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# ABSTRACTS

# Stabilization of Integro-Differential CNN Model Arising in Nano-Structures

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## 1 Introduction

Computational Nanotechnology has become an indispensable tool not only in predicting but also in engineering the properties of multi-functional nano-structured materials. The presence of nano-heterogeneities in these materials affects or disturbs their elastic field at the local and the global scale and thus greatly influences their mechanical properties. In this paper we shall study dynamical behaviour of 2D dynamic coupled problem in multifunctional nano-heterogeneous piezoelectric composites. More in detail, we shall present first modeling of two-dimensional anti-plane (SH) wave propagation problem in piezoelectric anisotropic solids containing nano-holes or nano-inclusions. Nano-heterogeneities are considered in two aspects as wave scatters provoking scattered and diffraction wave fields and also as stress concentrators creating local stress concentrations in the considered solid.There are no numerical results for dynamic behavior of bounded piezoelectric domain with heterogeneities under anti-plane load. Validation is done in [1] for infinite piezoelectric plane with a hole, in [3] for isotropic bounded domain with holes and inclusions and in [2] for piezoelectric plane with nano-hole or nano-inclusion.

In Section 2 we shall reduce the model under consideration to a system of integro-differntial equations (IDE) and we shall discretize it by Cellular Nonlinear/Nanoscale Network (CNN) architecture. Simulations and validation will be provided. Section 3 deals with feedback stabilization of the IDE CNN model together with simulations.

We shall state the model of piezoelectric solid with heterogeneities under time-harmonic anti-plane load. Let $G \in \mathbb{R}^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano–holes, nano–inclusions) subjected to time–harmonic load on the boundary $\partial G$. Note that heterogeneities are of macro size if their diameter is greater than $10^{-6}m$, while heterogeneities are of nano–size if their diameter is less than $10^{-7}m$.

The aim is to find the field in every point of $M = G \setminus I$, $I$ and to its dynamic behaviour. Using the methods of continuum mechanics the problem can be formulated in terms of boundary value problem for a system of 2-nd order differential equations, see [1, Chapter 2],

$$\begin{cases} \rho^N \dfrac{\partial^2 u_3}{\partial t^2} = c_{44}^N \Delta u_3^N + e_{15}^N \Delta u_4^N, \\ e_{15}^N \Delta u_3^N - \varepsilon_{15}^N \Delta u_4^N = 0, \end{cases} \tag{1}$$

where $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is Laplace operator with respect to $t$, $N = M$ for $x \in M$ and $N = I$ for $x \in I$; $u_3^N$ is mechanical displacement, $u_4^N$ is electric potential, $\rho^N$ is the mass density, $c_{44}^N > 0$ is the shear stiffness, $e_{15}^N \neq 0$ is the piezoelectric constant and $\varepsilon_{11}^N > 0$ is the dielectric permittivity.

We shall consider the case, when $I$ is a nano-hole or nano-inclusion and boundary conditions on $S$ are

$$t_j^M = \frac{\partial \sigma_{lj}^S}{\partial l} \ \text{ on } S, \ \text{ or } \ \tau_3^I + t_3^M = \frac{\partial \sigma_{l3}^S}{\partial l}\,, \quad \tau_4^I + t_4^M = \frac{\partial \sigma_{l4}^S}{\partial l}\,, \tag{2}$$

where $\sigma_{lj}^S$ is generalized stress [1], $j = 3, 4$, $l$ is the tangential vector. Then we shall study boundary value problem (BVP) (1) with boundary conditions (2).

## 2 Integro-differential CNN model

BVP $(1), (2)$ is reduced in [1] to integro-differential equation (IDE) using the Fourier transform and then applying the Gauss theorem [6]. In this paper we shall study the general form of IDE obtained in [1]. Let us consider the following system of IDE:

$$\frac{\partial u(x)}{\partial \tau} = D\, \frac{\partial^2 u}{\partial x^2} - C_1 \int\limits_S G(u(x))\, dx, \tag{3}$$

where $C_1$ is a constant depending on the $\rho^M$, $c_{44}^M > 0$, $e_{15}^M \neq 0$ and $\varepsilon_{11}^M > 0$, $D$ is diffusion coefficient, $u = (u_3, u_4)$, function $G(x)$ is a function of the displacement vectors $u_{3,4}$ and the traction $\tau_{3,4}$.

It is known [5] that some autonomous CNN represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate $A$-template. In our case the CNN model of IDE (3) is:

$$\frac{du_{ij}}{dt} = DA_1 * u_{ij} - C_1 \int\limits_S G(u_{ij})\, dt, \ \ 1 \le i \le n, \ \ j = 3, 4, \tag{4}$$

where $A_1$ is 1-dimensional discretized Laplacian template [5] $A_1 : \ (1, -2, 1)$, $*$ is convolution operator, $n = M \times M$ is the number of cells of the CNN architecture.

We develop the following algorithm for studying the dynamical behavior of CNN model (4) via describing function method [4]:

1. First, we apply double Fourier transform $F(s, z)$ to IDE CNN model (4)

   $$F(s, z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int\limits_{-\infty}^{\infty} f_k(t) \exp(-st)\, dt$$

   from continuous time $t$ and discrete space $k$ to continuous temporal frequency $\omega$, and continuous spatial frequency $\Omega$ such that $z = exp(I\Omega)$, $s = I\omega$, $I$ is the imaginary identity and therefore we obtain:

   $$sU(s, z) = D\big[z^{-1}U(s, z) - 2U(s, z) + zU(s, z)\big] - C_1 s^{-1} G(U(s, z)).$$

2. We express $U(s,z)$ as a function of $G(U(s,z))$:

$$U(s,z) = \frac{C_1}{sD(z^{-1} - 2 + z) - s^2}\, G(U)$$

and obtain the transfer function $H(s,z)$:

$$H(s,z) = \frac{C_1}{sD(z^{-1} - 2 + z) - s^2}\,.$$

According to the describing function technique [4], the transfer function can be expressed in terms of temporal frequency $\omega$ and spatial frequency $\Omega$:

$$H_\Omega(\omega) = \frac{C_1}{I\omega D(2\cos\Omega - 2) + \omega^2}\,.$$

3. We are looking for possible periodic solutions of our CNN model (4) in the form:

$$u_{ij}(t) = \xi(i\Omega + \omega t), \quad 1 \le i \le n, \quad j = 3, 4,$$

for some function $\xi : \mathbb{R} \to \mathbb{R}$ and for some spatial frequency $0 \le \Omega \le 2\pi$ and temporal frequency $\omega = \frac{2\pi}{T}$, where $T > 0$ is the minimal period.

4. According to the describing function technique [4] the following constraints hold:

$$\begin{aligned} \mathcal{R}(H_\Omega(\omega)) &= \frac{U_m}{Y_m}\,, \\ \mathcal{I}(H_\Omega(\omega)) &= 0. \end{aligned} \tag{5}$$

5. Thus (5) give us necessary set of equations for finding the unknowns $U_m$, $\Omega$ and $\omega$. As we mentioned before we are looking for a periodic wave solution of (4), therefore $U_m$ will determine approximate amplitude of the wave, and $T = \frac{2\pi}{\omega}$ will determine the wave speed. Now according to the describing function technique, if for a given value of $\Omega$ we can find the unknowns $(U_m, \omega)$, then we can predict the existence of a periodic solution of our CNN IDE (4) with an amplitude $U_m$ and period of approximately $T = \frac{2\pi}{\omega}$.

Following the above algorithm the next theorem has been proved:

**Theorem 1.** *CNN IDE* (4) *of the BVP* (1), (2) *with circular array of* $n = L \times L$ *cells has periodic solutions* $u_{ij}(t)$ *with a finite set of spatial frequencies* $\Omega = \frac{2\pi k}{n}$, $0 \le k \le n-1$ *and a period* $T = \frac{2\pi}{\omega}$.

Let us consider the square domain of piezoelectric solid $G_1G_2G_3G_4$ with a side $a$. For heterogeneities at nano–scale we have: the side of the square is $a = 10^{-7}m$; material parameters inside $I$ for hole are 0; material parameters on $S = \partial I$ for hole and for an inclusion are: $c^S_{44} = 0.1\, c^M_{44}$, $e^S_{15} = 0.1\, e^M_{15}$, $\varepsilon^S_{11} = 0.1\, \varepsilon^M_{11}$, $\rho^S = \rho^M$.

Then simulating our CNN IDE model (4) we obtain the following periodic wave solutions (see Figure 1).

The simulations of IDE CNN model are obtained by simulation system MATCNN applying 4th- order Runge–Kutta integration. In order to minimize the computational complexity and to maximize the significance of the mean square error only outputs of 4 cells are taken into account.

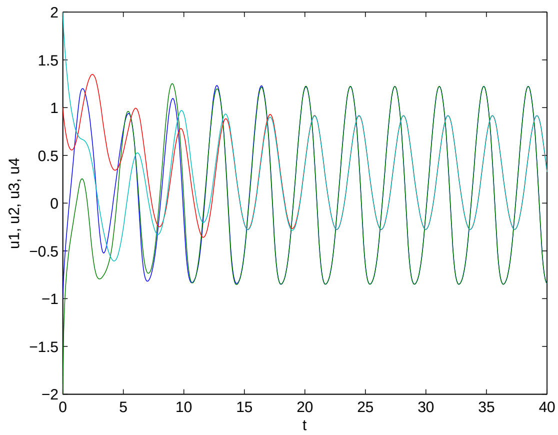


**Figure 1.** Simulation of IDE CNN model (4) with 4 cells

## 3 Stabilizing feedback control for IDE CNN model

Let us extend the IDE CNN model (4) by adding to each cell the local linear feedback:

$$\frac{du_{ij}}{dt} = D(u_{i-1j} - 2u_{ij} + u_{i+1j}) - C_1 \int\limits_S G(u_{ij})\,dt - ku_{ij}, \tag{6}$$

where $k$ is the feedback controls coefficient which is assumed to be equal for all cells. The problem is to prove that this simple and available for the implementation feedback can stabilize the IDE CNN model (4). In the following we present a proof of this statement and give sufficient condition on the feedback coefficient values which provide stability of the CNN nonlinear model (6). The following theorem holds:

**Theorem 2.** *Let the parameters of IDE CNN system and feedback coefficient $k$ (6) have positive values. Then its linearized model is asymptotically stable for all $k > 0$.*

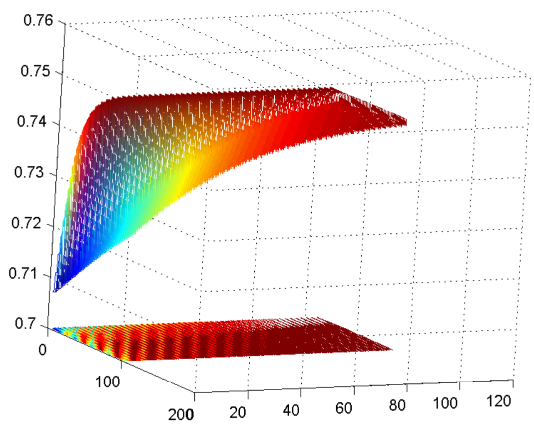


**Figure 2.** Simulation of stabilized IDE CNN model (6)

*Proof.* Define the quadratic Lyapunov function candidate $L(z)=\frac{1}{2}\,z^Tz$. Then its derivative along the linearized control IDE CNN is $\frac{dL(z)}{dt}=\frac{1}{2}\,z^T(J^T(k)+J(k))z=-z^TQ(k)z$. Therefore, $\frac{dL(z)}{dt}<0$ implies a positive definiteness of $Q(k)$. It can be shown that $Q(k)$ positive definiteness implies $k>0$. For verification of the above statement the eigenvalues of $J(k)$ were calculated related on the values of feedback coefficient $k$. Stability of the linear system requires that the eigenvalues $\lambda_j^i$, $i=1,\dots,4$ satisfy the inequality $\max\limits_i \operatorname{Re}\lambda_j^i<0$.

Simulations of the stabilized IDE CNN are in Figure 2. □

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# Recent Development of Boundary Value Problems of $q$-Difference and Fractional $q$-Difference Equations and Inclusions

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## 1 Introduction

The subject of $q$-calculus, also known as quantum calculus, rests on the concept of finite difference re-scaling. The formal work on $q$-difference equations dates back to the first quarter of twentieth century. The applications of $q$-calculus in several important disciplines like combinatorics, special functions, quantum mechanics, etc. led to the recent development of the subject. $q$-calculus is also regarded as a subfield of time scales calculus (unified setting for studying dynamic equations on both discrete and continuous domains). In this short note, we present some recent results on boundary value problems (BVP) of $q$-difference and fractional $q$-difference equations and inclusions.

## 2 BVP for $q$-difference equations and inclusions

We begin with some preliminary concepts of $q$-calculus.

**Definition 2.1.** Let $f$ be a function defined on a $q$-geometric set $I$, i.e., $qt \in I$ for all $t \in I$. For $0 < q < 1$, we define the $q$-derivative as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \ \ t \in I \setminus \{0\}, \ \ D_q f(0) = \lim_{t \to 0} D_q f(t).$$

For $t \geq 0$, we consider a set $J_t = \{tq^n : \ n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite $q$-integral of a function $f : J_t \to \mathbb{R}$ by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

For $a, b \in J_t$, we have

$$\int\limits_a^b f(s)\,d_q s = I_q f(b) - I_q f(a) = (1-q)\sum_{n=0}^{\infty} q^n \big[bf(bq^n) - af(aq^n)\big].$$

Consider the boundary value problem for a second order $q-$difference equation with non-separated boundary conditions

$$D_q^2 x(t) = f(t, x(t)), \;\; t \in I, \quad x(0) = \eta x(T), \quad D_q x(0) = \eta D_q x(T), \tag{2.1}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$, $I = [0, T] \cap \{q^n : \; n \in \mathbb{N}\} \cup \{0\}$, $T$ is a fixed constant and $\eta \neq 1$ is a fixed real number. By using a variety of fixed point theorems such as Banach's contraction principle, Leray–Schauder nonlinear alternative, Schauder fixed point theorem and Krasnoselskii's fixed point theorem, several results are proved for the problem (2.1) in [5], which are listed below.

**Theorem 2.2.** *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition*

$$|f(t,u) - f(t,v)| \le L|u - v|, \;\; \forall\, t \in I, \;\; u, v \in \mathbb{R},$$

*where $L$ is a Lipschitz constant. Then the boundary value problem* (2.1) *has a unique solution, provided*

$$L\Big(\frac{1}{1+q} + \frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \Big|\frac{\eta}{\eta-1}\Big|\Big)T^2 < 1.$$

**Theorem 2.3.** *Assume that:*

($H_1$) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0, T], \mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|) \;\; \text{for each} \;\; (t,u) \in I \times \mathbb{R};$$

($H_2$) *there exists a number $M > 0$ such that*

$$\|u\| \Big/ \Big(T\Big(1 + \frac{|\eta|(1 + |1-\eta|)}{(\eta-1)^2}\Big)\psi(M)\|p\|_{L^1}\Big) > 1.$$

*Then the BVP* (2.1) *has at least one solution.*

**Theorem 2.4.** *Assume that there exist constants*

$$0 \le c < 1 \Big/ \Big(\frac{1}{1+q} + \frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \Big|\frac{\eta}{\eta-1}\Big|\Big)$$

*and $N > 0$ such that $|f(t,u)| \le \frac{c}{T^2}\,|u| + N$ for all $t \in I$, $u \in C(I)$. Then the BVP* (2.1) *has at least one solution.*

**Theorem 2.5.** *Assume that there exists a constant $M_1$ such that*

$$|f(t,u)| \le M_1 \Big/ \Big(\frac{1}{1+q} + \frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \Big|\frac{\eta}{\eta-1}\Big|\Big)T^2, \;\; \forall\, t \in I, \;\; u \in [-M_1, M_1].$$

*Then the BVP* (2.1) *has at least one solution.*

**Theorem 2.6.** *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the following assumptions hold:*

$(H_3)$ $|f(t,u) - f(t,v)| \le L|u - v|$, $\forall\, t \in I$, $u, v \in \mathbb{R}$;

$(H_4)$ $|f(t,u)| \le \mu(t)$, $\forall\,(t,u) \in I \times \mathbb{R}$, *and* $\mu \in C(I, \mathbb{R}^+)$.

*If*

$$\Big(\frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \Big|\frac{\eta}{\eta-1}\Big|\Big)T^2 < 1,$$

*then the boundary value problem* (2.1) *has at least one solution on $I$.*

In [4], the authors discussed the existence and nonexistence of solutions for nonlinear second order $q$-integro-difference equation: $D_q^2 u(t) = f(t,u(t)) + I_q g(t,u(t))$, $f,g \in C(I \times \mathbb{R}, \mathbb{R})$ supplemented with non-separated boundary conditions given in (2.1). Similar results were proved for other classes of boundary value problems. The results for the second order $q$-difference equation $D_q^2 x(t) = f(t,x(t))$, $t \in I$, supplemented with non-separated boundary conditions $\alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1)$, $\alpha_2 x(1) - \beta_2 D_q x(1) = \gamma_2 x(\eta_2)$ were proved in [13], with three-point integral boundary conditions $\alpha x(\eta) + \beta D_r x(\eta) = 0$, $\int\limits_0^T x(s)\,d_p s = 0$ in [22], nonlocal and integral boundary conditions

$$x(0) = x_0 + g(x), \quad x(1) = \alpha \int\limits_\mu^\nu x(s)\,d_q s,$$

and

$$x(\xi) = g(x), \quad \alpha D_r x(\eta) + \beta \int\limits_\eta^T x(s)\,d_p s = 0$$

in [8] and [18], respectively. For results on inclusions, see [7] and [17].

Boundary value problems for nonlinear $q$-difference hybrid equations and inclusions were studied in [11]. In [11] the authors have investigated the problem:

$$D_q^2\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \ \ t \in I_q, \ \ x(0) = 0, \ \ x(1) = 0,$$

where $f \in C(I_q \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g : C(I_q \times \mathbb{R}, \mathbb{R})$ are such that $f(t,x(t)), g(t,x(t))$ are continuous at $t = 0, 1$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0,1\}$, $q \in (0,1)$ is a fixed constant. An existence result was established by using a fixed point theorem for the product of two operators under Lipschitz and Carathéodory conditions.

Agarwal *et al.* [3] discussed the existence, uniqueness and existence of extremal solutions for a nonlinear boundary value problem of $q$-difference equations with nonlocal $q$-integral boundary condition given by

$$D_q u(t) = f\big(t, u(t), u(\phi(t))\big), \quad u(0) = \lambda \int\limits_0^\eta g(s,u(s))\,d_q s + \mu, \ \ t \in I_q,$$

where $f \in C(I_q \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(I_q \times \mathbb{R}, \mathbb{R})$, $\phi \in C(I_q, I_q)$, $\eta \ge 0$, $\lambda, \mu \in \mathbb{R}$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0,1\}$, $q \in (0,1)$ is a fixed constant.

The notions of $q$-derivative and $q$-integral were extended on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$, let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define $q_k$-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

**Definition 2.7.** Let $f : J_k \to \mathbb{R}$ be a continuous function and let $t \in J_k$. Then we define the $q_k$-derivative of the function $f$ as

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, \quad t \neq t_k, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t).$$

We say that $f$ is $q_k$-differentiable on $J_k$ provided $D_{q_k} f(t)$ exists for all $t \in J_k$.

**Definition 2.8.** Let $f : J_k \to \mathbb{R}$ be a continuous function. Then the $q_k$-integral is defined by

$$\int\limits_{t_k}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + (1-q_k^n)t_k\big) \tag{2.2}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$, then the definite $q_k$-integral is defined by

$$\int\limits_{a}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + (1-q_k^n)t_k\big) - (1-q_k)(a-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n a + (1-q_k^n)t_k\big).$$

For more details on these two new notions, the interested reader is referred to the book [15].

Agarwal *et al.* [2] obtained the positive extremal solutions by the method of successive iterations for the nonlinear impulsive $q_k$-difference equations:

$$\begin{gathered} D_{q_k} u(t) = f(t, u(t)), \ \ 0 < q_k < 1, \ \ t \in J', \\ u(t_k) = I_k(u(t_k)), \ \ k = 1, 2, \dots, m, \quad u(0) = \lambda u(\eta) + d, \ \ \eta \in J_r, \ \ r \in \mathbb{Z}, \end{gathered}$$

where $D_{q_k}$ are $q_k$-derivatives ($k = 0, 1, 2, \dots, m$), $f \in C(J \times \mathbb{R}, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $J_r = (t_r, T]$, $0 \le \lambda < 1$, $d \ge 0$, $0 \le r \le m$ and $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ ($k = 1, 2, \cdots, m$), respectively.

# 3 BVP for fractional $q$-difference equations and inclusions

**Definition 3.1.** Let $\nu \ge 0$ and $h$ be a function defined on $[0, T]$. The fractional $q$-integral of Riemann–Liouville type is given by $(I_q^0 h)(t) = h(t)$ and

$$(I_q^\nu h)(t) = \frac{1}{\Gamma_q(\nu)} \int\limits_0^t (t - qs)^{(\nu-1)} h(s)\, d_q s, \ \ \nu > 0, \ \ t \in [0, T].$$

**Definition 3.2.** The fractional $q$-derivative of Riemann–Liouville type of order $\nu \ge 0$ is defined by $(D_q^0 h)(t) = h(t)$ and $(D_q^\nu h)(t) = (D_q^l I_q^{l-\nu} h)(t)$, $\nu > 0$, where $l$ is the smallest integer greater than or equal to $\nu$.

In recent years, several existence and uniqueness results were obtained. In [1], by applying Krasnoselskii's fixed point theorem, Leray–Schauder nonlinear alternative and Banach's contraction principle, the authors studied the existence and uniqueness of solutions for the following $q$-anti-periodic boundary value problem of sequential $q$-fractional integro-differential equations:

$$\begin{gathered} {}^cD_q^\alpha ({}^cD_q^\gamma + \lambda) x(t) = A f(t, x(t)) + B I_q^\rho g(t, x(t)), \ \ 0 \le t \le 1, \ \ 0 < q < 1, \\ x(0) = -x(1), \quad \big(t^{(1-\gamma)} D_q x(t)\big)\Big|_{t=0} = -D_q x(1), \end{gathered}$$

where ${}^cD_q^\alpha$ and ${}^cD_q^\gamma$ denote the fractional $q$-derivative of the Caputo type, $0 < \alpha, \gamma \le 1$, $I_q^\rho(\,\cdot\,)$ denotes Riemann–Liouville integral with $0 < \rho < 1$, $f$, $g$ are given continuous functions, $\lambda \in \mathbb{R}$ and $A$, $B$ are real constants.

In [12], the existence and uniqueness results were obtained for the following boundary value problem of nonlinear fractional $q$-difference equations with nonlocal and sub-strip type boundary conditions:

$$
\begin{gathered}
{}^cD_q^v x(t) = f(t, x(t)), \ \ t \in [0,1], \ \ 1 < v \le 2, \ \ 0 < q < 1, \\
x(0) = x_0 + g(x), \quad x(\xi) = b \int_\eta^1 x(s)\, d_q s, \ \ 0 < \xi < \eta < 1,
\end{gathered}
$$

where ${}^cD_q^v$ denotes the Caputo fractional $q$-derivative of order $v$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and $b$ is a real constant. In [6], the existence of solutions for nonlinear fractional $q$-difference integral equations with two fractional orders and nonlocal four-point boundary conditions were obtained, while the positive extremal solutions for nonlinear fractional differential equations on a half-line were discussed in [23]. For further results, see [9, 10, 14, 16, 19–21].

Finally, we emphasize that the Definition 2.1 does not remain valid for impulse points $t_k$, $k \in \mathbb{Z}$ such that $t_k \in (qt, t)$. On the other hand, this situation does not arise for impulsive equations on $q$-time scales as the domains consist of isolated points covering the case of consecutive points of $t$ and $qt$ with $t_k \notin (qt, t)$. Due to this reason, the subject of impulsive quantum difference equations on dense domains could not be studied. In [15], the authors modified the classical quantum calculus for obtaining the first and second order impulsive quantum difference equations on a dense domain $[0, T] \subset \mathbb{R}$ through the introduction of a new $q$-shifting operator defined by ${}_a\Phi_q(m) = qm + (1-q)a$, $m, a \in \mathbb{R}$. For details, see [15].

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# On the Cauchy Problem for Linear Systems of Generalized Ordinary Differential Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $t_0 \in \mathbb{R}$ and

$$I_{t_0} = I \setminus \{t_0\}.$$

Consider the linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \quad \text{for} \quad t \in I_{t_0}, \tag{1}$$

where $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $f = (f_k)_{k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n)$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be a diagonal matrix-function with continuous diagonal elements $h_k : I_{t_0} \to ]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of finding a solution $x \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n)$ of the system (1), satisfying the condition

$$\lim_{t\to t_0-} (H^{-1}(t)\, x(t)) = 0 \ \text{ and } \ \lim_{t\to t_0+} (H^{-1}(t)\, x(t)) = 0. \tag{2}$$

The analogous problem for systems of ordinary differential equations with singularities

$$\frac{dx}{dt} = P(t)\, x + q(t) \quad \text{for} \quad t \in I, \tag{3}$$

where $P \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_{t_0}, \mathbb{R}^n)$, are investigated in [5–7].

The singularity of system (3) is considered in the sense that the matrix $P$ and vector $q$ functions, in general, are not integrable at the point $t_0$. In general, the solution of the problem (3), (2) is not continuous at the point $t_0$ and, therefore, it is not a solution in the classical sense. But its restriction to every interval from $I_{t_0}$ is a solution of the system (3). In connection with this we give the example from [7].

Let $\alpha > 0$ and $\varepsilon \in ]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon-1\alpha}, \quad \lim_{t\to 0} \alpha(t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions to $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that one.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, [1–4, 8, 9].

We give sufficient conditions for the unique solvability of the problem (1), (2). The analogous results for the Cauchy problem for systems of ordinary differential equations with singularities belong to I. Kiguradze ([6, 7]).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$, $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$.

$O_{n\times m}$ (or $O$) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_{\mp}=\frac{1}{2}\,(|X|\mp X)$.

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x=(x_i)_{i=1}^{n}$.

If $X\in\mathbb{R}^{n\times n}$, then $X^{-1}$, $\det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant of $X$ and the spectral radius of $X$; $I_n$ is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components $x_{ij}$ $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$ for $t\in I$, where $v(x_{ij})(a)=0$, $v(x_{ij})(t)\equiv\bigvee\limits_a^t(x_{ij})$, and $a\in\mathbb{R}$ is some fixed point; $[X(t)]_+^v\equiv\frac{1}{2}(V(X)(t)+X(t))$, $[X(t)]_-^v\equiv\frac{1}{2}(V(X)(t)-X(t))$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X:[a,b]\to\mathbb{R}^{n\times m}$ at the point $t$ $(X(a-)=X(a),X(b+)=X(b))$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<\infty$).

$\mathrm{BV}_{loc}(J;D)$, where $J\subset\mathbb{R}$ is an interval and $D\subset\mathbb{R}^{n\times m}$, is the set of all $X:J\to D$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from $J$;

$\mathrm{BV}_{loc}(I_{t_0};D)$ is the set of all $X:I\to D$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from $I_{t_0}$;

$s_1$, $s_2$ and $s_c:\mathrm{BV}_{loc}(J;\mathbb{R})\to\mathrm{BV}_{loc}(J;\mathbb{R})$ are the operators defined by

$$s_1(x)(a)=s_2(x)(a)=0,\ \ s_0(x)=x(a);$$
$$s_1(x)(t)=s_1(x)(s)+\sum_{s<\tau\leq t}d_1x(\tau),\ \ s_2(x)(t)=s_2(x)(s)+\sum_{s\leq\tau<t}d_2x(\tau),$$
$$s_0(x)(t)=s_0(x)(s)+x(t)-x(s)-\sum_{j=1}^{2}(s_j(x)(t)-s_j(x)(s))\ \text{ for }\ s<t,$$

where $a\in J$ is an arbitrarily fixed point.

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\leq s<t\leq b$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\leq t}x(\tau)d_1g(\tau)+\sum_{s\leq\tau<t}x(\tau)d_2g(\tau),$$

where $\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$. So $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral [8,9]; We put

$$\int_{s\mp}^{t}x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int_{s\mp\delta}^{t}x(\tau)\,dg(\tau).$$

If $X \in \mathrm{BV}_{loc}(J;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_j X(t)) \neq 0$ for $t \in I$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(J;\mathbb{R}^{n\times m})$, then

$$\begin{aligned}\mathcal{A}(X,Y)(a) &= O_{n\times m},\\ \mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s) &= Y(t)-Y(s)+\sum_{s<\tau\le t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ &\quad -\sum_{s\le\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for } s<t.\end{aligned}$$

A vector-function $x : I_{t_0} \to \mathbb{R}^n$ is said to be a solution of the system (1) if $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ for every closed interval $[a,b]$ from $I_{t_0}$ and

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for } a\le s<t\le b.$$

We assume that

$$\det(I_n+(-1)^j d_jA(t)) \neq 0 \ \text{ for } t\in I_{t_0}\ (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $A\in \mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in \mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Let $A_0 \in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$. Then a matrix-function $C_0 : I_{t_0}\times I_{t_0} \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the generalized differential system

$$dx = dA_0(t)\cdot x, \tag{4}$$

if, for every interval and $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_0(\,\cdot\,,\tau) : I_{t_0} \to \mathbb{R}^{n\times n}$ to $J$ is the fundamental matrix of the system (4), satisfying the condition $C_0(\tau,\tau)=I_n$. Therefore, $C_0$ is the Cauchy matrix of (4) if and only if the restriction of $C_0$ to the every interval $J\times J$ is the Cauchy matrix of the system in the sense of definition given in [9].

We assume $I_{t_0}^- = ]-\infty,t_0[\cap I$, $I_{t_0}^+ = ]t_0,+\infty[\cap I$ and $I_{t_0}^-(\delta)=[t_0-\delta,t_0[\cap I_{t_0}$, $I_{t_0}^+(\delta)=]t_0,t_0+\delta]\cap I_{t_0}$, $I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)$ for every $\delta>0$.

**Theorem 1.** *Let there exist a matrix-function $A_0 \in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and constant matrices $B_0,\ B\in\mathbb{R}_+^{n\times n}$ such that*

$$\begin{gathered}\det\big(I_n+(-1)^j d_jA_0(t)\big)\neq 0 \ \textit{ for } t\in I_{t_0}\ (j=1,2),\\ r(B)<1,\end{gathered}\tag{5}$$

*and the estimates*

$$|C_0(t,\tau)| \le H(t)\,B_0\,H^{-1}(\tau)\ \textit{ for } t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0|;$$

$$\bigg|\int_{t_0\mp}^t |C_0(t,\tau)|\,dV(\mathcal{A}(A_0,A-A_0)(\tau))\cdot H(\tau)\bigg| \le H(t)B\ \textit{ for } t\in I_{t_0}^-(\delta) \textit{ and } t\in I_{t_0}^+(\delta),\ \textit{respectively,}$$

*hold for some $\delta>0$, where $C_0$ is the Cauchy matrix of the system (4). Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\bigg\|\int_{t_0\mp}^t H^{-1}(\tau)\,C_0(t,\tau)\,d\mathcal{A}(A_0,f)(\tau)\bigg\| = 0.$$

*Then the problem (1), (2) has a unique solution.*

**Theorem 2.** *Let there exist a matrix* $B=(b_{ik})_{i,k=1}^{n}\in\mathbb{R}_{+}^{n\times n}$ *such that the condition* (5) *and*

$$\big[(-1)^j d_j a_{ii}(t)\big]_- < 1 \ \text{ for } \ (-1)^j(t-t_0)>0 \ \ (j=1,2;\ i=1,\dots,n),$$

*hold, and the estimates*

$$|c_i(t,\tau)|\le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \text{ for } \ t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0| \ \ (i=1,\dots,n),$$

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau)h_i(\tau)\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg|\le b_{ii}(t)h_i(t)$$

$$\text{for } t\in I_{t_0}^-(\delta) \ \text{ and } t\in I_{t_0}^+(\delta), \text{ respectively } (i=1,\dots,n);$$

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau)h_k(\tau)\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg|\le b_{ik}(t)h_i(t)$$

$$\text{for } t\in I_{t_0}^-(\delta) \ \text{ and } t\in I_{t_0}^+(\delta), \text{ respectively } (i\ne k;\ i,k=1,\dots,n)$$

*hold for some* $b_0>0$ *and* $\delta>0$*. Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\int\limits_{t_0\mp}^{t}\frac{c_i(t,\tau)}{h_i(t)}\,dV(\mathcal{A}(a_{0ii},f_i))(\tau)=0 \ \ (i=1,\dots,n),$$

*where* $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$*, and* $c_i$ *is the Cauchy function of the equation* $dx=x\,da_{0ii}(t)$ *for* $i\in\{1,\dots,n\}$*. Then the problem* (1), (2) *has a unique solution.*

**Remark.** The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$, mentioned in the theorem, for $t,\tau\in I_{t_0}^-$ and $t,\tau\in I_{t_0}^+$, have the form

$$c_i(t,\tau)=\begin{cases}\exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\displaystyle\prod_{\tau<s\le t}(1-d_1a_{0ii}(s))^{-1}\prod_{\tau\le s<t}(1+d_2a_{0ii}(s)) & \text{for } t>\tau,\\ \exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\displaystyle\prod_{t<s\le\tau}(1-d_1a_{0ii}(s))\prod_{t\le s<\tau}(1+d_2a_{0ii}(s))^{-1} & \text{for } t<\tau.\end{cases}$$

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# On the Cauchy Problem for Linear Systems of Impulsive Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $t_0 \in \mathbb{R}$ and

$$I_{t_0} = I \setminus \{t_0\}.$$

Consider the linear system of impulsive equations with fixed and finite points of impulses actions

$$\frac{dx}{dt} = P(t)\,x + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_l\,x(\tau_l) + g_l \ \ (l = 1, 2, \dots), \tag{2}$$

where $P \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_{t_0}, \mathbb{R}^n)$, $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $g_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$, $\tau_l \in I_{t_0}$ $(l = 1, 2, \dots)$, $\tau_i \neq \tau_j$ if $i \neq j$ and $\lim\limits_{l\to\infty} \tau_l = t_0$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be a diagonal matrix-functions with continuous diagonal elements $h_k : I_{t_0} \to ]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of finding a solution $x : I_{t_0} \to \mathbb{R}^n$ of the system (1), (2), satisfying the condition

$$\lim_{t\to t_0} \big(H^{-1}(t)x(t)\big) = 0. \tag{3}$$

The analogous problem for the systems (1) of ordinary differential equations with singularities are investigated in [2–4].

The singularity of the system (1) is considered in the sense that the matrix $P$ and vector $q$ functions, in general, are not integrable at the point $t_0$. In general, the solution of the problem (1), (3) is not continuous at the point $t_0$ and, therefore, it is not a solution in the classical sense. But its restriction to every interval from $I_{t_0}$ is a solution of the system (1). In connection with this we give the example from [4].

Let $\alpha > 0$ and $\varepsilon \in ]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\,x}{t} + \varepsilon |t|^{\varepsilon - 1\alpha},$$
$$\lim_{t\to 0} (t^{\alpha} x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions to $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that equation.

We give sufficient conditions for the unique solvability of the problem (1), (2); (3). The analogous results belong to I. Kiguradze [3, 4] for the Cauchy problem for systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [1] (see, also the references herein).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R} = ]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ (or $O$) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^{n}$.

If $X \in \mathbb{R}^{n\times n}$, then $X^{-1}$, $\det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant of $X$ and the spectral radius of $X$; $I_n$ is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point $t$ $(X(a-) = X(a)$, $X(b+) = X(b))$.

$\widetilde{C}([a,b], D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\widetilde{C}_{loc}(I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}, D)$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}$ belong to $\widetilde{C}([a,b], D)$.

$L([a,b]; D)$ is the set of all integrable matrix-functions $X : [a,b] \to D$.

$L_{loc}(I_{t_0}; D)$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0}$ belong to $L([a,b], D)$.

A vector-function $x \in \widetilde{C}_{loc}(I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}, \mathbb{R}^n)$ is said to be a solution of the system (1), (2) if

$$x'(t) = P(t)\,x(t) + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}$$

and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1, 2, \dots)$ such that the equalities (2) hold.

We assume that

$$\det(I_n + G_l) \neq 0 \ \ (l = 1, 2, \dots).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P \in L_{loc}(I, \mathbb{R}^{n\times n})$ and $q \in L_{loc}(I, \mathbb{R}^n)$.

Let $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and $G_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$. Then a matrix-function $C_0 : I_{t_0} \times I_{t_0} \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \tag{4}$$

$$x(\tau_l+) - x(\tau_l-) = G_{0l}x(\tau_l) \ \ (l = 1, 2, \dots), \tag{5}$$

if for every interval $J \subset I_{t_0}$ and $\tau \in J$ the restriction of the matrix-function $C_0(\,\cdot\,,\tau) : I_{t_0} \to \mathbb{R}^{n\times n}$ to $J$ is the fundamental matrix of the system (4), (5) satisfying the condition $C_0(\tau,\tau) = I_n$. Therefore, $C_0$ is the Cauchy matrix of (4), (5) if and only if the restriction of $C_0$ on $J \times J$, for every interval $J \subset I_{t_0}$, is the Cauchy matrix of the system in the sense of definition given in [5].

We assume $I_{t_0}(\delta) = [t_0 - \delta, t_0 + \delta] \cap I_{t_0}$ for every $\delta > 0$.

**Theorem.** *Let there exist a matrix-function $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$ and $B_0, B \in \mathbb{R}_+^{n\times n}$ such that*

$$\det(I_n + G_{0l}) \neq 0 \ \ (l = 1, 2, \dots), \quad r(B) < 1,$$

*and the estimates*

$$|C_0(t,\tau)| \leq H(t) B_0 H^{-1}(\tau) \ \ \textit{for} \ \ t \in I_{t_0}(\delta), \quad (t - t_0)(\tau - t_0) > 0, \ \ |\tau - t_0| \leq |t - t_0|$$

*and*

$$\bigg|\int\limits_{t_0}^{t} \big|C_0(t,\tau)(P(\tau) - P_0(\tau))H(\tau)\big|\,d\tau\bigg| + \bigg|\sum_{l\in\mathcal{N}_{t_0,t}} \big|C_0(t,\tau_l)G_{0l}(I_n + G_{0l})^{-1}(G_l - G_{0l})\big|\bigg| \leq H(t)B \ \ \textit{for} \ \ t \in I_{t_0}(\delta)$$

*hold for some $\delta > 0$, where $C_0$ is the Cauchy matrix of the system* (4), (5). *Let, moreover,*

$$\lim_{t\to t_0}\bigg\|\int\limits_{t_0}^{t} H^{-1}(\tau)C_0(t,\tau)q(\tau)\,d\tau + \sum_{l\in\mathcal{N}_{t_0,t}} H^{-1}(\tau_l)C_0(t,\tau_l)G_{0l}(I_n + G_{0l})^{-1}g_l\bigg\| = 0.$$

*Then the problem* (1), (2); (3) *has the unique solution.*

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# Green–Samoilenko Function and Existence of Integral Sets of Linear Extensions of Differential Equations with Impulses

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We consider the following system of differential equations with impulsive perturbations [7, 9]

$$\frac{d\varphi}{dt} = a(t,\varphi), \quad \frac{dx}{dt} = P(t,\varphi)x + f(t,\varphi), \;\; t \neq \tau_i, \Delta x\big|_{t=\tau_i} = B_i(\varphi)x + I_i(\varphi), \tag{1}$$

where $t \in R$, $x \in R^n$, $\varphi \in \Im^m$, $\Im^m$ is an $m$-dimensional torus; $a(t,\varphi)$, $f(t,\varphi)$, $P(t,\varphi)$ are continuous (piecewise continuous with first-kind discontinuities at $t = \tau_i$) with respect to $t$, continuous and $2\pi$-periodic with respect to $\varphi_\nu$, $\nu = \overline{1,m}$, bounded for all $t \in R$, $\varphi \in \Im^m$ vector and matrix functions, respectively. Functions $B_i(\varphi)$ and $I_i(\varphi)$ are uniformly bounded with respect to $i \in Z$ matrices and vectors, $\det(E + B_i(\varphi)) \neq 0$ for any $\varphi \in \Im^m$. The sequence of the moments of impulsive perturbations $\{\tau_i\}$ is such that $\tau_i \to -\infty$ for $i \to -\infty$ and $\tau_i \to +\infty$ for $\tau_i \to +\infty$. We assume that there exists $\theta > 0$ such that for any $i \in Z$,

$$\tau_{i+1} - \tau_i \geq \theta > 0. \tag{2}$$

Function $a(t,\varphi)$ satisfies the Lipschitz condition with respect to $\varphi$ and

$$\sup_{t\in R} \big\| a(t,\varphi_1) - a(t,\varphi_2) \big\| \leq l \big\| \varphi_1 - \varphi_2 \big\| \tag{3}$$

holds uniformly with respect to $t \in R$. Additionally assume that functions $f(t,\varphi)$ and $I_i(\varphi)$ satisfy the following condition

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\| = M < \infty.$$

The problems of the existence of bounded solutions and integral sets for the system of the type (1) were considered in [1,2]. The problems of the persistence of integral sets under the perturbations of the right-hand side were considered in [3, 6]. In this paper, analogously to [4, 5, 8], we introduce the notion of Green–Samoilenko function of the problem on integral sets of differential equations with impulses and provide sufficient conditions for the existence of integral sets.

Consider the non-autonomous system of differential equations defined on the surface of the torus $\Im^m$

$$\frac{d\varphi}{dt} = a(t,\varphi) \tag{4}$$

and denote by $\varphi_t(\tau,\varphi)$ a solution of this system satisfying the initial condition $\varphi_\tau(\tau,\varphi) = \varphi$. From the compactness of the phase space of system (4) and the assumptions regarding function $a(t,\varphi)$, for any initial condition $\varphi_\tau(\tau,\varphi) = \varphi$, $\tau \in R$, $\varphi \in \Im^m$ the corresponding solution $\varphi_t(\tau,\varphi)$ exists and can be prolonged to the entire real axis $R$.

