounded piecewise-smooth domain from $R^3$, and each smooth part for $\partial\Omega$ belongs to class $C^{(1,\alpha)}$. We will say that the domain $\Omega$ is strictly convex at a point $x_0 \in \partial\Omega$ if for some neighborhood $\sigma = \{x : |x - x_0| < \varepsilon\}$ the intersection $\overline{\Omega} \cap \overline{\sigma}$ is a strict domain.

**Theorem 1.** *Let $\Omega_1$, $\Omega_2$ be a bounded simple-connected domain from $R^3$. Assume that there exists a smooth point $x_0 \in \partial\Omega_1$, $x_0 \notin \overline{\Omega}_2$, for which the domain $\Omega_1$ or $R^3 - \overline{\Omega}_1$ is strictly convex at a point $x_0$. Then the potentials*

$$v_1(x) = \int_{\Omega_1} \Gamma(x,y) dy, \quad v_2(x) = \int_{\Omega_2} \Gamma(x,y) dy \tag{1}$$

*do not coincide on $\Omega_\infty$ ( $\Omega = \Omega_1 \cup \Omega_2$, $\partial\Omega_i = \partial\overline{\Omega}_i$, $i = 1,2$.)*

**Proof.** Let us assume the contrary,i.e. $v_1(x) = v_2(x)$, $x \in \Omega_\infty$. Each smooth part $\partial\Omega_i$ $(i = 1, 2)$ belongs to $C^{(1,\alpha)}$. Denote $\sigma_1 = \{x : |x - x_0| < \varepsilon\} \cap \partial\Omega_1$, $x_0 = (x_1^0, x_2^0)$, $\sigma = \left\{x : |x - x_0| < \frac{\varepsilon}{2}\right\} \cap \partial\Omega_1$, $(\overline{\sigma}_1 \cap \overline{\Omega}_2 = \emptyset)$, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \subset S = \{x : |x| < r\}$. Rotate the coordinate system. After rotation the tangent plane at the point $x_0$ becomes parallel to the plane $x_2ox_3$. Pass the plan $P_1$ through the point $x_0$, for which $P_1 \perp x_2ox_3$, $P_1 \perp x_10x_2$. Let us consider the curve $l = P_1 \cap \sigma$, the equation of which has the form $x_3 = \tau(x_1, c_1)$, $x_2 = c_1 = const$, $x \in l$, $x_1^0 - \varepsilon_1 < x_1 < x_1^0$, $|\tau'(x_1^0, c)| = \infty$.

From equality (1) we obtain

$$\int\limits_{\Omega_1} U^\psi(x)dx = \int\limits_{\Omega_2} U^\psi(x)dx, \quad \psi \in C^{(1,\alpha)}(\partial S)$$

$$\int\limits_{\Omega_1} \frac{\partial U^\psi}{\partial x_3}dx = \int\limits_{\Omega_2} \frac{\partial U^\psi}{\partial x_3}dx \quad \psi \in C^{(1,\alpha)}(\partial S) \tag{2}$$

By the Green-Ostrogradski formula, from (2) we have

$$\int\limits_{\Omega_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x = \int\limits_{\Omega_2} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x \tag{3}$$

Let $\omega$ be the domain containing the surface $\sigma$.

For any $\psi \in C(\sigma)$ the following inequality is valid

$$||\psi||_{\{C^3(\sigma)^*\}} \le C_1||U^\psi||_{\{C_0^1(\omega)\}^*}, \tag{4}$$

where $C_0^1(\omega)$ - are finite functions from $C^1(\overline{\omega})$. It is obvious that the boundary function on $\sigma'$

$$g(x_1, x_2) \cdot x_3 = g(x_1x_2) \cdot \tau(x_1x_2)$$

$x_3 = \tau(x_1x_2)$ is the equation $\sigma' = \{(x_1x_2x_3) \in \sigma,\ x_3 < x_3^0\}$, $L_1 = \frac{\partial \delta_{x_1}}{\partial t} \cdot \delta_{x_2}$, $(x_1, x_2, x_3) \in \sigma'$, where $\delta_{x_1}, \delta_{x_2}$ are Dirac measures.

From this and the above reasoning we obtain for a ball $\left\{||U^\psi||_{\{C_0^1(\omega)\}^*} \le 1\right\}$ $\left(||\psi||_{\{C^3(\sigma)\}^*} \le C_1\right)$.

$$\frac{1}{C_1}\frac{\partial \delta_{x_1}}{\partial t} \times \delta_{x_2} \cdot x_3 \in \left\{||U^\psi||_{\{C_0^1(\omega)\}} \le 1\right\},\ U^{\psi_1}(x_1, x_2, x_3) = \frac{1}{C_1}\frac{\partial \delta_{x_1}}{\partial t} \times \delta_{x_2} \cdot x_3,\ (x_1x_2x_3) \in \sigma'.$$

$$\sup\left|\int\limits_{\sigma_1} U^\psi(x_1, x_2, x_3)\cos(\nu_x\hat{\ } x_3)dS_x\right| \ge C_2 \sup\left|\int\limits_{\sigma'} U^\psi(x_1, x_2)\tau(x_1x_2^0)dx_1dx_2\right| = \infty. \tag{5}$$

By virtue of (4) we have

$$\int\limits_{\sigma_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x = \int\limits_{\partial\Omega_2} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x - \int\limits_{\partial\Omega_1 - \sigma_1} U^\psi(x)\cos(\nu_x^\wedge x_3)dS_x \tag{6}$$

$$\sup_{\partial\Omega_2} |U^{\psi}(x)| \leq C_3, \quad \sup_{\partial\Omega_1 - \sigma_1} |U^{\psi}(x)| \leq C_4, \quad \psi \in \{C^3(\sigma)\}^*. \tag{7}$$

From (5), (6), (7) we obtain a contradiction.

Theorem 1 is proved.

**Theorem 2.** *Let $\Omega_1$ and $\Omega_2$ be simply connected bounded domains from the class $C^2$. Then the solution of an inverse problem is unique.*

**Proof.** Let us assume the contrary, i.e. that $v_1(x) = v_2(x)$, $x \in \Omega_\infty$, $\Omega_1 \neq \Omega_2$.

For the domains $\Omega_1$ and $\Omega_2$ the following alternatives are valid.

I) $\partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$ - is a finite number of smooth curves.

II) $\partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$ - contains some smooth surface $\sigma$.

Assume that alternative (I) is fulfilled. Consider the diameter of the domain $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$

$$d(\overline{\Omega}) = \max |x - y|, \quad x \in \overline{\Omega}, \ y \in \overline{\Omega}. \ \ d(\overline{\Omega}) = |x_0 - y_0|.$$

It is not difficult to see that in the neighborhood of point $x_0$ (or $y_0$) there exists a smooth point $z_0 \in \partial\overline{\Omega}_1$, $z_0 \notin \overline{\Omega}_2$, for which the domain $\Omega_1$ is strictly locally convex at a point $z_0$. Now is suffices to repeat the reasoning of Theorem 1.

Assume that alternative (II) is fulfilled. Consider the difference

$$(\overline{\Omega}_1 \cup \overline{\Omega}_2) - (\overline{\Omega}_1 \cap \overline{\Omega}_2) = \bigcup_1^N Q_i.$$

Since $\sigma \subset \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_\infty$, the complement of the closed set $F = \bigcup\limits_1^N \overline{Q}_i$ is a connected set (domain). Now assume that the potentials

$$v_1(x) = \int_{\Omega_1} \Gamma(x,y)dy, \quad v_2(x) = \int_{\Omega_2} \Gamma(x,y)dy$$

are considered on $\Omega_\infty$. Then we obtain

$$\int_{\overline{\Omega}_1 - (\overline{\Omega}_1 \cap \overline{\Omega}_2)} U^{\psi}(y)dy = \int_{\overline{\Omega}_2 - (\overline{\Omega}_1 \cap \overline{\Omega}_2)} U^{\psi}(y)dy \quad \psi \in C(\partial\Omega_\infty). \tag{8}$$

By virtue of Keldish theorem [2, ch. II] there exists a sequence of potentials for which we obtain

$$\lim_{n\to\infty} \int_{Q_1} [U^{\psi_n}(x) - 1]^2 dx = 0, \quad \lim_{n\to\infty} \int_{F - Q_1} [U^{\psi_n}(x)]^2 dx = 0.$$

$$\int_{Q_1} dy = 0, \quad |Q_1| = 0.$$

We have come to a contradiction. Theorem 2 is proved.

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# SOLUTION OF THE PROBLEMS OF ELASTOSTATICS FOR DOUBLE POROUS AN ELASTIC PLANE WITH A CIRCULAR HOLE. THE UNIQUENESS THEOREMS

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**Abstract**. In the present paper we solve explicitly, by means of absolutely and uniformly convergent series, the second boundary value problems of porous elastostatics for the plane with a circular hole.

**Keywords and phrases**: Porous medium, double porosity, boundary value problems, explicit solution, uniqueness theorems.

**AMS subject classification (2000):** 74F10; 74G10; 74G30.

## 1. Introduction

In the E.C. Aifantis theory of consolidation the elastic medium with double porosity is considered. For such a kind of media the problem is formulated under the following boundary conditions: the value of the displacement (or stress) vector and the value of pressures (or normal derivative pressures) of a liquid in pores are given. In the present work we solve explicitly, by means of absolutely and uniformly convergent series, the second boundary value problem of porous elastostatics for the plane with a circular hole. From the point of view of applications, very actual is the construction of solutions explicitly which allows one to perform numerical analysis of the problem under investigation.