Consider the following non-homogenous system of differential equations with impulsive perturbations

$$\begin{aligned} \frac{dx}{dt} &= P(t,\varphi_t(\tau,\varphi))x + f(t,\varphi_t(\tau,\varphi)), \;\; t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} &= B_i(\varphi_{\tau_i}(\tau,\varphi))x + I_i(\varphi_{\tau_i}(\tau,\varphi)) \end{aligned} \tag{5}$$

and the corresponding homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= P(t, \varphi_t(\tau, \varphi))x, \quad t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} &= B_i(\varphi_{\tau_i}(\tau, \varphi))x, \end{aligned} \tag{6}$$

and denote by $\Omega_s^t(\tau, \varphi)$ the fundamental matrix of (6). Due to continuous dependance of $\varphi_t(\tau, \varphi)$ on parameters $\tau \in R$ and $\varphi \in \Im^m$, the fundamental matrix $\Omega_s^t(\tau, \varphi)$ depends on these parameters also continuously.

**Lemma.** *For any $t, s, \tau, \sigma \in R$ and $\varphi \in \Im^m$ the following relation holds*

$$\Omega_s^t(\tau, \varphi_\tau(\sigma, \varphi)) = \Omega_s^t(\sigma, \varphi).$$

Let $C(t, \varphi)$ be continuous $2\pi$-periodic with respect to each of the component $\varphi_\nu$, $\nu = \overline{1, m}$, piecewise continuous with respect to $t \in R$, with first-kind discontinuities at the points $\{\tau_i\}$ matrix function. Denote

$$G(t, s, \varphi) = \begin{cases} \Omega_s^t(t, \varphi) C(s, \varphi_s(t, \varphi)), & s \leq t, \\ -\Omega_s^t(t, \varphi)\big[E - C(s, \varphi_s(t, \varphi))\big], & s > t \end{cases} \tag{7}$$

and call $G(t, s, \varphi)$ Green–Samoilenko function of the system

$$\frac{d\varphi}{dt} = a(t, \varphi), \quad \frac{dx}{dt} = P(t, \varphi)x, \quad t \neq \tau_i, \qquad \Delta x\big|_{t=\tau_i} = B_i(\varphi)x,$$

if there exists $K > 0$ such that for all $t, s \in R$, $\varphi \in \Im^m$

$$\int\limits_{-\infty}^{\infty} \|G(t, s, \varphi)\|\, ds + \sum_{i=\infty-}^{+\infty} \|G(t, \tau_i + 0, \varphi)\| \leq K. \tag{8}$$

Next, we recall the basic properties of Green–Samoilenko function $G(t, s, \varphi)$. From its definition it follows that Green–Samoilenko function is continuous for all $t, s \in R$, $t \neq s$, $\varphi \in \Im^m$, $2\pi$-periodic with respect to $\varphi_\nu$, $\nu = \overline{1, m}$, and

$$G(s + 0, s, \varphi) - G(s - 0, s, \varphi) = E.$$

Taking the above lemma into account, we get

$$G(t, s, \varphi_t(\tau, \varphi)) = \begin{cases} \Omega_s^t(t, \varphi) C(s, \varphi_s(\tau, \varphi)), & s \leq t, \\ -\Omega_s^t(t, \varphi)\big[E - C(s, \varphi_s(\tau, \varphi))\big], & s > t. \end{cases} \tag{9}$$

For $s = \tau$, we obtain

$$G(t, \tau, \varphi_t(\tau, \varphi)) = \begin{cases} \Omega_\tau^t(t, \varphi) C(\tau, \varphi), & \tau \leq t, \\ -\Omega_\tau^t(t, \varphi)[E - C(\tau, \varphi)], & \tau > t. \end{cases}$$

Matrix $G(t, \tau, \varphi_t(\tau, \varphi))$ consists from solutions to the homogeneous system (6) for $t \geq \tau$ and $t < \tau$, respectively.

Consider the relation

$$\int\limits_{-\infty}^{+\infty} G(t, s, \varphi) f(s, \varphi_s(t, \varphi))\, ds + \sum_{i=-\infty}^{+\infty} G(t, \tau_i + 0, \varphi) I_i(\varphi_{\tau_i}(t, \varphi)).$$

From (2) and (8) we get

$$\Big\| \int_{-\infty}^{+\infty} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{i=-\infty}^{+\infty} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)) \Big\| \\ \le \frac{2K}{\gamma} \sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \frac{2K}{1-e^{-\gamma\theta}} \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\|.$$

Finally denote

$$u(t,\varphi) = \int_{-\infty}^{+\infty} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{i=-\infty}^{+\infty} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)). \tag{10}$$

**Theorem 1.** *Let functions $a(t,\varphi)$, $f(t,\varphi)$, $P(t,\varphi)$ from system* (1) *be continuous with respect to $t$, continuous and $2\pi$-periodic with respect to $\varphi_\nu$, $\nu=\overline{1,m}$, bounded for all $t\in R$, $\varphi\in\Im^m$ vector and matrix functions, respectively. Let function $a(t,\varphi)$ satisfy condition* (3)*, functions $B_i(\varphi)$ and $I_i(\varphi)$ be uniformly bounded with respect to $i$ matrices and vectors, $\det(E+B_i(\varphi))\neq 0$ for any $\varphi\in\Im^m$. Let for the sequence of impulsive perturbations $\{\tau_i\}$ estimate* (2) *hold. Let also there exist Green–Samoilenko function $G(t,s,\varphi)$. Then formula* (10) *defines an integral set of system* (1) *and*

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|u(t,\varphi)\| \le \frac{2K}{\gamma} \sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \frac{2K}{1-e^{-\gamma\theta}} \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\|. \tag{11}$$

Now assume that the fundamental matrix $\Omega_s^t(\tau,\varphi)$ of system (6) satisfies the estimate

$$\|\Omega_s^t(\tau,\varphi)\| \le Ke^{-\gamma(t-s)} \tag{12}$$

for any $t\ge s\in R$, $\tau\in R$, $\varphi\in\Im^m$ and some $K\ge 1$, $\gamma>0$. In this case there exists Green–Samoilenko function of the following form

$$G(t,s,\varphi) = \begin{cases} \Omega_s^t(t,\varphi), & s<t, \\ 0, & s\ge t. \end{cases} \tag{13}$$

The corresponding integral set of system (1) gets the representation

$$x = u(t,\varphi) = \int_{-\infty}^{t} G(t,s,\varphi) f(s,\varphi_s(t,\varphi))\,ds + \sum_{\tau_i<t} G(t,\tau_i+0,\varphi) I_i(\varphi_{\tau_i}(t,\varphi)), \ \ t\in R, \ \ \varphi\in\Im^m. \tag{14}$$

**Theorem 2.** *Let system* (1) *satisfy the condition of Theorem* 1*. Let also the fundamental matrix $\Omega_s^t(\tau,\varphi)$ of system* (6) *satisfy inequality* (12)*. Then system* (1) *has an asymptotically stable integral set* (14) *and this set satisfies the following estimate*

$$\sup_{t\in R} \max_{\varphi\in\Im_m} \|u(t,\varphi)\| \le K_0 \Big[ \sup_{t\in R} \max_{\varphi\in\Im_m} \|f(t,\varphi)\| + \sup_{i\in Z} \max_{\varphi\in\Im_m} \|I_i(\varphi)\| \Big],$$

*where*

$$K_0 = \frac{K}{\gamma} + K \sup_{t\in R} \sum_{\tau_i<t} e^{-\gamma(t-\tau_i)}.$$

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# On Power-Law Asymptotic Behavior of Solutions to Weakly Superlinear Emden–Fowler Type Equations

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## 1 Introduction

Consider the equation

$$y^{(n)} = |y|^k \operatorname{sgn} y \tag{1.1}$$

with $k > 1$. Hereafter, we put $\gamma = \frac{k-1}{n}$ and $m = n - 1$.

**Definition 1.1.** A solution $y(x)$ of equation (1.1) will be said to be *n-positive* if it is maximally extended in both directions and eventually satisfies the inequalities

$$y(x) > 0, \ \ y'(x) > 0, \dots, y^{(m)}(x) > 0.$$

Note that if the above inequalities are satisfied by a solution of (1.1) at some point $x_0$, then they are also satisfied at any point $x > x_0$ in the domain of the solution. Moreover, such a solution, if maximally extended, must be a so-called blow-up solution (having a vertical asymptote at the right endpoint of its domain).

Immediate calculations show that equation (1.1) has $n$-positive solutions with exact power-law behavior, namely,

$$y(x) = C(x^* - x)^{-1/\gamma}, \ \text{ where } \ C^{k-1} = \prod_{j=0}^{m} \Big(j + \frac{1}{\gamma}\Big), \tag{1.2}$$

defined on $(-\infty, x^*)$ with arbitrary $x^* \in \mathbb{R}$. For $n = 1$ all $n$-positive solutions of (1.1) are defined by (1.2). For $n \in \{2, 3, 4\}$ it is known that any $n$-positive solution of (1.1) and even of more general equations is asymptotically equivalent, near the right endpoint of its domain, to the solution defined by (1.2) with appropriate $x^*$ (see [5] for $n = 2$, and [1–3] for $n \in \{3, 4\}$).

The natural hypothesis generalizing this statement for all $n > 4$ appears to be wrong (see [6] for sufficiently large $n$ and [4] for $n \in \{12, 13, 14\}$).

However, a weaker version of this statement for higher-order equations can be proved.

## 2 Main result

**Theorem 2.1.** *For any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, any $n$-positive solution of equation* (1.1) *is asymptotically equivalent, near the right endpoint of its domain, to a solution with exact power-law behavior.*

To prove this result, an auxiliary dynamical system is investigated on the $m$-dimensional sphere. To define it note that if a function $y(x)$ is a solution of equation (1.1), the same is true for the function

$$z(x) = Ay(A^\gamma x + B) \tag{2.1}$$

with any constants $A > 0$ and $B$. Any non-trivial solution $y(x)$ of equation (1.1) generates in $\mathbb{R}^n \setminus \{0\}$ the curve given parametrically by

$$(y(x), y'(x), y''(x), \dots, y^{(m)}(x)).$$

We can define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (2.1) with $A > 0$ generate equivalent curves, i.e. curves passing through equivalent points (maybe for different $x$). We assume the points $(y_0, y_1, y_2, \dots, y_m)$ and $(z_0, z_1, z_2, \dots, z_m)$ in $\mathbb{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a constant $\lambda > 0$ such that

$$z_j = \lambda^{n+j(k-1)} y_j, \;\; j \in \{0, 1, \dots, m\}.$$

The quotient space obtained is homeomorphic to the $m$-dimensional sphere

$$S^m = \big\{ y \in \mathbb{R}^n : \; y_0^2 + y_1^2 + y_2^2 + \dots + y_m^2 = 1 \big\}$$

having exactly one representative of each equivalence class since the equation

$$\lambda^{2n} y_0^2 + \lambda^{2(n+2(k-1))} y_1^2 + \dots + \lambda^{2(n+m(k-1))} y_m^2 = 1$$

has exactly one positive root $\lambda$ for any $(y_0, y_1, y_2, \dots, y_m) \in \mathbb{R}^n \setminus \{0\}$. Equivalent curves in $\mathbb{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulae using different independent variables. A unique common independent variable can be obtained from those ones by using a partition of unity.

Within the chart that covers the points corresponding to positive values of solutions and has the coordinate functions

$$u_j = y^{(j)} y^{-1-\gamma j}, \;\; j \in \{1, \dots, m\}, \tag{2.2}$$

the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - (1+\gamma) u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - (1+\gamma j) u_1 u_j, \quad j \in \{2, \dots, m-1\}, \\ \dfrac{du_m}{dt} = 1 - (1+\gamma m) u_1 u_m \end{cases} \tag{2.3}$$

with the independent variable

$$t = \int_{x_0}^{x} y(\xi)^\gamma \, d\xi.$$

The dynamical system described has some equilibrium points corresponding to the solutions of equation (1.1) with exact power-law behavior. One of them, which corresponds to the $n$-positive solutions with exact power-law behavior, can be found in terms of its $u_j$ coordinates denoted by $(a_1, \dots, a_m)$:

$$\begin{cases} a_{j+1} = (1+\gamma j) a_1 a_j = a_1^{j+1} \prod\limits_{l=1}^{j} (1+\gamma l), \quad j \in \{1, \dots, m-1\}, \\ a_1 = \Big( \prod\limits_{l=1}^{m} (1+\gamma l) \Big)^{-1/n}. \end{cases} \tag{2.4}$$

Instead of system (2.3) it is more convenient for our current purposes to use another one obtained by the substitution $\tau = a_1 t$, $u_j = a_j v_j$, $j \in \{1, \dots, m\}$:

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(1 - v_1 v_m). \end{cases} \tag{2.5}$$

The above equilibrium point has in the new chart all coordinates equal to 1.

**Lemma 2.1.** *There exist $\gamma_1 > 0$ and $r > 0$ such that for any real $\gamma \in [0, \gamma_1]$, the Jacobian matrix of system (2.5) at the point $(1, \dots, 1)$ has $m$ different eigenvalues with real parts less than $-r$.*

*Proof.* First, consider the mentioned Jacobian $m \times m$ matrix for $\gamma = 0$:

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & \dots & 0 & 0 \\ -1 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & -1 & 1 \\ -1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

We prove by mathematical induction that its characteristic polynomial is equal to

$$P_m(\lambda) = \frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda}. \tag{2.6}$$

For $m = 1$ this is proved immediately:

$$P_1(\lambda) = -2 - \lambda = -\frac{(1+\lambda)^2 - 1}{\lambda} = \frac{(1+\lambda)^{1+1} - 1}{(-1)^1 \lambda}.$$

If (2.6) is proved for some $m$, then $P_{m+1}(\lambda)$ can be calculated by expanding along the last row as follows:

$$\begin{aligned} P_{m+1}(\lambda) &= (-1) \cdot (-1)^m + (-1-\lambda) P_m(\lambda) \\ &= (-1)^{m+1} - (1+\lambda) \cdot \frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda} = \frac{(1+\lambda)^{m+2} - 1}{(-1)^{m+1} \lambda}. \end{aligned}$$

Now (2.6) is proved for $m + 1$, too.

The roots of the polynomial $P_m(\lambda)$ are equal to

$$\lambda_j = -1 + \cos \frac{2\pi j}{n} + i \sin \frac{2\pi j}{n}, \quad j \in \{1, \dots, m\},$$

with $j = 0$ excluded because of the denominator in (2.6). The real parts of the roots are less then or equal to $-2 \sin^2 \frac{\pi}{n}$. Since all roots of the polynomial are different and therefore simple, they depend continuously on the coefficients of the polynomial. Hence for sufficiently small $\gamma > 0$ the Jacobian matrix of system (2.5) at the point $(1, \dots, 1)$ has all eigenvalues with real part less than $-\sin^2 \frac{\pi}{n}$. □

**Lemma 2.2.** *If $\gamma = 0$, then any trajectory of system (2.5) passing through a point with positive $v_j$ coordinates tends to the equilibrium point $(1, \dots, 1)$.*

*Proof.* Trajectories of (2.5) passing through a point with positive $v_j$ coordinates correspond to $n$-positive solutions of equation (1.1). Trajectories of (2.5) with $\gamma = 0$ correspond to solutions of the linear equation $y^{(n)} = y$, which are all known exactly. They are

$$y(x) = C_0\, e^x + \sum_{j=1}^{\lfloor m/2 \rfloor} C_j e^{r_j x} \sin(\omega_j x + \varphi_j) + \widetilde{C} e^{-x}$$

with $r_j = \cos \frac{2\pi j}{n} < 1$, $\omega_j = \sin \frac{2\pi j}{n}$, and arbitrary constants $C_j$, $\varphi_j$, $\widetilde{C}$, though the last one must equal 0 whenever $n$ is odd. Such a solution is $n$-positive if and only if the constant $C_0$ is greater than 0. But in this case, all $v_j$ coordinates of the related trajectory, which are equal to $y^{(j)}/y$ whenever $\gamma = 0$, tend to 1. □

Up to the moment, we actually considered, for each $\gamma > 0$, its own dynamical system defined on its own quotient space homeomorphic to the $m$-dimensional sphere. In what follows, we need one sphere with a $\gamma$-parameterized dynamical system having an equilibrium point common for all $\gamma$ in consideration. Thus, the points $(y_0, y_1, \dots, y_m) \in \mathbb{R} \setminus \{0\}$ obtained while treating solutions of (1.1) with different $\gamma$ will generate the same point on the sphere $S^m$ if their corresponding coordinates have the same sign and the tuples

$$\bigg( |y| : \Big|\frac{y'}{a_1}\Big|^{\frac{1}{1+\gamma}} : \dots : \Big|\frac{y^{(j)}}{a_j}\Big|^{\frac{1}{1+\gamma j}} : \dots : \Big|\frac{y^{(m)}}{a_m}\Big|^{\frac{1}{1+\gamma m}} \bigg),$$

if considered as sets of projective coordinates, define the same point in the projective space $\mathbb{R}P^m$. In particular, for points corresponding to $n$-positive solutions this means that they have the same $v_j$ coordinates in the related charts. Hereafter, the domain consisting of all points with positive $v_j$ coordinates is denoted by $S^m_+$. The only equilibrium point in $S^m_+$, which has all $v_j$ coordinates equal to 1, is denoted by $v^*$.

**Lemma 2.3.** *There exist $\gamma_2 > 0$ and an open neighborhood $U$ of the point $v^*$ such that for any positive $\gamma < \gamma_2$, any trajectory of the global dynamical system passing through the closure $\overline{U}$ tends to $v^*$. If such a trajectory does not coincide with $v^*$, then it passes transversally, at some time, through the boundary $\partial U$.*

*Proof.* Now, once more, we choose other local coordinates to describe the dynamical system on $S^m_+$. First, we use a translation continuous in $\gamma$ to put the equilibrium point to 0. Then a linear complex transformation also continuous in $\gamma$ is used to make the linear part of the right-hand side to be a diagonal matrix. If the new complex coordinates are $w_j$, then our dynamical system can be written as

$$\frac{dw_j}{d\tau} = \lambda_j(\gamma) w_j + q_j(w, \gamma), \ \ j \in \{1, \dots, m\}, \tag{2.7}$$

with some functions $q_j(w, \gamma)$ quadratic in $w$ and continuous in $\gamma$. There exists a constant $Q > 0$ such that $|q_j(w, \gamma)|^2 \le Q|w|^2$ for all $j \in \{1, \dots, m\}$, all $w \in \mathbb{C}^m$, and all positive $\gamma \le \gamma_1$, where $|w|^2 = \sum\limits_{j=1}^m |w_j|^2$ and the constant $\gamma_1$ is taken from Lemma 2.1.

Now consider the quadratic function $|w|^2$ and note that

$$\frac{d|w|^2}{d\tau} = 2 \sum_{j=1}^m \operatorname{Re} \big( \lambda_j(\gamma) |w_j|^2 + q_j(w, \gamma) \overline{w}_j \big) < 2|w|^2 \big( -r + Q|w| \big)$$

with the constant $r > 0$ from Lemma 2.1.

Hence $\frac{d\log|w|^2}{d\tau} < -r$ if $|w| < -\frac{r}{2Q}$. Now, the equilibrium point $v^*$ has the neighborhood $U$ defined by the last inequality. For any trajectory passing through $\overline{U}$ we have $\log|w|^2 \to -\infty$ as $t \to \infty$, which means that all such trajectories tend to $v^*$. Since the function $\log|w|^2$ is defined for all points of $\overline{U} \setminus \{v^*\}$, the above estimate of $\frac{d\log|w|^2}{d\tau}$ proves the last statement of the current lemma. □

To complete the proof of the Theorem 2.1, consider the set difference of the closure $\overline{S^m_+}$ and the neighborhood $U$ from Lemma 2.3. This compact set will be denoted by $B$. Further, consider the function $f$ defined on $B$ and equal, for each point $b \in B$, to the time needed for the trajectory of the dynamical system with $\gamma = 0$ to reach $\partial U$ starting from $b$. This time is well-defined due to Lemma 2.2.

By the implicit function theorem, $f$ is a $C^1$ function. Its derivative along the trajectories with $\gamma = 0$ is equal to $-1$. Since the dynamical system depends continuously on $\gamma$, and $B$ is compact, there exists $\gamma_3 > 0$ such that for all $\gamma \in [0, \gamma_3)$, the derivative of $f$ along all trajectories with such $\gamma$ is less than to $-\frac{1}{2}$. This means that any trajectory with such $\gamma$ passing through $B$ must reach $\partial U$. Hence, due to Lemma 2.3, any trajectory with $\gamma \in [0, \min\{\gamma_2, \gamma_3\})$ starting from any point $b \in S^*_+$ must tend to the equilibrium point $v^*$, which corresponds to the $n$-positive solutions of equation (1.1) with exact power-law behavior (1.2). Putting $K = 1 + n\min\{\gamma_2, \gamma_3\}$ we complete the proof of Theorem 2.1. □

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# Exact Extreme Bounds of Mobility of the Lower and the Upper Bohl Exponents of the Linear Differential System Under Small Perturbations of its Coefficient Matrix

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1}$$

of dimension $n \geqslant 2$ with uniformly bounded $(\sup\{\|A(t)\| : \ t \geqslant 0\} < +\infty)$ and piecewise continuous on the semi axle $t \geqslant 0$ coefficient matrix. We denote by $\mathcal{X}(A)$ the set of all nonzero solutions to the system (1), and by $X_A(\,\cdot\,,\,\cdot\,)$ – its Cauchy matrix. Let $\mathcal{M}_n$ be the metric space of the systems (1) with the metric of uniform convergence of their coefficients on the semi axle. The lower $\underline{\beta}[x]$ and the upper $\overline{\beta}[x]$ Bohl exponents of a solution $x(\,\cdot\,) \in \mathcal{X}(A)$ are defined, respectively, by the formulas [3, pp. 171, 172], [5]

$$\underline{\beta}[x] = \varliminf_{t-\tau \to +\infty} \frac{1}{t-\tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|} \ \text{ and } \ \overline{\beta}[x] = \varlimsup_{t-\tau \to +\infty} \frac{1}{t-\tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|}\,,$$

and the quantities

$$\omega_0(A) = \varliminf_{t-\tau \to +\infty} \frac{1}{t-\tau} \ln \|X_A^{-1}(t,\tau)\|^{-1} \ \text{ and } \ \Omega^0(A) = \varlimsup_{t-\tau \to +\infty} \frac{1}{t-\tau} \ln \|X_A(t,\tau)\| \tag{2}$$

are called, respectively, the lower and the upper general exponents (they are also known as singular exponents) of the system (1) [3, p. 172].

The following obvious inequalities can't be in general case replaced by equalities [1]:

$$\omega_0(A) \leqslant \inf_{x \in \mathcal{X}(A)} \underline{\beta}[x] \ \text{ and } \ \sup_{x \in \mathcal{X}(A)} \overline{\beta}[x] \leqslant \Omega^0(A);$$

in particular, it is possible, that the exponents $\omega_0(A)$ and $\Omega^0(A)$ can not be implemented on any solution of the system (1).

R. E. Vinograd proved [5] the following equalities

$$\omega_0(A) = \lim_{\varepsilon \to +0} \inf_{\|Q\| \leqslant \varepsilon} \inf_{x \in \mathcal{X}(A+Q)} \underline{\beta}[x] \quad \text{and} \quad \Omega^0(A) = \lim_{\varepsilon \to +0} \sup_{\|Q\| \leqslant \varepsilon} \sup_{x \in \mathcal{X}(A+Q)} \overline{\beta}[x], \tag{3}$$

i.e., in other words, the lower (the upper) general exponent of the system (1) is the exact lower (upper) bound of the lower (the upper) Bohl exponents of the solutions $x(\,\cdot\,) \in \mathcal{X}(A)$ under arbitrary small perturbations of coefficient matrix of the system (1).

From the geometric point of view the lower $\omega_0(A)$ and the upper $\Omega^0(A)$ general exponents of the system (1) are asymptotically accurate when $t-\tau\to+\infty$, respectively, lower bound of the minor semi axis and upper bound of the major semi axis on a logarithmic scale of family of ellipsoids $E_{t,\tau}\overset{\text{def}}{=}\{\xi\in\mathbb{R}^n:\ \|X_A^{-1}(t,\tau)\xi\|=1\}$ (spectral matrix norm), which are generated by linear mappings $X_A(t,\tau)$, $t\geqslant\tau\geqslant0$. From this point of view it seems natural to consider along with the quantities (2) the quantities

$$\omega^0(A)=\overline{\lim_{t-\tau\to+\infty}}\,\frac{1}{t-\tau}\,\ln\|X_A^{-1}(t,\tau)\|^{-1}\ \text{ and }\ \Omega_0(A)=\underline{\lim_{t-\tau\to+\infty}}\,\frac{1}{t-\tau}\,\ln\|X_A(t,\tau)\|,\tag{4}$$

which give asymptotically accurate when $t-\tau\to+\infty$, respectively, upper bound of the minor semi axis and lower bound of the major semi axis on a logarithmic scale of family of ellipsoids $E_{t,\tau}$, and find out whether the values (4) are connected by equalities similar to (3) with the Bohl exponents of solutions to the pertubed systems.

The introduced exponents $\omega^0(A)$ and $\Omega_0(A)$ are called, respectively, the junior upper and the senior lower Bohl exponents of the system (1) (according to this terminology the exponents $\omega_0(A)$ and $\Omega^0(A)$ are called the junior lower and the senior upper Bohl exponents of the system (1)). The quantities (2) and (4) complement each other and give an asymptotically accurate two-sided estimates of variation of the norms $\|X_A(t,\tau)\|$ and $\|X_A^{-1}(t,\tau)\|$ when $t-\tau\to+\infty$. The exponents (4) were introduced in the review article by the authors [2], the motivation of their consideration was described above. In the paper [2] the authors, being based only on the formulas (3) and the mentioned above analogy of the quantities (2) and (4), gave without proof, due to the style of the mentioned paper, the following, similar to (3), formulas, which connect the exponents (4) of the system (1) and the Bohl exponents of perturbed systems

$$\omega^0(A)=\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\inf_{x\in\mathcal{X}(A+Q)}\overline{\beta}(x)\ \text{ and }\ \Omega_0(A)=\lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\sup_{x\in\mathcal{X}(A+Q)}\underline{\beta}[x]\tag{5}$$

considering that the proof of these equalities is completely analogous to the proof of the equalities (3) from paper [5], and even attributing it to the paper [5]. It appears that in general case the equalities (5) don't take place, as the following theorem shows.

**Theorem 1.** *The inequalities*

$$\omega^0(A)\geqslant\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\inf_{x\in\mathcal{X}(A+Q)}\overline{\beta}[x]\ \textit{ and }\ \Omega_0(A)\leqslant\lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\sup_{x\in\mathcal{X}(A+Q)}\underline{\beta}[x]\tag{6}$$

*are valid, and for every natural $n\geqslant2$ there exist such systems* (1) *for which each of these inequalities is strict.*

Let us denote by $\omega_*^0(A)$ and $\Omega_0^*(A)$ the right sides of the inequalities (6) respectively, in other words the exponent $\omega_*^0(A)$ is the exact lower bound of the upper Bohl exponents, and the exponent $\Omega_0^*(A)$ is the exact upper bound of the lower Bohl exponents of the solutions $x(\,\cdot\,)\in\mathcal{X}(A)$ under arbitrary small perturbations of coefficient matrix of the system (1). The exact expressions for the quantities $\omega_*^0(A)$ and $\Omega_0^*(A)$ using the Cauchy matrix of the system (1) are given in the following theorem.

**Theorem 2.** *The equalities*

$$\omega_*^0(A)=\lim_{T\to+\infty}\overline{\lim_{k-m\to+\infty}}\,\frac{1}{(k-m)T}\sum_{i=m+1}^{k}\ln\|X_A^{-1}(iT,(i-1)T)\|^{-1},$$

$$\Omega_0^*(A)=\lim_{T\to+\infty}\underline{\lim_{k-m\to+\infty}}\,\frac{1}{(k-m)T}\sum_{i=m+1}^{k}\ln\|X_A(iT,(i-1)T)\|,$$

*where $k,m\in\mathbb{N}$, are valid.*

The fact that the right sides of these equalities are correctly defined (i.e. that the outer limits in the right sides of these equalities exist), is established in the proof of Theorem 2.

The mentioned above theorem by R. E. Vinograd [5] (see the relations (2) and (3)) and Theorem 2 give the formulas for calculating, using the Cauchy matrix of the system (1), of the exact extreme bounds of variation (mobility) of the upper and the lower Bohl exponents of the solutions under small perturbations of its coefficient matrix. Consider how these exact bounds $\Omega^0(A)$, $\omega^0_*(A)$ and $\Omega^*_0(A)$, $\omega_0(A)$, as well as the quantities $\Omega_0(A)$ and $\omega^0(A)$, can vary themselves under small perturbations of the coefficient matrix of the system (1). Let us recall that a real-valued function, defined on a metric space $\mathcal{M}_n$, is called upwards stable (downwards stable), if it is upper (respectively, lower) semicontinuous function on this space.

The exponent $\Omega^0(\,\cdot\,)$ is upwards stable, and the exponent $\omega_0(\,\cdot\,)$ is downwards stable [3, p. 180], but they are both unstable in the opposite directions, if $n \geqslant 2$ [4]. The exponents $\Omega^*_0(A)$ and $\omega^0_*(A)$ possess the same properties, as the following theorems show, but neither $\Omega_0(A)$ nor $\omega^0(A)$ do.

**Theorem 3.** *The exponent $\Omega^*_0(\,\cdot\,)$ is upwards stable, and the exponent $\omega^0_*(\,\cdot\,)$ is downwards stable.*

**Theorem 4.** *If $n \geqslant 2$, the exponent $\Omega^*_0(\,\cdot\,)$ is downwards unstable, and the exponent $\omega^0_*(\,\cdot\,)$ is upwards unstable, i.e. for $n \geqslant 2$ there exist such systems $A \in \mathcal{M}_n$, for which the inequalities*

$$\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\Omega^*_0(A+Q) < \Omega^*_0(A) \quad \text{and} \quad \lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\omega^0_*(A+Q) > \omega^0_*(A)$$

*hold, respectively.*

**Theorem 5.** *Each of the exponents $\Omega_0(A)$ and $\omega^0(A)$ is neither upwards, nor downwards stable under small perturbations of the coefficient matrix.*

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# On the Baire Classes of the Sergeev Lower Frequency of Zeros, Signs, and Roots of Linear Differential Equations

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For a given positive integer $n$, by $\widetilde{\mathcal{E}}^n$ we denote the set of linear homogeneous $n$th-order differential equations

$$y^{(n)}+a_1(t)y^{(n-1)}+\dots+a_{n-1}(t)\dot{y}+a_n(t)y=0,\quad t\in\mathbb{R}_+\stackrel{\rm def}{=}[0,+\infty), \tag{1}$$

with continuous coefficients $a_i(\,\cdot\,)\colon\mathbb{R}_+\to\mathbb{R}$, $i=\overline{1,n}$. We identify the equation (1) and the row $a=a(\,\cdot\,)=(a_1(\,\cdot\,),\dots,a_n(\,\cdot\,))$ of its coefficients and hence denote the equation (1) by $a$ as well. By $S(a)$ we denote the solution set of the equation $a$, and by $S_*(a)$ we denote the set of its nonzero solutions, i.e. $S_*(a)=S(a)\setminus\{\mathbf{0}\}$.

Let $y(\,\cdot\,)$ be a real-valued function defined on some set $D\subset\mathbb{R}$. A point $t\in D$ is called a sign change point of a function $y(\,\cdot\,)$ if, in any neighborhood of that point, the function $y(\,\cdot\,)$ takes values of opposite signs. If $y(\,\cdot\,)$ is a continuous function, then a sign change point is its zero. If the function $y(\,\cdot\,)$ is defined in some neighborhood of its zero $t_0$, then the zero $t_0$ is referred to as a root of multiplicity $k$ of the function $y(\,\cdot\,)$ if at the point $t_0$ its first $k-1$ derivatives are zero and the $k$th derivative is nonzero.

Next, by $\varkappa$ we denote a symbol that takes values in the set of three elements $\{0,-,+\}$. For a function $y(\,\cdot\,)\colon\mathbb{R}_+\to\mathbb{R}$ and a number $t>0$, by $\nu^\varkappa(y(\,\cdot\,);t)$ we denote the following quantities for the function $y(\,\cdot\,)$ on the half-open interval $[0,t)$ depending on the value of $\varkappa$ : the number of zeros if $\varkappa=0$, the number of sign changes if $\varkappa=-$, and the sum of root multiplicities if $\varkappa=+$. If $t_0=0$ is a zero of the function $y(\,\cdot\,)$, then, for the computation of its multiplicity, all desired derivatives are considered to be right-sided. If the number of zeros or the number of sign changes or roots of the function $y(\,\cdot\,)$ on the half-open interval $[0,t)$ is infinite, then the corresponding values are considered to be equal to $+\infty$. It is easy to see that $\nu^\varkappa(y(\,\cdot\,);t)$ is a finite integer number for every symbol $\varkappa\in\{0,-,+\}$, nonzero solution $y(\,\cdot\,)$ of the equation (1), and $t>0$. Sergeev [7]– [9] introduced the following notion.

**Definition.** For any nonzero solution $y(\,\cdot\,)\in S_*(a)$ of the system $a$ the quantities

$$\hat{\nu}^{\varkappa}[y]\stackrel{\rm def}{=}\varlimsup_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y(\,\cdot\,);t)\ \text{ and }\ \check{\nu}^{\varkappa}[y]\stackrel{\rm def}{=}\varliminf_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y(\,\cdot\,);t) \tag{2}$$

are called the upper and lower characteristic frequencies, respectively, of zeros if $\varkappa=0$, signs if $\varkappa=-$, and roots if $\varkappa=+$.

Generally, the value of the quantities $\check{\nu}^{\varkappa}[y]$ and/or $\hat{\nu}^{\varkappa}[y]$ can be equal to $+\infty$. By $\overline{\mathbb{R}}$ we denote the extended numerical axis ($\overline{\mathbb{R}}=\mathbb{R}\sqcup\{-\infty,+\infty\}$) considered in the natural (ordinal) topology, and by $\overline{\mathbb{R}}_+$ we denote its nonnegative semiaxis.

For any $a\in\widetilde{\mathcal{E}}^n$, the asymptotic characteristics (2) define the mappings

$$\hat{\nu}^{\varkappa}[\,\cdot\,]\colon S_*(a)\to\overline{\mathbb{R}}_+\ \text{ and }\ \check{\nu}^{\varkappa}[\,\cdot\,]\colon S_*(a)\to\overline{\mathbb{R}}_+,\ \ \varkappa\in\{0,-,+\}, \tag{3}$$

acting by the rules $y \longmapsto \hat{\nu}^{\varkappa}[y]$ and $y \longmapsto \check{\nu}^{\varkappa}[y]$, respectively. Instead of the mappings (3), it is more convenient to consider the functions $\hat{\nu}^{\varkappa}(\,\cdot\,)$ and $\check{\nu}^{\varkappa}(\,\cdot\,)$, $\varkappa \in \{0,-,+\}$, respectively, which are defined as follows. Since, between the vector space $S(a)$ of solutions of an equation $a \in \widetilde{\mathcal{E}}^n$ and the vector space $\mathbb{R}^n$, there is a natural isomorphism $\iota : S(a) \to \mathbb{R}^n$ acting by the rule $y(\,\cdot\,) \longmapsto (y(0), \dot{y}(0), \dots, y^{(n-1)}(0))^\top$, it follows that the mappings (3) define the functions

$$\hat{\nu}^{\varkappa}(\,\cdot\,) \stackrel{\text{def}}{=} \hat{\nu}^{\varkappa}[\,\cdot\,] \circ \iota^{-1} \colon \mathbb{R}^n_* \to \overline{\mathbb{R}}_+ \quad \text{and} \quad \check{\nu}^{\varkappa}(\,\cdot\,) \stackrel{\text{def}}{=} \check{\nu}^{\varkappa}[\,\cdot\,] \circ \iota^{-1} \colon \mathbb{R}^n_* \to \overline{\mathbb{R}}_+, \quad \varkappa \in \{0,-,+\}, \tag{4}$$

where $\mathbb{R}^n_* \stackrel{\text{def}}{=} \mathbb{R}^n \setminus \{\mathbf{0}\}$. Conversely, since $\iota$ is a bijection, one can see that the functions (4) define the mappings (3). The functions (4) have the following advantage in comparison with the mappings (3): the domains of those functions coincide for all equations from the set $\widetilde{\mathcal{E}}^n$.

Since the functions (4) (and the mappings (3)) are constant on any one-dimensional linear subspace with the excluded zero, it follows that, instead of the functions $\hat{\nu}^{\varkappa}(\,\cdot\,)$ and $\check{\nu}^{\varkappa}(\,\cdot\,)$, $\varkappa \in \{0,-,+\}$, one can consider their restrictions to the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with center the origin. The function $\hat{\nu}^{\varkappa}(\,\cdot\,)$ (respectively, the function $\check{\nu}^{\varkappa}(\,\cdot\,)$) with $\varkappa = 0,-,+$ is referred [3], [4] to as the Sergeev upper (respectively, lower) frequency of zeros, signs, and roots of the equation (1), respectively. The image $\hat{\nu}^{\varkappa}(S_*(a))$ (respectively, the image $\check{\nu}^{\varkappa}(S_*(a))$) of the function $\hat{\nu}^{\varkappa}(\,\cdot\,)$ (respectively, the function $\check{\nu}^{\varkappa}(\,\cdot\,)$) is referred to as the upper (respectively, lower) frequency spectra of zeros if $\varkappa = 0$, signs if $\varkappa = -$, and roots if $\varkappa = +$.

The descriptions of the Baire classes and the spectra of the Sergeev upper frequency of zeros, signs, and roots of the equation (1) were provided in [2]. In this paper we present results on the Baire classes and structure of the spectra of the Sergeev lower frequency of zeros, signs, and roots of the equation (1).

To formulate our results let us briefly give some necessary notations and definitions. Let $f(\,\cdot\,)$ be a real- or $\overline{\mathbb{R}}$-valued function defined on some set $\mathcal{M}$. For each number $r \in \mathbb{R}$ and for a function $f(\,\cdot\,)$, the Lebesgue sets $[f > r]$ and $[f \geqslant r]$ are defined as the sets $[f > r] = \{t \in \mathcal{M} \colon f(t) > r\}$ and $[f \geqslant r] = \{t \in \mathcal{M} \colon f(t) \geqslant r\}$. The sets $[f < r]$ and $[f \leqslant r]$ have a similar meaning (the complements of the corresponding Lebesgue sets in $\mathcal{M}$), and $[f = r]$ is a level set of the function $f(\,\cdot\,)$. As usual, here we assume that $-\infty < r < +\infty$ for any $r \in \mathbb{R}$.

If $\mathcal{M}$ is a topological space, then its five first Borel classes of sets are known to be defined as follows [5, p. 192], [1, p. 108]. The zero class consists of closed and open sets (their classes are denoted by $F$ and $G$, respectively). The first class consists of sets of the type $G_\delta$ and the type $F_\sigma$ ($G_\delta$-sets and $F_\sigma$-sets) those are sets, which can be represented as countable intersections of open sets and countable unions of closed sets, respectively. The second class consists of sets of the type $F_{\sigma\delta}$ and the type $G_{\delta\sigma}$ ($F_{\sigma\delta}$-sets and $G_{\delta\sigma}$-sets) those are sets, which can be represented as countable intersections of $F_\sigma$-sets and countable unions of $G_\delta$-sets, respectively. Analogically, one can define sets of the type $G_{\delta\sigma\delta}$ and the type $F_{\sigma\delta\sigma}$, which form the third Borel class, and sets of the type $F_{\sigma\delta\sigma\delta}$ and the type $G_{\delta\sigma\delta\sigma}$ of the fourth Borel class.

Let $M$ and $N$ be some systems of subsets in $\mathcal{M}$. We say [5, pp. 223, 224] that a function $f(\,\cdot\,) \colon \mathcal{M} \to \mathbb{R}$ or $f(\,\cdot\,) \colon \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the class $(M, {}^*)$, or $f(\,\cdot\,)$ is a function of the class $(M, {}^*)$ if its Lebesgue set satisfies the condition $[f > r] \in M$ for any $r \in \mathbb{R}$. If $[f \geqslant r] \in N$ for any $r \in \mathbb{R}$, then we say that the function $f(\,\cdot\,)$ belongs to the class $({}^*, N)$, or $f(\,\cdot\,)$ is a function of the class $({}^*, N)$. If a function $f(\,\cdot\,)$ belongs to each of the classes $(M, {}^*)$ and $({}^*, N)$, then we say that it belongs to the class $(M, N)$, or it is a function of the class $(M, N)$. We say ( [5, pp. 248, 249]; for $\overline{\mathbb{R}}$-valued functions see [6, p. 383]) that the function $f(\,\cdot\,) \colon \mathcal{M} \to \mathbb{R}$ or $f(\,\cdot\,) : \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the first Baire class $\mathcal{B}_1$ if $f(\,\cdot\,) \in (F_\sigma, G_\delta)$, to the second Baire class $\mathcal{B}_2$ if $f(\,\cdot\,) \in (G_{\delta\sigma}, F_{\sigma\delta})$, and to the third Baire class $\mathcal{B}_3$ if $f(\,\cdot\,) \in (F_{\sigma\delta\sigma}, G_{\delta\sigma\delta})$.

A set $\mathcal{A} \in \mathbb{R}$ is called a Suslin set [5, p. 213], [6, p. 489] of the number line $\mathbb{R}$ if it is a continuous image of irrational numbers $\mathbb{I}$ with the subspace topology. The class of Suslin sets contains the class of Borel sets as a proper subclass, and at the same time it is a proper subclass of the class of Lebesgue measurable sets. A set $\mathcal{A} \in \overline{\mathbb{R}}$ is called a Suslin set of the extended real number line if it can be represented as an union of a Suslin set of $\mathbb{R}$ and some subset (including the empty subset) of two-element set $\{-\infty, +\infty\}$.

**Theorem.** *For any equation $a \in \widetilde{\mathcal{E}}^n$ its lower Sergeev frequency of zeros and signs belong to the class $(G_{\delta\sigma}, {}^*)$, and the lower frequency of roots belongs to the class $(F_\sigma, {}^*)$.*

It is quite interesting to compare this statement with the descriptions of the Baire classes of the Sergeev upper frequency of zeros, signs, and roots of the equation (1). Let us recall that for any equation $a \in \widetilde{\mathcal{E}}^n$ its upper Sergeev frequency of zeros and roots belong [3] to the class $({}^*, F_{\sigma\delta})$, and the lower frequency of signs belong to the class $({}^*, G_\delta, )$.

Since the image of any Baire function is [5, p. 255] a Suslin set, from the theorem it follows

**Corollary.** *For any equation $a \in \widetilde{\mathcal{E}}^n$ the lower frequency spectra $\check{\nu}^0(S_*(a))$, $\check{\nu}^-(S_*(a))$, and $\check{\nu}^+(S_*(a))$ of zeros, signs, and roots are Suslin sets of the nonnegative semi-axis $\overline{\mathbb{R}}_+$.*

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# On the Property of Separateness of the Angle Between Stable and Unstable Lineals of Solutions of Exponentially Dichotomous and Weak Exponentially Dichotomous Systems

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## 1

Denote by $\mathcal{M}_n$ the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geqslant 0, \tag{1}$$

where $n \geqslant 2$, with the piecewise continuous and uniformly bounded on the time half-line $t \geqslant 0$ coefficients matrix $A(\cdot\,) : [0, +\infty) \to \operatorname{End} \mathbb{R}^n$. Denote by $\mathcal{X}_A(\cdot\,)$ the linear space of solutions of system (1). Its subspaces we call further lineals to distinguish them from linear subspaces in $\mathbb{R}^n$. The angle between lineals $U(\cdot\,)$ and $V(\cdot\,)$ of the space $\mathcal{X}_A(\cdot\,)$ we call the function $\gamma(t)$ of the variable $t \geqslant 0$, which is defined by the equation $\gamma(t) = \angle(U(t), V(t))$, where $\angle(U(t), V(t))$ is the angle between subspaces $U(t)$ $V(t)$ of the space $\mathbb{R}^n$.

It is known [2, p. 236], [3, p. 10], system (1) in $\mathcal{M}_n$ is called an exponentially dichotomous system or a system with exponentially dichotomy on the time half-line if there exist positive constants $c_1$, $c_2$ and $\nu_1$, $\nu_2$ and a decomposition of the linear space $\mathcal{X}_A(\cdot\,)$ of its solutions into the direct sum $\mathcal{X}_A(\cdot\,) = L_A^-(\cdot\,) \oplus L_A^+(\cdot\,)$ of lineals, so that its solutions $x(\cdot\,)$ belonging to these lineals satisfy the following two conditions:

1) if $x(\cdot\,) \in L_A^-(\cdot\,)$, then $\|x(t)\| \leqslant c_1 \exp\{-\nu_1(t-s)\}\,\|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$;

2) if $x(\cdot\,) \in L_A^+(\cdot\,)$, then $\|x(t)\| \geqslant c_2 \exp\{\,\nu_2(t-s)\}\,\|x(s)\|$ for arbitrary $t \geqslant s \geqslant 0$.

In this definition the choice of norm in $\mathbb{R}^n$ does not play any role, because in a finite linear space any two norms are equivalent. The class of exponentially dichotomous $n$-dimensional systems is denoted by $\mathcal{E}_n$.

Condition of exponential dichotomy of system (1) means, in particular, that in any time segment the norm of any solution in $L_A^-(\cdot\,)$ uniformly decreases exponentially, and the norm of any solution in $L_A^+(\cdot\,)$ uniformly increases exponentially. If the system is exponentially dichotomous, its lineal $L_A^-(\cdot\,)$, called a stable lineal, is uniquely determined (it consists of all solutions, decreasing to zero at infinity), and any of lineals, complementary lineal $L_A^-(\cdot\,)$ to the space $\mathcal{X}_A(\cdot\,)$ of solutions, may be taken as a lineal $L_A^+(\cdot\,)$, called unstable lineal. Although in the above definition the case of zero dimension of one of subspaces is not excluded, i.e. one of the equalities $L_A^-(\cdot\,) = \{\mathbf{0}\}$ or $L_A^+(\cdot\,) = \{\mathbf{0}\}$ is possible, further we consider that each of the lineals $L_A^-(\cdot\,)$ and $L_A^+(\cdot\,)$ is nonzero.

We say that the lineals of solutions $U(\cdot\,)$ and $V(\cdot\,)$ of system (1) are separated if the angle between them is separated from zero: $\inf\{\gamma(t) : t \geqslant 0\} > 0$. It is well known [2, p. 237] that the stable lineal $L_A^-(\cdot\,)$ of an exponentially dichotomous system is separated from any of its unstable lineal $L_A^+(\cdot\,)$, i.e. for any unstable lineal $L_A^+(\cdot\,)$ there is the inequality

$$\inf\big\{\angle(L_A^-(t), L_A^+(t)) :\; t \geqslant 0\big\} > 0. \tag{2}$$

This property of finite-dimensional exponentially dichotomous systems is essential and must be included [2] in the definition of exponential dichotomy, when we extend the concept of exponential dichotomy of the finite-dimensional case to the case of Banach spaces, to preserve basic properties of finite-dimensional exponentially dichotomous system.

Nevertheless, the following theorem shows that the property of separateness from zero of the angle between stable and unstable lineals of exponentially dichotomous systems is not characteristic for such systems. More precisely, the angle between stable and unstable subspaces of exponentially dichotomous system is the same as can generally be the angle between separated subspaces of solutions of an arbitrary system (1) that is not exponentially dichotomous.

**Theorem 1.** *Let a system in $\mathcal{M}_n$ have separated lineals of solutions $U(\,\cdot\,)$ and $V(\,\cdot\,)$. Then there exists a system $A\in\mathcal{E}_n$ such that for its stable $L_A^-(\,\cdot\,)$ and unstable $L_A^+(\,\cdot\,)$ lineals for all $t\geqslant 0$ the equalities hold*

$$L_A^-(t)=U(t)\ \ \text{and}\ \ L_A^+(t)=V(t).$$

The following statement characterizes more fully the property of the angle between stable and unstable lineals of exponentially dichotomous systems and complements the above statement [2, p. 237] on the separateness of stable and unstable lineals of exponentially dichotomous systems.

**Theorem 2.** *For any system $A\in\mathcal{E}_n$ there exists a constant $c_A\in(0,\pi/2)$ such that for any of its unstable lineal $L_A^+(\,\cdot\,)$ for all sufficiently large $t\geqslant 0$ the inequality $\angle\{L_A^-(t),L_A^+(t)\}>c_A$ is true, i.e. there is a constant $c_A\in(0,\pi/2)$ such that the inequality*

$$\lim_{\tau\to+\infty}\inf_{t\geqslant\tau}\angle\big\{L_A^-(t),L_A^+(t)\big\}>c_A \tag{3}$$

*holds for any unstable lineal $L_A^+(\,\cdot\,)$.*

Obviously, inequality (3) enhances inequality (2). Inequality (3), if we denote by $\mathcal{U}_A$ the aggregate of unstable lineals of system $A\in\mathcal{E}n$, can be written as

$$\inf_{L_A^+(\,\cdot\,)\in\mathcal{U}_A}\lim_{\tau\to+\infty}\inf_{t\geqslant\tau}\angle\big\{L_A^-(t),L_A^+(t)\big\}>c_A.$$

## 2

In [1], it is introduced a generalization of the concept of exponentially dichotomous linear differential systems defined in a finite space, that consists in the refusal from the requirement of the uniformness of estimates for the norms of solutions under constants-multipliers in definition of an exponentially dichotomous system. In [1], such systems are referred to as weak exponentially dichotomous. In other words, system (1) in $\mathcal{M}_n$ is called a weak exponentially dichotomous system or a system with a weak exponential dichotomy on the half-line, if there exist positive constants $\nu_1$, $\nu_2$ and a decomposition of the linear space $\mathcal{X}_A(\,\cdot\,)$ of its solutions into the direct sum $\mathcal{X}_A(\,\cdot\,)=L_A^-(\,\cdot\,)\oplus L_A^+(\,\cdot\,)$ of lineals so that its solutions $x(\,\cdot\,)$ belonging to these lineals satisfy the following two conditions:

1′) if $x(\,\cdot\,)\in L_A^-(\,\cdot\,)$, then $\|x(t)\|\leqslant c_1(x)\exp\{-\nu_1(t-s)\}\,\|x(s)\|$ for arbitrary $t\geqslant s\geqslant 0$;

2′) if $x(\,\cdot\,)\in L_A^+(\,\cdot\,)$, then $\|x(t)\|\geqslant c_2(x)\exp\{\nu_2(t-s)\}\,\|x(s)\|$ for arbitrary $t\geqslant s\geqslant 0$,

where $c_1(x)$ and $c_2(x)$ are positive constants which, in general, depend on the choice of the solution $x(\,\cdot\,)$.

As can be seen, if we could choose, in the definition of a weak exponentially dichotomous system, the constants $c_1(x)$ and $c_2(x)$ which are the same for all solutions $x(\,\cdot\,)\in L_A^-(\,\cdot\,)$ and $x(\,\cdot\,)\in L_A^+(\,\cdot\,)$

respectively, then we get the definition of an exponentially dichotomous system. The class of $n$-dimensional weakly exponentially dichotomous systems is denoted by $W\mathcal{E}_n$. In [1], it is shown that for any $n \geqslant 2$, there is a proper inclusion $\mathcal{E}_n \subset W\mathcal{E}_n$. Just as for exponentially dichotomous systems, lineals $L^-_A(\,\cdot\,)$ and $L^+_A(\,\cdot\,)$ are called stable and unstable lineals of a system $A \in W\mathcal{E}_n$, and, just as in the case of exponentially dichotomous systems, for any system $A \in W\mathcal{E}_n$ its stable lineal $L^-_A(\,\cdot\,)$ is uniquely determined (it consists of all solutions decreasing to zero at infinity), and as a lineal $L^+_A(\,\cdot\,)$ may be taken any algebraic complement $L^-_A(\,\cdot\,)$ to the linear space $\mathcal{X}_A(\,\cdot\,)$ of solutions.

We can ask how significantly the properties of systems of the classes $\mathcal{E}_n$ and $W\mathcal{E}_n$ can differ. In particular, is it true that the unstable and stable lineals of a weak exponentially dichotomous system are separated? If the system $A \in W\mathcal{E}_2$, then, as is easy to show, it is either an exponentially dichotomous or its stable or unstable lineal is zero. That is why weak exponentially dichotomous system with unseparated angle between stable and unstable lineals of solutions should have the dimension of not less than 3. It turns out that for weak exponentially dichotomous system of dimension $n \geqslant 3$ incorrect is not only the property stated in Theorem 2 but also weaker property (2) of separateness of the angle between stable and unstable lineals of solutions as shown by

**Theorem 3.** *For any natural number $n \geqslant 3$ there exists the system $A \in W\mathcal{E}_n$ and such an unstable lineal $L^+_A(\,\cdot\,)$ of solutions that the angle between it and the stable lineal $L^-_A(\,\cdot\,)$ is not separated from zero, i.e.* $\inf\{\angle(L^-_A(t), L^+_A(t)) :\ t \geqslant 0\} = 0$.

Apparently, Theorem 3 can be enhanced: for any $n \geqslant 3$ there exist such systems in the $W\mathcal{E}_n \setminus \mathcal{E}_n$ that the angle between their stable and any unstable lineals is not separated from zero.

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# Periodic Reflecting Function of Linear Differential System with Incommensurable Periods of Homogeneous and Nonhomogeneous Parts

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Consider the differential system

$$\dot{x} = X(t,x), \ \ t \in \mathbb{R}, \ \ x = (x_1,\dots,x_n)^\top \in \mathbb{R}^n, \tag{1}$$

with continuous in all the variables and continuously differentiable right part over $x$. Let $\varphi(t;\tau,x)$ denote the general solution in the form of Cauchy system (1), that is $\varphi(t;\tau,x)$ – the solution of (1) with the initial condition $\varphi(\tau;\tau,x) = x$. Let $I_x$ be maximum symmetrical with respect to zero interval of existence of solution $\varphi(t;0,x)$. Let $D(X) := \{(t,\varphi(t;0,x)) \in \mathbb{R}^{n+1} : \ t \in I_x, \ x \in \mathbb{R}^n\}$. From the theorem on continuous dependence of solutions on the initial value and the definition of $D(X)$ it follows that $D(X)$ is the open domain in $\mathbb{R} \times \mathbb{R}^n$ which contains the hyperplane $t = 0$. Reflecting function of system (1) is called [3], [4, p. 11], [5, p. 62] the vector function $F : D(X) \to \mathbb{R}^n$, acting according to the rule $(t,x) \longmapsto \varphi(-t;t,x)$. In other words, for any solution $x(t)$ of this system, which exists on a symmetric interval $(-\xi,\xi)$, the identity $F(t,x(t)) \overset{t}{\equiv} x(-t)$ is valid for all $t \in (-\xi,\xi)$. This property can be taken [4, p. 16] for the definition of a reflecting function. From the definition of the reflecting function and the differentiability theorem on the initial value it follows that the reflecting function $F(t,x)$ of system (1) has partial derivatives $F_t$ and $F_x$ in the region $D(X)$.

Fundamentally important result of the theory of reflecting function is the following criterion [3], [4, pp. 11, 12], [5, pp. 63, 64]: the vector function $F = F(t,x) : D(X) \to \mathbb{R}^n$ is a reflecting function of system (1) if and only if it satisfies the initial condition $F(0,x) \equiv x$ and the system of equations in the partial derivatives

$$F_t + F_x X(t,x) + X(-t,F) = 0. \tag{2}$$

Equation (2) is called [4, p. 12], [5, p. 63] basic equation (the ratio) for the reflecting function. Methods have been developed which in some cases make it possible to find the reflecting function of system (1) without finding its solutions. Moreover, if we know only some of the properties of the reflecting function of the system, it is possible to investigate the behavior of its solutions without resorting to the construction of reflecting function [4–9].

Two systems are equivalent in the sense of the coincidence of reflecting functions [5, p. 75], if their reflecting functions are equal in a domain containing the hyperplane $t = 0$. Since the solutions of equivalent systems have a number of similar properties, the task of constructing classes of equivalent systems, and the choice of simple (for example, integrated into the final form) systems-representatives of these classes will be important and relevant.

In this article, the linear differential systems defined for all $t \in \mathbb{R}$ are discussed, and for them the domain $D(X)$ determination of reflecting function coincides with the extended phase space $\mathbb{R} \times \mathbb{R}^n$, then for such systems it is natural to study the conditions of coincidence of their reflecting functions in all extended phase space. Therefore, further as the equivalence of linear systems in the sense of the coincidence of their reflecting functions the coincidence of the reflecting functions of these systems throughout the extended phase space is understood.

In this article, the quasi-periodic two-frequency linear differential systems are discussed such that their homogeneous and nonhomogeneous parts are periodic with incommensurable periods, and the conditions of existence of the periodic reflecting functions in such systems are clarified.

**Theorem 1.** *For the linear nonhomogeneous differential system*

$$\dot{x} = A(t)x + f(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \tag{3}$$

*with continuous $n \times n$-matrix $A(t)$ and vector-function $f(t)$, to have the same reflecting function as the system*

$$\dot{x} = f(t), \tag{4}$$

*necessary and sufficient conditions are:*

1) *matrix-valued function $A(t)$ is odd;*

2) *there is the identity*

$$A(t) \int\limits_{t}^{-t} f(s)\, ds = 0 \ \text{ for all } \ t \in \mathbb{R}. \tag{5}$$

*At the same time, reflecting function $F(t,x)$ of these systems, is the vector-function*

$$F(t,x) = x + \int\limits_{t}^{-t} f(s)\, ds. \tag{6}$$

*Proof. Sufficiency.* The general solution in the form of the Cauchy system (4) is given by $\varphi(t;\tau,x) = x + \int\limits_{\tau}^{t} f(s)\, ds$. As a consequence of this presentation by the definition of the reflecting function we easily find that reflecting function $F(t,x)$ of system (4) is given by equation (6).

We will show that under the conditions 1) and 2) function (6) is the reflecting function of system (3). It's enough to make sure that function (6) satisfies the fundamental ratio (2) for reflecting function of system (3). Substituting in it function (6), after obvious equivalent transformations we obtain the identity:

$$A(t)x + A(-t)x + A(-t) \int\limits_{t}^{-t} f(t)\, dt \overset{t,x}{\equiv} 0. \tag{7}$$

Since under the conditions 1) and 2) of the theorem identity (7) is obviously true, then function (6) is the reflecting function of system (3). The sufficiency is proved.

*Necessity.* Let systems (3) and (4) are equivalent in the sense of coincidence of the reflecting functions. As it is shown above, system (4) has a reflecting function (6). Since function (6) is also the reflecting function of system (3), then for system (3) and this function the main identity (2) is satisfied. Hence we obtain identity (7). This identity is satisfied for all $t$ and $x$. Assuming in it $x = 0$ and replacing $-t$ onto $t$, one obtains the condition 2). Thus, the identity must be satisfied

$$(A(t) + A(-t))x \overset{t,x}{\equiv} 0. \tag{8}$$

Identity (8) means that the linear operator $A(t) + A(-t)$ is null, that is $A(t) = -A(-t)$ for all $t \in \mathbb{R}$.

Thus, the function $A(t)$ – odd, and as proved above, satisfies the condition 2). The necessity, and thus the theorem is proved. ☐

**Corollary 1.** *If matrix $A(t)$ is nonsingular for all $t \in \mathbb{R}$, then systems* (3) *and* (4) *have the same reflecting function if and only if the matrix-valued function $A(\cdot)$ and the vector function $f(\cdot)$ are odd. In this case, reflecting function of systems* (3) *and* (4) *will be the function $F(t,x) = x$.*

If the set of those $t \in \mathbb{R}$, in which matrix $A(t)$ is non-singular, not coincides with the $\mathbb{R}$, then condition 2) of the theorem does not necessarily mean oddness of the vector-function $f(\cdot)$ which is confirmed by the following example.

**Example 1.** Consider the system

$$\dot{x} = A(t)x + f(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2,$$

in which matrix of coefficients $A(t)$ is odd and has zero determinant for all $t \in \mathbb{R}$. Let

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

We will assume that $a_1^2(t) + a_2^2(t) \neq 0$ for any $t \in \mathbb{R}$. According to Theorem 1, the given system has the same reflecting function as the system $\dot{x} = f(t)$ if and only if identity (5) is satisfied. From this identity we obtain

$$a_1(t) \int_t^{-t} f_1(s)\,ds \equiv -a_2(t) \int_t^{-t} f_2(s)\,ds, \quad a_3(t) \int_t^{-t} f_1(s)\,ds \equiv -a_4(t) \int_t^{-t} f_2(s)\,ds. \tag{9}$$

We will find all vector-functions $f(t) = (f_1(t), f_2(t))^\top$, for which these identities are satisfied. Since $\det A(t) = 0$ for all $t \in \mathbb{R}$ and the first row of the matrix $A(t)$ is nonzero then its second row is proportional to the first one, and then, for the validity of these identities it is necessary and sufficient the first of them to be valid.

Since the vector $(a_1(t), a_2(t))^\top$ is nonzero, then the first identity in (9) is performed, if and only if for some function $h(t)$ satisfies the identities

$$\int_t^{-t} f_1(s)\,ds \equiv -a_2(t)h(t), \quad \int_t^{-t} f_2(s)\,ds \equiv a_1(t)h(t). \tag{10}$$

In order identities (10) to be carried out, it is necessary the function $h(t)$ to be even (as left sides in (10) and functions $a_1(t)$, $a_2(t)$ are odd) and that the functions $a_1(t)h(t)$ and $a_2(t)h(t)$ have been continuously differentiable (as left sides in (10) – continuously differentiable functions).

We will show that these conditions are sufficient for the existence of functions $f_1(t)$, $f_2(t)$, which satisfy (10). Fix some even function $h(t)$, for which the right sides in (10) – continuously differentiable functions. Denote $-a_2(t)h(t)$ through $g_1(t)$. Then the first identity in (10) takes the form $\int\limits_t^{-t} f_1(s)\,ds \equiv g_1(t)$. Differentiating it on $t$, we obtain

$$f_1(t) + f_1(-t) \equiv -\dot{g}_1(t). \tag{11}$$

The function $\dot{g}_1(t)$ is even, as a derivative of an odd function, and it is continuous. We will seek solution of the functional equation (11) in the form of

$$f_1(t) = -\frac{\dot{g}_1(t)}{2} + r_1(t), \tag{12}$$

where $r_1(t)$ is an unknown continuous function. Replacing in (11) the function $f_1(t)$ by the given representation, we obtain the identity $r_1(t) + r_1(-t) \equiv 0$ in view of parity of $\dot{g}_1(t)$, that is $r_1(t)$ – an odd function. Conversely, it is easy to see that the function of the form (12) with an odd continuous function $r_1(t)$ satisfies the first identity in (10). Indeed,

$$\int\limits_t^{-t} f_1(s)ds \equiv \int\limits_t^{-t} \Big(-\frac{\dot{g}_1(s)}{2} + r_1(s)\Big)ds = g_1(t) + \int\limits_t^{-t} r_1(s)ds = g_1(t) = -a_2(t)h(t).$$

Similarly, if we denote the function $a_1(t)h(t)$ via $g_2(t)$, a solution of the second functional equation in (10) we find in the form of

$$f_2(t) = -\frac{\dot{g}_2(t)}{2} + r_2(t), \tag{13}$$

where $g_2(t) \equiv a_1(t)h(t)$, and $r_2(t)$ – arbitrary odd function. Thus, the solution of the problem on the description of the set of vector-functions $f(t) = (f_1(t), f_2(t))^\top$, $t \in \mathbb{R}$, satisfy (9) and it is reduced to the problem of the description of the set of even functions $h(t)$, $t \in \mathbb{R}$, for which both functions $a_1(t)h(t)$ and $a_2(t)h(t)$ would be continuously differentiable.

As we see, the vector function $f(t) = (f_1(t), f_2(t))^\top$, the components of which are built up, and given by equalities (12), (13), generally speaking, is not odd, whatever the elements of a degenerate odd matrix $A(t)$ would be , the first row of which for all $t \in \mathbb{R}$ is nonzero ($a_1^2(t) + a_2^2(t) \neq 0$ for all $t \in \mathbb{R}$).

**Remark 1.** Considered example gives a partial solution for the following problem, formulated by E. A. Barabanov: for a linear homogeneous differential system $\dot{x} = A(t)x$ in terms of its coefficient matrix $A(t)$ to describe all those its nonhomogeneous perturbations $f(t)$, at which the reflecting functions of systems $\dot{y} = A(t)y + f(t)$ and $\dot{z} = f(t)$ coincide.

**Corollary 2.** *Let the matrix $A(t)$ have period $\omega_1$, and the vector function $f(t)$ – period $\omega_2$. For system* (3) *to have an $\omega_2$-periodic on $t$ reflecting function* (6) *it is necessary and sufficient the fulfillment of conditions* 1) *and* 2) *of Theorem* 1 *and the equality*

$$\int\limits_0^{\omega_2} f(s)\,ds = 0. \tag{14}$$

**Remark 2.** In the case 3 when numbers $\omega_1$ and $\omega_2$ are incommensurable, Corollary 2 gives sufficient condition for the existence of $\omega_2$-periodic on $t$ reflecting function in a quasi-periodic system (3).

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# Fully Linearized Difference Scheme for Generalized Rosenau Equation

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We consider the generalized Rosenau equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\lambda\,\frac{\partial (u)^m}{\partial x}+\mu\,\frac{\partial^5 u}{\partial x^4\partial t}=0,\ \ (x,t)\in Q, \tag{1}$$

together with the initial and boundary conditions

$$u(x,0)=\varphi(x),\ \ x\in[a,b],\quad u(a,t)=u(b,t)=\frac{\partial^2 u(a,t)}{\partial x^2}=\frac{\partial^2 u(b,t)}{\partial x^2}=0,\ \ t\in[0,T]. \tag{2}$$

Here $\lambda$ and $\mu$ are positive constants, $m\ge 2$ is a positive integer, and $Q=(a,b)\times(0,T)$.

In this article, two-level scheme is constructed to find the values of the unknown function on the first level, besides the term $\partial(u)^m/\partial x$ is approximated by the offered in [1] way. For the upper levels, as in [2], the known approximation are used for derivatives.

The domain $\overline{Q}$ is divided into rectangular grid by the points $(x_i,t_j)=(a+ih,j\tau)$, $i=0,1,2,\dots,n$, $j=0,1,\dots,J$, where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function $U$ at the node $(x_i,t_j)$ is denoted by $U_i^j$, that is $U_i^j=U(x_i,t_j)$.

We define the difference quotients (forward, backward, and central, respectively) in $x$ and $t$ directions as follows:

$$(U_i^j)_x:=\frac{U_{i+1}^j-U_i^j}{h},\quad (U_i^j)_{\overline{x}}:=\frac{U_i^j-U_{i-1}^j}{h},\quad (U_i^j)_{\overset{\circ}{x}}:=\frac{1}{2}\big((U_i^j)_x+(U_i^j)_{\overline{x}}\big),$$
$$(U_i^j)_t:=\frac{U_i^{j+1}-U_i^j}{\tau},\quad (U_i^j)_{\overline{t}}:=\frac{U_i^j-U_i^{j-1}}{\tau},\quad (U_i^j)_{\overset{\circ}{t}}:=\frac{1}{2}\big((U_i^j)_t+(U_i^j)_{\overline{t}}\big).$$

We approximate the problem (1), (2) by the difference scheme

$$(U_i^j)_{\overset{\circ}{t}}+\frac{1}{2}\,(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\frac{\lambda m}{2(m+1)}\,\Lambda U_i^j+\mu(U_i^j)_{\overline{x}x\overline{x}x\overset{\circ}{t}}=0, \tag{3}$$
$$i=1,2,\dots,n-1,\ \ j=1,2,\dots,J-1,$$

$$(U_i^0)_t+\frac{1}{2}\,(U_i^1+U_i^0)_{\overset{\circ}{x}}+\frac{\lambda m}{2(m+1)}\,\Lambda U_i^0+\mu(U_i^0)_{\overline{x}x\overline{x}xt}=0,\ \ i=1,2,\dots,n-1, \tag{4}$$

$$U_i^0=\varphi(x_i),\quad U_0^j=U_n^j=(U_0^j)_{\overline{x}x}=(U_n^j)_{\overline{x}x}=0\ \ i=0,1,\dots,n,\ \ j=0,1,\dots,n, \tag{5}$$

where

$$\Lambda U_i^j:=(U_i^j)^{m-1}(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\big((U_i^j)^{m-1}(U_i^{j+1}+U_i^{j-1})\big)_{\overset{\circ}{x}},\ \ j=1,2,\dots,J-1,$$
$$\Lambda U_i^0:=(U_i^0)^{m-1}(U_i^1+U_i^0)_{\overset{\circ}{x}}+\big((U_i^0)^{m-1}(U_i^1+U_i^0)\big)_{\overset{\circ}{x}},\ \ i=1,2,\dots,n-1.$$

The obtained algebraic equations are linear with respect to the values of unknown function for each new level.

An a priori estimate of a solution of the difference scheme (3)–(5) is obtained with the help of energy inequality method, from which follows a uniquely solvability of the scheme.

In the equality of the obtained discrete conservation law the initial energy depends explicitly only on initial data.

Stability and second order convergence of difference scheme is proved without any restriction on discretization parameters $\tau$, $h$.

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# Asymptotic Behavior of Some Special Classes of Solutions of Essentially Nonlinear $n$-th Order Differential Equations

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The following differential equation

$$y^{(n)} = \alpha_0 p(t) \exp\big(R\big(|\ln|y^{(n-1)}||\big)\big) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}) \tag{1}$$

is considered. In (1) $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[$[1] $\to ]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to ]0, +\infty[$ $(i = 0, \dots, n)$ are continuous functions, $R : ]0, +\infty[ \to ]0, +\infty[$ is continuously differentiable function, $Y_i \in \{0, \pm\infty\}$, $\Delta_{Y_i}$ is either the interval $[y_i^0, Y_i[$[2], or the interval $]Y_i, y_i^0]$. We suppose also that $R$ is a regularly varying on infinity function of index $\mu$, $0 < \mu < 1$, every $\varphi_i(z)$ is a regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index $\sigma_i$ and $\sum\limits_{i=0}^{n-1} \sigma_i \neq 1$.

We call the measurable function $\varphi : \Delta_Y \to ]0, +\infty[$ a regularly varying as $z \to Y$ of index $\sigma$ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma,$$

where $Y \in \{0, \pm\infty\}$, $\Delta_Y$ is some one-sided neighbourhood of $Y$. If $\sigma = 0$, such function is called a slowly varying.

It follows from the results of monograph [5] that regularly varying functions have the following properties.

$M_1$: Function $\varphi(z)$ is regularly varying of index $\sigma$ as $z \to Y$ if and only if it admits the representation

$$\varphi(z) = z^\sigma \theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

$M_2$: If function $L : \Delta_{Y^0} \to ]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to ]0, +\infty[$ is slowly varying as $z \to Y$.

$M_3$: If function $\varphi : \Delta_Y \to ]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then $\varphi$ is regularly varying as $z \to Y$ of index $\sigma$.

[1] If $\omega > 0$, we take $a > 0$.
[2] If $Y_i = +\infty$ $(Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to ]0;+\infty[$ satisfies the condition $S$ if for any continuous differentiable function $L : \Delta_{Y_i} \to ]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following condition takes place

$$\theta(zL(z)) = \theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

We call defined on $[t_0,\omega[\subset [a,\omega[$ solution $y$ of the equation (1) the $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t_0,\omega[\to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,\dots,n-1), \quad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0.$$

In regular cases $\lambda^0_{n-1} \in R\setminus\{0,\frac12,\frac23,\dots,\frac{n-2}{n-1}\}$, the $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions of the equation (1) have been established in [3]. Such $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions are regularly varying functions as $t\uparrow\omega$ of indexes different from $\{0,1,\dots,n-1\}$.

The cases $\lambda_0 \in \{0,\frac12,\frac23,\dots,\frac{n-2}{n-1}\}$ are singular. Such solutions are regularly varying functions as $t\uparrow\omega$ of indexes $\{0,1,\dots,n-1\}$, so such solutions or some of their derivatives are slowly varying functions as $t\uparrow\omega$. Therefore for investigation of such solutions we must put additional conditions on functions $\varphi_0,\dots,\varphi_{n-1}$ and on the function $p$. The case $\lambda_0 = 0$ is of the most difficult ones. It is presented in this work. The case was investigated before [1,4] only when $R(z)\equiv 1$ and the function $\varphi_{n-1}(z)|z|^{-\sigma_{n-1}}$ satisfies the condition $S$. For equations of type (1), that contain, for example, functions like $\exp(|\ln|y||^\mu)$ $(0<\mu<1)$, the asymptotic representations of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$-solutions were not established before. Let us notice that function $\exp(R(|\ln|z||))$ does not satisfy the condition $S$.

Now we need the following subsidiary notations.

$$\gamma_0 = 1-\sum_{j=0}^{n-1}\sigma_j, \quad C = \frac{1}{1-\sigma_{n-1}}, \quad \eta = \prod_{j=0}^{n-3}((n-i-2)!)^{\sigma_i}, \quad \gamma = \sum_{i=0}^{n-3}(i+2-n)\sigma_i,$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,\dots,n-1),$$

$$Q(t) = -\pi_\omega(t)\Big|\frac{(1-\sigma_{n-1})}{\eta}|\pi_\omega(t)|^{-\gamma} I_0(t)\theta_{n-1}\big(y^0_{n-1}|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}\big)\Big|^{\frac{1}{1-\sigma_{n-1}}} \operatorname{sign} y^0_{n-1},$$

$$I_0(t) = \int_{A^0_\omega}^t p(\tau)\,d\tau, \quad I_1(t) = \int_{A^1_\omega}^t \frac{Q(\tau)}{\pi_\omega(\tau)}\,d\tau,$$

$$A^0_\omega = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)\,d\tau < +\infty, \end{cases} \qquad A^1_\omega = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega \Big|\frac{Q(\tau)}{\pi_\omega(\tau)}\Big|\,d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega \Big|\frac{Q(\tau)}{\pi_\omega(\tau)}\Big|\,d\tau < +\infty. \end{cases}$$

The following conclusions take place.

**Theorem 1.** *Let in equation* (1) $\sigma_{n-1} \ne 1$, *the function* $\theta_{n-1}$ *satisfy the condition* $S$ *and*

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|I(t)||)I_1(t)I_0'(t)}{I_0(t)I_1'(t)} = 0.$$

*We suppose also that there exists the finite or infinite limit*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_0(t)}\,. \tag{2}$$

*Then the following conditions are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$-solutions of equation* (1),

$$\lim_{t\uparrow\omega}\frac{I_1'(t)I_0(t)}{p(t)I_1(t)}=0,\quad \lim_{t\uparrow\omega}y^0_{n-1}|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}=Y_{n-1},$$
$$\lim_{t\uparrow\omega}y^0_{n-2}|I_1(t)|^{\frac{1-\sigma_{n-1}}{\gamma_0}}=Y_{n-2},\quad \lim_{t\uparrow\omega}y^0_i|\pi_\omega(t)|^{n-i-2}=Y_i,$$
$$\alpha_0y^0_{n-1}(1-\sigma_{n-1})I_0(t)>0,\quad (1-\sigma_{n-1})\gamma_0y^0_{n-2}I_1(t)<0,$$
$$y^0_iy^0_{i+1}\pi_\omega(t)(n-i-2)>0\ \text{ as }\ t\in[a,\omega[\,.$$

*Here* $i=0,\dots,n-3$.

*For any such solution the following asymptotic representations take place as* $t\uparrow\omega$

$$\frac{y^{(n-1)}(t)}{\exp(R(|\ln|y^{(n-1)}(t)||))\prod\limits_{j=0}^{n-1}\varphi_j(y^{(j)}(t))}=\alpha_0(1-\sigma_{n-1})I_0(t)[1+o(1)], \tag{3}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}=\frac{I_1'(t)(1-\sigma_{n-1})}{\gamma_0I_1(t)}\,[1+o(1)],\quad \frac{y^{(i)}(t)}{y^{(n-2)}(t)}=\frac{[\pi_\omega(t)]^{n-i-2}}{(n-i-2)!}\,[1+o(1)], \tag{4}$$
$$i=0,\dots,n-3.$$

**Theorem 2.** *Let in equation* (1) $\sigma_{n-1}\neq1$, *the function* $\theta_{n-1}$ *satisfy the condition* $S$ *and*

$$\lim_{t\uparrow\omega}\frac{I_0(t)Q'(t)}{R'(|\ln|I(t)||)Q(t)I_0'(t)}=0.$$

*We suppose also that there exists the finite or infinite limit* (2). *Then the following conditions are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$-solutions of equation* (1),

$$\lim_{t\uparrow\omega}\frac{I_0(t)}{p(t)R'(|\ln|I_0(t)||)}=0,\quad \lim_{t\uparrow\omega}y^0_{n-1}|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}=Y_{n-1},$$
$$\lim_{t\uparrow\omega}y^0_{n-2}\Big|\frac{Q(t)}{R'(|\ln|I_0(t)||)}\Big|^{\frac{1-\sigma_{n-1}}{\gamma_0}}=Y_{n-2},\quad \lim_{t\uparrow\omega}y^0_i|\pi_\omega(t)|^{n-i-2}=Y_i,$$
$$\alpha_0y^0_{n-1}(1-\sigma_{n-1})I_0(t)>0,\quad (1-\sigma_{n-1})\gamma_0Q(t)y^0_{n-2}y^0_{n-1}>0,$$
$$y^0_iy^0_{i+1}\pi_\omega(t)(n-i-2)>0\ \text{ as }\ t\in[a,\omega[\,.$$

*Here* $i=0,\dots,n-3$.

*For any such solution the representation* (3), *the second representation in* (4) *and the following asymptotic representation*

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}=\frac{I_1'(t)}{\gamma_0R'(|\frac{1-\sigma_{n-1}}{\ln|I_0(t)|}|)}\,[1+o(1)]$$

*take place as* $t\uparrow\omega$.

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# On a Four-Point Boundary Value Problem for Second Order Linear Functional Differential Equations

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The multi-point and nonlocal boundary value problems for ordinary and functional differential equations have been studied by many authors in recent years, see [1–20] and references therein. Nonlocal boundary value problems arise in many applications and can be used for modeling [2, 9, 11, 18].

In the resonance and non-resonance cases, many authors (see, for instance, [2, 3, 5, 6, 10–12, 14, 15, 18, 20]) consider, firstly, the boundary value problem for a linear ordinary differential equation. They established the existence of a unique solution, investigate the properties of the Green function, then apply the results to non-linear equations.

Motivated by the above work, in this paper, we consider a four-point boundary value problem for linear second order functional differential equation at resonance. We obtain sharp sufficient conditions for the existence and uniqueness of solutions. So, the results of many previous works on multi-point boundary value problems can be extended in the case of this four-point problem.

Let us define some sets and functions:

$$\Omega \equiv \big\{(b,c):\ 0 \le b \le c \le 1\big\},\quad \Omega_1 \equiv \Big\{(b,c)\in\Omega:\ c \ge 3b-1,\ c \ge \frac{b+1}{3}\Big\},$$
$$\Omega_2 \equiv \Big\{(b,c)\in\Omega:\ c < \frac{b+1}{3}\Big\},\quad \Omega_3 \equiv \{(b,c)\in\Omega : c < 3b-1\}$$

(it is clear that $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$),

$$d_2(b,c) \equiv \sqrt{(3b-1-c)(1+c-b)},\quad d_3(b,c) \equiv \sqrt{(1+b-3c)(1+c-b)},$$
$$\omega_2(b,c) \equiv \Big[\frac{b-d_2(b,c)}{2}, \frac{b+d_2(b,c)}{2}\Big],\quad \omega_3(b,c) \equiv \Big[\frac{1+c-d_3(b,c)}{2}, \frac{1+c+d_3(b,c)}{2}\Big],$$
$$h_2(b,c,t) \equiv \frac{2}{t^2}\Big(\frac{b(1+c-b)}{((1+c)/2-t)^2}-1\Big),\ \ t\in\omega_2,$$
$$h_3(b,c,t) \equiv \frac{2}{(1-t)^2}\Big(\frac{(1-c)(1+c-b)}{(t-b/2)^2}-1\Big),\ \ t\in\omega_3.$$

Let

$$M(b,c) \equiv \begin{cases} \dfrac{32}{(1+c-b)^2} & \text{if } (b,c)\in\Omega_1;\\ \min\limits_{t\in\omega_2(b,c)} h_2(b,c,t) & \text{if } (b,c)\in\Omega_2;\\ \min\limits_{t\in\omega_3(b,c)} h_3(b,c,t) & \text{if } (b,c)\in\Omega_3. \end{cases}$$

**Definition.** A linear operator $T$ from the space of all continuous real functions $\mathbf{C}[0,1]$ into the space of all integrable functions $\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

**Theorem 1.** *Let $0 < b \le c < 1$, $p \in \mathbf{L}[0,1]$ be a non-negative function, $h : [0,1] \to [0,1]$ be a measurable function.*

*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = p(t)x(h(t)) + f(t), & t \in [0,1], \\ x(0) = x(b), \quad x(c) = x(1), \end{cases} \tag{1}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le M(b,c), \ \ p \not\equiv 0, \ \ p \not\equiv M(b,c).$$

**Remark.** The constant $M(b,c)$ is the best one. If $p(t) \equiv P > M(b,c)$, then there exists a measurable function $h : [0,1] \to [0,1]$ such that problem (1) has no a unique solution.

Theorem 1 can be transferred to a more general case.

**Theorem 2.** *Let $0 < b \le c < 1$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \ \ t \in [0,1], \\ x(0) = x(b), \quad x(c) = x(1), \end{cases} \tag{2}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le M, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv M.$$

We can get some simple corollaries about the solvability of problem (2) for different $b$ and $c$ satisfying the condition $0 < b \le c < 1$. The cases $b = 0$ or $c = 1$ correspond to the boundary value conditions $\dot{x}(0) = 0$ and $\dot{x}(1) = 0$. These cases can be dealt by the similar way.

**Corollary 1.** *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ x(0) = x\Big(\frac{1}{2}\Big) = x(1), \end{cases}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le 32, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv 32.$$

**Corollary 2.** *Let $b \in (0, 1/2)$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(0) = x(b), \quad x(1-b) = x(1), \end{cases}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le \frac{8}{(1-b)^2}\,, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv \frac{8}{(1-b)^2}\,.$$

**Corollary 3.** *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*
*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ \dot{x}(0) = 0, \quad x(0) = x(1) \quad (\text{or } \dot{x}(1) = 0, \quad x(0) = x(1)), \end{cases}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]}(T1)(t) \le 11 + 5\sqrt{5}, \;\; T1 \not\equiv 0, \;\; T1 \not\equiv 11 + 5\sqrt{5}.$$

**Corollary 4.** *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*
*Then the boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ \dot{x}(0) = 0, \quad \dot{x}(1) = 0, \end{cases}$$

*has a unique solution for every $f \in \mathbf{L}[0,1]$ if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]}(T1)(t) \le 8, \;\; T1 \not\equiv 0, \;\; T1 \not\equiv 8.$$

The constants in Theorem 2 and all corollaries are sharp.