## 2. Basic equations

We consider the plane $D$ with a circular hole. Let $R$ be the radius of the boundary $S$. Find a regular vector $U(u(x), p_1(x), p_2(x))$, satisfying in $D$ a system of equations [1,2]:

$$\begin{aligned} &\mu\Delta(u(x)) + (\lambda+\mu) graddiv(u(x)) = grad[\beta_1 p_1(x) + \beta_2 p_2(x)],\\ &(m_1\Delta - k)p_1(x) + kp_2(x) = 0, \qquad\qquad (1)\\ &kp_1(x) + (m_2\Delta - k)p_2(x) = 0, x \in D \end{aligned}$$

and on the circumference $S$ one of the following conditions:

$$\begin{aligned} &I. u(z) = f(z), \quad \partial_n p_1 = f_3(z), \quad \partial_n p_2(z) = f_4(z);\\ &II. P(\partial_z, n)U(z) = f(z), \quad p_1(z) = f_3(z), \quad p_2(z) = f_4(z), \end{aligned} \qquad (2)$$

where $\lambda, \mu, m_1, m_2, \beta_1, \beta_2$ are the known elastic and physical constants, $k, m_i > 0, i = 1,2[1,2]$; $u(x) = (u_1(x)), u_2(x))$ is the displacement of the point $x$;$n(z) = (n_1(z), n_2(z))$, $z = (z_1, x_2) \in S$, $p_1$ is the fluid pressure within the primary pores and $p_2$ is the fluid pressure within the secondary pores; $\Delta$ is the Laplace operator; $f(z) = (f_1(z), f_2(z))$, $f_3(z), f_4(z)$ are the given functions on the circumference $S$;

$$P(\partial_x, n)U(x) = T(\partial_x, n)u(x) - n(x)[\beta_1 p_1(x) + \beta_2 p_2(x)] \tag{3}$$

is the stress vector of the theory of poroelasticity; $T(\partial_x, n)u(x) = \mu\partial_n u(x) + \lambda n(x) div(u(x)) + \mu \sum_{i=1}^{\infty} n_i(x) grad u_i(x)$ is the stress vector of the theory of elasticity; $\partial_n = \dfrac{\partial}{\partial n}$; $\partial_k = \dfrac{\partial}{\partial x_k}$, $k = 1, 2$.

Vector $U(x)$ satisfies the following conditions at infinity:

$$U(x) = O(1), \quad \partial_k U(x) = O(1), \quad k = 1, 2. \tag{4}$$

We will study separately the following problems:

1. Find in a plane $D$ solution $u(x)$ of equation $(1)_1$, if on the circumference $S$ there are given the values: a) of the vector $u$ - problem $A_1$; b) of the vector $P(\partial_z, n)u(z)$ - problem $A_2$.

2. Find in a plane $D$ solutions $p_1(x)$ and $p_2(x)$ of the system of equations $(1)_2$ and $(1)_3$, if on the circumference $S$ there are given the values: a) of the function $p_1$ and the vector $p_2$ - problem $B_1$; b)of the derivates $\partial_n p_1(z)$ and $\partial_n p_2(z)$ - problem $B_2$.

Thus the above-formulated problems of poroelastostatics can be considered as a union of two problems: I - $(A_1, B_2)$ and II - $(A_2, B_1)$.

### 3. Uniqueness theorems

For regular solutions of equation $(1)_1$ and equations $(1)_2$ and $(1)_3$ Green's formulas:

$$\int_D [E(u(x), u(x)) - (\beta_1 p_1(x) + \beta_2 p_2)(x) div u(x)]dx = \int_S u(y) P(\partial_y, n(y)) d_y S; \tag{5}$$

$$\int_D [m_1 \mid grad p_1 \mid^2 + m_2 \mid grad p_2 \mid^2 + k(p_2 - p_1)^2]dx$$
$$= \int_S [m_1 p_1(y) \partial_n p_1(y) + m_2 p_2(y) \partial_n p_2(y)] d_y S \tag{6}$$

are valid, where

$$E(u, u) = (\lambda + \mu)(\partial_1 u_1 + \partial_2 u_2)^2 + \mu(\partial_1 u_1 - \partial_2 u_2)^2 + \mu(\partial_2 u_1 + \partial_1 u_2)^2$$

is a nonnegative quadratic form under the condition that $\lambda + \mu > 0$, $\mu > 0$ .

**Problems B**. Since $m_i$, $k > 0$, therefore in the case of homogeneous boundary conditions (2) the product $p_i \partial_n p_i$ vanishes. Let $p_1$ and $p_2$ be differences of two different solutions of problems $B_1$ and $B_2$. By virtue of equality (6), the following theorems are valid.