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# On Baire Classes of Lyapunov Invariants

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For a given $n \in \mathbb{N}$ let us denote by $\mathcal{M}^n$ the set of linear systems of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1}$$

where $A$ is a piecewise continuous matrix function (which we identify with the respective system) and by $\widehat{\mathcal{M}}^n$ the subset of $\mathcal{M}^n$ comprising systems with bounded coefficients.

The set $\mathcal{M}^n$ is endowed with the *uniform* and *compact-open* topologies defined respectively by the metrics

$$\rho_U(A, B) = \sup_{t \in \mathbb{R}^+} \min \big\{ \|A(t) - B(t)\|, 1 \big\}, \quad \rho_C(A, B) = \sup_{t \in \mathbb{R}^+} \min \big\{ \|A(t) - B(t)\|, 2^{-t} \big\},$$

with $\|\cdot\|$ being a matrix norm (e.g., the spectral one). The resulting topological spaces will be denoted by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$. Similar notation will be used for their subspaces.

As early as 1928, O. Perron [9] (see also [4, 1.4]) discovered that for $n \geq 2$ the largest Lyapunov exponent is not upper semi-continuous as a functional on the space $\widehat{\mathcal{M}}^n_U$. He also suggested sufficient conditions for a system (1) to be a point of continuity of all the Lyapunov exponents in the uniform topology, which is commonly used in the study of the effect of perturbations on one or the other property of a system.

Further development of the theory of linear systems has led to introduction of a whole range of asymptotic behaviour characteristics, many of which proved to be discontinuous with respect to the uniform topology.

In a seminal work [7] V. M. Millionshchikov proposed using the Baire classification of functions to describe the dependence of those characteristics on the system coefficients. Motivated by parametric families of systems, V. M. Millionshchikov actively studied the compact-open topology on $\mathcal{M}^n$ and systematically tried to get rid of the assumption that the coefficients of (1) are bounded.

Let us introduce a piece of useful notation. Let $M$ be a metric space and $F$ be a set of functions $f : M \to \overline{\mathbb{R}}$. Define for each countable ordinal $\alpha$ the set $[F]_\alpha$ by transfinite induction as follows:

1) $[F]_0 = F$;

2) $[F]_\alpha$ is the set of functions $f : M \to \overline{\mathbb{R}}$ representable in the form

$$f(x) = \lim_{k \to \infty} f_k(x), \ \ x \in M,$$

where functions $f_k$, $k \in \mathbb{N}$, belong to the sets $[F]_\xi$ with $\xi < \alpha$.

**Definition 1** ([5, § 31.IX])**.** Let $M$ be a metric space and $\alpha$ be a countable ordinal. The $\alpha$-th *Baire class* $\mathfrak{F}_\alpha(M)$ is defined by $\mathfrak{F}_\alpha(M) = [C(M)]_\alpha$, $C(M)$ being the set of continuous functions $f : M \to \overline{\mathbb{R}}$. The class $\mathfrak{F}^0_\alpha(M) = \mathfrak{F}_\alpha(M) \setminus \bigcup\limits_{\xi < \alpha} \mathfrak{F}_\xi(M)$ is called the $\alpha$-th *exact* Baire class. For convenience, let us denote by $\mathfrak{F}^0_{\omega_1}(M)$ the set of functions which do not belong to any of the classes $\mathfrak{F}_\alpha(M)$, $\alpha \in [0, \omega_1)$ (here and subsequently, $\omega_1$ is the first uncountable ordinal).

V. M. Millionshchikov proved [8] that the Lyapunov exponents belong to the class $\mathfrak{F}_2(\mathcal{M}^n_C) \subset \mathfrak{F}_2(\mathcal{M}^n_U)$. Later M. I. Rakhimberdiev [10] proved that for $n \geq 2$ they do not belong to the class $\mathfrak{F}_1(\widehat{\mathcal{M}}^n_U) \supset \mathfrak{F}_1(\widehat{\mathcal{M}}^n_C)$. Therefore, for $n \geq 2$ the Lyapunov exponents (and their restrictions to $\widehat{\mathcal{M}}^n$) belong to the second exact Baire classes on both spaces $\mathcal{M}^n_C$ and $\mathcal{M}^n_U$ ($\widehat{\mathcal{M}}^n_C$ and $\widehat{\mathcal{M}}^n_U$, respectively).

Investigations in this vein have been continued by V. M. Millionshchikov himself, his students and followers. It was established by efforts of several authors [2, 11] that the minorants of the Lyapunov exponents belong to the class $\mathfrak{F}_3(\widehat{\mathcal{M}}^n_C)$, and A. N. Vetokhin proved [14] that they do not belong to the class $\mathfrak{F}_2(\widehat{\mathcal{M}}^n_C)$. Thus they belong to the third exact class on the space $\widehat{\mathcal{M}}^n_C$ (at the same time, they are known to belong to the first exact class on the space $\widehat{\mathcal{M}}^n_U$).

The natural question arises: for which $\alpha, \beta, \gamma, \delta \in [0, \omega_1]$ there exists an asymptotic invariant [1] from $\mathfrak{F}^0_\gamma(\mathcal{M}^n_U) \cap \mathfrak{F}^0_\delta(\mathcal{M}^n_C)$ such that its restriction to $\widehat{\mathcal{M}}^n$ belongs to $\mathfrak{F}^0_\alpha(\widehat{\mathcal{M}}^n_U) \cap \mathfrak{F}^0_\beta(\widehat{\mathcal{M}}^n_C)$?

Let us make the notion of asymptotic invariant more precise for the purposes of this paper (see the discussion of this notion in [6, § 2]).

**Definition 2** ([3, Chapter IV, § 2])**.** Systems $A, B \in \mathcal{M}^n$ are said to be *weakly Lyapunov equivalent* if they possess fundamental matrices $X(\cdot)$ and $Y(\cdot)$ such that

$$\sup_{t \in \mathbb{R}^+} \big( \|X(t)Y^{-1}(t)\| + \|Y(t)X^{-1}(t)\| \big) < \infty.$$

A functional taking equal values at any weakly Lyapunov equivalent systems is called a *weak Lyapunov invariant.*

**Proposition 1** ([13])**.** *Classes $\mathfrak{F}^0_1(\mathcal{M}^n_C)$ and $\mathfrak{F}^0_1(\widehat{\mathcal{M}}^n_C)$ do not contain any weak Lyapunov invariants.*

Let us note that the index of the exact Baire class of a function on a space is not less than that of its restriction to a subspace and also that the index of the exact Baire class of a function on $\mathcal{M}^n_C$ is not less than that on $\mathcal{M}^n_U$ (since the uniform topology is finer).

The following theorem states that a quadruple of the indices of the exact Baire classes with respect to the compact-open and uniform topologies containing a weak Lyapunov invariant and its restriction to $\widehat{\mathcal{M}}^n$ is subject to no restrictions except the natural ones mentioned above and those implied by Proposition 1.

**Theorem 1.** *Let ordinals $\alpha, \beta, \gamma, \delta \in [0, \omega_1]$ be given. Then a weak Lyapunov invariant satisfying the conditions*

1) $\varphi \in \mathfrak{F}^0_\gamma(\mathcal{M}^n_U) \cap \mathfrak{F}^0_\delta(\mathcal{M}^n_C)$;

2) $\varphi\big|_{\widehat{\mathcal{M}}^n} \in \mathfrak{F}^0_\alpha(\widehat{\mathcal{M}}^n_U) \cap \mathfrak{F}^0_\beta(\widehat{\mathcal{M}}^n_C)$,

*exists if and only if*

$$\alpha \leq \min\{\beta, \gamma\}, \quad \max\{\beta, \gamma\} \leq \delta, \quad \beta \neq 1, \quad \delta \neq 1.$$

**Definition 3** ([12])**.** Let $\mathcal{M} \subset \mathcal{M}^n$. We say that a functional $\varphi : \mathcal{M} \to \overline{\mathbb{R}}$ *has a compact support* if there exists $T > 0$ such that $\varphi(A) = \varphi(B)$ whenever $A, B \in \mathcal{M}$ coincide on the interval $[0, T]$. The set of all functionals on $\mathcal{M}$ with compact support is denoted by $\mathfrak{C}(\mathcal{M})$.

**Remark 1.** In the abstract [12] functionals with compact support are called boundedly dependent.

Suppose that a functional defined on a subspace of $\mathcal{M}^n_C$ is the repeated pointwise limit of a sequence of continuous ones. As noted in [12], the desire to compute the values of those based only on information on the system on finite time intervals naturally leads to the requirement that their supports be compact.

**Definition 4.** Let $\mathcal{M} \subset \mathcal{M}_C^n$. Define the $\alpha$-th *formula class* $\mathfrak{C}_\alpha(\mathcal{M})$ by (cf. [12])

$$\mathfrak{C}_\alpha(\mathcal{M}) = [\mathfrak{F}_0(\mathcal{M}) \cap \mathfrak{C}(\mathcal{M})]_\alpha, \ \ \alpha \in [0, \omega_1).$$

**Proposition 2** ([12]). *Let $\mathcal{M} \subset \mathcal{M}_C^n$. Then $\mathfrak{C}_\alpha(\mathcal{M}) \subset \mathfrak{F}_\alpha(\mathcal{M}) \subset \mathfrak{C}_{\alpha+1}(\mathcal{M})$ for all $\alpha \in [0, \omega_1)$. Moreover, for $\mathcal{M} = \mathcal{M}_C^n$ and $\alpha = 0$ the first inclusion is strict.*

Let a functional defined on a subspace of $\mathcal{M}_C^n$ be the repeated limit of a sequence of continuous ones. The next theorem states that the latter could be chosen to have compact support.

**Theorem 2.** *Let $\mathcal{M} \subset \mathcal{M}_C^n$. Then $\mathfrak{C}_\alpha(\mathcal{M}) = \mathfrak{F}_\alpha(\mathcal{M})$ for all $\alpha \in [1, \omega_1)$.*

The case $\alpha = 0$ is totally different as the next theorem shows.

**Theorem 3.** *Let $\mathcal{M} \subset \mathcal{M}_C^n$. Then $\mathfrak{C}_0(\mathcal{M}) = \mathfrak{F}_0(\mathcal{M})$ if and only if there exists $T > 0$ such that $A = B$ whenever $A, B \in \mathcal{M}$ coincide on the interval $[0, T]$.*

It appears that, generally speaking, one cannot decrease the number of limits in a formula for a weak Lyapunov invariant by allowing the prelimit functionals with compact support to be discontinuous.

**Theorem 4.** *Let $\mathcal{M} \supset \{A \in \mathcal{M}^n : \sup\limits_{t \geq 0} \|A(t)\| \leq 1\}$ be endowed with the compact-open topology. Then for all $\alpha \in [1, \omega_1)$ there exists a weak Lyapunov invariant $\varphi \in \mathfrak{F}_{\alpha+1}(\mathcal{M}) \setminus [\mathfrak{C}(\mathcal{M})]_\alpha$.*

For $\alpha = 1$ the statement of the above theorem can be strengthened: no nontrivial weak Lyapunov invariant is the limit of a sequence of functionals with compact support.

**Theorem 5.** *If $\mathcal{M} \in \{\widehat{\mathcal{M}_C^n}, \mathcal{M}_C^n\}$, then $[\mathfrak{C}(\mathcal{M})]_1$ does not contain weak Lyapunov invariants except constants.*

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# The Asymptotic Properties of Slowly Varying Solutions of Second Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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The aim of the work is to find necessary and sufficient conditions of existence of sufficiently wide special class of solutions of second order differential equations with regularly and rapidly varying nonlinearities and to obtain asymptotic representations for such solutions and their derivatives of the first order.

Second order differential equations with power and exponential nonlinearities play an important role in development of the qualitative theory of differential equations. Such equations also have a lot of applications in practice. It happens, for example, when we study the distribution of electrostatic potential in a cylindrical volume of plasma of products of burning.

The corresponding equation may be reduced to the following one:

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

In the work of V. M. Evtuhov and N. G. Drik [3], some results on asymptotic behavior of solutions of such equations have been obtained.

Exponential nonlinearities form a special class of rapidly varying nonlinearities. The consideration of the last ones is necessary for some models. All this makes the topic of our research actual.

Our investigations need establishment of the next class of functions.

We call the measurable function $\varphi : \Delta_Y \to ]0, +\infty[$ a regularly varying as $y \to Y$, $z \in \Delta_Y$ of index $\sigma$ [1] if for every $\lambda > 0$ we have

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{\varphi(\lambda y)}{\varphi(y)} = \lambda^{\sigma}.$$

Here $Y \in \{0, \pm\infty\}$, $\Delta_Y$ is some one-sided neighbourhood of $Y$. If $\sigma = 0$, such function is called slowly varying.

The function $\varphi : [s, +\infty[ \to ]0, +\infty[$ $(s > 0)$ is called a rapidly varying function [1] of the $+\infty$ order on infinity if this function is measurable and

$$\lim_{y \to \infty} \frac{\varphi(\lambda y)}{\varphi(y)} = \begin{cases} 0 & \text{at } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ +\infty & \text{at } \lambda > 1. \end{cases}$$

It is called a rapidly varying function of the $-\infty$ order on infinity if

$$\lim_{y \to \infty} \frac{\varphi(\lambda y)}{\varphi(y)} = \begin{cases} +\infty & \text{if } 0 < \lambda < 1, \\ 1 & \text{at } \lambda = 1, \\ 0 & \text{if } \lambda > 1. \end{cases}$$

The function $\varphi(y)$ is called a rapidly varying function of zero order if $\varphi(\frac{1}{y})$ is a rapidly varying function of $+\infty$ order. An exponential function is a special case of the last ones.

The differential equation

$$y'' = \alpha_0 p(t)\varphi(y),$$

with a rapidly varying function $\varphi$, was investigated in the work of V. M. Evtuhov and V. M. Kharkov [4]. But in the mentioned work the introduced class of solutions of the equation depends on the function $\varphi$. This is not convenient for practice.

The more general class of equations of such type is established in this work.

Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[ \to ]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to ]0, +\infty[$ $(i \in \{0, 1\})$ – are continuous functions, $Y_i \in \{0, \pm\infty\}$, $\Delta_{Y_i}$ – is one-sided neighborhood of $Y_i$.

Furthermore, we assume that function $\varphi_1$ is a regularly varying function as $y \to Y_1$ $(y \in \Delta_{Y_1})$ of the order $\sigma_1$, and function $\varphi_0$ is twice continuously differentiable and satisfies the following limit relations

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

From conditions (2) it can be proved that $\varphi_0$ and its derivatives of the first order are rapidly varying function as $y \to Y_0$ $(y \in \Delta_{Y_0})$.

The main aim of our research is the development of methods of establishing asymptotic representations of solutions of such differential equations in order to receive a new class of mentioned equations.

We use a lot of methods of mathematical analysis, linear algebra, analytic geometry, theory of homogeneous differential equations in our work. Some special methods of investigation of equations of the mentioned type, being developed by the superiors, are also used.

We call solution $y$ of the equation (1) defined on $[t_0, \omega[ \subset [a, \omega[$, the $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t, \omega[ \to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In this work we consider $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) in case $\lambda_0 = 0$. Because of the properties of these solutions (see, eg., [2]) all of them are slowly varying functions as $t \uparrow \omega$. Therefore the case $\lambda_0 = 0$ is one of the most difficult for research. The problem of investigation $P_\omega(Y_0, Y_1, 0)$-solutions for equations with rapidly varying functions is difficult by the fact that composition of rapidly and regularly varying functions may be as rapidly, as regularly, as slowly varying function as the argument tents to the singular point.

We have obtained the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of equation (1) and find asymptotic representations of these solutions and their derivatives of the first order.

Now we need the following notations

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\qquad \theta_1(y)=\varphi_1(y)|y|^{-\sigma_1},$$

$$I(t)=\operatorname{sign}(y_1^0)\times\int\limits_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$\Phi_0(y)=\int\limits_{A_\omega^0}^{y}|\varphi_0(s)|^{\frac{1}{\sigma_1-1}}\,ds,\quad A_\omega^0=\begin{cases} y_0^0 & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}|\varphi_0(y)|^{\frac{1}{\sigma_1-1}}\,dy=+\infty,\\ Y_0 & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}|\varphi_0(y)|^{\frac{1}{\sigma_1-1}}\,dy<+\infty,\end{cases}$$

$$\operatorname{sign}\varphi_0(y)=f_1 \ \text{ as } \ y\in\Delta_{Y_0},\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y).$$

The inferior limits of the integrals are chosen in such forms that the corresponding integrals tend either to 0 or to $\infty$ as $t\uparrow\omega$ and $y\to Y_0$, $y\in\Delta_{Y_0}$, correspondingly.

Note some necessary definitions.

**Definition 1.** Let $Y\in\{0,\pm\infty\}$, $\Delta_Y$ – is some one-sided neighborhood of $Y$. The continuously differentiable function $L:\Delta_Y\to]0,+\infty[$ is called normaliyed slowly varying function [5] as $y\to Y$ $(y\in\Delta_Y)$ if

$$\lim_{\substack{y\to Y_1\\ y\in\Delta_{Y_i}}}\frac{yL'(y)}{L(y)}=0.$$

**Definition 2.** We say that a slowly varying as $y\to Y$ $(y\in\Delta_Y)$ function $\theta:\Delta_Y\to]0,+\infty[$ satisfies the condition $S$ if for any normaliyed slowly varying function $L:\Delta_{Y_i}\to]0,+\infty[$ the following condition takes place

$$\theta(yL(y))=\theta(y)(1+o(1)) \ \text{ as } \ y\to Y \ \ (y\in\Delta_Y).$$

**Remark 1.** The following statement is true

$$\Phi(y)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)] \ \text{ as } \ y\to Y_0 \ \ (y\in\Delta_{Y_0}).$$

From this as $y\in\Delta_{Y_0}$, we have

$$\operatorname{sign}(\varphi_0'(y)\Phi(y))=\operatorname{sign}(\sigma_1-1).$$

**Remark 2.** Because of conditions (2) on the function $\varphi_0$, we have that $z_1 \in \{0, +\infty\}$ and

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\Phi''(y) \cdot \Phi(y)}{(\Phi'(y))^2} = 1.$$

The following conclusions take place for equation (1).

**Theorem 1.** *Let $\sigma_1 \neq 1$. Then for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of the equation* (1) *such that the following finite or infinite limit exists*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) y''(t)}{y'(t)}\,,$$

*it's necessary the following conditions*

$$f_1 I(t)(\sigma_1 - 1) > 0, \quad \alpha_0 \pi_\omega(t) y_1^0 < 0 \ \ as \ \ t \in [a, \omega[\,, \tag{3}$$

$$\lim_{t \uparrow \omega} \frac{y_1^0}{|\pi_\omega(t)|} = Y_1, \quad \lim_{t \uparrow \omega} I(t) = Z_1, \quad \lim_{t \uparrow \omega} \frac{I'(t)\pi_\omega(t)}{\Phi'(\Phi^{-1}(I(t)))\Phi^{-1}(I(t))} = 0 \tag{4}$$

*to be fulfilled.*

*If the function $\theta_1$ satisfies the condition $S$, the following finite or infinite limit exists $\lim\limits_{t \uparrow \omega} \frac{\pi_\omega(t) I'(t)}{I(t)}$, the function $\frac{\pi_\omega(t) \cdot I'(t)}{I(t)}$ is a normalized slowly varying function as $t \uparrow \omega$, the function $(\frac{\Phi'(y)}{\Phi(y)})$ is a regularly varying function of the order $\gamma_0$ as $y \to Y_0$ $(y \in \Delta_{Y_0})$, $(\gamma_0 + 1) < 0$ as $Y_0 = 0$, and $(\gamma_0 + 1) > 0$ in another case, and*

$$\lim_{t \uparrow \omega} \Big| \frac{\pi_\omega(t) I'(t)}{I(t)} \Big| < +\infty$$

*or*

$$\pi_\omega(t) \cdot I(t) \cdot I'(t)(1 - \sigma_1) > 0, \ \ when \ \ t \in [a, \omega[\,,$$

*then* (3), (4) *are sufficient conditions for the existence of such solutions for the equation* (1). *For every $P_\omega(Y_0, Y_1, 0)$-solution the following asymptotic representations take place as $t \uparrow \omega$*

$$\Phi(y(t)) = I(t)[1 + o(1)], \quad \frac{y'(t)\Phi'(y(t))}{\Phi(y(t))} = \frac{I'(t)}{I(t)}\,[1 + o(1)].$$

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# The Dirichlet Problem for a Class of Anisotropic Mean Curvature Equations

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## 1 Introduction

We are concerned with the study of the existence, uniqueness, regularity and boundary behaviour of the solutions of the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $b > 0$ are given constants and $\Omega$ is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial\Omega$.

Problem (1.1) has been recently introduced in order to describe the geometry of the human cornea. We refer to [13–17] for the derivation of the model, further discussions on the subject and an additional bibliography. It should however be pointed out that in [13,14,16,17] a simplified version of (1.1) has been investigated, where the curvature operator, $\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$, is replaced by its linearization around 0, $\operatorname{div}(\nabla u) = \Delta u$, and, furthermore, $\Omega$ is supposed to be either an interval in $\mathbb{R}$, or a disk in $\mathbb{R}^2$. In [2,3] we have instead considered the complete model (1.1) and we have proved the existence of a unique classical solution for any choice of the positive parameters $a$, $b$, but still assuming that $\Omega$ is an interval in $\mathbb{R}$, or a ball in $\mathbb{R}^N$. Some numerical experiments for approximating the solution of the 1-dimensional problem have also been performed in [2,15]. Later on, in [4], we tackled the quite challenging problem in arbitrary Lipschitz domains and we proved, for all $a, b > 0$, the existence and the uniqueness of a generalized solution, which is regular in the interior and attains the Dirichlet boundary data under an additional condition that relates the values of the parameters with the geometry of the domain. The necessity of considering generalized solutions in this context is dictated by the possible occurrence of solutions which are singular at the boundary, namely solutions that are regular in the interior, but do not attain the Dirichlet condition at some points of the boundary, where in addition the normal derivative blows up. We refer to the survey paper [5] for a thorough discussion of this matter. The following notions of solution for problem (1.1), partially inspired by [6, 7, 9–12, 19], are therefore introduced.

**Definition 1.1.** A function $u \in W^{1,1}(\Omega)$ is a *generalized* solution of (1.1) if the following conditions hold:

- $\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^N(\Omega)$;
- $u$ satisfies the equation in (1.1) a.e. in $\Omega$;
- for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,
  - either $u(x) = 0$,
  - or $u(x) > 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,, \nu\Big](x) = -1$,
  - or $u(x) < 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,, \nu\Big](x) = 1$,

  where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu] \in L^\infty(\partial\Omega)$ is the weakly defined trace on $\partial\Omega$ of the component of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal $\nu$ to $\Omega$ (cf. [1]).

A generalized solution $u$ of (1.1) is *classical* if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u(x) = 0$ on $\partial\Omega$.
A generalized solution $u$ of (1.1) is *singular* if it is not classical.

The concept of generalized solution expressed by Definition 1.1 looks rather natural in the frame of (1.1) and can heuristically be interpreted as follows: the solution $u$ is not required to satisfy the homogeneous Dirichlet boundary condition at all points of $\partial\Omega$, but at any point of $\partial\Omega$ where the zero boundary value is not attained the unit upper normal $\mathcal{N}(u)$ to the graph of $u$ equals either the unit outer normal $(\nu, 0)$, or the unit inner normal $(-\nu, 0)$, according to the sign of $u$; in this case, roughly speaking, the graph of the solution might be smoothly continued by vertical segments up to the zero level. This kind of boundary behaviour of solutions of the $N$-dimensional prescribed mean curvature equation has already been observed and discussed in [6, 7, 10, 12]. With reference to Definition 1.1 we can state various existence, uniqueness and regularity results, which are the contents of the next sections.

## 2 Radially symmetric solutions

Since the equation in (1.1) is invariant under orthogonal transformations, it is natural to look for radially symmetric solution whenever the domain is either a ball, or a spherical shell. However the solvability patterns in the two cases are quite different.

### Classical solutions on balls

Let $B = B(x_0, R)$ be the open ball in $\mathbb{R}^N$ of center $x_0$ and radius $R$.

**Theorem 2.1.** *For every $a > 0$, $b > 0$, there exists a unique generalized solution $u$ of (1.1), with $\Omega = B$, which is radially symmetric and classical, with $u \in C^2(\overline{B})$. Moreover, there exists a function $v \in C^2([0,R])$, with $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$, such that*

- $0 < v(t) < b/a$ *for all* $t \in [0, R[\,$;
- $v'(t) < 0$ *for all* $t \in \,]0, R]$;
- $v''(t) < 0$ *for all* $t \in [0, R]$.

### Singular solutions on thick shells

Let $S = S_{r,R}(x_0) = \{x \in \mathbb{R}^N \mid r < |x - x_0| < R\}$ be the spherical shell centered at $x_0$ and having radii $r$, $R$, with $0 < r < R$.

**Theorem 2.2.** *For any given $N \geq 2$, $a > 0$ and $r > 0$, there exist $R^* > 0$ and $b^* > 0$ such that, for all $R > R^*$ and $b > b^*$, there is a unique generalized solution $u$ of (1.1), with $\Omega = S$, which is radially symmetric, singular and satisfies*

$$u \in C^2(S \cup \partial B), \quad u(x) = 0 \ \text{ if } |x - x_0| = R,$$
$$u(x) > 0 \ \text{ if } \Big[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nu\Big](x) = -1 \ \text{ if } |x - x_0| = r.$$

### Classical solutions on thin shells

It is worth observing that the conclusions of Theorem 2.2 fail if $R$ is not bounded away from $r$.

**Theorem 2.3.** *For any given $N \geq 2$, $a > 0$, $b > 0$ and $r > 0$, there exists $R_* > 0$ such that, for all $R \in \,]r, R_*[\,$, there is a unique generalized solution $u$ of (1.1), with $\Omega = S$, which is radially symmetric and classical, with $u \in C^2(\overline{S})$.*

## 3 Small classical solutions on arbitrary domains

If $\Omega$ is an arbitrary bounded regular domain in $\mathbb{R}^N$, the existence of a maximal connected two-dimensional branch of classical solutions, which emanates from the line of trivial solutions, can be established.

**Theorem 3.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in \,]0, 1[\,$. Then, there exists a set*

$$\mathcal{S} = \bigcup_{a>0} \big(\{a\} \times [0, b_\infty(a)[\big) \subseteq \mathbb{R}_0^+ \times \mathbb{R}^+$$

*such that, for any $(a, b) \in \mathcal{S} \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$, problem (1.1) has a unique generalized solution $u = u(a, b) \in C^{2,\alpha}(\overline{\Omega})$, which is classical, asymptotically stable, smoothly depends on the parameters $(a, b)$ in the topology of $C^{2,\alpha}(\overline{\Omega})$, and satisfies, for every $a > 0$,*

$$\lim_{b \to 0} \|u(a, b)\|_{C^{2,\alpha}} = 0$$

*and, in case $b_\infty(a) < +\infty$,*

$$\limsup_{b \to b_\infty(a)} \|\nabla u(a, b)\|_\infty = +\infty.$$

# 4 Generalized solutions on arbitrary domains

The proof of the existence of generalized solutions is conceptually delicate and technically elaborate. It requires the study, in the space of bounded variation functions, of a suitable action functional, involving an anisotropic area term, whose minimizers give raise, via a change of variables, to the generalized solutions. The interior regularity of these bounded variation minimizers is obtained by combining a delicate approximation scheme with a "local" existence result basically due to Serrin [18] and the classical gradient estimates of Ladyzhenskaya and Ural'tseva [8].

**Theorem 4.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every $a > 0$, $b > 0$, there exists a unique generalized solution $u$ of problem (1.1), which also satisfies:*

- *$u \in C^\infty(\Omega)$;*
- *the set of points $x_0 \in \partial\Omega$, where $u$ is continuous and satisfies $u(x_0) = 0$, is non-empty;*
- *$0 < u(x) < b/a$ for all $x \in \Omega$;*
- *$u$ minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional*

$$\int\limits_{\Omega} e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int\limits_{\Omega} e^{-bz}\Big(z+\frac{1}{b}\Big)\,dx + \frac{1}{b}\int\limits_{\partial\Omega} |e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

**Remarks.** The second conclusion of Theorem 4.1 can be further specified as follows: $u$ is continuous at $x_0$ and satisfies $u(x_0) = 0$ at any point $x_0 \in \partial\Omega$ where an exterior sphere condition holds with radius $r \geq (N-1)\,b/a$ (i.e., there exists a point $y \in \mathbb{R}^N$ such that the open ball $B(y,r)$ of center $y$ and radius $r$ satisfies $B(y,r)\cap\Omega = \varnothing$ and $x_0 \in \overline{B(y,r)}\cap\partial\Omega$). Clearly, an exterior sphere condition, with arbitrary radius, holds at all points $x_0 \in \partial\Omega$ belonging to the boundary of the convex hull of $\overline{\Omega}$. The last conclusion of Theorem 4.1 also shows that all generalized solutions of (1.1) enjoy some form of stability.

# 5 Classical versus singular solutions

Combining the previous results yields a rather complete picture of the structure of the solution set of problem (1.1).

**Theorem 5.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in ]0,1[$. Then, for every $a > 0$, either for all $b > 0$ problem (1.1) has a unique generalized solution, which is classical, or there exists $b^* = b^*(a) \in ]0,+\infty[$ such that*

- *if $b \in ]0,b^*]$, then problem (1.1) has a unique generalized solution $u$, which is classical;*
- *if $b \in ]b^*,+\infty[$, then problem (1.1) has a unique generalized solution $u$, which is singular.*

*In addition, the following conclusions hold:*

- *the map $a \longmapsto b^*(a)$ is non-decreasing, with $\inf\limits_{a>0} b^*(a) > 0$;*
- *the map $(a,b) \longmapsto u(a,b)$ is continuous from $\mathbb{R}^+_0 \times \mathbb{R}^+$ to $L^\infty(\Omega)$;*
- *for any $a > 0$, the map $b \longmapsto u(a,b)$ is increasing in the sense that if $b_1 < b_2$, then $u(a,b_1) < u(a,b_2)$ in $\Omega$;*
- *for any $b > 0$, the map $a \longmapsto u(a,b)$ is decreasing in the sense that if $a_1 < a_2$, then $u(a_1,b) > u(a_2,b)$ in $\Omega$.*

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# Oscillation and Nonoscillation Results for Half-Linear Equations with Deviated Argument

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This is an enlarged abstract of the joint work with Alois Kufner and Komil Kuliev [**?**]. We introduce oscillatory and nonoscillatory criteria for half-linear equations with deviated argument and dedicate it to the 100 birthday anniversary of Professor A. Bitsadze. Our method relies on the *weighted Hardy inequality*.

Let us consider the *half-linear equation* with *deviated argument*

$$\big(r(t)|u'(t)|^{p-2}u'(t)\big)' + c(t)|u(\tau(t))|^{p-2}u(\tau(t)) = 0, \quad t \in (0,\infty), \tag{1}$$

where $p > 1$, $c : [0,\infty) \to (0,\infty)$ is continuous, $c \in L^1(0,\infty)$, $r : [0,\infty) \to (0,\infty)$ is continuously differentiable, $\tau : [0,\infty) \to \mathbb{R}$ is continuously differentiable and increasing function satisfying $\lim\limits_{t\to\infty} \tau(t) = \infty$.

Assume that (1) has at least one nonzero *global solution* defined on the entire interval $(0,\infty)$. We say that a global solution of (1) is *nonoscillatory* (at $\infty$) if there exists $T > 0$ such that $u(t) \neq 0$ for all $t > T$. Otherwise, it is called *oscillatory*, i.e., there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $u(t_n) = 0$ for all $n \in \mathbb{N}$. We let $p' = \frac{p}{p-1}$.

**Theorem 1** (nonoscillatory criterion)**.** *Let*

$$\limsup_{t\to\infty} \bigg(\int_0^t r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} < \frac{(p-1)}{p^{p'}} \tag{2}$$

*and*

$$\limsup_{t\to\infty} \bigg(\int_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} < \frac{(p-1)}{p^{p'}}\,. \tag{3}$$

*Then every global solution of* (1) *is nonoscillatory.*

**Theorem 2** (oscillatory criterion)**.** *Let one of the following three cases occur:*

(i) *There exists $T > 0$ such that for all $t \geq T$ we have $\tau(t) \geq t$ and*

$$\limsup_{t\to\infty} \bigg[\bigg(\int_0^t r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} + \bigg(\int_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int_{\tau(t)}^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}}\bigg] > 1.$$

(ii) *There exists $T > 0$ such that for all $t \geq T$ we have $\tau(t) \leq t$ and*

$$\limsup_{t\to\infty} \bigg(\int_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1.$$

(iii) *For any $T > 0$ the function $\tau(t) - t$ changes sign in $(T, \infty)$ and either*

$$\liminf_{\substack{t\to\infty\\ t>\tau(t)}} \bigg(\int\limits_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1$$

*or*

$$\liminf_{\substack{t\to\infty\\ t<\tau(t)}} \bigg[\bigg(\int\limits_0^{t} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} + \bigg(\int\limits_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_{\tau(t)}^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}}\bigg] > 1.$$

*Then every global solution of* (1) *is oscillatory.*

A typical example of $\tau = \tau(t)$ is a linear function

$$\tau(t) = t - \tau, \ \ \tau \geq 0 \ \text{ is fixed.}$$

Then (1) is half-linear equation with the *delay* given by fixed parameter $\tau \geq 0$. For this, rather special case, (2) implies (3), and only the case (ii) of Theorem 2 occurs. Hence we have the following corollary concerning the equation

$$\big(r(t)|u'(t)|^{p-2}u'(t)\big)' + c(t)|u(t-\tau)|^{p-2}u(t-\tau) = 0, \ \ t \in (0, \infty). \tag{4}$$

**Corollary 3** (equation with delay)**.** *Let* (2) *hold. Then every global solution of* (4) *with the delay $\tau \geq 0$ is nonoscillatory. On the other hand, let*

$$\limsup_{t\to\infty} \bigg(\int\limits_0^{t-\tau} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1.$$

*Then every global solution of* (4) *with the delay $\tau \geq 0$ is oscillatory.*

**Remark 4.** Let us note that nonoscillatory criteria are rare in the literature even for the linear equations with the delay. Oscillatory criteria for solutions of half-linear equations with the delay are presented in recent papers [3]–[**?**], [8] and [**?**]. The methodology in these articles is based on the so-called Riccati technique and the assumptions are different than those of ours. In particular, if $\tau(t) = t$ in (1), we have the "classical" half-linear equation considered e.g. in [1, Chapter 3]. Then oscillatory criterion in Corollary 3 (with $\tau = 0$) recovers [1, Theorem 3.1.2]. On the other hand, nonoscillatory criterion in Corollary 3 (with $\tau = 0$) recovers [1, Theorem 3.1.3]. The approach in [1, Chapter 1] is based also on the *Riccati technique*. In contrast with works on half-linear equations with the delay mentioned above, we present both oscillatory and nonoscillatory criteria and our method relies on the *weighted Hardy inequality*. Similar approach to that of ours was used in [9] to prove oscillation and nonoscillation results for solutions of higher order half-linear equations, but without the deviated argument. For the completeness, we refer also to the papers [**?**], [**?**] and [12] which deal with the half-linear equations with the deviated argument in the case $r(t) = 1$.

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# On Asymptotic Behavior of Solutions to Second-Order Regular and Singular Emden–Fowler Type Differential Equations with Negative Potential

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## 1 Introduction

Consider the second-order Emden–Fowler type differential equation

$$y'' - p(x,y,y')|y|^k \operatorname{sgn} y = 0, \quad k > 0, \quad k \neq 1, \tag{1}$$

where the function $p(x,u,v)$ defined on $\mathbb{R} \times \mathbb{R}^2$ is positive, continuous in $x$, Lipschitz continuous in $u$, $v$.

Asymptotic classification of all solutions to equation (1) in the case $p = p(x)$ was described by I. T. Kiguradze and T. A. Chanturia in [13]. Asymptotic classification of non-extensible solutions to similar third- and fourth-order differential equations was obtained by I. V. Astashova (see [1,3–5]). Asymptotic classification of solutions to equation (1) for the bounded function $p(x,u,v)$ is contained in [8,9].

Sufficient conditions providing $\lim\limits_{x\to a} |y'(x)| = +\infty$, $a \in \mathbb{R}$, were obtained in [13]. However, the question of separating two cases

$$\lim_{x\to a} |y(x)| = +\infty \ \text{ and } \ \lim_{x\to a} |y(x)| < +\infty \tag{2}$$

remained open. The answer on this question for $p(x,u,v) = \widetilde{p}(x)|v|^\lambda$, $\lambda \neq 1$ was considered in [11].

Asymptotic behavior of non-extensible solutions to equation (1) for unbounded function $p(x,u,v)$ is investigated in [6,7,10]. By using methods described in [1,2], conditions on function $p(x,u,v)$ and initial data providing the existence of a vertical asymptote to related solution (i.e. the first case of (2)) are obtained. Other conditions on $p(x,u,v)$ and initial data sufficient for the second case of (2) are considered. Solutions satisfying the second condition of (2) are called *black hole* solutions (see [12]).

## 2 Asymptotic classification of solutions to Emden–Fowler type differential equations with bounded negative potential

Let us use the notation

$$\alpha = \frac{2}{k-1}\,, \quad C(\widetilde{p}) = \Big(\frac{\alpha(\alpha+1)}{\widetilde{p}}\Big)^{\frac{1}{k-1}} = \Big(\frac{\widetilde{p}(k-1)^2}{2(k+1)}\Big)^{\frac{1}{1-k}}.$$

**Definition 2.1.** A solution $y(x)$ to (1) is called *positive Kneser solution on* $(x_0;+\infty)$ if it satisfies the conditions $y(x) > 0$, $y'(x) < 0$ at $x > x_0$.

**Definition 2.2.** A solution $y(x)$ to (1) is called *negative Kneser solution on* $(x_0;+\infty)$ if it satisfies the conditions $y(x)<0$, $y'(x)>0$ at $x>x_0$.

**Definition 2.3.** A solution $y(x)$ to (1) is called *positive Kneser solution on* $(-\infty;x_0)$ if it satisfies the conditions $y(x)>0$, $y'(x)>0$ at $x<x_0$.

**Definition 2.4.** A solution $y(x)$ to (1) is called *negative Kneser solution on* $(-\infty;x_0)$ if it satisfies the conditions $y(x)<0$, $y'(x)<0$ at $x<x_0$.

**Theorem 2.1.** *Suppose $k>1$. Let the function $p(x,u,v)$ be continuous in $x$, Lipschitz continuous in $u$, $v$ and satisfying inequalities*

$$0<m\le p(x,u,v)\le M<+\infty. \tag{3}$$

*Let there also exist the following limits of $p(x,u,v)$:*

1) $P_+$ *as* $x\to+\infty$, $u\to 0$, $v\to 0$,

2) $P_-$ *as* $x\to-\infty$, $u\to 0$, $v\to 0$,

*and for any $c\in\mathbb{R}$,*

3) $P_c^+$ *as* $x\to c$, $u\to+\infty$, $v\to\pm\infty$,

4) $P_c^-$ *as* $x\to c$, $u\to-\infty$, $v\to\pm\infty$.

*Then all non-extensible solutions to (1) are divided into the following nine types according to their asymptotic behavior:*

0. *Defined on the whole axis trivial solution $y_0(x)\equiv 0$.*

1–2. *Defined on $(b,+\infty)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:*

$$y_1(x)=C(P_b^+)(x-b)^{-\alpha}(1+o(1)),\ \ x\to b+0,\quad y_1(x)=C(P_+)x^{-\alpha}(1+o(1)t),\ \ x\to+\infty,$$
$$y_2(x)=-C(P_b^-)(x-b)^{-\alpha}(1+o(1)t),\ \ x\to b+0,\quad y_2(x)=-C(P_+)x^{-\alpha}(1+o(1)),\ \ x\to+\infty.$$

3–4. *Defined on $(-\infty,a)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:*

$$y_3(x)=C(P_a^+)(a-x)^{-\alpha}(1+o(1)),\ \ x\to a-0,\quad y_3(x)=C(P_-)|x|^{-\alpha}(1+o(1)),\ \ x\to-\infty,$$
$$y_4(x)=-C(P_a^-)(a-x)^{-\alpha}(1+o(1)),\ \ x\to a-0,\quad y_4(x)=-C(P_-)|x|^{-\alpha}(1+o(1)),\ \ x\to-\infty.$$

5–6. *Defined on $(a,b)$ positive and negative solutions with power asymptotic behavior near domain boundaries:*

$$y_5(x)=C(P_a^+)(x-a)^{-\alpha}(1+o(1)),\ x\to a+0,\ \ y_5(x)=C(P_b^+)(b-x)^{-\alpha}(1+o(1)),\ x\to b-0,$$
$$y_6(x)=-C(P_a^-)(x-a)^{-\alpha}(1+o(1)),\ x\to a+0,\ \ y_6(x)=-C(P_b^-)(b-x)^{-\alpha}(1+o(1)),\ x\to b-0.$$

7–8. *Defined on $(a,b)$ solutions with different signs and power asymptotic behavior near domain boundaries:*

$$y_7(x)=C(P_a^+)(x-a)^{-\alpha}(1+o(1)),\ x\to a+0,\ \ y_7(x)=-C(P_b^-)(b-x)^{-\alpha}(1+o(1)),\ x\to b-0,$$
$$y_8(x)=-C(P_a^-)(x-a)^{-\alpha}(1+o(1)),\ x\to a+0,\ \ y_8(x)=C(P_b^+)(b-x)^{-\alpha}(1+o(1)),\ x\to b-0.$$

**Definition 2.5** (see [5])**.** A solution $y : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MU-solution* if the following conditions hold:

(i) the equation has no solution equal to $y$ on some subinterval of $(a, b)$ and not equal to $y$ at some point of $(a, b)$;

(ii) either there is no solution defined on another interval containing $(a, b)$ and equal to $y$ on $(a, b)$ or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of $(a, b)$.