**Theorem 1.** *The difference of two arbitrary solutions of problem $B_1$ is equal to zero: $p_1(x) = p_2(x) = 0$.*

**Theorem 2.** *The difference of two arbitrary solutions of problem $B_2$ may differ only by an arbitrary constant $p_1(x) = p_2(x) = c$.*

**Problems A**. Let $(u', p_1', p_2')$ and $(u'', p_1'', p_2'')$ be two different solutions of any of the problems I, II. Then the differences $u = u' - u''$, $p_1 = p_1' - p_1''$ and $p_2 = p_2' - p_2''$ are the solutions of the corresponding homogeneous problems.

Taking into account Theorems 1 and 2, and formula (5), under the homogeneous boundary conditions for the problems $I$ and $II$, we obtain $E(u, u) = 0$. The solution of the above equation has the form

$$u_1(x) = -cx_2 + q_1, \quad u_2(x) = cx_1 + q_2, \tag{7}$$

where $c$, $q_1$ and $q_2$ are arbitrary constants.

Taking into account conditions (4) and formulas (7), we obtain:
$u_1(x) = u_2(x) = 0$ - for problem $A_1$;
$u_1(x) = q_1, \quad u_2(x) = q_2$ - for problem $A_2$;
The following theorems are valid.

**Theorem** 3. *The difference of two arbitrary solutions of problem $I$ is the vector $U(u_1(x), u_2(x), p_1(x), p_2(x))$, where $u_1 = u_2 = 0$, $p_1 = p_2 = c$;*

**Theorem** 4. *The difference of two arbitrary solutions of problem $II$ is the vector $U(u_1(x), u_2(x), p_1(x), p_2(x))$, where $u_1(x) = q_1, \quad u_2(x) = q_2$ and $p_1 = p_2 = 0$.*

## 4. Solutions of the problems

On the basis of the system $[(1)_2, (1)_3]$, we can write $m_1 m_2 \triangle(\triangle + \lambda_0^2) p_i = 0$, $i = 1, 2$. Solutions of these equations are represented in the form

$$p_1(x) = a_1\varphi_1(x) + a_2\varphi_2(x), \quad p_2(x) = a_3\varphi_1(x) + a_4\varphi_2(x), \tag{8}$$

where

$$\lambda_0^2 = -\frac{k(m_1 + m_2)}{m_1 m_2}, \quad a_1 = a_3 = \frac{2}{m_1 + m_2}, \quad a_2 = -\frac{m_1 - m_2}{m_1(m_1 + m_2)},$$

$$a_4 = -\frac{m_1 - m_2}{m_2(m_1 + m_2)}; \quad \triangle\varphi_1 = 0, \quad (\triangle + \lambda_0^2)\varphi_2 = 0,$$

Taking into account (8), we write

$$\beta_1 p_1 + \beta_2 p_2 = a\varphi_2 + b\varphi_1, \tag{9}$$

where

$$a = (\beta_1 + \beta_2)a_1, \quad b = \beta_1 a_2 + \beta_2 a_4. \tag{10}$$

**Problem B$_1$.** The functions $\varphi_1$ and $\varphi_2$ in formulas (8) are unknown. From the conditions (2), for problem $B_1$ we can write

$$\varphi_1(z) = \frac{d_1(z)}{d} \equiv \Omega_1(z), \quad \varphi_2(z) = \frac{d_2(z)}{d} \equiv \Omega_2(z), \quad z \in S, \tag{11}$$

where

$$d=a_1a_4-a_2^2,\quad d_1(z)=a_4f_3(z)-a_2f_4(z),\quad d_2(z)=a_1f_4(z)-a_2f_3(z).$$

Taking into account (11), for the harmonic function $\varphi_1(x)$ we have:

$$\varphi_1(x)=\sum_{m=0}^{\infty}\Big(\frac{R}{r}\Big)^m(A_m\cos m\psi+B_m\sin m\psi),\tag{12}$$

where

$$r^2=x_1^2+x_2^2,\ \ x=(x_1,x_2)=(r,\psi),\quad A_0=\frac{1}{2\pi}\int\limits_0^{2\pi}\Omega_1(\theta)d\theta,$$

$$A_m=\frac{1}{\pi}\int\limits_0^{2\pi}\Omega_1(\theta)\cos m\theta d\theta,\quad B_m=\frac{1}{\pi}\int\limits_0^{2\pi}\Omega_1(\theta)\sin m\theta d\theta.$$

Taking into account (8), the values in the plane of metaharmonic function $\varphi_2(x)$ can be represented as follows [3]:

$$\varphi_2(x)=K_0(\lambda_0 r)C_0+\sum_{m=1}^{\infty}K_m(\lambda_0 r)(C_m\cos m\psi+D_m\sin m\psi),\tag{13}$$

where $K_m(\lambda_0 r)$ is the modified Hancel ' s function of an imaginary argument,

$$C_m=\frac{1}{\pi}\int_0^{2\pi}\Omega_2(\theta)\cos m\theta d\theta,\ \ D_m=\frac{1}{\pi}\int_0^{2\pi}\Omega_2(\theta)\sin m\theta d\theta,\ \ m=0,1,\ldots.\tag{14}$$

Using now formulas (8), with regard to (12) and (13), we can find values of the functions $p_1(x)$ and $p_2(x)$.

**Problem B$_2$.** Taking into account formulas (8), the boundary conditions of problem $B_2$ can be rewritten as

$$\partial_R\varphi_1(z)=F_1(z),\quad \partial_R\varphi_2(z)=F_2(z),\quad z\in S,\tag{15}$$

where $F_1(z)=\dfrac{1}{d}[a_4f_3(z)-a_2f_4(z)]$, $\quad F_2(z)=\dfrac{1}{d}[a_1f_4(z)-a_2f_3(z)]$, $\quad \partial_R\equiv\partial_n$.

Then the harmonic function $\varphi_1(x)$ can be represented in the form of a series:

$$\varphi_1(x)=c_0-\sum_{m=1}^{\infty}\frac{R}{m}\left(\frac{R}{r}\right)^m(A_m\cos m\psi+B_m\sin m\psi),\tag{16}$$

where $c_0$ is an arbitrary constant, $A_m=\dfrac{1}{\pi}\int\limits_0^{2\pi}F_1(\theta)\cos m\theta d\theta$ and $B_m=\dfrac{1}{\pi}\int\limits_0^{2\pi}F_1(\theta)\sin m\varphi d\theta$.

Expanding the function $F_2(z)$ into Fourier series and substituting (13) into (15), we obtain the representation of the metaharmonic function $\varphi_2(x)$ in the plane in the form

$$\varphi_2(x) = \frac{1}{\lambda_0} \sum_{m=1}^{\infty} \frac{K_m(\lambda_0 r)}{K'_m(\lambda_0 R)} \, (\alpha_m \cos m\psi + \beta_m \sin m\psi), \tag{17}$$

where $\alpha_m$ and $\beta_m$ are the Fourier coefficients of the function $F_2(z)$,

$$K'_m(\zeta) = \partial_\zeta K_m(\zeta), \quad \partial_r K_m(\lambda_0 r) = \lambda_0 K'_m(\lambda_0 r).$$

**Problem $\mathbf{A}_1$.** A solution of equation $(1)_1$ is sought in the form of a sum

$$u(x) = v_0(x) + v(x), \tag{18}$$

where $v_0$ is a particular solution of equation $(1)_1$, and $v$ is a general solution of the corresponding homogeneous equation $(1)_1$. Direct checking shows that $v_0$ has the form

$$v_0(x) = \frac{1}{\lambda + 2\mu} grad\Big[ -\frac{a}{\lambda_0^2}\varphi_2(x) + b\varphi_0(x)\Big], \tag{19}$$

where $a$ and $b$ are defined by formulas (10), and $\varphi_0$ is a biharmonic function: $\triangle\varphi_0 = \varphi_1$.

A solution $v(x) = (v_1, v_2)$ of the homogeneous equation corresponding to $(1)_1$ is sought in the form

$$v_1(x) = \partial_1[\Phi_1(x) + \Phi_2(x)] - \partial_2\Phi_3(x), \quad v_2(x) = \partial_2[\Phi_1(x) + \Phi_2(x)] + \partial_1\Phi_3(x), \tag{20}$$

where

$$\begin{aligned} &\triangle\Phi_1(x) = 0, \quad \triangle\triangle\Phi_2(x) = 0, \quad \triangle\triangle\Phi_3(x) = 0, \\ &(\lambda + 2\mu)\partial_1\triangle\Phi_2(x) - \mu\partial_2\triangle\Phi_3(x) = 0, \\ &(\lambda + 2\mu)\partial_2\triangle\Phi_2(x) + \mu\partial_1\triangle\Phi_3(x) = 0, \end{aligned} \tag{21}$$

$\Phi_1$, $\Phi_2$, $\Phi_3$ are the scalar functions.

Taking into account (18) and relying on the condition $(2)_I$, we can write

$$v(z) = \Psi(z), \tag{22}$$

where $\Psi(z) = f(z) - v_0(z)$ is the known vector; $v_0$ is defined by formula (19), and $\varphi_1$ and $\varphi_2$ by equalities (11). The value of the function $\varphi_0$ is defined by means of the equation $\triangle\varphi_0 = \varphi_1$; it has the form

$$\varphi_0(x) = \frac{R^2}{4} \sum_{m=2}^{\infty} \frac{1}{1-m} \Big(\frac{R}{r}\Big)^{m-2} (A_m \cos m\psi + B_m \sin m\psi) + \frac{A_0}{4} r^2, \tag{23}$$

where $A_m$ and $B_m$ are defined in (12).