**Theorem 2.2.** *Suppose $0 < k < 1$. Let the function $p(x, u, v)$ be continuous in $x$, Lipschitz continuous in $u$, $v$ and satisfying inequalities* (3)*. Let there also exist the following limits of $p(x, u, v)$:*

1) $P_{++}$ *as* $x \to +\infty$, $u \to +\infty$, $v \to +\infty$;

2) $P_{+-}$ *as* $x \to +\infty$, $u \to -\infty$, $v \to -\infty$;

3) $P_{-+}$ *as* $x \to -\infty$, $u \to +\infty$, $v \to -\infty$;

4) $P_{--}$ *as* $x \to -\infty$, $u \to -\infty$, $v \to +\infty$,

*and for any $c \in \mathbb{R}$ denote $P_c = p(c, 0, 0)$.*

*Then all MU–solutions to equation* (1) *are divided into the following eight types according to their asymptotic behavior:*

1–2. *Defined on semi-axis $(b, +\infty)$ positive and negative solutions tending to zero with their derivatives as $x \to b + 0$ with power asymptotic behavior near domain boundaries:*

$$y_1(x) = C(P_b)(x-b)^{-\alpha}(1+o(1)), \ \ x \to b+0, \quad y_1(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty,$$
$$y_2(x) = -C(P_b)(x-b)^{-\alpha}(1+o(1)), \ \ x \to b+0, \quad y_2(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty.$$

3–4. *Defined on semi-axis $(-\infty, a)$ positive and negative solutions tending to zero with their derivatives as $x \to a - 0$ with power asymptotic behavior near domain boundaries:*

$$y_3(x) = C(P_a)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_3(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_4(x) = -C(P_a)(a-x)^{-\alpha}(1+o(1)), \ \ x \to a-0, \quad y_4(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

5–6. *Defined on the whole axis solutions with same signs and power asymptotic behavior near domain boundaries:*

$$y_5(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_5(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_6(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_6(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

7–8. *Defined on the whole axis solutions with different signs and power asymptotic behavior near domain boundaries:*

$$y_7(x) = C(P_{++})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_7(x) = -C(P_{--})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y_8(x) = -C(P_{+-})x^{-\alpha}(1+o(1)), \ \ x \to +\infty, \quad y_8(x) = C(P_{-+})|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty.$$

## 3 Asymptotic behavior of solutions to Emden–Fowler type differential equations with unbounded negative potential

**Lemma 3.1.** *Suppose $k > 1$. Let $p(x,u,v)$ be continuous in $x$, Lipschitz continuous in $u, v$, and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation* (1) *satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ at some point $x_0$. Then there exists $x^* \in (x_0,+\infty)$ or respectively $x_* \in (-\infty, x_0)$, such that*

$$\lim_{x\to x^*-0} |y'(x)| = +\infty \quad \text{or respectively} \quad \lim_{x\to x_*+0} |y'(x)| = +\infty. \tag{4}$$

**Lemma 3.2.** *Suppose $0 < k < 1$. Let $p(x,u,v)/|v|$ be continuous in $x$, Lipschitz continuous in $u, v$, for $v \neq 0$ and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation* (1) *satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ but not $y(x_0) = y'(x_0) = 0$ at some point $x_0$. Then there exists $x^* \in (x_0,+\infty)$ or respectively $x_* \in (-\infty, x_0)$ providing* (4).

Using the substitutions $x \longmapsto -x$, $y(x) \longmapsto -y(x)$ we obtain an equation of the same type as (1). That is why we investigate asymptotic behavior of non-extensible positive solutions to equation (1) near the right domain boundary only.

**Theorem 3.1.** *Suppose there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the function $p = p(x,u,v)$ has the representation $p = h(u)g(v)$, where the functions $h(u)$, $g(v)$ are continuous and bounded below by a positive constant, and for $0 < k < 1$ function $p$ additionally satisfies the conditions of Lemma* 3.2*. Then for any non-extensible solution $y(x)$ to equation* (1) *with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of* (2) *the line $x = x^*$ is a vertical asymptote if and only if*

$$\int_{v_0}^{+\infty} \frac{v}{g(v)}\, dv = +\infty. \tag{5}$$

**Theorem 3.2.** *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x,u,v)$ satisfies the conditions of Lemma* 3.1 *or respectively Lemma* 3.2*. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x,u,v) \leq f(x,u)g(v)$ holds, where the function $f(x,u)$ is continuous, the function $g(v)$ is continuous, bounded below by a positive constant and satisfies the condition*

$$\int_{v_0}^{+\infty} \frac{dv}{g(v)} = +\infty. \tag{6}$$

*Then for any non-extensible solution $y(x)$ to equation* (1) *with initial data satisfying inequalities $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and with the first property of* (2) *the line $x = x^*$ is a vertical asymptote.*

**Theorem 3.3.** *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x,u,v)$ satisfies the conditions of Lemma* 3.1 *or respectively Lemma* 3.2*. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x,u,v) \leq g(v)$ holds, where the function $g(v)$ is continuous and satisfies the condition* (6)*. Then for any non-extensible solution $y(x)$ to equation* (1) *with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of* (2) *the line $x = x^*$ is a vertical asymptote.*

**Theorem 3.4.** *Suppose for $k > 1$ or $0 < k < 1$ the function $p(x,u,v)$ satisfies the conditions of Lemma* 3.1 *or respectively Lemma* 3.2*. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x,u,v) \geq g(v)$ holds, where the function $g(v)$ is continuous, bounded below* *by a positive constant and doesn't satisfy the condition* (5). *Then for any non-extensible solution* $y(x)$ *to equation* (1) *with initial data* $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ *and the first property of* (2) *we have*

$$0 < \lim_{x \to x^* - 0} y(x) < +\infty, \quad x^* - x_0 < \frac{1}{y^k(x_0)} \int\limits_{y'(x_0)}^{+\infty} \frac{dv}{g(v)}\,.$$

**Theorem 3.5.** *Suppose* $k > 0$, $k \neq 1$. *Let the function* $p(x, u, v)$ *be continuous in* $x$, *Lipschitz continuous in* $u$, $v$. *Let there exist constants* $u_0 > 0$, $v_0 > 0$ *such that for* $u > u_0$, $v > v_0$ *the inequality* $p(x, u, v) \leq C|v|^{-\alpha}$ *holds. Then any non-extensible solution* $y(x)$ *to equation* (1) *with initial data* $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ *can be extended to* $(x_0, +\infty)$ *and*

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} y(x) = +\infty.$$

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# Asymptotic Behaviour of Solutions of One Class of Third-Order Ordinary Differential Equations

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We consider the differential equation

$$y''' = \alpha_0 p(t) y L(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[ \to ]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $L : \Delta_{Y_0} \to ]0, +\infty[$ is a continuous function slowly varying as $y \to Y_0$, $Y_0$ is equal to either 0 or $\pm\infty$, and $\Delta_{Y_0}$ is a one-sided neighborhood of $Y_0$.

In the case where $L(y) \equiv 1$, Eq. (1) is a linear third-order differential equation. The asymptotic behavior of its solutions as $t \to +\infty$ (the case $\omega = +\infty$) is investigated in details (see, for example, the monograph [2, Ch. I, § 6, pp. 175–194]).

In the paper [1], the conditions for the existence and asymptotic representations as $t \uparrow \omega$ of all possible types of $P_\omega(Y_0, \lambda_0)$-solutions were established for the second-order differential equation with the same kind of right-hand side.

**Definition.** We say that a solution $y$ of Eq. (1) is a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[ \subset [a, \omega[$ and satisfies the conditions

$$y : [t_0, \omega[ \to \Delta_{Y_0}, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \ \pm\infty \end{cases} \quad (k = 1, 2), \quad \lim_{t \uparrow \omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0.$$

Further, without loss of generality, we assume that

$$\Delta_{Y_0}(b) = \begin{cases} [b, Y_0[\,, & \text{if } \Delta_{Y_0} - \text{left neighborhood } Y_0, \\ ]Y_0, b], & \text{if } \Delta_{Y_0} - \text{right neighborhood } Y_0, \end{cases}$$

where a number $b \in \Delta_{Y_0}$ is chosen such that the inequalities

$$|b| < 1 \ \text{ when } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ when } \ Y_0 = +\infty \ \ (Y_0 = -\infty),$$

are fulfilled and introduce numbers by setting

$$\mu_0 = \operatorname{sign} b, \ \ \mu_1 = \begin{cases} 1, & \text{if } \Delta_{Y_0} - \text{left neighborhood } Y_0, \\ -1, & \text{if } \Delta_{Y_0} - \text{right neighborhood } Y_0, \end{cases}$$

respectively, defining the signs of the $P_\omega(Y_0, \lambda_0)$-solution and its first derivative at some left neighborhood $\omega$.

Besides, we introduce the following auxiliary functions

$$\Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{sL(s)}\,,\quad \Phi_2(y)=\int\limits_{B_2}^{y}\frac{ds}{sL^{\frac{1}{3}}(s)}\,,$$

$$I_1(t)=\int\limits_{A_1}^{t}p(\tau)\,d\tau,\quad I_2(t)=\frac{\alpha_0(\lambda_0-1)^2}{\lambda_0}\int\limits_{A_2}^{t}\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad I_3(t)=\frac{\alpha_0(2\lambda_0-1)^{\frac{2}{3}}}{\lambda_0^{\frac{1}{3}}}\int\limits_{A_3}^{t}p^{\frac{1}{3}}(\tau)\,d\tau,$$

where each of the limits of integration $B_i\in\{Y_0;b\}$ $(i=1,2)$ $(A_i\in\{\omega;a\}$ $(i=1,2,3))$ is chosen so that the corresponding integral tends either to zero or to $\pm\infty$ at $y\to Y_0$ (respectively, at $t\uparrow\omega$), as well as the numbers

$$\mu_i^*=\begin{cases}1, & \text{if } B_i=b,\\ -1, & \text{if } B_i=Y_0\end{cases}\quad (i=1,2).$$

Since the functions $\Phi_i$ $(i=1,2)$ are strictly monotone on the interval $\Delta_{Y_0}$ and the area of their values are intervals

$$\Delta_{Z_i}=\begin{cases}[c_i,Z_i[\,, & \text{if } \mu_0>0,\\ ]Z_i,c_i], & \text{if } \mu_0<0,\end{cases}\quad\text{where } c_i=\Phi_i(b),\quad Z_i=\lim_{y\to Y_0}\Phi_i(y)\ \ (i=1,2),$$

so there exist continuously differentiable and strictly monotone inverse functions for them $\Phi_i^{-1}$ : $\Delta_{Z_i}\to\Delta_{Y_0}$, for which $\lim\limits_{z\to Z_i}\Phi_i^{-1}(z)=Y_0$ $(i=1,2)$.

By the properties of slowly varying functions (see [3]), there exists a continuously differentiable function $L_1:\Delta_{Y_0}\to]0,+\infty[$ slowly varying as $y\to Y_0$ such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{L(y)}{L_1(y)}=1\ \text{ and }\ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{yL_1'(y)}{L_1(y)}=0. \tag{2}$$

We also say that a function $L$ slowly varying as $y\to Y_0$ satisfies the $S_1$ if the function $L(\mu_0\exp z)$ is a regularly varying function when $z\to Z_0$ of any index $\gamma$, where $Z_0=+\infty$ in the case when $Y_0=\pm\infty$, and $Z_0=-\infty$ in the case when $Y_0=0$, so it can be represented in the form

$$L(\mu_0\exp z)=|z|^\gamma L_0(z),$$

where $L_0$ is continuous in the neighborhood of $Z_0$ and slowly varying function as $z\to Z_0$.

**Theorem 1.** *Let the function $L(\Phi_1^{-1}(z))$ be regularly varying as $z\to Z_1$ of index $\gamma$ and $\lambda_0\in\mathbb{R}\setminus\{0,1\}$. Then for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation* (1) *it is necessary and, if*

$$(2\lambda_0^2+2\lambda_0-1)\big[(2\lambda_0^2+2\lambda_0-1)(\gamma+1)+\lambda_0\big]\neq 0,$$

*it is sufficient that following conditions*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_1(t)}=-2,\qquad \frac{\lambda_0^2}{(\lambda_0-1)^2}\lim_{t\uparrow\omega}I_2(t)=Z_1,\quad \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)L\big(\Phi_1^{-1}(I_2(t))\big)=\frac{\alpha_0\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^3}\,,$$

*and inequalities*

$$\alpha_0\lambda_0\mu_0\mu_1>0,\quad \mu_0\mu_1\mu_1^*I_2(t)>0\ \ \textit{as}\ \ t\in[a,\omega[$$

*are satisfied. Moreover, each of these solutions admit the following asymptotic representations*

$$\Phi_1(y(t)) = I_2(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y'(t)}{y(t)} = \frac{\alpha_0(\lambda_0-1)^2}{\lambda_0}\,\pi_\omega^2(t)p(t)L\big(\Phi_1^{-1}(I_2(t))\big)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y''(t)}{y'(t)} = \frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

**Theorem 2.** *Let the function $L(\Phi_2^{-1}(z))$ be regularly varying as $z\to Z_2$ of index $\gamma$ and $\lambda_0\in\mathbb{R}\setminus\{0;\frac{1}{2};1\}$. Then for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the equation* (1) *it is necessary and, if*

$$(2\lambda_0^2+2\lambda_0-1)\Big[2\lambda_0^2+2\lambda_0-1+\frac{\gamma}{3}(2\lambda_0^2-\lambda_0-1)\Big]\neq 0,$$

*it is sufficient that following conditions*

$$\lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac{1}{3}}(t)L^{\frac{1}{3}}\big(\Phi_2^{-1}(I_3(t))\big) = \frac{\alpha_0[\lambda_0(2\lambda_0-1)]^{\frac{1}{3}}}{\lambda_0-1}\,, \quad \frac{|\lambda_0|^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}\lim_{t\uparrow\omega} I_3(t) = Z_2$$

*and inequalities*

$$\alpha_0\lambda_0\mu_0\mu_1>0, \quad \mu_0\mu_1\mu_2^* I_3(t)>0 \ \text{ as } \ t\in\,]a,\omega[$$

*are satisfied. Moreover, each of these solutions admit the following asymptotic representations*

$$\Phi_2(y(t)) = I_3(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{(3-k)\lambda_0+k-2}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega \ \ (k=1,2),$$

**Theorem 3.** *Let the function $L(\Phi_2^{-1}(z))$ be regularly varying as $z\to Z_2$ of index $\gamma$. Then for the existence of $P_\omega(Y_0,1)$-solutions of the equation* (1) *it is necessary and, if function $p:[a,\omega[\,\to\,]0,+\infty[$ – is continuously differentiable and there is the finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega}\frac{\Big(p^{\frac{1}{3}}(t)L_1^{\frac{1}{3}}\big(\Phi_2^{-1}\big(\frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}I_3(t)\big)\big)\Big)'}{p^{\frac{2}{3}}(t)L_1^{\frac{2}{3}}\big(\Phi_2^{-1}\big(\frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}I_3(t)\big)\big)}\,,$$

*where $L_1:\Delta_{Y_0}\to\,]0,+\infty[$ is continuously differentiable and slowly varying function as $y\to Y_0$ with properties* (2)*, it is sufficient, that*

$$\lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac{1}{3}}(t)L^{\frac{1}{3}}\bigg(\Phi_2^{-1}\Big(\frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}I_3(t)\Big)\bigg)=\infty, \quad \frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}\lim_{t\uparrow\omega}I_3(t)=Z_2$$

*and the following inequalities*

$$\alpha_0\mu_0\mu_1>0, \quad \alpha_0\lambda_0\mu_2^* I_3(t)>0 \ \text{ as } \ t\in\,]a,\omega[$$

*are satisfied. Moreover, each of these solutions admit the following asymptotic representations*

$$\Phi_2(y(t)) = \frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}\,I_3(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \alpha_0 p^{\frac{1}{3}}(t)L^{\frac{1}{3}}\bigg(\Phi_2^{-1}\Big(\frac{\lambda_0^{\frac{1}{3}}}{(2\lambda_0-1)^{\frac{2}{3}}}\,I_3(t)\Big)\bigg)[1+o(1)] \ \text{ as } \ t\uparrow\omega \ \ (k=1,2).$$

**Theorem 4.** *Let $L$ satisfy the $S_1$. Then for the existence of $P_\omega(Y_0,\pm\infty)$-solutions of the equation* (1) *it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t)>0 \;\; \text{when} \;\; t\in\,]a,\omega[\,, \quad \mu_0\lim_{t\uparrow\omega}|\pi_\omega(t)|=Y_0, \tag{3}$$

$$\lim_{t\uparrow\omega}p(t)\pi_\omega^3(t)L(\mu_0\pi_\omega^2(t))=0, \quad \int\limits_{a_1}^{\omega}p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau=+\infty, \tag{4}$$

*where $a_1\in[a,\omega[$ such that $\mu_0\pi_\omega^2(t)\in\Delta_{Y_0}$ when $t\in[a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations*

$$\ln|y(t)|=2\ln|\pi_\omega(t)|+\frac{\alpha_0}{2}\int\limits_{a_1}^{t}p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau[1+o(1)] \;\; \text{as} \;\; t\uparrow\omega, \tag{5}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)}=\frac{3-k}{\pi_\omega(t)}\,[1+o(1)] \;\; \text{as} \;\; t\uparrow\omega \;\; (k=1,2). \tag{6}$$

**Theorem 5.** *Let $L$ satisfies the $S_1$. Then for the existence of $P_\omega(Y_0,0)$-solutions of the equation* (1) *for which there is the finite or equal to $\pm\infty$, $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y'''(t)}{y''(t)}$, it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t)>0 \;\; \text{when} \;\; t\in\,]a,\omega[\,, \quad \mu_0\lim_{t\uparrow\omega}|\pi_\omega(t)|=Y_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_1(t)}=-2, \tag{7}$$

$$\lim_{t\uparrow\omega}p(t)\pi_\omega^3(t)L\big(\mu_0|\pi_\omega(t)|\big)=0, \quad \int\limits_{a_1}^{\omega}p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau=+\infty, \tag{8}$$

*where $a_1\in[a,\omega[$ such that $\mu_0|\pi_\omega(t)|\in\Delta_{Y_0}$ when $t\in[a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations*

$$\ln|y(t)|=\ln|\pi_\omega(t)|-\alpha_0\int\limits_{a_1}^{t}p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau\,[1+o(1)] \;\; \text{as} \;\; t\uparrow\omega, \tag{9}$$

$$\frac{y'(t)}{y(t)}=\frac{1+o(1)}{\pi_\omega(t)}\,, \quad \frac{y''(t)}{y'(t)}=-\alpha_0p(t)\pi_\omega^2(t)L\big(\mu_0|\pi_\omega(t)|\big)\,[1+o(1)] \;\; \text{as} \;\; t\uparrow\omega. \tag{10}$$

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# On Estimates for the First Eigenvalue of Some Sturm–Liouville Problems with Dirichlet Boundary Conditions and a Weighted Integral Condition

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## 1 Introduction

In this paper, a problem is considered whose origin was the Lagrange problem. It is a problem on finding the form of the firmest column of given volume. The Lagrange problem was the source for different extremal eigenvalue problems. One of them is the eigenvalue problem for second-order differential equations with an integral condition on the potential.

The Dirichlet problem for the equation $y'' + \lambda Q(x)y = 0$ with non-negative summable on $[0,1]$ function $Q(x)$ satisfying $\int\limits_0^1 Q^\gamma(x)\,dx = 1$, as $\gamma \in \mathbb{R}$, $\gamma \neq 0$, was considered in [1]. The Dirichlet problem for the equation $y'' - Q(x)y + \lambda y = 0$ with a real integrable on $(0,1)$ by Lebesgue function $Q$ was considered in [8] for $\gamma \geqslant 1$.

In this paper, the problems of that kind are considered under different integral conditions, in particular, if the integral condition contains a weight function. The purpose of research is to give methods of finding the sharp estimates for the first eigenvalue of Sturm–Liouville problems with Dirichlet boundary conditions for those values of the integral condition parameters for which the estimates are finite, and to prove attainability of those estimates.

Consider the Sturm–Liouville problem

$$y'' + \sigma Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $\sigma = \pm 1$, and $Q$ belongs to the set $T_{\alpha,\beta,\gamma}$ of all real–valued locally integrable functions on $(0,1)$ with non–negative values such that the following integral condition holds

$$\int\limits_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \gamma \neq 0. \tag{3}$$

A function $y$ is a *solution* to problem $(1),(2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative $y'$ is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

A function $y \in H_0^1(0,1)$ is called a *weak solution* to equation (1) if for any function $\psi \in C_0^\infty(0,1)$ the following equality

$$\int\limits_0^1 (y'\psi' + \sigma\, Q(x)y\psi)\,dx = \lambda \int\limits_0^1 y\psi\,dx$$

holds.

We give estimates for

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q),\quad M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q).$$

For any function $Q\in T_{\alpha,\beta,\gamma}$ by $H_Q$ we denote the closure of the set $C_0^\infty(0,1)$ in the norm

$$\|y\|_{H_Q}=\bigg(\int_0^1 y'^2\,dx+\int_0^1 Q(x)y^2\,dx\bigg)^{\frac{1}{2}}.$$

For any function $Q\in T_{\alpha,\beta,\gamma}$ it is proved (see, for example, [5,6]) that

$$\lambda_1(Q)=\inf_{y\in H_Q\setminus\{0\}}R[Q,y],\ \text{ where }\ R[Q,y]=\frac{\int_0^1(y'^2-\sigma Q(x)y^2)\,dx}{\int_0^1 y^2\,dx}\,.$$

Previous results are published in [2–7]. Results of this type can be useful to give methods of finding the sharp estimates for eigenvalues in cases of non-differentiable functionals.

# 2 Main results

## 2.1 Estimates for $\sigma=-1$

By Friedrichs' inequality for any function $Q\in T_{\alpha,\beta,\gamma}$ we obtain

$$\inf_{y\in H_Q\setminus\{0\}}\frac{\int_0^1 y'^2\,dx+\int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\geqslant\inf_{y\in H_Q\setminus\{0\}}\frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx}\geqslant\inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx}=\pi^2.$$

Consequently, for any $\alpha,\beta,\gamma\in\mathbb{R}$, $\gamma\neq 0$, we have

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}}R[Q,y]\geqslant\pi^2.$$

If $\gamma>0$, then it is proved that $m_{\alpha,\beta,\gamma}=\pi^2$ (see, for example, [5,6]).

Put $\gamma<0$. For any positive function $Q\in T_{\alpha,\beta,\gamma}$ by the Hölder inequality we have

$$\int_0^1 Q(x)y^2\,dx\geqslant\bigg(\int_0^1 x^{\frac{\alpha}{1-\gamma}}\,(1-x)^{\frac{\beta}{1-\gamma}}\,|y|^{\frac{2\gamma}{\gamma-1}}]\,dx\bigg)^{\frac{\gamma-1}{\gamma}}.\tag{4}$$

Consider the subspace $B_{\alpha,\beta,\gamma}$ of functions in the space $H_0^1(0,1)$ such that

$$\|y\|^2_{B_{\alpha,\beta,\gamma}}=\int_0^1 y'^2\,dx+\bigg(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\bigg)^{\frac{\gamma-1}{\gamma}}<+\infty.$$

By inequality (4) we have $H_Q \subset B_{\alpha,\beta,\gamma} \subset H_0^1(0,1)$. Put $m = \inf\limits_{y\in B_{\alpha,\beta,\gamma}\setminus\{0\}} G[y]$, where

$$G[y] = \frac{\int\limits_0^1 y'^2\,dx + \Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y^2\,dx}\,.$$

Since

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \geqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \geqslant \inf_{y\in B_{\alpha,\beta,\gamma}\setminus\{0\}} G[y] = m,$$

it follows that

$$m_{\alpha,\beta,\gamma} = \inf_{Q\in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \geqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \geqslant \inf_{y\in B_{\alpha,\beta,\gamma}\setminus\{0\}} G[y] = m.$$

The following two theorems prove that $m_{\alpha,\beta,\gamma} = m$.

Consider the set

$$\Gamma = \left\{ y \in B_{\alpha,\beta,\gamma} \mid \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx = 1 \right\}.$$

**Theorem 2.1.** *If $\gamma < 0$, then there exists a non-negative function $u \in \Gamma$ such that $G[u] = m$, moreover, for $\gamma < -1$ $u$ is a weak solution to the equation*

$$u'' + mu = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}}.$$

**Theorem 2.2.** *Suppose that $\gamma < 0$ and the function $u$ satisfies the conditions of Theorem 2.1. Then there exists a sequence $Q_n(x) \in T_{\alpha,\beta,\gamma}$ such that $R[Q_n,u] \to G[u] = m$ as $n \to \infty$ and $m_{\alpha,\beta,\gamma} = m$.*

**Remark 2.1.** In the case of $\gamma < 0$, inequalities for $m_{\alpha,\beta,\gamma} = m$ can be found, for example, in [5,6].

**Theorem 2.3** (see [2,6,7])**.** *For $M_{\alpha,\beta,\gamma}$ the following estimates hold:*

1. *If $\gamma < 0$ or $0 < \gamma < 1$, then we have $M_{\alpha,\beta,\gamma} = \infty$.*
2. *If $\gamma \geqslant 1$, then we have $M_{\alpha,\beta,\gamma} < \infty$, moreover:*

   1) *If $\gamma > 1$, then there is a function $Q_* \in T_{\alpha,\beta,\gamma}$ and a positive on $(0,1)$ function $u \in H_{Q_*}$ such that $R[Q_*,u] = G[u] = m$ and $M_{\alpha,\beta,\gamma} = m > \pi^2$. The function $u$ satisfies the equation*

   $$u'' + mu = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}}$$

   *and the condition*

   $$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1.$$

   *In the case of $\gamma > 1$, $\alpha = \beta = 0$, $m$ is the solution of the system of the equations*

   $$\begin{cases} \displaystyle\int\limits_0^H \frac{du}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}} = \frac{1}{2}\,, \\ \displaystyle\int\limits_0^H \frac{u^p}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}}\,du = \frac{1}{2}\,, \end{cases}$$

*where* $H = \max\limits_{x\in[0,1]} u(x)$, $p = \frac{2\gamma}{\gamma-1}\, s$.

2) *If* $\gamma \geqslant 1$ *and* $\alpha, \beta > \gamma$, *then we have* $M_{\alpha,\beta,\gamma} \leqslant R[\frac{1}{y_1^2}, y_1]$, *where* $y_1(x) = x^{\frac{\alpha}{2\gamma}}(1-x)^{\frac{\beta}{2\gamma}}$.

3) *If* $\beta \leqslant \gamma < \alpha$ *and* $y_2(x) = x^{\frac{\alpha}{2\gamma}} \sin \pi(1-x)$, *then we have*

$$M_{\alpha,\beta,\gamma} \leqslant \frac{\int\limits_0^1 {y_2'}^2\,dx + \pi^2\big(\frac{\gamma-1}{3\gamma-\beta-1}\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y_2^2\,dx} \quad \textit{for } \gamma > 1,$$

$$M_{\alpha,\beta,\gamma} \leqslant \frac{\int\limits_0^1 {y_2'}^2\,dx + \pi^2}{\int\limits_0^1 y_2^2\,dx} \quad \textit{for } \gamma = 1.$$

*If* $\alpha \leqslant \gamma < \beta$ *and* $y_3(x) = (1-x)^{\frac{\beta}{2\gamma}} \sin \pi x$, *then we have*

$$M_{\alpha,\beta,\gamma} \leqslant \frac{\int\limits_0^1 {y_3'}^2\,dx + \pi^2\big(\frac{\gamma-1}{3\gamma-\beta-1}\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y_3^2\,dx} \quad \textit{for } \gamma > 1,$$

$$M_{\alpha,\beta,\gamma} \leqslant \frac{\int\limits_0^1 {y_3'}^2\,dx + \pi^2}{\int\limits_0^1 y_3^2\,dx} \quad \textit{for } \gamma = 1.$$

4) *If* $\gamma \geqslant 1$, *then*

   (a) *for* $\alpha > \gamma$, $\beta \leqslant 0$ *and* $y_2(x) = x^{\frac{\alpha}{2\gamma}} \sin \pi(1-x)$ *we have* $M_{\alpha,\beta,\gamma} \leqslant R[\frac{1}{y_2^2}, y_2]$;

   (b) *for* $\beta > \gamma$, $\alpha \leqslant 0$ *and* $y_3(x) = (1-x)^{\frac{\beta}{2\gamma}} \sin \pi x$ *we have* $M_{\alpha,\beta,\gamma} \leqslant R[\frac{1}{y_3^2}, y_3]$.

5) *If* $\gamma = 1 \geqslant \alpha > 0 \geqslant \beta$ *or* $\gamma = 1 \geqslant \beta > 0 \geqslant \alpha$, *then* $M_{\alpha,\beta,\gamma} \leqslant 2\pi^2$.

6) *If* $\gamma = 1 \geqslant \alpha$, $\beta > 0$, *then* $M_{\alpha,\beta,\gamma} \leqslant 3\pi^2$.

7) *If* $\gamma = 1$, $\alpha, \beta \leqslant 0$, *then* $M_{\alpha,\beta,\gamma} \leqslant \frac{5}{4}\,\pi^2$. *If* $\gamma = 1$, $\alpha = \beta = 0$, *then there exist functions* $Q_*(x) \in T_{0,0,1}$ *and* $u \in H_0^1(0,1)$ *such that*

$$M_{0,0,1} = R[Q_*, u] = \frac{\pi^2}{2} + 1 + \frac{\pi}{2}\,\sqrt{\pi^2 + 4}\,.$$

**Remark 2.2.** In the case of $\gamma > 1$, inequalities for $M_{\alpha,\beta,\gamma} = m$ can be found, for example, in [6,7]. In the case of $\gamma = 1$, attainability of sharp estimates for $M_{\alpha,\beta,1}$ were proved in [10].

## 2.2 Estimates for $\sigma = 1$

**Theorem 2.4.** 1. *For any* $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, *we have* $M_{\alpha,\beta,\gamma} \leqslant \pi^2$.

2. *If $\gamma > 1$, then $M_{0,0,\gamma} = \pi^2$ and there exist functions $Q_*(x) \in T_{0,0,\gamma}$ and $u \in H_0^1(0,1)$ such that $m_{0,0,\gamma} = R[Q_*, u] \geqslant \frac{\pi^2}{2}$ .*

3. *If $\gamma = 1$, then $M_{0,0,1} = \pi^2$, $m_{0,0,1} = \lambda_*$, where $\lambda_* \in (0, \pi^2)$ is the solution to the equation $2\sqrt{\lambda} = \operatorname{tg}(\frac{\sqrt{\lambda}}{2})$. Here $m_{0,0,1}$ is attained at $Q(x) = \delta(x - \frac{1}{2})$.*

4. *If $\frac{1}{2} \leqslant \gamma < 1$, then for any $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, we have $m_{\alpha,\beta,\gamma} = -\infty$, $M_{0,0,\gamma} = \pi^2$.*

5. *If $\frac{1}{3} \leqslant \gamma < 1/2$, then for any $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, we have $m_{\alpha,\beta,\gamma} = -\infty$, $M_{0,0,\gamma} \leqslant \pi^2$.*

6. *If $0 < \gamma < \frac{1}{3}$ , then for any $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, we have $m_{\alpha,\beta,\gamma} = -\infty$, $M_{0,0,\gamma} < \pi^2$.*

7. *If $\gamma < 0$, then for any $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, we have $m_{\alpha,\beta,\gamma} = -\infty$, $M_{0,0,\gamma} < \pi^2$, and there exist functions $Q_*(x) \in T_{0,0,\gamma}$ and $u \in H_0^1(0,1)$ such that $M_{0,0,\gamma} = R[Q_*, u]$.*

**Remark 2.3.** The result $M_{0,0,\gamma} < \pi^2$ for $0 < \gamma < 1/2$ was obtained in [9].

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# Invariant Tori and Dichotomy of Linear Extension of Dynamical Systems

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## 1 Introduction and preliminaries

We consider a system of differential equations defined in the direct product of a torus $\mathcal{T}_m$, $m \in \mathbb{N}$ and an Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$,

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{1.1}$$

where $(\varphi_1, \dots, \varphi_m)^\intercal \in \mathcal{T}_m$, $(x_1, \dots, x_n)^\intercal \in \mathbb{R}^n$, $a \in \mathcal{C}^1(\mathcal{T}_m)$ is an $m$-dimensional vector function, $A, f \in \mathcal{C}(\mathcal{T}_m)$ are $n \times n$ square matrix and $n$-dimensional vector function respectively; $\mathcal{C}^r(\mathcal{T}_m)$ stands for the space of continuously differentiable up to the order $r$ $2\pi$-periodic with respect to each of the variables $\varphi_j$, $j = 1, \dots, m$ functions defined on the surface of the torus $\mathcal{T}_m$. The problem of the existence and construction of invariant toroidal manifold

$$x = u(\varphi) \in \mathcal{C}(\mathcal{T}_m), \ \ \varphi \in \mathcal{T}_m$$

of the system (1.1) for any inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$ can be solved using a notion of Green–Samoilenko function [7]. The existence of such a function is sufficient for the existence of non-trivial invariant torus for system (1.1). In particular, Green–Samoilenko function exists if for any $\varphi \in \mathcal{T}_m$ the system

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x \tag{1.2}$$

is exponential dichotomous on the entire real axis $\mathbb{R} = (\infty, +\infty)$. This means that there exist a projection matrix $C(\varphi) = C^2(\varphi)$ and constants $K \geq 1$, $\alpha > 0$ that do not depend on $\varphi$, $\tau$ such that the following inequalities

$$\begin{aligned} \big\|\Omega_0^t(\varphi)C(\varphi)\Omega_\tau^0(\varphi)\big\| &\leq Ke^{-\alpha(t-\tau)}, \ \ t \geq \tau, \\ \big\|\Omega_0^t(\varphi)(I - C(\varphi))\Omega_\tau^0(\varphi)\big\| &\leq Ke^{-\alpha(\tau-t)}, \ \ \tau \geq t \end{aligned} \tag{1.3}$$

are satisfied for any $t, \tau \in \mathbb{R}$. Here $\Omega_\tau^t(\varphi)$ is $(n \times n)$-dimensional fundamental matrix of the system (1.2) such that $\Omega_\tau^\tau(\varphi) \equiv I_n$; $\varphi_t(\varphi)$ is a solution of the initial value problem $\frac{d\varphi}{dt} = a(\varphi)$, $\varphi_0(\varphi) = \varphi$.

In recent papers [3, 5, 6] some particular classes of system (1.1) were distinguished for which the corresponding homogenous equations possess Green–Samoilenko function. These are the systems whose matrix $A(\varphi)$ becomes Hurwitz matrix for $\varphi$-s from the non-wandering set of dynamical system $\frac{d\varphi}{dt} = a(\varphi)$. We recall here the definition of non-wandering set.

**Definition 1.1.** A point $\varphi$ is called wandering if there exist its neighbourhood $U(\varphi)$ and a positive number $T > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = 0 \ \text{ for } \ t \geq T.$$

Let $W$ be a set of all wandering points of dynamical system and $\Omega = \mathcal{T}_m \setminus W$ be a set of non-wandering points. From the compactness of a torus it follows that the set $\Omega$ is nonempty and compact.

Analogously to [5, 6], in this paper we also consider the case when matrix $A(\varphi)$ is a constant matrix in non-wandering set $\Omega$: $A(\varphi)|_{\varphi\in\Omega} = \widetilde{A}$. However we do not require the real parts of all eigenvalues of matrix $\widetilde{A}$ to be negative in order to guarantee the existence of invariant toroidal manifold for system (1.1).

## 2 Main results

To state the main result of the paper we recall that system (1.2) possesses exponential dichotomy property on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ if there exist projection matrices $C_+(\varphi) = C_+^2(\varphi)$ and $C_-(\varphi) = C_-^2(\varphi)$ and constants $K_1, K_2 \geq 1$, $\alpha_1, \alpha_2 > 0$ that do not depend on $\varphi$, $\tau$ such that for any $\varphi \in \mathcal{T}_m$ the following inequalities

$$\begin{gathered}
\big\|\Omega_0^t(\varphi)C_+(\varphi)\Omega_\tau^0(\varphi)\big\| \leq K_1 e^{-\alpha_1(t-\tau)}, \ \ t \geq \tau, \\
\big\|\Omega_0^t(\varphi)(I - C_+(\varphi))\Omega_\tau^0(\varphi)\big\| \leq K_1 e^{-\alpha_1(\tau-t)}, \ \ \tau \geq t, \ \ \forall\, t,\tau \in \mathbb{R}_+, \\
\big\|\Omega_0^t(\varphi)C_-(\varphi)\Omega_\tau^0(\varphi)\big\| \leq K_2 e^{-\alpha_2(t-\tau)}, \ \ t \geq \tau, \\
\big\|\Omega_0^t(\varphi)(I - C_-(\varphi))\Omega_\tau^0(\varphi)\big\| \leq K_2 e^{-\alpha_2(\tau-t)}, \ \ \tau \geq t, \ \ \forall\, t,\tau \in \mathbb{R}_-
\end{gathered} \tag{2.1}$$

are satisfied.

**Theorem 2.1.** *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set $\Omega$:*

$$A(\varphi)|_{\varphi\in\Omega} = \widetilde{A},$$

*and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. Then for any $\varphi \in \mathcal{T}_m$ the corresponding homogenous system $\frac{dx}{dt} = A(\varphi_t(\varphi))x$ is exponential dichotomous on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$, e.g. there exist projection matrices $C_+(\varphi)$ and $C_-(\varphi)$ such that the inequalities* (2.1) *are satisfied and*

$$C_\pm(\varphi_t(\varphi)) = \Omega_0^t(\varphi)C_\pm(\varphi)\Omega_t^0(\varphi), \quad C_\pm^2(\varphi) = C_\pm(\varphi).$$

For example, the conditions of Theorem 2.1 are satisfied in the case when the real parts of all eigenvalues of constant matrix $\widetilde{A}$ are nonzero.

Denote by $D(\varphi) = C_+(\varphi) - (I - C_-(\varphi))$ an $(n \times n)$-dimensional matrix. Let $D^+(\varphi)$ be its Moore–Penrose pseudoinverse [2], and $P_{N(D)}(\varphi)$ and $P_{N(D^*)}(\varphi)$ be $(n \times n)$-orthoprojector matrices

$$\begin{gathered}
P_{N(D)}^2(\varphi) = P_{N(D)}(\varphi) = P_{N(D)}^*(\varphi), \\
P_{N(D^*)}^2(\varphi) = P_{N(D^*)}(\varphi) = P_{N(D^*)}^*(\varphi)
\end{gathered}$$

that project $\mathbb{R}^n$ onto the kernel $N(D) = \ker D(\varphi)$ and co-kernel $N(D^*) = \ker D^*(\varphi)$ of the matrix $D(\varphi)$:

$$P_{N(D^*)}(\varphi) = I - D(\varphi)D^+(\varphi), \quad P_{N(D)}(\varphi) = I - D^+(\varphi)D(\varphi).$$

Theorem 2.1 states that exponential dichotomy on $\mathbb{R}$ property of a "limit system" $\frac{dx}{dt} = \widetilde{A}x$ implies the exponential dichotomy on semiaxes $\mathbb{R}_+$, $\mathbb{R}_-$ for the system $\frac{dx}{dt} = A(\varphi_t(\varphi))x$. Combination of this result with [1, 4] immediately leads to the following corollaries.

**Corollary 2.1.** *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set $\Omega$:*

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A},$$

*and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. Then system* (1.1) *has an invariant toroidal manifold if and only if the inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$ satisfies the following constraint*

$$P_{N(D^*)}(\varphi) \int\limits_{-\infty}^{+\infty} C_-(\varphi)\Omega^0_\tau(\varphi) f(\varphi_\tau(\varphi))\, d\tau = 0.$$

**Corollary 2.2.** *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set $\Omega$:*

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A},$$

*and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. If additionally for any $\varphi \in \mathcal{T}_m$ matrices $\widetilde{A}$ and $(A(\varphi) - \widetilde{A})$ commute then system* (1.1) *has an invariant toroidal manifold for any inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$.*

## 3 Conclusions and discussion

New results that are presented in this paper allow to investigate qualitative behavior of solutions of a class of nonlinear systems that have a simple structure of limit sets and recurrent trajectories. Additionally they can be used to prove the persistence of a stable invariant toroidal manifold under the perturbation of the right-hand side of (1.1) in the case when this perturbation is sufficiently small only in non-wandering set $\Omega$, but not on the whole surface of the torus $\mathcal{T}_m$.

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# Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities in Some Sense Near to Regularly Varying

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')f(y,y') \tag{1}$$

is considered, where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\, \to ]0,+\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to ]0,+\infty[$ are continuous functions, $f : \Delta_{Y_0} \times \Delta_{Y_1} \to ]0,+\infty[$ is a continuously differentiable function, $Y_i \in \{0, \pm\infty\}$ $(i = 0,1)$, $\Delta_{Y_i}$ is a one-sided neighborhood of $Y_i$. We suppose also that each of the functions $\varphi_i(z)$ $(i = 0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order $\sigma_i$, $\sigma_0 + \sigma_1 \neq 1$, $\sigma_1 \neq 0$ and the function $f$ satisfies the condition

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(v_0,v_1)}{f(v_0,v_1)} = 0 \ \text{ uniformly in } \ v_j \in \Delta_{Y_j}, \ \ j \neq k, \ \ k,j = 0,1.$$

A lot of works (see, e.g., [1, 3]) were devoted to the establishing of asymptotic representation of solutions of equations of the form (1), in which $f \equiv 1$. In this research the right part of (1) was either in explicit form or asymptotically represented as the product of features, each of which depends only on $t$, or only on $y$, or only on $y'$. Let us notice that it played an important role in the research. Therefore, the general case of equation (1) can contain nonlinearities of another types, for example, $e^{|\gamma \ln |y| + \mu \ln |y'||^\alpha}$, $0 < \alpha < 1$, $\gamma, \mu \in \mathbb{R}$.

**Definition.** The solution $y$ of equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$ solution if it is defined on $[t_0, \omega[\, \subset [a, \omega[$ and

$$\lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0,1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

The $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0 - 1}$ if $\lambda_0 \in R \setminus \{0,1\}$. The asymptotic properties and necessary and sufficient conditions of the existence of such solutions are obtained (see, [2]).

The cases $\lambda_0 \in \{0,1\}$ and $\lambda_0 = \infty$ are special. $P_\omega(Y_0, Y_1, 1)$-solutions of equation (1) are rapidly varying functions as $t \uparrow \omega$. The cases $\lambda_0 = 0$ and $\lambda_0 = \infty$ are most difficult for establishing because in these cases such solutions or their derivatives are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties end existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) in special cases are presented in this work. Now we need the next definition.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to ]0;+\infty[$ satisfies the condition $S$ if for any continuous differentiable function $L : \Delta_{Y_i} \to ]0;+\infty[$ such that

$$\lim_{\substack{z \to Y_i \\ z \in \Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following condition takes place

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

We need the following subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t-\omega & \text{as } \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,1),$$

$$I(t) = \alpha_0 \int_{A_\omega}^{t} p(\tau)\,d\tau, \quad A_\omega = \begin{cases} a & \text{if } \int_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int_a^\omega p(\tau)\,d\tau < +\infty, \end{cases}$$

$$J_1(t) = \int_{B_\omega^1}^{t} |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau, \quad B_\omega^1 = \begin{cases} b_1 & \text{if } \int_{b_1}^\omega |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau = +\infty, \\ \omega & \text{if } \int_{b_1}^\omega |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau < +\infty, \end{cases}$$

$$J_2(t) = \int_{B_\omega^2}^{t} |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau, \quad B_\omega^2 = \begin{cases} b_2 & \text{if } \int_{b_2}^\omega |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau = +\infty, \\ \omega & \text{if } \int_{b_2}^\omega |I(\tau)|^{\frac{1}{\sigma_0}}\,d\tau < +\infty, \end{cases}$$

$$J_3(t) = \int_{B_\omega^1}^{t} \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^3 = \begin{cases} b_3 & \text{if } \int_{b_3}^\omega \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau = +\infty, \\ \omega & \text{if } \int_{b_3}^\omega \left| I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\right|^{\frac{1}{1-\sigma_1}}\,d\tau < +\infty, \end{cases}$$

$$I_0(t) = \alpha_0 \int_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)|y_0^0\big)\,d\tau,$$

$$A_\omega^0 = \begin{cases} b & \text{if } \int_b^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt = +\infty, \\ \omega & \text{if } \int_b^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt < +\infty, \end{cases}$$

where $b \in [a,\omega[$ is chosen so that $|\pi_\omega(t)| \operatorname{sign} y_0^0 \in \Delta_{Y_0}$ as $t \in [b,\omega[$.

**Theorem 1.** *Let $\sigma_1 \neq 1$. Then for the existence of $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following conditions are necessary*

$$y_0^0 \alpha_0 > 0, \quad y_1^0 I(t)(1-\sigma_0-\sigma_1) > 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{2}$$

$$\lim_{t\uparrow\omega} y_0^0 |J_1(t)|^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0 |J_1(t)|^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{J_1(t) I'(t)}{J_1'(t) I(t)} = 1 - \sigma_1. \tag{3}$$

*If*

$$\sigma_1 \neq 2 \ \text{ or } \ (\sigma_1 - 1)(\sigma_0 + \sigma_1 - 1) > 0,$$

*conditions* (2), (3) *are sufficient for the existence of such solutions of equation* (1).

*For $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following asymptotic representations take place as $t \uparrow \omega$*

$$\frac{y(t)|y(t)|^{-\frac{\sigma_0}{1-\sigma_1}}}{\big(f(y(t), y'(t))\Theta_0(y(t))\Theta_1(y'(t))\big)^{\frac{1}{1-\sigma_1}}} = J_1(t)\,\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1-\sigma_0|^{\frac{1}{1-\sigma_1}}[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = \frac{J_1(t)(1-\sigma_0-\sigma_1)}{J_1'(t)(1-\sigma_1)}\,[1+o(1)].$$

**Theorem 2.** *Let $\sigma_1 = 1$. Then for the existence of $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following conditions are necessary*

$$y_0^0 \alpha_0 > 0, \quad \sigma_0 y_1^0 I(t) < 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{4}$$

$$\lim_{t\uparrow\omega} y_0^0 |J_2'(t)|^{-1} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0 |J_2(t)|^{-1} = Y_1, \quad \lim_{t\uparrow\omega} \frac{J_2(t) I'(t)}{J_2'(t) I(t)} = \sigma_0. \tag{5}$$

*If $\sigma_0 I(t) < 0$, conditions* (4), (5) *are sufficient for the existence of such solutions of equation* (1). *For $P_\omega(Y_0, Y_1, 1)$-solutions of equation* (1) *the following asymptotic representations take place as $t \uparrow \omega$*

$$|y'(t)|\big(f(y(t), y'(t))\Theta_0(y(t))\Theta_1(y'(t))\big)^{\frac{1}{\sigma_0}} = |\sigma_0|^{-\frac{1}{\sigma_0}}|J_2(t)|^{-1}[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = -\frac{J_2(t)}{J_2'(t)}\,[1+o(1)].$$

**Theorem 3.** *Let in equation* (1) *the function $f$ be of the type $f(y, y') = \exp(R(|\ln|yy'||))$, the function $R : ]0, +\infty[ \to ]0, +\infty[$ be continuously differentiable with monotone derivative and regularly varying on infinity of the order $\mu$, $0 < \mu < 1$. Let, moreover, $\varphi_1(y')$ satisfy the condition $S$ and the following conditions take place*

$$\lim_{t\uparrow\omega} \frac{R(|\ln|\pi_\omega(t)||) J_3(t)}{\pi_\omega(t) \ln|\pi_\omega(t)| J_3'(t)} = 0.$$

*Then for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of equation* (1) *the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega} y_0^0 |J_3(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} \frac{J_3'(t)}{y_1^0 |J(t)|} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) I'(t)}{I(t)} = \sigma_1 - 1,$$
$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0 \ \text{ as } \ t \in \,]a, \omega[\,, \quad \frac{y_0^0 y_1^0 (1-\sigma_1) J_3(t)}{1-\sigma_0-\sigma_1} > 0 \ \text{ as } \ t \in \,]b, \omega[\,.$$

*For such solutions the following asymptotic representations take place as* $t \uparrow \omega$

$$\frac{y(t)}{|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}} J_3(t)[1+o(1)],$$

$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)J_3(t)}{(1-\sigma_1)J_3'(t)}\,[1+o(1)].$$

**Theorem 4.** *Let in equation* (1) *the function* $f$ *be of the type* $f(y,y') = \exp(R(|\ln|yy'||))$, *the function* $R : ]0,+\infty[\to]0,+\infty[$ *be continuously differentiable with monotone derivative and regularly varying on infinity of the order* $\mu$, $0<\mu<1$. *Then for the existence of* $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1) *the following conditions are necessary*

$$Y_0 = \begin{cases} \pm\infty & \text{if } \omega = +\infty, \\ 0 & \text{if } \omega < +\infty, \end{cases} \quad \pi_\omega(t) y_0^0 y_1^0 > 0 \ \text{ as } \ t \in [a,\omega[\,. \tag{6}$$

*If* $\varphi_0$ *satisfies the condition* $S$ *and*

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)} = 0,$$

*then along with* (6) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of equation* (1)*:*

$$y_1^0(1-\sigma_0-\sigma_1)I_0(t) > 0 \ \text{ as } \ t \in [b,\omega[\,,$$

$$\lim_{t\uparrow\omega} y_1^0 |I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I_0'(t)}{I_0(t)} = 0.$$

*For such solutions the following asymptotic representations take place as* $t \uparrow \omega$

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

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# Bounded Solutions to Systems of Nonlinear Functional Differential Equations

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Consider the system of functional differential equations

$$x'(t) = F(x)(t) \tag{1}$$

where $F : C_{loc}(\mathbb{R};\mathbb{R}^n) \to L_{loc}(\mathbb{R};\mathbb{R}^n)$ is a continuous operator satisfying the local Carathéodory conditions, i.e., there exists a function $\psi : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing in the second argument such that $\psi(\cdot,r) \in L_{loc}(\mathbb{R};\mathbb{R})$ for $r \in \mathbb{R}_+$ and for any $x \in C_0(\mathbb{R};\mathbb{R}^n)$ the inequality

$$\|F(x)(t)\| \le \psi\big(t,\|x\|\big) \quad \text{for a.e. } t \in \mathbb{R}$$

is fulfilled.

By a solution to the system (1) we understand a vector-valued function $x \in AC_{loc}(\mathbb{R};\mathbb{R}^n)$ satisfying the equality (1) almost everywhere in $\mathbb{R}$. By a bounded solution to the system (1) it is understood a solution $x$ to the system (1) that satisfies

$$\sup\big\{\|x(t)\| : \ t \in \mathbb{R}\big\} < +\infty.$$

To formulate our results, we need to introduce the following definition (the complete list of notation and symbols is given at the end of this text). Let $\sigma \in \{-1,1\}$ and put

$$I_\sigma(t) = \begin{cases} ]-\infty,t] & \text{if } \sigma = 1,\\ [t,+\infty[ & \text{if } \sigma = -1 \end{cases} \quad \text{for } t \in \mathbb{R}.$$

A linear continuous operator $\ell : C_{loc}(\mathbb{R};\mathbb{R}) \to L_{loc}(\mathbb{R};\mathbb{R})$ is called a $\sigma$-Volterra operator if for arbitrary $t \in \mathbb{R}$ and $v \in C_{loc}(\mathbb{R};\mathbb{R})$ such that $v(s) = 0$ for $s \in I_\sigma(t)$, the equality $\ell(v)(s) = 0$ for a.e. $s \in I_\sigma(t)$ is fulfilled.

**Theorem 1.** *Let the inequality*

$$\mathcal{D}(\sigma)\operatorname{Sgn}(v(t))\big[F(v)(t) - \mathcal{D}(h(t))v(t) + g_0(v)(t)\big] \le p(|v|)(t) + \eta(t,\|v\|) \quad \textit{for a.e. } t \in \mathbb{R} \tag{2}$$

*be fulfilled for any $v \in C_0(\mathbb{R};\mathbb{R}^n)$, where $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1,1\}$ $(i = 1,\dots,n)$, $h \in L_{loc}(\mathbb{R};\mathbb{R}^n)$,*

$$g_0(v)(t) \overset{def}{=} (g_{0i}(v_i)(t))_{i=1}^n \quad \textit{for a.e. } t \in \mathbb{R}, \ \ v \in C_{loc}(\mathbb{R};\mathbb{R}^n)$$
$$\mathcal{D}(\sigma)g_0 \in \mathcal{P}_n(\mathbb{R}), \ \ p \in \mathcal{P}_n(\mathbb{R}), \tag{3}$$

*each $g_{0i}$ is a $\sigma_i$-Volterra operator, and $\eta \in K_{loc}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}^n_+)$ satisfies*

$$\lim_{r \to +\infty} \frac{1}{r} \int_a^b \|\eta(s,r)\|\, ds = 0 \tag{4}$$

*for every interval $[a,b]$. Let, moreover, there exist functions $\beta, \gamma \in AC_{loc}(\mathbb{R}; \mathbb{R}^n)$ such that*

$$\begin{gathered} \beta(t) > 0, \ \ \gamma(t) > 0 \quad \text{for } t \in \mathbb{R}, \ \ \|\gamma\| < +\infty, \\ \mathcal{D}(\sigma)\big[\beta'(t) - \mathcal{D}(h(t))\beta(t) + g_0(\beta)(t)\big] \leq 0 \quad \text{for a.e. } t \in \mathbb{R}, \\ \mathcal{D}(\sigma)\big[\gamma'(t) - \mathcal{D}(h(t))\gamma(t) - \mathcal{D}(\sigma)p(\gamma)(t)\big] \geq 0 \quad \text{for a.e. } t \in \mathbb{R}. \end{gathered}$$

*Let, in addition, for every $i \in \{1, \dots, n\}$,*

$$G_i(t,r) \overset{def}{=} \lim_{\tau \to -\sigma_i \infty} \sigma_i \int_\tau^t \exp\bigg(\int_s^t h_i(\xi)\, d\xi\bigg) \eta_i(s,r)\, ds < +\infty \quad \text{for } t \in \mathbb{R}, \ \ r \in \mathbb{R}_+, \tag{5}$$

$$H_i(t) \overset{def}{=} \lim_{\tau \to -\sigma_i \infty} \gamma_i(\tau) \exp\bigg(\int_\tau^t h_i(s)\, ds\bigg) > 0 \quad \text{for } t \in \mathbb{R}, \tag{6}$$

*and*

$$\limsup_{r \to +\infty} \frac{G_i(t,r)}{rH_i(t)} < \frac{1}{\|\gamma\|} \quad \text{uniformly for } t \in \mathbb{R}. \tag{7}$$

*Then (1) has at least one bounded solution.*

**Theorem 2.** *Let the inequality*

$$\begin{aligned} \mathcal{D}(\sigma) \operatorname{Sgn}(v(t))\big[F(v)(t) - \mathcal{D}(h(t))v(t) - \ell_0(v)(t) + g_0(v)(t)\big] \\ \leq p(|v|)(t) + \eta(t, \|v\|) \quad \text{for a.e. } t \in \mathbb{R} \end{aligned}$$

*be fulfilled for any $v \in C_0(\mathbb{R}; \mathbb{R}^n)$, where $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, $h \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$, (3) and*

$$\mathcal{D}(\sigma)\ell_0 \in \mathcal{P}_n(\mathbb{R}), \quad \mathcal{D}(\sigma)\big[\ell_0 - g_0\big] \in \mathcal{P}^\sigma_n(\mathbb{R}; h)$$

*hold, and $\eta \in K_{loc}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}^n_+)$ satisfies (4) for every interval $[a,b]$. Let, moreover, there exist a function $\gamma \in AC_{loc}(\mathbb{R}; \mathbb{R}^n)$ such that*

$$\begin{gathered} \gamma(t) > 0 \quad \text{for } t \in \mathbb{R}, \ \ \|\gamma\| < +\infty, \\ \mathcal{D}(\sigma)\big[\gamma'(t) - \mathcal{D}(h(t))\gamma(t) - \ell_0(\gamma)(t) - \mathcal{D}(\sigma)p(\gamma)(t)\big] \geq 0 \quad \text{for a.e. } t \in \mathbb{R}. \end{gathered}$$

*Let, in addition, (6)–(7) be fulfilled for every $i \in \{1, \dots, n\}$. Then (1) has at least one bounded solution.*

Consider the nonlinear differential system with argument deviation

$$\begin{aligned} x_i'(t) = h_i(t)x_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(\tau_{ij}(t)) - \sum_{j=1}^n g_{ij}(t)x_j(\mu_{ij}(t)) \\ + f_i(t, x(t), x(\nu_1(t)), \dots, x(\nu_m(t))) \ \ (i = 1, \dots, n), \end{aligned} \tag{8}$$

where $h = (h_i)_{i=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$, $P = (p_{ij})_{i,j=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^{n \times n})$, $G = (g_{ij})_{i,j=1}^n \in L_{loc}(\mathbb{R}; \mathbb{R}^{n \times n})$, $f = (f_i)_{i=1}^n \in K_{loc}(\mathbb{R} \times \mathbb{R}^{(m+1)n}; \mathbb{R}^n)$, $t_{ij}, \mu_{ij}, \nu_k : \mathbb{R} \to \mathbb{R}$ $(i, j = 1, \dots, n;\ k = 1, \dots, m)$ are locally essentially bounded functions, and $x = (x_i)_{i=1}^n$. Then Theorems 1 and 2 imply in particular the following corollaries.

**Corollary 1.** *Let the inequality*

$$\operatorname{Sgn}(v(t))f(t,v(t),v(\nu_1(t)),\dots,v(\nu_m(t)))\leq q(t)\quad \textit{for a.e. } t\in\mathbb{R} \tag{9}$$

*be fulfilled for any* $v\in C_0(\mathbb{R};\mathbb{R}^n)$, $q\in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$. *Let, moreover,*

$$P(t)\geq\Theta,\quad G(t)\geq\Theta\quad \textit{for a.e. } t\in\mathbb{R}, \tag{10}$$

$$g_{ij}(t)=0\quad \textit{for a.e. } t\in\mathbb{R}\ \ (i\neq j;i,j=1,\dots,n), \tag{11}$$

$$g_{ii}(t)\big[\mu_{ii}(t)-t\big]\leq 0\quad \textit{for a.e. } t\in\mathbb{R}\ \ (i=1,\dots,n), \tag{12}$$

*and*

$$\int\limits_{\mu_{ii}(t)}^{t} g_{ii}(s)\exp\bigg(-\int\limits_{\mu_{ii}(s)}^{s} h_i(\xi)\,d\xi\bigg)\,ds\leq\frac{1}{e}\quad \textit{for a.e. } t\in\mathbb{R},\ \ (i=1,\dots,n),$$

$$\int\limits_{t}^{\tau_{ij}(t)}\widetilde{p}(s)\,ds\leq\frac{1}{e}\quad \textit{for a.e. } t\in\mathbb{R}\ \ (i,j=1,\dots,n), \tag{13}$$

*where*

$$\widetilde{p}(t)\overset{def}{=}\max\bigg\{\sum_{k=1}^{n}p_{ik}(t)\exp\bigg(\int\limits_{t}^{\tau_{ik}(t)}\widetilde{h}(s)\,ds\bigg)\ :\ i=1,\dots,n\bigg\}\quad \textit{for a.e. } t\in\mathbb{R}, \tag{14}$$

$$\widetilde{h}(t)\overset{def}{=}\max\big\{h_i(t):i=1,\dots,n\big\}\quad \textit{for a.e. } t\in\mathbb{R}. \tag{15}$$

*Let, in addition,*

$$\sup\bigg\{\int\limits_{0}^{t}\big[\widetilde{h}(s)+e\widetilde{p}(s)\big]\,ds:\ t\in\mathbb{R}\bigg\}<+\infty,\quad \int\limits_{-\infty}^{0}\widetilde{p}(s)\,ds<+\infty, \tag{16}$$

$$\int\limits_{-\infty}^{+\infty}q(s)\exp\bigg(-\int\limits_{0}^{s}h_i(\xi)\,d\xi\bigg)\,ds<+\infty\ \ (i=1,\dots,n). \tag{17}$$

*Then* (8) *has at least one bounded solution.*

**Corollary 2.** *Let the inequality* (9) *be fulfilled for any* $v\in C_0(\mathbb{R};\mathbb{R}^n)$, $q\in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$. *Let, moreover,* (10) *hold,*

$$p_{ik}(t)\exp\bigg(\int\limits_{\mu_{ik}(t)}^{\tau_{ik}(t)}h_k(s)\,ds\bigg)\geq g_{ik}(t),\quad g_{ik}(t)\big[\tau_{ik}(t)-\mu_{ik}(t)\big]\geq 0\quad \textit{for a.e. } t\in\mathbb{R}\ \ (i,k=1,\dots,n),$$

*and let* (13) *be fulfilled, where* $\widetilde{p}$ *is given by* (14) *and* (15). *Let, in addition,* (16) *and* (17) *hold. Then* (8) *has at least one bounded solution.*

**Corollary 3.** *Let the inequality*

$$\mathcal{D}(\sigma)\operatorname{Sgn}(v(t))f\big(t,v(t),v(\nu_1(t)),\dots,v(\nu_m(t))\big)\leq q(t)\quad \textit{for a.e. } t\in\mathbb{R} \tag{18}$$

*be fulfilled for any* $v \in C_0(\mathbb{R};\mathbb{R}^n)$, $q \in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$, *where* $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. *Let, moreover,*

$$\mathcal{D}(\sigma)P(t) \geq \Theta, \quad \mathcal{D}(\sigma)G(t) \geq \Theta \quad \text{for a.e. } t \in \mathbb{R}, \tag{19}$$

(11) *and* (12) *hold, and*

$$\int_{-\infty}^{\infty} |g_{ii}(s)| \exp\bigg(-\int_{\mu_{ii}(s)}^{s} h_i(\xi)\,d\xi\bigg)\,ds < 1 \ \ (i=1,\dots,n).$$

*Furthermore, let there exist* $A=(a_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}_+$ *such that* $r(A)<1$ *and*

$$\int_{-\infty}^{+\infty} |p_{ij}(s)| \exp\bigg(\int_0^{\tau_{ij}(s)} h_j(\xi)\,d\xi - \int_0^{s} h_i(\xi)\,d\xi\bigg)\,ds \leq a_{ij} \ \ (i,j=1,\dots,n). \tag{20}$$

*Let, in addition,*

$$\sup\bigg\{\int_0^t h_i(s)\,ds : \ t \in \mathbb{R}\bigg\} < +\infty \ \ (i=1,\dots,n) \tag{21}$$

*and* (17) *hold. Then* (8) *has at least one bounded solution.*

**Corollary 4.** *Let* (18) *be fulfilled for any* $v \in C_0(\mathbb{R};\mathbb{R}^n)$, $q \in L_{loc}(\mathbb{R};\mathbb{R}^n_+)$, *where* $\sigma \in \mathbb{R}^n$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$. *Let* (19) *hold and, moreover,*

$$\sigma_i p_{ik}(t) \exp\bigg(\int_{\mu_{ik}(t)}^{\tau_{ik}(t)} h_k(s)\,ds\bigg) \geq \sigma_i g_{ik}(t), \quad \sigma_i\sigma_k g_{ik}(t)\big[\tau_{ik}(t)-\mu_{ik}(t)\big] \geq 0 \ \ (i,k=1,\dots,n)$$

*for a.e.* $t \in \mathbb{R}$. *Furthermore, let there exist* $A=(a_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}_+$ *such that* $r(A)<1$ *and* (20) *hold. Let, in addition,* (21) *and* (17) *hold. Then* (8) *has at least one bounded solution.*

## Notation

If $x=(x_i)_{i=1}^n \in \mathbb{R}^n$, then

$$\mathcal{D}(x) = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}, \quad \operatorname{Sgn}(x) = \mathcal{D}(\operatorname{sgn} x), \quad \text{where} \quad \operatorname{sgn} x = (\operatorname{sgn} x_i)_{i=1}^n.$$

$\Theta$ is a zero matrix, $r(X)$ is a spectral radius of the matrix $X$.

$C_{loc}(\mathbb{R};\mathbb{R}^n)$ is a space of continuous functions $x:\mathbb{R}\to\mathbb{R}^n$ with a topology of uniform convergence on every compact interval.

$C_0(\mathbb{R};\mathbb{R}^n)$ is a Banach space of bounded continuous functions $x:\mathbb{R}\to\mathbb{R}^n$ endowed with a norm

$$\|x\| = \sup\big\{\|x(t)\| : \ t \in \mathbb{R}\big\}.$$

$AC_{loc}(\mathbb{R};\mathbb{R}^n)$ is a set of locally absolutely continuous functions $x:\mathbb{R}\to\mathbb{R}^n$.

$L_{loc}(\mathbb{R};\mathbb{R}^n)$ is a space of locally Lebesgue integrable vector-valued functions $p:\mathbb{R}\to\mathbb{R}^n$ with a topology of convergence in mean on every compact interval.

$L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ is a space of locally Lebesgue integrable matrix-valued functions $P:\mathbb{R}\to\mathbb{R}^{n\times n}$.

$\mathcal{P}_n(\mathbb{R})$ is a set of linear continuous operators $\ell : C_{loc}(\mathbb{R};\mathbb{R}^n)\to L_{loc}(\mathbb{R};\mathbb{R}^n)$ that transforms non-negative functions into the set of non-negative functions.

$\mathcal{P}_n^{\sigma}(\mathbb{R};h)$, where $h\in L_{loc}(\mathbb{R};\mathbb{R}^n)$ and $\sigma=(\sigma_i)_{i=1}^n\in\mathbb{R}^n$, $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, is a set of linear continuous operators $\ell : C_{loc}(\mathbb{R};\mathbb{R}^n)\to L_{loc}(\mathbb{R};\mathbb{R}^n)$ such that

$$\ell(x)(t)\geq 0 \quad \text{for a.e. } t\in\mathbb{R},$$

whenever $x\in AC_{loc}(\mathbb{R};\mathbb{R}^n)$ satisfies

$$x(t)\geq 0 \quad \text{for } t\in\mathbb{R}, \quad \mathcal{D}(\sigma)\big[x'(t)-\mathcal{D}(h(t))x(t)\big]\geq 0 \quad \text{for a.e. } t\in\mathbb{R}.$$

$K([a,b]\times A;B)$, where $A\subseteq\mathbb{R}^m$ and $B\subseteq\mathbb{R}^n$, is a set of functions $f:[a,b]\times A\to B$ satisfying the Carathéodory conditions, i.e.,

(i) $f(\cdot,x):[a,b]\to B$ is a measurable function for every $x\in A$,

(ii) $f(t,\cdot):A\to B$ is a continuous function for almost all $t\in[a,b]$,

(iii) for every $r>0$ there exists a function $q_r\in L([a,b];\mathbb{R}_+)$ such that

$$\|f(t,x)\|\leq q_r(t) \quad \text{for a.e. } t\in[a,b], \ \ x\in A, \ \ \|x\|\leq r.$$

$K_{loc}(\mathbb{R}\times A;B)$, where $A\subseteq\mathbb{R}^m$ and $B\subseteq\mathbb{R}^n$, is a set of functions $f:\mathbb{R}\times A\to B$ such that $f\in K([a,b]\times A;B)$ for every compact interval $[a,b]$.

# Existence of Optimal Control on an Infinitive Interval for Systems of Differential Equations with Pulses at Non-Fixed Times

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We consider two problems of optimal control for systems of differential equations with pulse action

$$\begin{gathered} \dot{x} = A(x,t) + B(x,t)u, \ \ x \notin S, \\ \Delta x\big|_{x\in S} = g(x), \\ x(0) = x_0. \end{gathered} \tag{1}$$

In the first problem for the system (1) the quality criteria is the following

$$J(u) = \int\limits_0^\infty \nu(t)L(t,x(t),u(t))\,dt \to \inf, \tag{2}$$

where $S$ – some hypersurface in the space $R^d$, $x_0 \in R^d$ – a fixed vector, $t \in [0,\infty)$, $x \in R^d$, $L(t,x,u)$ – a limited function, $u \in U \subset R^m$, $U$ – a closed, convex set in the space $R^m$, $0 \in U$, $A(x,t)$ – $d$-dimensional vector function, $B(x,t)$ – $d \times m$-dimensional matrix, $g$ – $d$-dimensional vector function.

In the second problem for the system (1) we consider the quality criteria

$$J(u) = \int\limits_0^\theta \nu(t)L(t,x(t),u(t))\,dt \longrightarrow \inf, \tag{3}$$

where $t \in [0,\infty)$, $x \in D$, $D$ – a limited area in the space $R^d$, $D \cap S$ – is not empty, $x_0 \in R^d$ – a fixed vector, $\theta$ – a moment of leaving the solution $x(t)$ the area $D$.

We consider the problem (1), (2) with the following conditions: functions $A(x,t)$, $B(x,t)$ are continuous for a set of variables $t \in [0,\infty)$, $x \in R^d$, $g(x)$ is continuous by $x \in R^d$ and the condition of Lipschitz is satisfied, there is a constant $H > 0$ such that for any $x_1, x_2 \in R^d$, $t \geq 0$ and $u \in U$ the conditions:

$$\big|A(t,x_1) - A(t.x_2)\big| \leq H|x_1 - x_2|, \quad \big\|B(t,x_1) - B(t.x_2)\big\| \leq H|x_1 - x_2| \tag{4}$$

hold.

Functions $L(t,x,u)$, $L_x(t,x,u)$ and $L_u(t,x,u)$ are continuous for a set of variables, for any $t \in [0,\infty)$, $x \in R^d$ and $u \in U$, the following conditions are satisfied:

1) $L(t,x,u) \geq 0$ for any $t \in [0,\infty)$, $x \in R^d$ and $u \in U$;

2) there are constants $R > 0$ and $p > 2$ such that for any $t \in [0,\infty)$, $x \in R^d$ and $u \in U$, the inequality
$$L(t,x,u) \geq R(1 + |u|^p)$$
is fulfilled;

3) there is $M > 0$ such that for any $t \in [0,\infty)$, $x \in R^d$ and $u \in U$,
$$|L_x(t,x,u)| + |L_u(t,x,u)| \leq M(1 + |u|^{p-1});$$

4) $L(t,x,u)$ is convex by $u$ for any fixed $t \in [0,\infty)$, $x \in R^d$.

For the problem $(1),(3)$ conditions are similar to the problem $(1),(2)$ for $x \in D$.
Acceptable for problems $(1),(2)$ and $(1),(3)$ are such controls $u = u(t)$ that:

(a) $u(t) \in L_p([0,\infty))$, $u(t) \in U$, $t \in [0,\infty)$;

(b) there is a constant $C_1 > 0$ which does not depend on $u(t)$ and the following condition holds:
$$\int\limits_0^\infty |u(t)|^p\,dt \leq C_1.$$

The set of acceptable controls will be named acceptable for (1), (2) and (1), (3) and will be denoted by $F$.

We assume that the hypersurface $S$ is a compact set and is given by $s(x) = 0$, where $s$ is a continuous function.

Let $\tau_u^k$ be moments in which the solution $x(t,u)$ hit on the hypersurface $S$.

**Theorem 1.** *Let the system* (1) *with the quality criteria* (2)*, for functions* $A(x,t)$*,* $B(x,t)$*,* $\nu(t)$ *and* $L(t,x,u)$ *satisfy the condition* (4) *and* 1)–3)*, the function* $\nu(t) \in L_1([0,\infty))$*,* $0 \leq \nu(t) \leq 1$ *for any* $t \geq 0$*. Then the problem* $(1),(2)$ *has a solution in the set of acceptable controls* $F$*.*

**Theorem 2.** *Let the system* (1) *with the quality criteria* (3)*, for functions* $A(x,t)$*,* $B(x,t)$*,* $\nu(t)$ *and* $L(t,x,u)$ *satisfy the condition of Theorem* 1 *for* $t \geq 0$*,* $x \in D$*. Then the problem* $(1),(3)$ *has a solution in the set of acceptable controls* $F$*.*

**Proof for the problem** $(1),(2)$**.** Since $J(u) \geq 0$, then there exists a non-negative lower bound $m$ of values $J(u)$. Let $u_n$ be the sequence of acceptable controls such that: $J(u_n) \to m$, $n \to \infty$. Namely,
$$J(u_n) = \int\limits_0^\infty \nu(t)L(t,x_n(t),u_n(t))\,dt \longrightarrow m, \quad n \to \infty,$$
where $x_n(t)$ are solutions of the system (1) which correspond to controls $u_n(t)$.

The condition (b) guarantees a weak compactness of the sequence $u_n(t)$. Thus the sequence $u_n(t)$ converge weakly to $u^*(t) \in L_p([0,\infty))$. It is easy to show that $u^*(t) \in U$ for almost all $t \in [0,\infty)$.

We take an arbitrary $T > 0$ and fix. Since in the interval $[0,T]$ all the conditions of the Theorem 1 are fulfilled, then there exists $x_T^*(t)$ – the solution of the system (1) at $[0,T]$, which correspond to control $u^*(t)$ and $x_n(t) \rightrightarrows x_T^*(t)$, $n \to \infty$ for any $t \in [0,T]$.

We show that there is a subsequence of functions $x_{n_n}(t)$ which pointwise converges to the function $x^*(t)$ for any $t \in [0,\infty)$.

For $T = 1$ there exists the subsequence $x_{n_1}(t)$ of the sequence $x_{n_n}(t)$, $n \geq 1$ such that $x_{n_1}(t) \rightrightarrows x_1^*(t)$ for any $t \in [0, 1]$.

For $T = 2$ there exists the subsequence $x_{n_2}(t)$ of the sequence $x_{n_1}(t)$, $n \geq 1$ such that $x_{n_2}(t) \rightrightarrows x_2^*(t)$ for any $t \in [0, 2]$, where $x_2^*(t) = x_1^*(t)$, $t \in [0, 1]$.

Similarly, for any natural $N$ there exists the subsequence $x_{n_N}(t)$ of the sequence $x_{n_{N-1}}(t)$ such that $x_{n_N}(t) \rightrightarrows x_N^*(t)$ for any $t \in [0, N]$, where $x_N^*(t) = x_{N-1}^*(t)$, $t \in [0, N-1]$.

Using the diagonal method of this sequences, we can distinguish the following subsequence $x_{n_n}(t)$, $n \geq 1$

$$x_{1_1}(t), x_{2_2}(t), x_{3_3}(t), \dots, x_{n_n}(t), \dots .$$

This sequence pointwise converges to the function $x^*(t)$ for any $t \in [0, \infty)$.

Similarly to [3], it can be shown that the control $u^*(t)$ is optimal for the problem (1), (2), that $J(u^*) = m$.

**Proof for the problem (1), (3).** The proof of Theorem 2 is similar to the proof of Theorem 1, but it must be taken into account the moment of coming out the solution of the area.

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# Non-Lipschitz Lower Sigma-Exponents of Linear Differential Systems

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For investigation of exponential stability and instability of perturbed linear differential systems

$$\dot{y} = A(t)y + Q(t)y, \;\; y \in R^n, \;\; t \geq 0, \tag{$1_{A+Q}$}$$

with bounded piecewise-constant coefficients, characteristic exponents $\lambda_1(A+Q) \leq \cdots \leq \lambda_n(A+Q)$ and exponentially decreasing sigma-perturbations $Q$ satisfying the condition

$$\lambda[Q] \equiv \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \|Q(t)\| \leq -\sigma < 0,$$

the use is made of the so-called higher [3, 4]

$$\nabla_\sigma(A) \equiv \sup_{\lambda[Q]\leq-\sigma} \lambda_n(A+Q), \;\; \sigma > 0,$$

and lower [5–7]

$$\Delta_\sigma(A) \equiv \inf_{\lambda[Q]\leq-\sigma} \lambda_1(A+Q), \;\; \sigma > 0 \tag{2}$$

sigma-exponents. And if for the first of them the calculation algorithm by the Cauchy matrix $X_A(t,\tau)$ of the initial system $(1_A)$ is constructed [3, 4] and fully described [1, 2, 8] as the function of a parameter $\sigma > 0$ (with the properties of boundedness, concavity and coincidence with the constant $\sigma$ greater than some $\sigma_0 \geq 0$), then for the second, lower sigma-exponent $\Delta_\sigma(A)$, there is nothing.

In works [6, 7] devoted to the investigation of the lower sigma-exponent $\Delta_\sigma(A)$, relying only on its definition (2), the author constructed lower sigma-exponents of linear differential systems $(1_A)$ of general Lipschitz on the interval $(0,+\infty)$ type, more general compared to the higher sigma-exponents. In particular, they are not only convex or only concave functions in the whole domain $(0,+\infty)$ of their definition. Indeed, for every nondecreasing function $f : (0,+\infty) \to R$ coinciding with the constant on some interval $[\sigma_0,+\infty)$ (the lower sigma-exponent of any system $(1_A)$ possesses these obvious properties) and satisfying the Lipschitz condition on the interval $(0,\sigma_0$, the existence of the linear differential system $(1_A)$ with a lower sigma-exponent $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$ is proved.

There arises the question whether there exist lower sigma-exponents $\Delta_\sigma(A)$ of linear non-Lipschitz type systems, that is not satisfying in parameter $\sigma > 0$ Lipschits condition on the whole interval $(0,+\infty)$ with a finite Lipschitz constant $L > 0$. The positive answer is contained in the following

**Theorem.** *Any nondecreasing function*

$$f : [0,+\infty) \to [c_0, c_1] \subset (-\infty,+\infty),$$

*coinciding with the constant $c_1$ on some interval $[\sigma_1,+\infty)$ and satisfying the Lipschitz condition*

$$0 \leq f(\xi_2) - f(\xi_1) < L(\sigma_0)(\xi_2 - \xi_1), \;\; 0 < \sigma_0 \leq \xi_1 < \xi_2 \leq \sigma_1,$$

*on any interval $[\sigma_0,\sigma_1]$ with the Lipschitz constant $L(\sigma_0) \leq const/\sigma_0$, $\sigma_0 > 0$, is a lower sigma-exponent $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$, of some linear differential system $(1_A)$ with a piecewise-continuous bounded on the time semi-axis $[0,+\infty)$ matrix of coefficients $A(t)$.*

**Remark.** Such satisfying conditions of the theorem (and not satisfying the Lipschitz on the whole interval $(0,+\infty)$ condition with one finite Lipschitz constant $L>0$) are, for example, the functions

$$f(\sigma)=\begin{cases}\sigma^{\alpha}, & \sigma\in[0,\sigma_1],\\ \sigma_1^{\alpha}, & \sigma>\sigma_1,\ \ \alpha\in(0,1).\end{cases}$$

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# Unique Solvability and Additive Averaged Rothe's Type Scheme for One Nonlinear Multi-Dimensional Integro-Differential Parabolic Problem

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The paper is devoted to the existence and uniqueness of a solution of the initial-boundary problem for one nonlinear multi-dimensional integro-differential equation of parabolic type. Construction and study of the additive averaged Rothe's type scheme is also given. The studied equation is based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [10]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of magnetic field, $\theta$ is temperature, $c_\nu$ and $\nu_m$ characterize correspondingly heat capacity and electroconductivity of the medium.

The system (1), (2) is complex and its investigation and numerical resolution still yield for special cases (see, for example, [6] and the references therein).

In [1], the Maxwell's system (1), (2) were proposed to integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t |\operatorname{rot} H|^2\, d\tau\right)\operatorname{rot} H\right], \tag{3}$$

where $a = a(S)$ is dependent on coefficients $c_\nu$, $\nu_m$ and is defined for $S \in [0, \infty)$.

Making certain physical assumptions in mathematical description of the above-mentioned process in [12], a new integro-differential model is constructed which represents a generalization of the system (3)

$$\frac{\partial H}{\partial t} = a\left(\int_\Omega \int_0^t |\operatorname{rot} H|^2\, dx\, d\tau\right)\Delta H. \tag{4}$$

Principal characteristic peculiarity of systems (3) and (4) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral of time and space variables. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

The literature on the questions of existence, uniqueness, and regularity of solutions to the models of above types is very rich. In [1–5, 11–13], the solvability of the initial-boundary value problems for (3) type models in scalar cases is studied using a modified version of the Galerkin's method and compactness arguments that are used in [14, 16] for investigation elliptic and parabolic equations. The uniqueness of solutions is investigated also in works [1–5, 11–13]. The asymptotic behavior of solutions is discussed in [4, 6, 9] and in a number of other works as well. Note also that to numerical resolution of (3) and (4) type one-dimensional models were devoted many works as well (see, e.g., [5–7, 9] and the references therein).

Many authors study the Rothe's scheme, semi-discrete scheme with space variable, finite element and finite difference approximation for a integro-differential models (see, for example, [5–9, 14, 15]).

It is very important to study decomposition analogs for above-mentioned multi-dimensional differential and integro-differential models as well. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [14, 15] and the references therein).

This paper dedicated to the existence and uniqueness of solutions of initial-boundary value problem. Investigations are given in usual Sobolev spaces. Main attention is also paid to investigation of Rothe's type additive averaged scheme. In this paper we shall focus our attention to (4) type multi-dimensional integro-differential scalar equation.

Let $\Omega$ is bounded domain in the $n$-dimensional Euclidean space $R^n$ with sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0,T)$ of the variables $(x,t) = (x_1, x_2, \dots, x_n, t)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n}\bigg(1 + \int\limits_{\Omega}\int\limits_{0}^{t}\Big|\frac{\partial U}{\partial x_i}\Big|^2\,dx\,d\tau\bigg)\frac{\partial^2 U}{\partial x_i^2} = f(x,t), \quad (x,t) \in Q, \tag{5}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{6}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{7}$$

where $T$ is a fixed positive constant, $f$ is a given function of its arguments.

Since problem (5)–(7) similar to problems considered in [4], where investigation of (3) type multi-dimensional scalar equations is given and at first is discussed unique solvability and asymptotic behavior of (5) type models as well, we can follow the same procedure used there. Using modified version of the Galerkin's method and compactness arguments [16], [14] the following statement can be proved.

**Theorem 1.** *If*

$$f \in W_2^1(Q), \quad f(x,0) = 0,$$

*then there exists a unique solution $U$ of problem* (5)–(7) *satisfying the properties:*

$$U \in L_4(0,T; \overset{\circ}{W}{}_4^1(\Omega)) \cap L_2(0,T; W_2^2(\Omega)), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\sqrt{T-t}\,\frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad i = 1, \dots, n.$$

The proof of the formulated theorem is divided into several steps. One of the basic step is to obtain necessary a priori estimates.

Using the scheme of investigation as in, e.g., [4, 6, 9], it is not difficult to get the result of exponentially asymptotic behavior of solution as $t \to \infty$ for (5) equation with $f(x,t) \equiv 0$ and homogeneous boundary (6) and nonhomogeneous initial (7) conditions.

On $[0,T]$ let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0, 1, \dots, J$, with $\tau = 1/J$.