In view of (21), we can represent the harmonic function $\Phi_1$ and biharmonic functions $\Phi_2$ and $\Phi_3$ in the form

$$\begin{aligned}
\Phi_1(x) &= \sum_{m=0}^{\infty}\Big(\frac{R}{r}\Big)^m (X_{m_1}\cdot \nu_m(\psi)),\\
\Phi_2(x) &= R^2\sum_{m=0}^{\infty}\Big(\frac{R}{r}\Big)^{m-2} (X_{m_2}\cdot \nu_m(\psi)),\\
\Phi_3(x) &= R^2\frac{\lambda+2\mu}{\mu}\sum_{m=0}^{\infty}\Big(\frac{R}{r}\Big)^{m-2} (X_{m_2}\cdot s_m(\psi)),
\end{aligned} \tag{24}$$

where $X_{mk}$ are the unknown two-component vectors, $k=1,2$;

$$\nu_m(\psi)=(\cos m\psi,\sin m\psi),\quad s_m(\psi)=(-\sin m\psi,\cos m\psi),\quad x=(r,\psi), x\in D.$$

Substituting (24) into (20), the condition (22) for every $m$ results in a system of linear algebraic equations whose solution is written as follows:

$X_{01}=\dfrac{\alpha_0 R}{4},\quad X_{02}=\dfrac{\beta_0 R}{4},$

$X_{m1}=\dfrac{R(\alpha_m+\beta_m)}{2m(\lambda+3\mu)}[2\mu+(\lambda+\mu)m]-\dfrac{R\alpha_m}{m},\quad X_{m2}=\dfrac{\mu(\alpha_m+\beta_m)}{2(\lambda+3\mu)R},$

$m=1,2,...$; $\alpha_m$ and $\beta_m$ are the Fourier coefficients of, respectively, the normal and tangential components of the function $\Psi(z)=f(z)-v_0(z)$, $z\in S$.

Thus the solution of problem $A_1$ is represented by the sum (21) in which $v(x)$ is defined by means of formula (23), and $v_0(x)$ by formula (22).

**Problem $\mathbf{A}_2$.** Taking into account (3) and (9), the boundary condition $(2)_{II}$ can be rewritten as

$$T(\partial_z,n)v(z)=\Psi(z),\quad z\in S, \tag{25}$$

where

$$\Psi(z)=f(z)+n(z)[a\varphi_2(z)+b\varphi_1(z)]-T(\partial_z,n)v_0(z)$$

is the known vector, $\Psi=(\Psi_1,\Psi_2)$.

We substitute (24) first into (23) and then into (25). For the unknowns $X_{m1}$ and $X_{m2}$ we obtain a system of algebraic equations:

$$2(\lambda+2\mu)X_{01}=\frac{\alpha_0}{2},\quad 2(\lambda+2\mu)=\frac{\beta_0}{2},$$

$$m[\lambda+2\mu(m+1)]X_{m1}+\{(\lambda+2\mu)(1-m)(2-m+\frac{\lambda+2\mu}{\mu}m)$$

$$-\lambda mR^2[m+\frac{\lambda+2\mu}{\mu}(2-m)]\}X_{m2}=\alpha_m R^2,$$

$$-m(1+2\mu)X_{m1}+R^2[m(3-2m)+\frac{\lambda+2\mu}{\mu}(m^2-3m+2)]X_{m2}=\beta_m\frac{R^2}{\mu},$$

$m=1,2,...$; $\alpha_m$ and $\beta_m$ are the Fourier coefficients of, respectively, the normal and tangential components of the function $\Psi(z)=f(z)+n(z)[a\varphi_2(z)+b\varphi_1(z)]-T(\partial_z,n)v_0(z)$;

$v_0$ is defined by means of formula (19) in which $\varphi_0(x)$ for problem $B_1$ has the form (23) and for problem $B_2$ the form

$$\varphi_0(x) = \frac{R^3}{4}\sum_{m=2}^{\infty}\frac{1}{m(1-m)}\Big(\frac{R}{r}\Big)^{m-2}(A_m\cos m\psi + B_m\sin m\psi),$$

where $A_m$ and $B_m$ are defined in (16).

Conditions: $f, p_1, p_2 \in C^3(S)$ - in problem $A_1$ and conditions: $f, p_1, p_2 \in C^2(S)$ in problem $A_2$, ensure absolutely and uniformly convergence of series obtained for $v(x)$ and $v_0(x)$ and also, (8).