Coming back to problem (5)–(7), let us construct additive averaged Rothe's type scheme:

$$\begin{gathered}\eta_i\,\frac{u_i^{j+1} - u^j}{\tau} = \bigg(1 + \tau\sum_{k=1}^{j+1}\int\limits_{\Omega}\Big|\frac{\partial u_i^k}{\partial x_i}\Big|^2\,dx\bigg)\frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{j+1},\\ u_i^0 = u^0 = 0, \quad i = 1, \dots, n, \quad j = 0, 1, \dots, J-1,\end{gathered} \tag{8}$$

with homogeneous boundary conditions, where $u_i^j(x)$, $j=1,\dots,J$, is a solution of problem (8) and the following notations are introduced:

$$u^j(x)=\sum_{i=1}^n \eta_i u_i^j(x), \quad \sum_{i=1}^n \eta_i=1, \;\; \eta_i>0, \quad \sum_{i=1}^n f_i^{j+1}(x)=f^{j+1}(x)=f(x,t_{j+1}),$$

where $u^j$ denotes approximation of exact solution $U$ of problem (5)–(7) at $t_j$. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

**Theorem 2.** *If problem* (5)–(7) *has sufficiently smooth solution, then the solution of problem* (8) *converges to the solution of problem* (5)–(7) *and the following estimate is true*

$$\|U^j-u^j\|=O(\tau^{1/2}), \;\; j=1,\dots,J.$$

Using early investigated finite difference and finite element schemes for one-dimensional (5) type models (see, for example, [5–7, 9]) now we can reduce numerical resolution of the multi-dimensional integro-differential model (5) to one-dimensional ones. It is very important to construct and investigate studied in this note type models for more general type nonlinearities and for (5) type multi-dimensional systems as well.

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# Structure and Asymptotic Behavior of Nonoscillatory Solutions of First-order Cyclic Functional Differential Systems

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We consider first-order cyclic functional differential systems of the type

$$x'(t)+p(t)\varphi_\alpha(y(k(t)))=0, \quad y'(t)+q(t)\varphi_\beta(x(l(t)))=0, \tag{A}$$

under the assumption that

(a) $\alpha$ and $\beta$ are positive constants;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[0,\infty)$;

(c) $k(t)$ and $l(t)$ are positive continuous functions on $[0,\infty)$ tending to $\infty$ as $t\to\infty$;

(d) $\varphi_\gamma(u)=|u|^\gamma \operatorname{sgn} u=|u|^{\gamma-1}u$, $\gamma>0$, $u\in\mathbf{R}$.

Let $T>0$ be a fixed point on the real line. Define $T_0$ by

$$T_0=\min\Big\{T,\inf_{t\ge T}k(t),\inf_{t\ge T}l(t)\Big\}.$$

By a *solution* of system (A) on $[T,\infty)$ we mean a vector function $(x(t),y(t))$ which is defined on $[T_0,\infty)$ and satisfies (A) for all $t\in[T,\infty)$. Such a solution is called *oscillatory* (or *nonoscillatory*) if both components of it are oscillatory (or nonoscillatory) in the usual sense. It is clear that (A) admits no oscillatory solutions, so that all nontrivial solutions of (A), if exist, are nonoscillatory.

Let $(x(t),y(t))$ be a nonoscillatory solution of (A). Since (A) implies that $x(t)$ and $y(t)$ are eventually monotone, the two cases may occur: either (Case I) $x(t)y(t)>0$ or (Case II) $x(t)y(t)<0$ for all large $t$. In either case the limits $x(\infty)=\lim\limits_{t\to\infty}x(t)$ and $y(\infty)=\lim\limits_{t\to\infty}y(t)$ exist in the extended real numbers.

Suppose that $x(t)y(t)>0$ for all large $t$. Then, $|x(t)|$ and $|y(t)|$ are eventually decreasing, and so there are the following three possibilities for the combination $(x(\infty),y(\infty))$:

I(i) $0 < |x(\infty)| < \infty, \ \ 0 < |y(\infty)| < \infty;$

I(ii) (a) $0 < |x(\infty)| < \infty$, $|y(\infty)| = 0$, or

(b) $|x(\infty)| = 0$, $0 < |y(\infty)| < \infty$;

I(iii) $|x(\infty)| = 0, \ \ |y(\infty)| = 0.$

Suppose that $x(t)y(t) < 0$ for all large $t$. In this case $|x(t)|$ and $|y(t)|$ are eventually increasing, and there are the following three possibilities for the combination $(|x(\infty)|, |y(\infty)|)$:

II(i) $|x(\infty)| < \infty, \ \ |y(\infty)| < \infty;$

II(ii) (a) $|x(\infty)| < \infty$, $|y(\infty)| = \infty$, or

(b) $|x(\infty)| = \infty$, $|y(\infty)| < \infty$;

II(iii) $|x(\infty)| = \infty, \ \ |y(\infty)| = \infty.$

The existence of nonoscillatory solutions of the four types I(i), I(ii), II(i) and II(ii) can be completely characterized as shown in the following theorems.

**Theorem 1.** *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) > 0$ *for all large* $t$ *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = const \neq 0,$$

*if and only if*

$$\int_0^\infty p(t)\,dt < \infty \ \ \text{and} \ \ \int_0^\infty q(t)\,dt < \infty.$$

**Theorem 2.** *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) > 0$ *for all large* $t$ *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = 0,$$

*if and only if*

$$\int_0^\infty q(t)\,dt < \infty \ \ \text{and} \ \ \int_0^\infty p(t)\rho(k(t))^\alpha\,dt < \infty,$$

*where*

$$\rho(t) = \int_t^\infty q(s)\,ds.$$

**Theorem 3.** *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) < 0$ *for all large* $t$ *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = const \neq 0,$$

*if and only if*

$$\int_0^\infty p(t)\,dt < \infty \ \ \text{and} \ \ \int_0^\infty q(t)\,dt < \infty.$$

**Theorem 4.** *System* (A) *has a solution* $(x(t),y(t))$ *such that* $x(t)y(t)<0$ *for all large* $t$ *and*

$$\lim_{t\to\infty}|x(t)|=const\neq 0,\quad \lim_{t\to\infty}|y(t)|=\infty,$$

*if and only if*

$$\int_0^\infty q(t)\,dt=\infty \ \ and \ \int_0^\infty p(t)Q(k(t))^\alpha\,dt<\infty,$$

*where*

$$Q(t)=\int_0^t q(s)\,ds.$$

Note that the theorems concerning the cases I(iib) and II(iib) could be formulated automatically from Theorems 2 and 4, respectively.

The solutions of types I(iii) and II(iii) seem to be extremely difficult to analyze, and for the present we have to content ourselves with seeking *regularly varying solutions* for system (A) in which $\alpha\beta<1$, $p(t)$ and $q(t)$ are regularly varying and $k(t)$ and $l(t)$ are regularly varying of index 1. By a regularly varying solution of system (A) we here mean a nonoscillatory solution $(x(t),y(t))$ of (A) such that both $|x(t)|$ and $|y(t)|$ are regularly varying in the sense of Karamata. If $|x|\in \mathrm{RV}(\rho)$ and $|y|\in\mathrm{RV}(\sigma)$, we write $(x,y)\in\mathrm{RV}(\rho,\sigma)$, and call $(x(t),y(t))$ a regularly varying solution of index $(\rho,\sigma)$.

In the following theorems it is assumed that $p\in\mathrm{RV}(\lambda)$ and $q\in\mathrm{RV}(\mu)$ and they have the expressions

$$p(t)=t^\lambda L(t),\quad q(t)=t^\mu M(t),\ \ L,M\in\mathrm{SV},$$

and that $k(t)$ and $l(t)$ satisfy

$$\lim_{t\to\infty}\frac{k(t)}{t}=\gamma,\quad \lim_{t\to\infty}\frac{l(t)}{t}=\delta,$$

for some positive constants $\gamma$ and $\delta$, respectively.

First we look for regularly varying solutions of type I(iii). It is clear that $(x,y)\in\mathrm{RV}(\rho,\sigma)$ is of type I(iii) (i.e., $x(\infty)=y(\infty)=0$) if $(\rho,\sigma)$ falls into one of the three cases:

(a) $\rho<0$, $\sigma<0$,

(b) $\rho=0$, $\sigma<0$, or $\rho<0$, $\sigma=0$,

(c) $\rho=\sigma=0$.

We are able to deal with the cases (a) and (b) exhaustively. Our result for the case (a) follows.

**Theorem 5.** *Let* $\alpha\beta<1$. *Suppose that* $\lambda$ *and* $\mu$ *satisfy*

$$\lambda+1+\alpha(\mu+1)<0,\quad \beta(\lambda+1)+\mu+1<0,$$

*and define* $\rho$ *and* $\sigma$ *by*

$$\rho=\frac{\lambda+1+\alpha(\mu+1)}{1-\alpha\beta}\,,\quad \sigma=\frac{\beta(\lambda+1)+\mu+1}{1-\alpha\beta}\,.$$

*Then system* (A) *possesses a nonoscillatory solution* $(x(t),y(t))$ *of type* I(iii) *which satisfies* $x(t)y(t)>0$ *for all large* $t$ *and belongs to the class* $\mathrm{RV}(\rho,\sigma)$. *The asymptotic behavior of the components* $x(t)$ *and* $y(t)$ *are governed by the precise decay laws:*

$$|x(t)|\sim t^\rho\bigg[\Big(\frac{\gamma^{\alpha\sigma}L(t)}{-\rho}\Big)\Big(\frac{\delta^{\beta\rho}M(t)}{-\sigma}\Big)^\alpha\bigg]^{\frac{1}{1-\alpha\beta}},\quad |y(t)|\sim t^\sigma\bigg[\Big(\frac{\gamma^{\alpha\sigma}L(t)}{-\rho}\Big)^\beta\Big(\frac{\delta^{\beta\rho}M(t)}{-\sigma}\Big)\bigg]^{\frac{1}{1-\alpha\beta}},$$

*as* $t\to\infty$.

As for the case (b) it suffices to present the result for solutions belonging to $\mathrm{RV}(0,\sigma)$ with $\sigma<0$, from which, as is easily seen, the result for solutions in $\mathrm{RV}(\rho,0)$ with $\rho<0$ can be formulated almost automatically.

**Theorem 6.** *Let $\alpha\beta<1$. Suppose that $\lambda$ and $\mu$ satisfy*

$$\lambda=-1-\alpha(\mu+1),\ \ \mu<-1.$$

*Suppose moreover that for any $a>0$*

$$\int\limits_a^\infty t^{-1}L(t)M(t)^\alpha\,dt=\int\limits_a^\infty p(t)(tq(t))^\alpha\,dt<\infty.$$

*Put $\sigma=\mu+1$. Then system* (A) *possesses a nonoscillatory solution $(x(t),y(t))$ of type* I(iii) *which satisfies $x(t)y(t)>0$ for all large $t$ and belongs to the class $\mathrm{RV}(0,\sigma)$. The asymptotic behavior of the components $x(t)$ and $y(t)$ are governed by the precise decay laws:*

$$|x(t)|\sim\bigg[(1-\alpha\beta)\gamma^{\alpha\sigma}\int\limits_t^\infty s^{-1}L(s)\Big(\frac{M(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}},$$

$$|y(t)|\sim t^\sigma\frac{M(t)}{-\sigma}\bigg[(1-\alpha\beta)\gamma^{\alpha\sigma}\int\limits_t^\infty s^{-1}L(s)\Big(\frac{M(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}},$$

*as $t\to\infty$.*

In order to handle solutions of type II(iii) of (A) we note that if $(x(t),y(t))$ is a solution of (A) of that type, then $(-x(t),y(t))$ and $(x(t),-y(t))$ are solutions of the "dual" system

$$X'(t)-p(t)\varphi_\alpha(Y(k(t)))=0,\quad Y'(t)-q(t)\varphi_\beta(X(l(t)))=0, \tag{B}$$

satisfying $X(t)Y(t)>0$ for all large $t$ and $|X(\infty)|=|Y(\infty)|=\infty$. Then the desired results for the cases (a) and (b) of II(iii) could easily be obtained from Theorems 3.1 and 3.2 established for (B) in the paper [1]. Their formulations may be omitted.

Some of the above-mentioned results for system (A) seem to be new even (A) is reduced to the ordinary differential system

$$x'+p(t)\varphi_\alpha(y)=0,\quad y'+q(t)\varphi_\beta(x)=0. \tag{C}$$

For the pioneering systematic investigation of first-order ordinary differential systems including (C) the reader is referred to the book of Mirzov [2].

It should be noticed that the results obtained for system (A) find applications to systems of the form

$$x'(g(t))+p(t)\varphi_\alpha(y(k(t)))=0,\quad y'(h(t))+q(t)\varphi_\beta(x(l(t)))=0,$$

as well as to scalar equations of the form

$$\big(p(t)\varphi_\alpha(x'(g(t)))\big)'+q(t)\varphi_\beta(x(l(t)))=0.$$

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# On the Solvability of the Mixed Problem for the Semilinear Wave Equation with a Nonlinear Boundary Condition

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In the plane of independent variables $x$ and $t$ in the domain $D_T:\ 0<x<l,\ 0<t<T$ consider the mixed problem of finding the solution $u(x,t)$ of semilinear wave equation of the form

$$u_{tt}-u_{xx}+g(u)=f(x,t),\ \ (x,t)\in D_T, \tag{1}$$

satisfying the initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\leq x\leq l, \tag{2}$$

and boundary conditions

$$u_x(0,t)=F[u(0,t)],\quad u_x(l,t)=\alpha(t)u(l,t),\ \ 0\leq t\leq T, \tag{3}$$

where $f$, $\varphi$, $\psi$, $\alpha$, $g$ and $F$ are given, and $u$ is an unknown real functions.

Let the following conditions of smoothness

$$\begin{gathered} f\in C^1(\overline{D}_T),\ \ g,F\in C^1(\mathbb{R}),\\ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \alpha\in C^1([0,T]) \end{gathered} \tag{4}$$

be fulfilled. It is assumed that the second order conditions of agreement are fulfilled at the points $(0,0)$ and $(l,0)$.

Note that nonlinear boundary condition of the form given in (3) arises, for example, in description of the process of longitudinal oscillations of a spring in case of elastic fixing of one of its ends when the tension does not comply with linear Hooke's law and is nonlinear function of shift, and also in description of processes in the distributed self-oscillatory systems.

Consider the conditions

$$\begin{gathered} \int\limits_0^s g(s_1)\,ds_1\geq -M_1s^2-M_2,\quad \int\limits_0^s F(s_1)\,ds_1\geq -M_3\ \ \forall\,s\in\mathbb{R},\\ \alpha(t)\leq 0,\ \ \alpha'(t)\geq 0,\ \ 0\leq t\leq T, \end{gathered} \tag{5}$$

where $M_i:=const\geq 0$, $1\leq i\leq 3$.

The following theorem is valid.

**Theorem.** *Let the conditions* (4), (5) *be fulfilled. Then there exists a unique classical solution of the problem* (1)–(3).

**Remark 1.** In the case when at least one of the conditions (5), imposed on nonlinear functions $g$ and $F$, is violated, as the following particular case shows, the solution $u$ of considering problem can be explosive, i.e. there exists a number $T^*>0$ such that the problem (1)–(3) has a unique solution, besides

$$\lim_{T\to T^*-0}\|u\|_{C(\overline{D}_T)}=\infty. \tag{6}$$

Thus, in particular, it follows that the problem under consideration does not have a solution in the domain $D_T$ for $T\geq T^*$.

Indeed, consider the case of the problem (1)–(3) when functions $f$, $g$, $\alpha$ equal zero, besides $\varphi \in C^2([0,l])$, $\varphi(0) > 0$, $\psi \in C^1([0,l])$ and $F(s) = -\delta|s|^\lambda s$, $\delta := const > 0$, $\lambda := const > 0$, $s \in \mathbb{R}$, and the corresponding conditions of agreement are fulfilled. Then in the case $\psi = -\varphi'$ the solution $u$ of this problem in the domain $D_T$ for $T = T^*$ is given by the formula

$$u(x,t) = \begin{cases} \varphi(x-t), & (x,t) \in \Delta_1 \cap \{t < T^*\}, \\ \mu_1(t-x), & (x,t) \in \Delta_2 \cap \{t < T^*\}, \\ \varphi(2l-x-t) - \varphi(l) + \varphi(x-t), & (x,t) \in \Delta_3 \cap \{t < T^*\}, \\ \mu_1(t-x) + \varphi(2l-x-t) - \varphi(x+t-l), & (x,t) \in \Delta_4 \cap \{t < T^*\}. \end{cases} \tag{7}$$

Here

$$\mu_1(t) = \frac{\varphi(0)}{[1-\delta\lambda\varphi^\lambda(0)t]^{\frac{1}{\lambda}}}, \quad 0 \le t < T^* := \frac{1}{\delta\lambda\varphi^\lambda(0)} < l, \tag{8}$$

and

$$\Delta_1 := \Delta OO_1C, \ \ \Delta_2 := \Delta OO_1A, \ \ \Delta_3 := \Delta CO_1B, \ \ \Delta_4 := \Delta O_1AB$$

are right-angled triangles, where

$$O = O(0,0), \ \ A = A(0,l), \ \ B = B(l,l), \ \ C = C(l,0), \ \ O_1 = O_1\Big(\frac{l}{2}, \frac{l}{2}\Big).$$

From (7), (8) it follows that the solution of problem (1)–(3) is explosive, i.e. the equality (6) holds. Therefore, in this case, at the problem statement we should require that $T < T^*$.

**Remark 2.** In fact, the formula (7) allows continuation of the solution of considering problem from the domain $D_{T^*}$ into the domain $D_l \cap \{t < x + T^*\}$, besides, this solution $u(x,t)$ will rise indefinitely at approaching of the point $(x,t)$ from the domain $D_l \cap \{t < x + T^*\}$ to the characteristics $t - x = T^*$, to which adjoins this domain with a part of its boundary.

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# Linear Stochastic Functional Differential Equations: Stability and N. V. Azbelev's $W$-Method

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The $W$-method, in its present form, was proposed by N. V. Azbelev, but according to his comment in [2] it goes back to G. Fubini and F. Tricomi. The method described originally a way to regularize boundary value problems for deterministic differential equations (see e.g. [2,3]). Later on the method has been developed, generalized and applied in the stability theory for determinsitic [1,4,5] and stochastic [6–9] functional differential equations.

Below we describe general principles of the W-method in connection with stochastic functional differential equations.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete $\sigma$-subalgebras of $\mathcal{F}$. By $E$ we denote the expectation on this probability space.

The space $k^n$ consists of all $n$-dimensional, $\mathcal{F}_0$-measurable random variables, and $k = k^1$ is a commutative ring of all scalar $\mathcal{F}_0$-measurable random variables.

By $Z := (z_1, \dots, z_m)^T$ we denote an $m$-dimensional semimartingale (see e.g. [11]). A popular example of such $Z$ is the vector Brownian motion (the Wiener process).

We consider the homogeneous stochastic hereditary equation

$$dx(t) = (V_h x)(t)dZ(t), t \geq 0, \tag{1}$$

equipped with two extra conditions

$$x(s) = \varphi(s), \quad s < 0, \tag{1a}$$

$$x(0) = x_0. \tag{1b}$$

Here $V_h$ is a $k$-linear Volterra operator (see below), which is defined in certain linear spaces of vector stochastic processes, $\varphi$ is an $\mathcal{F}_0$-measurable stochastic process, $x_0 \in k^n$.

By $k$-linearity of the operator $V_h$ we mean the following property:

$$V_h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V_h x_1 + \alpha_2 V_h x_2$$

holding for all $\mathcal{F}_0$-measurable, bounded and scalar random values $\alpha_1$, $\alpha_2$ and all stochastic processes $x_1$, $x_2$ belonging to the domain of the operator $V_h$.

The solution of the initial value problem (1), (1a), (1b) will be denoted by $x(t, x_0, \varphi)$, $t \in (-\infty, \infty)$. Below the solution is always assumed to exist and be unique for an appropriate choice of $\varphi(s)$, $x_0$.

The following kinds of stochastic Lyapunov stability are well-known:

**Definition 1.** For a given real number $p$ $(0 < p < \infty)$ we call the zero solution of the homogeneous equation (1)

- $p$-stable (w.r.t. the initial data, i.e. w.r.t. $x_0$ and the "prehistory" function $\varphi$) if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ implies $E|x(t, x_0, \varphi)|^p \le \varepsilon$ for all $t \ge 0$ and all (admissible) $\varphi$, $x_0$;
- asymptotically $p$-stable (w.r.t. the initial data) if it is $p$-stable and, in addition, any $\varphi$, $x_0$ such that $E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ satisfies $\lim\limits_{t\to+\infty} E|x(t, x_0, \varphi)|^p = 0$;
- exponentially $p$-stable (w.r.t. the initial data) if there exist positive constants $\overline{c}$, $\beta$ such that the inequality

$$E|x(t, x_0, \varphi)|^p \le \overline{c}\Big(E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p\Big) \exp\{-\beta s\}$$

  holds true for all $t \ge 0$ and all $\varphi$, $x_0$.

To be able to link stochastic Lyapunov stability and the $W$-method, we need to represent (1), (1a) as a functional differential equation. Let $x(t)$ be a stochastic process on the real semiaxis $(t \in [0, +\infty))$ and $x_+(t)$ be a stochastic process on the entire real axis $(t \in (-\infty, +\infty))$ coinciding with $x(t)$ for $t \ge 0$ and equalling 0 for $t < 0$, while $\varphi_-(t)$ be a stochastic process on the axis $(t \in (-\infty, +\infty))$ coinciding with $\varphi(t)$ for $t < 0$ and equalling 0 for $t \ge 0$. Then the stochastic process $x_+(t) + \varphi_-(t)$, defined for $t \in (-\infty, +\infty)$ will be a solution of the problem (1), (1a), (1b) if $x(t)$ $(t \in [0, +\infty))$ satisfies the initial value problem

$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t), \quad t \ge 0, \tag{2}$$

$$x(0) = x_0, \tag{2a}$$

where

$$(Vx)(t) := (V_h x_+)(t), \quad f(t) := (V_h \varphi_-)(t) \ \text{ for } \ t \ge 0.$$

Indeed, by linearity $V_h(x_+ + \varphi_-) = V_h(x_+) + V_h(\varphi_-) = Vx + f$, which gives (2). Note that $f$ is uniquely defined by the stochastic process $\varphi$, "the prehistory function". Let us also observe that the initial value problem (2), (2a) is equivalent to the initial value problem (1), (1a), (1b) only for $f$, which have representation $f = V_h\varphi'$, where $\varphi'$ is an arbitrary extension of the function $\varphi$ to the real axis $(-\infty, \infty)$.

In the sequel the following linear spaces of stochastic processes will be used:

- $L^n(Z)$ consists of all predictable $n \times m$-matrix stochastic processes on $[0, +\infty)$, the rows of which are locally integrable w.r.t. the semimartingale $Z$ (see e.g. [11]);
- $D^n$ consists of all $n$-dimensional stochastic processes on $[0, +\infty)$, which can be represented as

$$x(t) = x(0) + \int_0^t H(s)\,dZ(s),$$

  where $x(0) \in k^n$, $H \in L^n(Z)$.

Let $B$ be a linear subspace of the space $L^n(Z)$ equipped with some norm $\|\cdot\|_B$. For a given positive and continuous function $\gamma(t)$ $(t \in [0, \infty))$ we define $B^\gamma = \{f : \ f \in B, \ \gamma f \in B\}$. The latter space becomes a linear normed space if we put $\|f\|_{B^\gamma} := \|\gamma f\|_B$.

We will also need the following linear subspaces of "the space of initial values" $k^n$ and "the space of solutions" $D^n$:

$$k_p^n = \big\{\alpha : \ \alpha \in k^n, \ E|\alpha|^p < \infty\big\}, \quad M_p^\gamma = \Big\{x : \ x \in D^n, \ \sup_{t\geq 0} E|\gamma(t)x(t)|^p < \infty\Big\}, \ \ M_p^1 = M_p.$$

For $1 \leq p < \infty$ the linear spaces $k_p^n$, $M_p^\gamma$ become normed spaces if we define

$$\|\alpha\|_{k_p^n} = \big(E|\alpha|^p\big)^{1/p}, \quad \|x\|_{M_p^\gamma} = \sup_{t\geq 0} \big(E|\gamma(t), x(t)|^p\big)^{1/p}.$$

In the sequel, we will always assume that the operator $V : D^n \to L^n(Z)$ in the equation (2) is a $k$-linear Volterra operator, $f \in L^n(Z)$ and $x_0 \in k^n$. Recall that $V : D^n \to L^n(Z)$ is said to be *Volterra* if for any (random) stopping time $\tau$, $\tau \in [0, +\infty)$ a.s. and for any stochastic processes $x, y \in D^n$ the equality $x(t) = y(t)$ $(t \in [0, \tau]$ a.s.) implies the equality $(Vx)(t) = (Vy)(t)$ $(t \in [0, \tau]$ a.s.).

A solution of (2), (2$a$) is a stochastic process from the space $D^n$ satisfying the equation

$$x(t) = x_0 + (Fx)(t), \ \ t \geq 0,$$

where

$$(Fx)(t) = \int_0^t \big[(Vx)(s) + f(s)\big] \, dZ(s)$$

is a $k$-linear Volterra operator in the space $D^n$ and the integral is understood as a stochastic one w.r.t. the semimartingale $Z$ (see e.g. [11]).

Below $x_f(t, x_0)$ stands for the solution of the initial value problem (2), (2a).

**Definition 2.** Let $1 \leq p < \infty$. We say that the equation (2) is input-to-state stable (ISS) w.r.t. the pair $(M_p^\gamma, B^\gamma)$ if there exists $\overline{c} > 0$, for which $x_0 \in k_p^n$ and $f \in B^\gamma$ imply the relation $x_f(\,\cdot\,, x_0) \in M_p^\gamma$ and the following estimate:

$$\|x_f(\,\cdot\,, x_0)\|_{M_p^\gamma} \leq \overline{c}\big(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma}\big).$$

This definition says that the solutions belong to $M_p^\gamma$ whenever $f \in B^\gamma$ and $x_0 \in k_p^n$ and that they continuously depend on $f$ and $x_0$ in the appropriate topologies. The choice of the spaces is closely related to the kind of stability we are interested in.

The following result describes connections between Lyapunov stability of the zero solution of the equation (1) and input-to-state stability of the equation (2) with the operator $V$ which is constructed from the operator $V_h$ in (1).

**Theorem 3.** *Let $\gamma(t)$ $(t \geq 0)$ be a positive continuous function and $1 \leq p < \infty$. Assume that the equation* (2) *is constructed from* (1), (1a) *and $f(t) \equiv (V_h\varphi_-)(t) \in B^\gamma$ whenever $\varphi$ satisfies the condition $\operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \infty$, and $\|f\|_{B^\gamma} \leq K \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p$ for some constant $K > 0$.*

1) *If $\gamma(t) = 1$ $(t \geq 0)$ and the equation* (2) *is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$, then the zero solution of* (1) *is $p$-stable.*

2) *If $\gamma(t) = \exp\{\beta t\}$ $(t \geq 0)$ for some $\beta > 0$ and the equation* (2) *is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$, then the zero solution of* (1) *is exponentially $p$-stable.*

3) *If $\lim\limits_{t\to+\infty} \gamma(t) = +\infty$, $\gamma(t) \geq \delta > 0$, $t \in [0, +\infty)$ $(t \geq 0)$ for some $\delta$, and the equation* (2) *is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$, then the zero solution of* (1) *is asymptotically $p$-stable.*

The main idea of the $W$-method is to convert the given property of Lyapunov stability – via the property of ISS – into the property of invertibility of a certain regularized operator in a suitable functional space. This operator can be constructed with the help of an auxiliary equation. The latter is similar to the equation (2), but it is "simpler", so that the required ISS property is already established for this equation:

$$dx(t) = \big[(Qx)(t) + g(t)\big]dZ(t), \quad t \geq 0, \tag{3}$$

where $Q : D^n \to L^n(Z)$ is a $k$-linear Volterra operator, and $g \in L^n(Z)$. For the equation (3) it is always assumed the existence and uniqueness assumption, i. e. that for any $x(0) \in k^n$ there is the only (up to a $P$-equivalence) solution $x(t)$ satisfying (3), so that we have the following representation:

$$x(t) = U(t)x_0 + (Wg)(t), \quad t \geq 0, \tag{4a}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and $W$ is the corresponding Cauchy operator for the equation (3).

Now, let us rewrite the equation (2) in the following way:

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]dZ(t), \quad t \geq 0,$$

or

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t), \quad t \geq 0.$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation

$$((I-\Theta)x)(t) = U(t)x(0) + (Wf)(t).$$

**Theorem 4.** *Given a weight $\gamma$ (i. e. a positive continuous function defined for $t \geq 0$), let us assume that the equation* (2) *and the reference equation* (3) *satisfy the following conditions:*

1) *the operators $V$, $Q$ act continuously from $M_p^\gamma$ to $B^\gamma$;*

2) *the reference equation* (3) *is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$.*

*If now the operator $I - \Theta : M_p^\gamma \to M_p^\gamma$ has a bounded inverse in this space, then the equation* (2) *is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$.*

*Proof.* Under the above assumptions we have that $U(\,\cdot\,)x_0 \in M_p^\gamma$ whenever $x_0 \in k_p^n$ and also that

$$x_f(t,x_0) = \big((I-\Theta)^{-1}(U(\,\cdot\,)x_0)\big)(t) + \big((I-\Theta)^{-1}Wf\big)(t) \ \ (t \geq 0)$$

for an arbitrary $x_0 \in k_p^n$, $f \in B^\gamma$. Taking the norms and using the assumptions put on the reference equation, we, as in the previous theorem, obtain the inequality

$$\|x_f(\,\cdot\,,x_0)\|_{M_p^\gamma} \leq \bar{c}\big(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma}\big),$$

where $x_0 \in k_p^n$, $f \in B^\gamma$. Thus, the equation (2) is ISS w.r.t. the pair $(M_p^\gamma, B^\gamma)$. □

The choice of the space $B$ and the weight $\gamma$ depend on the asymptotic property one is studying.

In the theorem below we use the universal constants $c_p$ $(1 \leq p < \infty)$ from the Burkholder–Davis–Gandy inequalities to estimate stochastic integrals, see e.g. [11].

**Theorem 5.** *The zero solution of the equation*

$$dx(t) = \Big(a\xi(t)x(t) + b\xi(t)x\Big(\frac{t}{\tau_0}\Big)\Big)dt + c\sqrt{\xi(t)}\,x\Big(\frac{t}{\tau_1}\Big)d\mathcal{B}(t) \ \ (t \geq 0),$$

*where* $\xi(t) = I_{[0,r]}(t) + tI_{[r,\infty]}(t)$, $t \geq 0$ ($I_A(t)$ *is the indicator of* $A$), $\mathcal{B}(t)$ *is the standard scalar Brownian motion,* $a$, $b$, $c$, $\tau_0$, $\tau_1$, $r$ *are real numbers* ($\tau_0 > 1$, $\tau_1 > 1$), *is asymptotically* $2p$*-stable* (*with respect to* $x_0$, *as* $\varphi$ *is not needed in this case*) *if there exists* $\alpha > 0$ *for which*

$$|a + b + \alpha| + c_p|c|\sqrt{0.5\alpha} + \big(|ab| + b^2\big)\delta_0 + c_p|bc|\sqrt{\delta_0} < \alpha,$$

*where*

$$\delta_0 = \max\big\{\log\tau_0, (1 - \tau_0^{-1})r\big\}.$$

The proof of the result can be found in [8].

The $W$-method is also proven to be efficient in the difficult case of stochastic differential equations with impulses, see [10].

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# Lyapunov Exponents of Parametric Families of Linear Differential Systems

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Consider parametric family of $n$-dimensional ($n \geq 2$) linear differential systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \ \ \mu \in B, \tag{1}$$

whose solutions continuously depend on parameter $\mu \in B$, and $B$ is metric space. Denote class of all such systems by $\mathcal{S}_n^*$. By $\mathcal{S}_n$ we denote subclass of $\mathcal{S}_n^*$ of such systems that for any $\mu \in B$ coefficient matrix $A(\,\cdot\,,\mu)$ is bounded over all $t \geq 0$. We identify family (1) and it's coefficient matrix and therefore write $A \in \mathcal{S}_n^*$ or $A \in \mathcal{S}_n$. For any $A \in \mathcal{S}_n^*$ and $\mu \in B$ by $A_\mu$ we denote differential system of family (1) with fixed parameter $\mu$.

For any family $A \in \mathcal{S}_n^*$ let $\lambda_1(\mu) \leq \cdots \leq \lambda_n(\mu)$ be Lyapunov exponents of system $A_\mu$. Lyapunov exponents $\lambda_i(\mu)$, $i = \overline{1,n}$, are real numbers for all families $A \in \mathcal{S}_n$, therefore we consider $\lambda_i(\,\cdot\,)$ as functions $B \to \mathbb{R}$. For families $A \in \mathcal{S}_n^*$, generally speaking, Lyapunov exponents $\lambda_i(\mu)$, $i = \overline{1,n}$ can take improper values, therefore we consider $\lambda_i(\,\cdot\,)$ as functions $B \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty,+\infty\}$.

All statements given below are true in essentially more general case of Lyapunov exponents of families of morphisms of Millionshtchikov bundles and generalized Millionshtchikov bundles. Nevertheless we use the more familiar language of Lyapunov exponents of parametric families (1).

Lyapunov exponents of families $A \in \mathcal{S}_n$ as functions $B \to \mathbb{R}$ are completely described using Baire characterization. V. M. Millionschikov [5] proved that every function $\lambda_k(\,\cdot\,) : B \to \mathbb{R}$ is a function of the second Baire class. M. I. Rakhimberdiev [7] proved that the number of Baire class in the statement above cannot be reduced. A. N. Vetokhin [8], [9] in special spaces of differential systems proved that Lyapunov exponents considered as functions of systems belong to the Baire class $({}^*, G_\delta)$. Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [1, pp. 223, 224] if for each $r \in \mathbb{R}$ the pre-image of the interval $[r,+\infty)$ under the mapping $f$ is a $G_\delta$-set, i.e. can be represented as a countable intersection of open sets. A complete description of Lyapunov exponents of families $A \in \mathcal{S}_n$ as functions $B \to \mathbb{R}$ was announced if [2] and presented in [3]. For any positive integer $n$ and metric space $B$ set $(f_1(\,\cdot\,),\dots,f_n(\,\cdot\,))$ of functions $B \to \mathbb{R}$ coincides with set of Lyapunov exponents $(\lambda_1(\,\cdot\,),\dots,\lambda_n(\,\cdot\,))$ of some family $A \in \mathcal{S}_n$ if and only if all these functions belong to the Baire class $({}^*, G_\delta)$, have upper semi-continuous minorant and satisfy inequalities $f_1(\mu) \leq \cdots \leq f_n(\mu)$ for all $\mu \in B$.

Consider the same problem of description of Lyapunov exponents of families $A \in \mathcal{S}_n^*$ as functions $B \to \overline{\mathbb{R}}$. V. M. Millionschikov [6] proved that every function $\lambda_k(\,\cdot\,) \colon B \to \overline{\mathbb{R}}$ is a function of the second Baire class. A complete solution of this problem is given by the following theorem.

**Theorem 1.** *For any positive integer $n$ and metric space $B$ set $(f_1(\,\cdot\,),\dots,f_n(\,\cdot\,))$ of functions $B \to \overline{\mathbb{R}}$ coincides with set of Lyapunov exponents $(\lambda_1(\,\cdot\,),\dots,\lambda_n(\,\cdot\,))$ of some family $A \in \mathcal{S}_n^*$ if and only if all these functions belong to the Baire class $({}^*, G_\delta)$ and satisfy inequalities $f_1(\mu) \leq \cdots \leq f_n(\mu)$ for all $\mu \in B$.*

Here for functions $B \to \overline{\mathbb{R}}$ we use the same definition of the Baire class $(\,^*\,, G_\delta)$: function $B \to \overline{\mathbb{R}}$ is referred to as a function of the class $(\,^*\,, G_\delta)$ if for each $r \in \overline{\mathbb{R}}$ the preimage of the segment $[r, +\infty]$ under the mapping $f$ is a $G_\delta$-set.

Consider family $A \in \mathcal{S}_n$. For every Lyapunov exponent $\lambda_i(\,\cdot\,)$ consider set $M_i$ of all points $\mu \in B$ at which function $\lambda_i(\,\cdot\,)$ is upper (lower) semi-continuous. Set $(M_1, M_2, \dots, M_n)$ we call the set of upper (lower) semi-continuity of Lyapunov exponents of family $A$. V. M. Millionschikov [6] proved that if parameter space $B$ is full metric space, then upper semi-continuity is Baire typical for all Lyapunov exponents i.e. for any $A \in \mathcal{S}_n$ and $i = \overline{1,n}$ the set $M_i$ of upper semi-continuity contains dense $G_\delta$-subset. A. N. Vetokhin showed that sets of lower semi-continuity can be empty.

Sets of upper semi-continuity and lower semi-continuity of families $A \in \mathcal{S}_n$ are completely described in [4]. In the case of Lyapunov exponents of families $A \in \mathcal{S}_n^*$ the description of upper and lower semi-continuity sets turned out to be the same. This description is given in the next theorem.

**Theorem 2.** *For any positive integer $n$ and full metric space $B$ set $(M_1, \dots, M_n)$ of subsets of space $B$ is the set of upper semi-continuity of Lyapumov exponents of some family $A \in \mathcal{S}_n^*$ if and only if every $M_i$, $i = \overline{1,n}$ is dense $G_\delta$-set, and the set of lower semi-continuity of Lyapumov exponents of some family $A \in \mathcal{S}_n^*$ if and only if every $M_i$, $i = \overline{1,n}$ is $F_{\sigma\delta}$-set which contains all isolated points of space $B$.*

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# On Some Sufficient Conditions for the $\xi$-Exponential Asymptotical Stability in the Lyapunov Sense of Systems of Linear Impulsive Equations

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Consider the linear system of impulsive equations

$$\frac{dx}{dt} = Q(t)x + q(t) \ \text{ for } \ t \in \mathbb{R}_+, \tag{1}$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) + g_j \ \ (j = 1, 2, \dots), \tag{2}$$

where $Q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^n)$, $G_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$, $g_j \in \mathbb{R}^n$ $(j = 1, 2, \dots)$, $t_j \in \mathbb{R}_+$ $(j = 1, 2, \dots)$, $0 < t_1 < t_2 < \cdots$, $\lim\limits_{j\to+\infty} t_j = +\infty$.

We use the following notation and definitions.

$\mathbb{R} = ]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n\times n}$, then $X^{-1}$, $\det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant of $X$ and the spectral radius of $X$; $I_n$ is the identity $n \times n$-matrix.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$\widetilde{C}([a, b], D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}(I \setminus T, D)$, where $T = \{t_1, t_2, \dots\}$, is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I \setminus \{\tau_l\}_{l=1}^m$ belong to $\widetilde{C}([a, b], D)$.

$L([a, b]; D)$ is the set of all integrable matrix-functions $X : [a, b] \to D$.

$L_{loc}(I; D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I_{t_0}$ belong to $L([a, b], D)$.

By a solution of the impulsive system $(1), (2)$ we understand a continuous from the left vector function $x : \mathbb{R}_+ \to \mathbb{R}^n$, $x \in \widetilde{C}_{loc}(\mathbb{R}_+ \setminus T; \mathbb{R}^n)$, satisfying the system (1) a.e on $]t_j, t_{j+1}[$, and the equality (2) at the point $t_j$ for every $j \in \{1, 2, \dots\}$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi \in \widetilde{C}_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, be a continuous from the left nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty.$$

**Definition 1.** The solution $x_0$ of the system $(1), (2)$ is said to be $\xi$-exponentially asymptotically stable if there is $\eta > 0$ such that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every solution $x$ of the system $(1), (2)$ satisfying the condition

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp\big(\eta(\xi(t) - \xi(t_0))\big) \ \text{ for } \ t \geq t_0$$

holds.

**Definition 2.** The system (1), (2) is said to be $\xi$-exponentially asymptotically stable if every its solution is $\xi$-exponentially asymptotically stable.

**Definition 3.** The pair $(Q, \{G_l\}_{l=1}^{\infty})$, where $Q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$, is $\xi$-exponentially asymptotically stable if the corresponding to this pair homogeneous impulsive system

$$\frac{dx}{dt} = Q(t)x \ \text{ for } \ t \in \mathbb{R}_+,$$
$$x(t_j+) - x(t_j-) = G_j x(t_j-) \ \ (j = 1, 2, \dots)$$

is stable in the same sense.

**Theorem.** *Let $Q = (q_{ik})_{i,k=1}^{n} \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j = (g_{jik})_{i,k=1}^{n} \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$ be such that the conditions*

$$1 + g_{jii} \neq 0 \ \ (i = 1, \dots, n; \ j = 1, 2, \dots),$$
$$r(H) < 1, \tag{3}$$

$$\sup\bigg\{(\xi(t) - \xi(\tau))^{-1}\bigg(\int_{\tau}^{t} q_{ii}(s)\,ds + \sum_{\tau \leq t_j < t} \ln|1 + g_{jii}|\bigg) : \ t \geq \tau \geq t^*, \ \ \xi(t) \neq \xi(\tau); \ \ t, \tau \in \mathbb{R}_+ \setminus T\bigg\} < -\gamma \ \ (i = 1, \dots, n) \tag{4}$$

*and*

$$\int_{t^*}^{t} \exp\bigg(\gamma(\xi(t) - \xi(\tau)) + \int_{\tau}^{t} q_{ii}(s)\,ds\bigg)|q_{ik}(\tau)| \prod_{\tau \leq t_j < t} |1 + g_{jii}|\,d\tau + \sum_{t^* \leq t_l < t} \exp\bigg(\gamma(\xi(t) - \xi(t_l)) + \int_{t_l}^{t} q_{ii}(s)\,ds\bigg)|g_{lik}| \prod_{t_l < t_j < t} |1 + g_{jii}| \leq h_{ik},$$
$$\text{for } \ t \in [t^*, +\infty[\setminus T \ \ (i \neq k; \ \ i, k = 1, \dots, n)$$

*hold, where $\gamma > 0$, $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$, $H = (h_{ik})_{i,k=1}^{n}$ matrix, where $h_{ii} = 0$ $(i = 1, \dots, n)$. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is $\xi$-exponentially asymptotically stable.*

**Corollary.** *Let $Q = (q_{ik})_{i,k=1}^{n} \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j = (g_{jik})_{i,k=1}^{n} \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$ be such that the conditions* (3), (4),

$$-1 < g_{jii} \leq 0 \ \ (i = 1, \dots, n; \ \ j = 1, 2, \dots),$$
$$q_{ii}(t) \leq 0 \ \ (i = 1, \dots, n),$$
$$|q_{ik}(t)| \leq -h_{ik} q_{ii}(t) \ \ (i \neq k; \ \ i, k = 1, \dots, n),$$
$$|g_{jik}| < -h_{ik} g_{jii}(1 + g_{jii}) \ \ (i \neq k; \ \ i, k = 1, \dots, n; \ \ j = 1, 2, \dots)$$

*hold a.e on the interval $[t^*, +\infty[$, where $\gamma > 0$, $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$, $h_{ii} = 0$ $(i = 1, \dots, n)$, and $H = (h_{ik})_{i,k=1}^{n}$. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is $\xi$-exponentially asymptotically stable.*

The questions on the Lyapunov stability in this and other sense are investigated in [1, 3] (see, also the references therein) for linear impulsive systems, and analogous questions in [2] (see, also the references therein) for ordinary differential systems.