Having solved problems $A_1$, $A_2$, $B_1$ and $B_2$, we can write solutions of the initial problems I and II.

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მოხსენებები, ტომი 36-37, 2010-2011

# ქართული რეზიუმეები

## წრფივ კერძოწარმოებულიან დიფერენციალურ განტოლებათა სისტემები თხელ პრიზმულ არეებში

გ. ჯაიანი

სტატია ეძღვნება განზომილების რედუქციის მეთოდს კერძოწარმოებულიან დიფერენციალურ განტოლებათა სისტემებისათვის სასაზღვრო და საწყის-სასაზღვრო ამოცანების ამოსახსნელად, საზოგადოდ, არალიფშიცურ თხელ პრიზმულ არეებში.

## მესამე და მეოთხე სასაზღვრო ამოცანების ამოხსნა ორგვარი ფოროვნობის მქონე სფეროსათვის და სივრცისათვის სფერული ღრუთი

მ. ბაშალეიშვილი, ლ. ბიწაძე

ნაშრომში აგებულია მესამე და მეოთხე სასაზღვრო ამოცანების ამონახსნები, აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით, ორგვარი ფოროვნობის მქონე სფეროსათვის და უსასრულო სივრცისათვის სფერული ღრუთი.

## წინწასწრებულ არგუმენტიანი ემდენ-ფაულერის განზოგადოებული დიფერენციალური განტოლების ამონახსნების ასიმპტოტური ყოფაქცევა

რ. კოპლარაძე, გ. კვინიკაძე, გ. გიორგაძე

განხილულია ემდენ-ფაულერის განზოგადოებული დიფერენციალური განტოლება. შესწავლილია ამონახნების ოსცილაციური თვისებები. დადგენილია რხევადობის, ნულისაკენ მისწრაფების და შემოუსაზღვრელი ამონახსნების არსებობის საკმარისი პირობები.

## სტატიკის ძირითადი სასაზღვრო ამოცანების არსებობის და ერთადერთობის თეორემები ორგვარი ფოროვნობის მქონე სხეულისათვის

მ. ბაშელეიშვილი, ლ. ბიწაძე

ნაშრომში განხილულია სტატიკის განტოლებები ორგვარი ფოროვნობის მქონე სხეულებისათვის. დამტკიცებულია ძირითადი სასაზღვრო ამოცანების ამონახსნის ერთადერთობის თეორემები. აგებულია ამონახსნების ძირითადი ფუნდამენტური მატრიცი. შედგენილია მარტივი და ორმაგი ფენის პოტენციალები და შესწავლილია მათი თვისებები. პოტენციალთა მეთოდის გამოყენებით დამტკიცებულია ძირითადი სასაზღვრო ამოცანების ამონახსნის არსებობის თეორემები სასრული და უსასრულო არეებისათვის.

## ოპტიმალური საწყისი მონაცემების არსებობა და ინტეგრალური ფუნქციონალის მინიმუმის უწყვეტობა შეშფოთებების მიმართ ერთი კლასის ნეიტრალური დიფერენციალური განტოლებისათვის

თ. თადუმაძე

სტატიაში მოყვანილია ოპტიმალური საწყისი მონაცემების არსებობის თეორემები. გამოკვლეულია ინტეგრალური ფუნქციონალის მინიმუმის უწყვეტობის საკითხი განტოლების მარჯვენა მხარისა და ინტეგრადის შეშფოთების მიმართ.

## განსაკუთრებული მართვების ოპტიმალურობის აუცილებელი პირობები მართვის ამოცანაში ვოლტერას ტიპის ორგანზომილებია სხვაობიანი განტოლებისათვის

რ. ამიროვა, კ. მანსიმოვი

მიღებულია ოპტიმალურობის აუცილებელი პირობა დისკრეტული მაქსიმუმის პრინციპის ფორმით ოპტიმალური მართვის ამოცანისათვის, რომელიც აღიწერება ვოლტერას ტიპის ორგანზომილებიან სხვაობიანი განტოლებით. გამოკვლეულია დისკრეტული მაქსიმუმის პირობის გადაგვარების შემთხვევა.

## ოპტიმალურობის მეორე რიგის აუცილებელი პირობები მართვის ორპარამეტრიან საფეხურებიან დისკრეტულ ამოცანებში

კ. მანსიმოვი, მ. ნასიაიტი

განხილულია ოპტიმალური მართვის საფეხურებიანი ამოცანა, რომელიც აღიწერება ორგანზომილებიანი დისკრეტული სისტემებით. მიღებულია ოპტიმალურობის პირველი და მეორე რიგის აუცილებელი პირობები იმ შემთხვევისთვის, როცა მართვის არე ღია სიმრავლეა.

## წამახვილებული პრიზმული გარსების ორ მოდელთან დაკავშირებული ერთი შენიშვნა

გ. ჯაიანი

ჩატარებულია სასაზღვრო ამოცანების დასმის თავისებურებათა შედარებითი ანალიზი პრიზმული გარსების იერარქიული მოდელების ნულოვან მიახლოებაში, როცა პირით ზედაპირებზე მოცემულია ან ძაბვის ან გადაადგილების ვექტორები.

## მუდმივი დაგვიანების და წყვეტილი საწყისი პირობის ეფექტები დაგვიანებულ არგუმენტიანი სამართი ფუნქციონალურ-დიფერენციალური განტოლების ამონახსნის ვარიაციის ფორმულებში

თ. თადუმაძე, ნ. გორგოძე

დაგვიანებულ არგუმენტიანი სამართი არაწრფივი ფუნქციონალურ-დიფერენციალური განტოლებისათვის, წყვეტილი საწყისი პირობით, დამტკიცებულია ამონახსნის ვარიაციის ფორმულები, საწყისი მომენტის, დაგვიანების პარამეტრის, საწყისი ვექტორის, საწყისი და მართვის ფუნქციების შეშფოთებების მომართ. ამონახსნის ვარიაციის ფორმულებში გამოვლენილია დაგვიანების შეშფოთების და წყვეტილი საწყისი პირობის ეფექტები.

## პოტენციალთა თეორიის შებრუნებული ამოცანის ამონახსნის ერთადერთობის შესახებ სამგანზომილებიან სივრცეში

ჯ. კაპანაძე

სტატიაში განხილულია მოცულობითი პოტენციალისათვის შებრუნებული ამოცანის ამონახსნის ერთადერთობის საკითხი. აღნიშნული ამოცანა არაკორექტულია, რის გამოც ამოცანის ამონახსნის ერთადერთობის დამტკიცებას მნიშვნელოვანი ადგილი უკავია. სტატიაში შემოღებულია უბან-უბან გლუვი შემოსაზღვრული, მარტივადბმული არის საზღვრის გლუვ წერტილში ლოკალურად მკაცრად ამოზნექილობის ცნება. დამტკიცებულია, რომ თუ $\Omega_1$ არე $x_0 \in \partial\Omega_1$ წერტილში მკაცრად ამოზნექილია $(x_0 \notin \bar{\Omega}_2)$, მაშინ შებრუნებულ ამოცანას ერთადერთი ამონახსენი აქვს. გარდა ამისა, განხილულია შებრუნებული ამოცანა შემოსაზღვრული მარტივადბმული არეებისათვის $C^2$ კლასიდან.

# ელასტოსტატიკის სასაზღვრო ამოცანების ამოხსნა ორგვარი ფოროვნობის მქონე დრეკადი სიბრტყისათვის წრიული ხვრელით

ი. ცაგარელი, მ. სვანაძე

ნაშრომში ცხადად, აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით ამოხსნილია ფოროელასტოსტატიკის ორი ამოცანა შემდეგი სასაზღვრო პირობებით: 1) მოცემულია გადაადგილების ვექტორისა და ფორებში წნევათა ნორმალური მდგენელების მნიშვნელობები; 2) მოცემულია ძაბვის ვექტორისა და ფორებში წნევათა მნიშვნელობები. დადგენილია განსახილველი ამოცანების ამონახსნთა ერთადერთობის საკითხები.

## C O N T E N T S

## ავტორთა საყურადღებოდ!

ჟურნალში გამოსაქვეყნებელი სტატია უნდა წარმოადგინოთ ინგლისურად, ელექტრონული და ამობეჭდილი სახით, აკრეფილი TEX (plain, LATEX, AMSTEX, AMSLATEX) პროგრამის გამოყენებით.

დაიცავით შემდეგი მოთხოვნები:

- ასოების ზომა და სტრიქონებს შორის მანძილი – 12 pt;
- ტექსტის სიგრძე – 227 mm, სიგანე – 155 mm;
- პირველ სტრიქონებში აიკრიფოს ნაშრომის სათაური, შემდეგ - ავტორის გვარი და ინიციალი;
- აბსტრაქტი - არაუმეტეს 1/4 გვერდისა;
- საკვანძო სიტყვები და ფრაზები;
- ინდექს(ებ)ი MSC 2010 წლის კლასიფიკატორის მიხედვით;
- ციტირებული ლიტერატურის სია მოთავსდეს სტატიის ბოლოს ტექსტში მითითებული თანმიმდევრობით (დაინომროს კვადრატულ ფრჩხილებში არაბული ციფრებით, წიგნის შემთხვევაში გარდა ზოგიერთი გამონაკლისისა უნდა დაერთოს გვერდის ნომრები მაგ.: ([2], გვ.30)). სიის შედგენისას გთხოვთ იხელმძღვანელოთ Mathematical Reviews სტანდარტით:

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