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# On the Solvability of One Multidimensional Boundary Value Problem for a Semilinear Hyperbolic Equation

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Consider the semilinear hyperbolic equation of the type

$$L_f u := \square^2 u + f(u) = F, \tag{1}$$

where $f : R \to R$ is a given continuous nonlinear function, $F$ is a given and $u$ is an unknown real function,

$$\square := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,, \ \ n \geq 2.$$

Let $D$ be a convex domain in the space $R^{n+1}$ of variables $x_1, \dots, x_n, t$ with piecewise – smooth boundary $S = \partial D$, consisting of smooth $n$-dimensional manifolds $S_1, S_2, \dots, S_{m_0}, S_{m_0+1}, \dots, S_m$ whose $S_i$, $i = 1, \dots, m_0$, are manifolds of spatial and temporal types, and $S_{m_0+1}, \dots, S_m$ are characteristic manifolds.

For the equation (1), we consider the boundary value problem: find in the domain $D$ a solution $u = u(x_1, \dots, x_n, t)$ of that equation according to the boundary conditions:

$$u|_S = 0; \quad \frac{\partial u}{\partial \nu}\Big|_{S_i} = 0, \ \ i = 1, \dots, m_0, \tag{2}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to $\partial D$.

Assume

$$\overset{\circ}{C}{}^k(D, \partial D) := \Big\{ u \in C^k(D) : \ \ u\big|_S = 0; \ \frac{\partial u}{\partial \nu}\Big|_{S_i} = 0, \ \ i = 1, \dots, m_0 \Big\}, \ \ k \geq 2.$$

Let $u \in \overset{\circ}{C}{}^4(D, \partial D)$ be a classical solution of the problem (1), (2). Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in \overset{\circ}{C}{}^2(D, \partial D)$ and integrating the obtained equality by parts over the domain $D$, we obtain

$$\int\limits_D \square\, u\, \square\, \varphi \, dx \, dt + \int\limits_D f(u) \varphi \, dx \, dt = \int\limits_D F \varphi \, dx \, dt. \tag{3}$$

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\square}(D)$ as the completion with respect to the norm

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\square}(D)} = \int\limits_D \Big[ u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 + (\square\, u)^2 \Big] \, dx \, dt$$

of the classical space $\overset{\circ}{C}{}^2(D, \partial D)$.

Consider the following conditions imposed on the function $f = f(u)$:

$$f \in C(R), \quad |f(u)| \leq M_1 + M_2|u|^\alpha, \quad u \in R, \tag{4}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,. \tag{5}$$

Let $F \in L_2(D)$. We take the equality (3) as a basis for our definition of the generalized solution $u$ of the problem (1), (2): the function $u \in \overset{\circ}{W}{}^1_{2,\square}(D)$ is said to be a weak generalized solution of the problem (1), (2) if for any function $\varphi \in \overset{\circ}{W}{}^1_{2,\square}(D)$ the integral equality (3) is valid.

**Theorem.** *Let $f$ be a monotone function and satisfy the conditions* (4), (5) *and* $uf(u) \geq 0 \ \forall\, u \in R$. *Then for any $F \in L_2(D)$ the problem* (1), (2) *has a unique weak generalized solution in the space* $\overset{\circ}{W}{}^1_{2,\square}(D)$.

As the examples show, if the conditions imposed on the nonlinear function $f$ are violated, then the problem (1), (2) may not have a solution.

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# On Proper Oscillatory Solutions of Higher Order Emden–Fowler Type Differential Systems

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On the interval $\mathbb{R}_+ = [0, +\infty[$, we consider the differential system

$$u_1^{(n_1)} = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn}(u_2), \quad u_2^{(n_2)} = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn}(u_1), \tag{1}$$

where

$$n_1 + n_2 \text{ is even}, \ \ \lambda_1 > 0, \ \ \lambda_1\lambda_2 > 1,$$

and $p_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions such that

$$p_1(t) \geq 0, \ \ p_2(t) \leq 0 \ \text{ for } \ t \in \mathbb{R}_+.$$

If $n_1 = 1$, $n_2 = n - 1$, $\lambda_1 = 1$, $\lambda_2 = \lambda$, $p_1(t) \equiv 1$ and $p_2(t) \equiv p(t)$, then system (1) is equivalent to the Emden–Fowler type differential equation

$$u^{(n)} = p(t)|u|^{\lambda} \operatorname{sgn}(u).$$

Therefore this system may naturaly be called as Emden–Fowler type differential system.

A nontrivial solution $(u_1, u_2)$ of system (1) defined on some infinite interval $[t_0, +\infty[\subset \mathbb{R}_+$ is said to be **proper**.

A proper solution $(u_1, u_2)$ of (1) is said to be **oscillatory** if its components $u_1$ and $u_2$ change sign in any neighbourhood of $+\infty$.

We have established the necessary and sufficient conditions for the oscillation of all proper solutions of system (1) and also the conditions guaranteeing the existence of a multiparametric family of proper oscillatory solutions of that system.

Such results were known earlier only in the cases where $n_1 = n_2 = 1$ or $p_1(t) \equiv 1$ and $\lambda_1 = 1$ (see [1, 2] and the references therein).

**Theorem 1.** *If the conditions*

$$\int_0^{+\infty} p_1(t)\,dt = +\infty, \tag{2}$$

$$\int_0^{+\infty} t^{n_2-1}\bigg[\int_0^t (t-s)^{n_1-1}\Big(\frac{s}{t}\Big)^{(n_2-1)\lambda_1} p_1(s)\,ds\bigg]^{\lambda_2} p_2(t)\,dt = -\infty, \tag{3}$$

$$\lim_{x\to+\infty} \int_0^x t^{n_1-1}\bigg[\int_t^x (s-t)^{n_2-1}|p_2(s)|\,ds\bigg]^{\lambda_1} p_1(t)\,dt = +\infty \tag{4}$$

*are fulfilled, then every proper solution of system* (1) *is oscillatory.*

*If*

$$\liminf_{t\to+\infty} \frac{\int\limits_0^t (t-s)^{n_1-1} s^{(n_2-1)\lambda_1} p_1(s)\,ds}{t^{(n_2-1)\lambda_1} \int\limits_0^t (t-s)^{n_1-1} p_1(s)\,ds} > 0, \tag{5}$$

*then* (3) *takes the form*

$$\int\limits_0^{+\infty} t^{n_2-1} \bigg[ \int\limits_0^t (t-s)^{n_1-1} p_1(s)\,ds \bigg]^{\lambda_2} p_2(t)\,dt = -\infty. \tag{6}$$

**Theorem 2.** *Let conditions* (2) *and* (5) *be fulfilled. Then for the oscillation of all proper solutions of system* (1)*, it is necessary and sufficient that equalities* (4) *and* (6) *be satisfied.*

**Corollary 1.** *Let there exist numbers $t_0 > 0$, $r_i > 0$ $(i = 1, 2)$, $\mu_1 \le 1$ and $\mu_2$ such that*

$$r_1 \le t^{\mu_1} p_1(t) \le r_2, \quad r_1 \le -t^{\mu_2} p_2(t) \le r_2 \ \text{ for } \ t \ge t_0. \tag{7}$$

*Then for the oscillation of all proper solutions of system* (1)*, it is necessary and sufficient that the inequality*

$$\mu_2 \le \frac{n_1 - \mu_1}{\lambda_1} + n_2 \tag{8}$$

*be fulfilled.*

Theorems 1 and 2 leave the question on the existence of proper solutions of system (1) open. The answer to this question gives the following theorem.

**Theorem 3.** *If $n_1$ is even and $n_2 = n_1$, then system* (1) *has $n_1$-parametric family of proper solutions satisfying the condition*

$$\int\limits_0^{+\infty} \Big( p_1(t)|u_2(t)|^{1+\lambda_1} + p_2(t)|u_1(t)|^{1+\lambda_2} \Big)\,dt < +\infty.$$

From Corollary 1 and Theorem 3 it follows

**Corollary 2.** *Let $n_2 = n_1$, $n_1$ be even and there exist numbers $t_0 > 0$, $r_2 > r_1 > 0$, $\mu_1 \le 1$ and $\mu_2$ such that inequalities* (7) *and* (8) *are fulfilled. Then system* (1) *has $n_1$-parametric family of proper oscillatory solutions.*

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# Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order

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Let $m_1,\dots,m_n$ be positive integers. In the $n$-dimensional box $\Omega=[0,\omega_1]\times\cdots\times[0,\omega_n]$ for the linear hyperbolic equation

$$u^{(\mathbf{m})}=\sum_{\alpha<\mathbf{m}}p_\alpha(\mathbf{x})u^{(\alpha)}+q(\mathbf{x}) \tag{1}$$

consider the boundary conditions

$$h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \text{ for }\ \widehat{\mathbf{x}}_i\in\Omega_i\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{2}$$

Here $\mathbf{x}=(x_1,\dots,x_n)$, $\widehat{\mathbf{x}}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $\Omega_i=[0,\omega_1]\times\cdots\times[0,\omega_{i-1}]\times[0,\omega_{i+1}]\times\cdots\times[0,\omega_n]$, $\mathbf{m}=(m_1,\dots,m_n)$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\mathbf{m}_{1\cdots k}=(m_1,\dots,m_k,0,\dots,0)$ ($\mathbf{m}_{1\cdots k}=(0,\dots,0)$ if $k=0$), $\widehat{\mathbf{m}}_i=\mathbf{m}-\mathbf{m}_i$ and $\mathbf{m}_i=(0,\dots,m_i,\dots,0)$ are multi-indices,

$$u^{(\alpha)}(\mathbf{x})=\frac{\partial^{\alpha_1+\cdots+\alpha_n}u(\mathbf{x})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,,$$

$p_\alpha\in C(\Omega)$ $(\alpha<\mathbf{m})$, $q\in C(\Omega)$, $\varphi_{ik}\in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k=1,\dots,m_i;\ i=1,\dots,n)$, and $h_{ik}:C^{m_i-1}([0,\omega_i])\to C^{\widehat{\mathbf{m}}_{i+1\cdots n}}(\Omega_i)$ $(k=1,\dots,m_i;\ i=1,\dots,n)$ are bounded linear operators.

Two-dimensional initial-boundary value problems were studied in [1–3].

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u\in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})}=\sum_{\alpha<\mathbf{m}}p_\alpha(\mathbf{x})u^{(\alpha)}, \tag{$1_0$}$$

$$h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)(\widehat{\mathbf{x}}_i)\\ =0\ \text{ for }\ \widehat{\mathbf{x}}_i\in\Omega_i\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{$2_0$}$$

**Remark 1.** Even if $h_{ik}:C^{m_i-1}([0,\omega_i])\to\mathbb{R}$ are bounded linear functionals, conditions (2) are not equivalent to the conditions

$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=\varphi_{ik}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n),$$

since the latter require the additional consistency conditions

$$h_{ik}(\varphi_{jl})=h_{jl}(\varphi_{ik})\ \ (k=1,\dots,m_i;\ \ l=1,\dots,m_j;\ \ i,j=1,\dots,n).$$

However, the homogeneous conditions ($2_0$) are equivalent to the homogeneous conditions

$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=0\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n).$$

We make use of following notations and definitions.

$\operatorname{supp}\alpha = \{i \mid \ \alpha_i > 0\}$, $\|\alpha\| = |\alpha_1| + \cdots + |\alpha_n|$.

$\alpha = (\alpha_1, \dots, \alpha_n) < \beta = (\beta_1, \dots, \beta_n) \Longleftrightarrow \alpha_i \le \beta_i$ $(i = 1, \dots, n)$ and $\alpha \ne \beta$.

$\alpha = (\alpha_1, \dots, \alpha_n) \le \beta = (\beta_1, \dots, \beta_n) \Longleftrightarrow \alpha < \beta$, or $\alpha = \beta$.

$\mathbf{m}_{i_1\cdots i_k} = (\alpha_1, \dots, \alpha_n)$, where $\alpha_{i_j} = m_{i_j}$ $(j = 1, \dots, k)$ and $\alpha_j = 0$ if $j \notin \{i_1, \dots, i_k\}$.

$\widehat{\alpha} = \mathbf{m} - \alpha$, $\widehat{\mathbf{m}}_{i_1\cdots i_k} = \mathbf{m} - \mathbf{m}_{i_1\cdots i_k}$.

$\mathbf{x}_{i_1\cdots i_l} = (x_{i_1}, \dots, x_{i_l})$, $\Omega_{i_1\cdots i_l} = [0, \omega_{i1}] \times \cdots \times [0, \omega_{i_l}]$.

$\widehat{\mathbf{x}}_{i_1\cdots i_l} = (x_{j_1}, \dots, x_{j_{n-l}})$, $\widehat{\Omega}_{i_1\cdots i_l} = [0, \omega_{j_1}] \times \cdots \times [0, \omega_{i_{n-l}}]$, where $j_1 < j_2 < \cdots < j_{n-l}$, and $\{j_1, \dots, j_{n-l}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_l\}$.

$C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\alpha)}$, $\alpha \le \mathbf{m}$, with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\alpha \le \mathbf{m}} \|u^{(\alpha)}\|_{C(\Omega)}.$$

**Definition 1.** Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ and $q \in C(\Omega)$, and its solution $u$ admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big( \sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\Omega_i)} + \|q\|_{C(\Omega)} \Big), \tag{3}$$

where $M$ is a positive constant independent of $q$ and $\varphi_{ik}$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$.

In the domain $\Omega_{i_1\cdots i_l}$ consider the homogeneous boundary value problem depending on the parameter $\widehat{\mathbf{x}}_{i_1\cdots i_l} \in \Omega_{i_1\cdots i_l}$

$$v^{(\mathbf{m}_{i_1\cdots i_l})} = \sum_{\alpha < \mathbf{m}_{i_1\cdots i_l}} p_{\widehat{\mathbf{m}}_{i_1\cdots i_l}+\alpha}(\mathbf{x}) v^{(\alpha)}, \tag{$1_{i_1\cdots i_l}$}$$

$$h_{i_jk}\big(v^{(\mathbf{m}_{i_1\cdots i_{j-1}})}(x_1, \dots, x_{i_j-1}, \bullet, x_{i_j+1}, \dots, x_n)\big)(\widehat{\mathbf{x}}_{i_j}) = 0 \ \text{ for } \ \widehat{\mathbf{x}}_{i_j} \in \Omega_{i_j} \ \ (k = 1, \dots, m_{i_j};\ \ j = 1, \dots, l). \tag{$2_{i_1\cdots i_l}$}$$

**Definition 2.** Problem $(1_{i_1\cdots i_l}), (2_{i_1\cdots i_l})$ is called an *associated problem of level $l$*.

Associated problems of level $n-1$ can be written in the relatively simpler form

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\alpha < \widehat{\mathbf{m}}_j} p_{\mathbf{m}_j+\alpha}(\mathbf{x}) v^{(\alpha)}, \tag{$1_j$}$$

$$h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big)(\widehat{\mathbf{x}}_i) = 0 \ \text{ for } \ \widehat{\mathbf{x}}_i \in \Omega_i \quad (k = 1, \dots, m_i,\ i \ne j). \tag{$2_j$}$$

Associated problems of level $n-1$ play a principal role in well-posedness of problem (1), (2).

**Theorem 1.** *Problem* (1), (2) *has Fredholm property if and only if each associated homogeneous problem* $(1_{i_1\cdots i_l}), (2_{i_1\cdots i_l})$ *has only the trivial solution for every* $\widehat{\mathbf{x}}_{i_1\cdots i_l} \in \Omega_{i_1\cdots i_l}$.

**Theorem 2.** *Problem* $(1),(2)$ *is well-posed if and only if problem* $(1_0),(2_0)$ *has only a trivial solution, and each associated homogeneous problem* $(1_{i_1\cdots i_l}),(2_{i_1\cdots i_l})$ *has only the trivial solution for every* $\widehat{\mathbf{x}}_{i_1\cdots i_l}\in\Omega_{i_1\cdots i_l}$.

**Theorem 2′.** *Problem* $(1),(2)$ *is well-posed if and only if problem* $(1_0),(2_0)$ *has only a trivial solution, and each associated homogeneous problem* $(1_j),(2_j)$ *of the level* $n-1$ *is well-posed for every* $x_j\in[0,\omega_j]$ $(j=1,\dots,n)$.

In case where the coefficients $p_\alpha$ are smooth functions, estimate (3) is not the most precise estimate for a solution of problem $(1),(2)$. Consider the equation

$$u^{(\mathbf{m})}=\sum_{\alpha<\mathbf{m}}p_\alpha(\mathbf{x})u^{(\alpha)}+q^{(\beta)}(\mathbf{x}). \tag{$1_\beta$}$$

**Theorem 3.** *Let problem* $(1),(2)$ *be well posed,* $p_\alpha\in C^{\mathbf{m}}(\Omega)$ $(\alpha<\mathbf{m})$, $\beta\le\mathbf{m}$ *and* $q\in C^\beta(\Omega)$. *Then the solution* $u$ *of the problem* $(1_\beta),(2)$ *admits the estimate*

$$\|u\|_{C(\Omega)}\le M\Big(\sum_{i=1}^n\sum_{k=1}^{m_i}\|\varphi_{ik}\|_{C(\Omega_i)}+\|q\|_{C(\Omega)}\Big), \tag{4}$$

*where* $M$ *is a positive constant independent of* $q$ *and* $\varphi_{ik}$ $(k=1,\dots,m_i;\ i=1,\dots,n)$.

Now consider the following particular cases of conditions (2):

(I) Characteristic value problem:

$$u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{5}$$

(II) Initial-Boundary value problems with $n-1$ initial conditions:

$$\begin{gathered} h_{1k}\big(u(\,\bullet\,,x_2,\dots,x_n)\big)(\widehat{\mathbf{x}}_1)=\varphi_{1k}(\widehat{\mathbf{x}}_1),\\ u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ 2=1,\dots,n). \end{gathered} \tag{6}$$

(III) Initial-Boundary value problems with $n-l$ initial conditions:

$$\begin{gathered} h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}(x_1,\dots,x_{i-1},\,\bullet\,,x_{i+1},\dots,x_n)\big)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,l),\\ u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=l+1,\dots,n). \end{gathered} \tag{7}$$

**Corollary 1.** *Then problem* $(1),(5)$ *is well-posed.*

**Corollary 2.** *Problem* $(1),(6)$ *is well-posed if and only if the problem*

$$z^{(m_1)}=\sum_{k=0}^{m_1-1}p_{(k,m_2,\dots,m_n)}(\mathbf{x})z^{(k)},$$

$$h_1(z)(x_2,\dots,x_n)=0$$

*has only the trivial solution for every* $(x_2,\dots,x_n)\in[0,\omega_2]\times\cdots\times[0,\omega_n]$.

**Corollary 3.** *Problem* (1), (7) *is well-posed if and only if the problem*

$$v^{(m_1,\dots,m_l)} = \sum_{\alpha<(m_1,\dots,m_l)} p_{\alpha+(m_{l+1},\dots,m_n)}(\mathbf{x})w^{(\alpha)},$$
$$h_1\big(w(\,\bullet\,,x_2,\dots,x_l)\big)(\widehat{\mathbf{x}}_1)=0,\dots,h_l\big(w^{(m_1,\dots,m_{l-1},0)}(x_1,\dots,x_{l-1},\,\bullet\,)\big)(\widehat{\mathbf{x}}_l)=0$$

*is well-posed for every* $(x_{l+1},\dots,x_n)\in[0,\omega_{l+1}]\times\cdots\times[0,\omega_n]$.

Consider the particular case of equation (1)

$$u^{(\mathbf{2},\dots,\mathbf{2})} = \sum_{\alpha\in\mathcal{E}} p_\alpha(\mathbf{x}_\alpha)u^{(\alpha)} + q(\mathbf{x}), \tag{8}$$

where

$$\mathcal{E}=\Big\{(\alpha_1,\dots,\alpha_n)<(2,\dots,2)\ \big|\ \ \alpha_k=0,\ \text{ or }\ \alpha_k=2\ \ (k=1,\dots,n)\Big\},$$

and

$$\mathbf{x}_\alpha=(x_{i_1},\dots,x_{i_k}),\quad \{i_1,\dots,i_k\}=\operatorname{supp}\widehat{\alpha}.$$

For equation (8) consider the Dirichlet and periodic boundary conditions:

$$\begin{gathered} u(0,x_2,\dots,x_n)=0,\quad u(\omega_1,x_2,\dots,x_n)=0,\\ \vdots\\ u(x_1,\dots,x_{n-1},0)=0,\quad u(x_1,\dots,x_{n-1},\omega_n)=0, \end{gathered} \tag{9}$$

and

$$\begin{gathered} u^{(i,0,\dots,0)}(0,x_2,\dots,x_n)=u^{(i,0,\dots,0)}(\omega_1,x_2,\dots,x_n)\ \ (i=0,1)\\ \vdots\\ u^{(0,\dots,0,i)}(x_1,\dots,x_{n-1},0)=u^{(0,\dots,0,i)}(x_1,\dots,x_{n-1},\omega_n)=0\ \ (i=0,1). \end{gathered} \tag{10}$$

**Corollary 4.** *Let*

$$(-1)^{n+\frac{\|\alpha\|}{2}}\,p_\alpha(\mathbf{x}_\alpha)\le 0\ \ \textit{for}\ \ \alpha\in\mathcal{E}. \tag{11}$$

*Then problem* (8), (9) *is well-posed.*

**Corollary 5.** *Let*

$$(-1)^{n+\frac{\|\alpha\|}{2}}\,p_\alpha(\mathbf{x}_\alpha)< 0\ \ \textit{for}\ \ \alpha\in\mathcal{E}. \tag{12}$$

*Then problem* (8), (10) *is well-posed.*

**Remark 2.** In Corollary 5 strict inequality (12) cannot be replaced by the non-strict inequality (11). Indeed, consider the equation

$$u^{(2,\dots,2)} = (-1)^{n-1}\sum_{i=1}^{n} u_{x_ix_i} + (-1)^n u + q(x_1,\dots,x_{n-1}). \tag{13}$$

Equation (13) satisfies conditions (11) but does not satisfy (12). For problem (13), (10), all associate problems of level $n-1$ have only trivial solutions. However, none of them is well-posed, because all associate problems of level less than $n-1$ have nontrivial solutions. Let us show ill-posedness of problem (13), (10) directly, without applying Theorem 2 (ill-posedness of problem (13), (10) follows immediately from Theorem 2).

Indeed, assume that problem (13), (10) has a solution $u$. One can easily verify that $u$ is a unique solution of problem (13), (10), and thus is independent of $x_n$. Therefore, $u$ satisfies the equation

$$\sum_{i=1}^{n-1} u_{x_i x_i} - u = q(x_1, \dots, x_{n-1}). \tag{14}$$

From the theory of elliptic equations it is well-known, that if $q \in C(\widehat{\Omega}_n)$, then, generally speaking, $u$ is not a classical solution, i.e., it does not belong $C^2(\widehat{\Omega}_n)$, and thus does not belong to $C^{2,\dots,2}(\widehat{\Omega}_n)$.

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# On Well-Posed Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables

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In the rectangle $\Omega = [0,a] \times [0,b]$ consider the nonlinear hyperbolic equation

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big), \tag{1}$$

$$l_j(u(\,\cdot\,,y))(y) = \varphi_j(y) \ \ (j = 1, \dots, m), \quad h_k(u^{(m,0)}(x,\,\cdot\,))(x) = \psi_k^{(m)}(x) \ \ (k = 1, \dots, n), \tag{2}$$

where

$$u^{(j,k)} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}\,,$$

$f : \Omega \times \mathbb{R}^{n+m+mn} \to \mathbb{R}$ is a continuous function, $\varphi_j \in C^n([0,b])$, $\psi_k \in C^m([0,a])$, $l_j : C^{m-1}([0,a] \to C^n([0,b]))$ and $h_k : C^{n-1}[0,b] \to C([0,a])$ are bounded linear operators.

Initial-boundary value problems for linear hyperbolic equations and systems were studied in [1] and [2]. Initial-periodic problems for nonlinear hyperbolic systems were studied in [3].

$C^{m,n}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j = 0, \dots, m;\ k = 0, \dots, n)$, with the norm

$$\|u\|_{C^{m,n}(\Omega)} = \sum_{j=0}^{m} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}.$$

$\widetilde{C}^{m,n}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j = 0, \dots, m;\ k = 0, \dots, n;\ j + k < m + n)$, with the norm

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)} = \sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}.$$

If $z \in \widetilde{C}^{m,n}(\Omega)$ and $r > 0$, then

$$\widetilde{\mathcal{B}}^{m,n}(z;r) = \big\{\zeta \in \widetilde{C}^{m,n}(\Omega) :\ \|\zeta - z\|_{\widetilde{C}^{m,n}} \le r\big\}.$$

Let $\mathbf{v} = (v_0, \dots, v_{n-1})$, $\mathbf{w} = (w_0, \dots, w_{m-1})$ and $\mathbf{z} = (z_{m-1\,n-1}, \dots, z_{0\,0})$. For a function $f(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})$ that is is continuously differentiable with respect to $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, set:

$$f_{mk}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) = \frac{\partial f(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})}{\partial v_k} \ \ (k = 0, \dots, n-1),$$

$$f_{jn}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) = \frac{\partial f(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})}{\partial w_j} \ \ (j = 0, \dots, m-1),$$

$$f_{jk}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) = \frac{\partial f(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})}{\partial z_{j\,k}} \ \ (j = 0, \dots, m-1;\ \ k = 0, \dots, n-1),$$

$$p_{jk}[u](x,y) = f_{jk}\Big(x, y, u^{(m,0)}(x,y), \dots, u^{(m,n-1)}(x,y), u^{(0,n)}(x,y), \dots, u^{(m-1,n)}(x,y), \\ u^{(m-1,n-1)}(x,y), \dots, u(x,y)\Big) \ \ (j = 0, \dots, m;\ \ k = 0, \dots, n).$$

**Definition 1.** Let $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ be continuously differentiable with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$. We say that problem (1), (2) to is $(u_0,r)$-well-posed, if:

(I) it has a solution $u_0(x,y)$;

(II) in the neighborhood $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$ $u_0$ is the unique solution;

(III) for an arbitrary $\varepsilon>0$ there exists $\delta>0$ such that for any $\widetilde{\varphi}_j inC^n([0,b])$, $\widetilde{\psi}_k\in C([0,a])$, and $\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ satisfying the inequalities

$$\|\varphi_j-\widetilde{\varphi}_j\|_{C^n([0,b])}<\delta \ \ (j=1,\dots,m), \quad \|\psi_k-\widetilde{\psi}_k\|_{C([0,a])}<\delta \ \ (k=1,\dots,n),$$
$$\big|f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big|+\big\|f_{\mathbf{v}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}_{\mathbf{v}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big\|$$
$$+\big\|f_{\mathbf{w}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}_{\mathbf{w}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big\|+\big\|f_{\mathbf{z}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}_{\mathbf{z}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big\|<\delta$$

In the neighborhood $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$ the problem

$$u^{(m,n)}=\widetilde{f}\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{$\widetilde{1}$}$$
$$l_j(u(\,\cdot\,,y))(y)=\widetilde{\varphi}_j(y) \ \ (j=1,\dots,m), \quad h_k(u^{(m,0)}(x,\,\cdot\,))(x)=\widetilde{\psi}_k^{(m)}(x) \ \ (k=1,\dots,n) \tag{$\widetilde{2}$}$$

has a unique solution $\widetilde{u}$ and

$$\|u-\widetilde{u}\|_{C^{m,n}(\Omega)}<\varepsilon.$$

Following [4] introduce the definition.

**Definition 2.** Problem (1), (2) is called well-posed if it is $(u_0,r)$-well-posed for every $r>0$.

First consider the linear case, i.e., the equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}+q(x,y). \tag{3}$$

**Theorem 1.** *The linear problem* (3), (2) *is well-posed if and only if:*

(i) *the problem*

$$\zeta^{(n)}=\sum_{i=0}^{n-1}p_{mk}(x,y)\zeta^{(i)}; \quad h_k(\zeta)(x)=0 \ \ (k=1,\dots,n) \tag{4}$$

*has only the trivial solution for every $x\in[0,a]$;*

(ii)

$$\xi^{(m)}=\sum_{i=0}^{m-1}p_{jn}(x,y)\xi^{(i)}; \quad l_j(\zeta)(x)=0 \ \ (j=1,\dots,m) \tag{5}$$

*has only the trivial solution for every $y\in[0,b]$;*

(iii) *the homogeneous problem*

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}, \tag{$3_0$}$$
$$l_j(u(\,\cdot\,,y))(y)=0 \ \ (j=1,\dots,m), \quad h_k(u^{(m,0)}(x,\,\cdot\,))(x)=0 \ \ (k=1,\dots,n) \tag{$2_0$}$$

*has only the trivial solution.*

**Theorem 2.** *The $f$ be a continuously differentiable function with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, and let problem* (1), (2) *be $(u_0, r)$-well-posed for some $r > 0$. Then problem* $(3_0)$, $(2_0)$ *is well-posed, where*

$$p_{jk}(x,y) = p_{jk}[u_0](x,y) \ \ (j = 0,\dots,m; \ \ k = 0,\dots,n).$$

**Theorem 3.** *Let $f$ be a continuously differentiable function with respect to the phase variables $v$, $w$ and $z$, and there exist functions $P_{ijk} \in C(\Omega)$ such that:*

($A_0$)

$$P_{1jk}(x,y) \le f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \le P_{2jk}(x,y) \ \ \text{for} \ \ (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}$$
$$(j = 0,\dots,m; \ \ k = 0,\dots,n; \ \ j+k < m+n);$$

($A_1$) *for every $x \in [0,a]$ and arbitrary measurable functions $p_{mk} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1mk}(x,y) \le p_{mk}(x,y) \le P_{2mk}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (k = 0,\dots,n-1), \tag{6}$$

*problem* (3) *has only the trivial solution;*

($A_2$) *for every $y \in [0,b]$ and arbitrary measurable functions $p_{jn} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1jn}(x,y) \le p_{jn}(x,y) \le P_{2jn}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (j = 0,\dots,m-1), \tag{7}$$

*problem* (5) *has only the trivial solution;*

($A_3$) *for arbitrary measurable functions $p_{jk} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1jk}(x,y) \le p_{jk}(x,y) \le P_{2jk}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (j=0,\dots,m, \ k=0,\dots,n; \ j+k<m+n), \tag{8}$$

*problem* $(3_0)$, $(2_0)$ *has only the trivial solution.*

*Then problem* (1), (2) *is well-posed.*

Consider the "perturbed" equation

$$\begin{aligned} u^{(m,n)} &= f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big) \\ &\quad + \gamma(x,y,u^{(m-1,n-1)},\dots,u). \end{aligned} \tag{$1_\gamma$}$$

**Theorem 4.** *Let $f$ satisfy all of the conditions of Theorem* 3, *and $\gamma(x,y,\mathbf{z})$ be an arbitrary continuous function such that*

$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{|\gamma(x,y,\mathbf{z})|}{\|\mathbf{z}\|} = 0 \tag{9}$$

*uniformly on $\Omega$. Then problem* $(1_\gamma)$, (2) *has at least one solution.*

**Corollary 1.** *Let problem* $(3_0)$, $(2_0)$ *be well-posed, and $\gamma(x,y,\mathbf{z})$ be an arbitrary continuous function satisfying condition* (9) *uniformly on $\Omega$. Then the equation*

$$\begin{aligned} u^{(m,n)} &= \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)} \\ &\quad + \gamma(x,y,u^{(m-1,n-1)},\dots,u) \end{aligned}$$

*has at least one solution satisfying conditions* (2).

The initial-boundary conditions

$$u^{(j-1,0)}(0,y)=\varphi_j(y)\ \ (j=1,\dots,m),\quad h_k(u^{(m,0)}(x,\cdot))(x)=\psi_k^{(m)}(x)\ \ (k=1,\dots,n) \tag{10}$$

are the particular case of (2).

**Theorem 5.** *Let $f$ be a continuously differentiable function with respect to the phase variables $\mathbf{v}$ and $\mathbf{w}$, and let there exist a constant $M$ and functions $P_{1mk}\in C(\Omega)$ satisfying conditions $(\mathsf{A}_1)$ of Theorem 3, such that*

$$\begin{gathered}P_{1mk}(x,y)\le f_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\le P_{2mk}(x,y)\\ \text{for }\ (x,y,\mathbf{v},\mathbf{w},\mathbf{z})\in\Omega\times\mathbb{R}^{n+m+mn}\ \ (k=0,\dots,n-1),\\ \big|f(x,y,\mathbf{0},\mathbf{w},\mathbf{z})\big|\le M\big(1+\|\mathbf{w}\|+\|\mathbf{z}\|\big).\end{gathered} \tag{11}$$

*Then problem* (1), (10) *is solvable. Moreover, if $f$ is locally Lipschitz continuous with respect to $\mathbf{z}$, then problem is well-posed.*

**Remark 1.** In Theorems 3–5 continuous differentiability of the function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ with respect to $\mathbf{v}$ and $\mathbf{w}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorems more cumbersome. However, without Lipshcitz continuity problem (1), (2) may not have a classical solution at all.

Indeed, in the rectangle $[0,1]\times[0,2]$ consider the characteristic value problem

$$u_{xy}=\frac{3}{2}\,u_y^{\frac{1}{3}}, \tag{12}$$

$$u(0,y)=\frac{1}{2}\,(y-1)^2\ \text{ for }\ y\in[0,2],\quad u_x(x,0)=0\ \text{ for }\ x\in[0,1]. \tag{13}$$

It has a unique *absolutely continuous* solution

$$u(x,y)=\frac{1}{2}+\int\limits_0^y \operatorname{sgn}(t-1)(x+|t-1|)^{\frac{3}{2}}\,dt,$$

which is not a classical solution because $u_y(x,y)=\operatorname{sgn}(y-1)(x+|y-1|)^{\frac{3}{2}}$ is discontinuous along the line $y=1$.

**Remark 2.** In Theorem 5 condition $(\mathsf{A}_1)$ cannot be weakened. Indeed, in the rectangle $[0,2\pi]\times[0,1]$ consider the initial-periodic problem

$$u_{xy}=3p(u^2)u_x+\cos x, \tag{14}$$

$$u(0,y)=0\ \text{ for }\ y\in[0,1],\quad u_x(x,0)=u_x(x,1)\ \text{ for }\ x\in[0,2\pi], \tag{15}$$

where $p\in C^\infty(\mathbb{R})$, $p(z)z>0$ for $z\neq 0$ and

$$p(z)=\begin{cases} z & \text{if } |z|<2,\\ 3\operatorname{sgn} z & \text{if } |z|>3.\end{cases}$$

Although the righthand side of the equation is smooth, problem (14), (15) has a unique *absolutely continuous* but not continuously differentiable solution solution $u(x)=\sin^{\frac{1}{3}} x$.

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# Uniqueness of a Solution and Convergence of Finite Difference Scheme for One System of Nonlinear Integro-Differential Equations

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We consider one-dimensional analog of the following system which arises in the mathematical modeling of process of an electromagnetic field penetration into a substance [11]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\Big(\int_0^t |\operatorname{rot} H|^2\,d\tau\Big)\operatorname{rot} H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that system (1) is obtained by the reduction of the well-known Maxwell's equations to the integro-differential form [2]. There are many works devoted to the investigation of the particular cases of system (1) (see, for example, [1–10, 12–14, 16] and the references therein).

Let us consider the following magnetic field

$$H = (0, U, V),$$

where

$$U = U(x,t), \quad V = V(x,t).$$

Then from (1) we get the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\,\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\,\frac{\partial V}{\partial x}\right], \tag{2}$$

where

$$S(x,t) = \int_0^t \left[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\right] d\tau. \tag{3}$$

In [13], some generalization of system of type (1) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by so-called averaged integro-differential model, (2), (3) type analog of which have the following form:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad \frac{\partial V}{\partial t} = a(S)\,\frac{\partial^2 V}{\partial x^2}\,, \tag{4}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\right] dx\,d\tau. \tag{5}$$

The existence of solutions of the corresponding initial-boundary value problems for the models of type (2), (3) and (4), (5) are studied in many works (see, for example, [1–5, 12–14, 16] and the references therein).

Our aim is to study the existence and uniqueness of solutions and discrete analog for the initial-boundary value problem with mixed boundary conditions for system (4), (5) in case $a(S) = (1+S)^p$, $0 < p \le 1$.

Thus, in the domain $[0,1] \times [0,\infty)$ let us consider the following initial-boundary value problem:

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= \Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial^2 U}{\partial x^2}\,,\\
\frac{\partial V}{\partial t} &= \Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial^2 V}{\partial x^2}\,,
\end{aligned}
\tag{6}
$$

$$
U(0,t) = V(0,t) = 0, \quad \frac{\partial U}{\partial x}\Big|_{x=1} = \frac{\partial V}{\partial x}\Big|_{x=1} = 0, \ \ t \ge 0, \tag{7}
$$

$$
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \ \ x \in [0,1], \tag{8}
$$

where $0 < p \le 1$; $U_0$ and $V_0$ are given functions.

The following statement takes place.

**Theorem 1.** *If $0 < p \le 1$, $U_0$, $V_0 \in H^2(0,1)$ and conditions of coincidence are fulfilled, then there exists unique solution $(U,V)$ of problem (6)–(8) such that: $U, V \in L_2(0,\infty; H^2(0,1))$, $U_{xt}, V_{xt} \in L_2(0,\infty; L_2(0,1))$.*

We use usual $L_2(0,1)$ and Sobolev spaces $H^2(0,1)$.

The existence part of the Theorem 1 is proved using Galerkin's modified method and compactness arguments as in [15, 18] for nonlinear parabolic equations and as it is carried out for the case of one component magnetic field in works [2–4].

As to uniqueness of a solution, we assume that there exist two different $(U_1,V_1)$ and $(U_2,V_2)$ solutions of problem (6)–(8) and introduce the differences $Z = U_2 - U_1$ and $W = V_2 - V_1$. To show that $Z = W \equiv 0$ the following identity, analogue of Hadamard formula, is mainly used:

$$
\begin{aligned}
&\bigg\{\Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U_2}{\partial x}\Big)^2 + \Big(\frac{\partial V_2}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial U_2}{\partial x}\\
&\qquad - \Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U_1}{\partial x}\Big)^2 + \Big(\frac{\partial V_1}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial U_1}{\partial x}\bigg\}\Big(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x}\Big)\\
&\quad + \bigg\{\Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U_2}{\partial x}\Big)^2 + \Big(\frac{\partial V_2}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial V_2}{\partial x}\\
&\qquad - \Big(1 + \int_0^t \int_0^1 \Big[\Big(\frac{\partial U_1}{\partial x}\Big)^2 + \Big(\frac{\partial V_1}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p \frac{\partial V_1}{\partial x}\bigg\}\Big(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x}\Big)\\
&\quad = \int_0^1 \frac{d}{d\mu}\bigg(1 + \int_0^t \int_0^1 \Big\{\Big[\frac{\partial U_1}{\partial x} + \mu\Big(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x}\Big)\Big]^2 + \Big[\frac{\partial V_1}{\partial x} + \mu\Big(\frac{\partial V_2}{\partial x} - \frac{\partial U_1}{\partial x}\Big)\Big]^2\Big\}\,dx\,d\tau\bigg)^p
\end{aligned}
$$

$$\times \Big[\frac{\partial U_1}{\partial x}+\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]\,d\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)$$
$$+\int\limits_0^1\frac{d}{d\mu}\left(1+\int\limits_0^t\int\limits_0^1\Big\{\Big[\frac{\partial U_1}{\partial x}+\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2+\Big[\frac{\partial V_1}{\partial x}+\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2\Big\}\,dx\,d\tau\right)^p$$
$$\times \Big[\frac{\partial V_1}{\partial x}+\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\Big)\Big]\,d\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\Big).$$

Now, let us consider the finite difference scheme for problem (6)–(8). On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M$; $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at $(x_i,t_j)$ is designed by $(u_i^j,v_i^j)$ and the exact solution to problem (6)–(8) by $(U_i^j,V_i^j)$. We will use the following known notations [17]:

$$r^j_{x,i}=\frac{r^j_{i+1}-r^j_i}{h}\,,\quad r^j_{\overline{x},i}=\frac{r^j_i-r^j_{i-1}}{h}\,.$$

Introduce inner products and norms:

$$(r^j,g^j)=h\sum_{i=1}^{M-1}r_i^jg_i^j,\quad (r^j,g^j]=h\sum_{i=1}^{M}r_i^jg_i^j,$$
$$\|r^j\|=(r^j,r^j)^{1/2},\quad \|r^j]|=(r^j,r^j]^{1/2}.$$

For problem (6)–(8), let us consider the following finite difference scheme:

$$\begin{gathered}\frac{u_i^{j+1}-u_i^j}{\tau}-\Big(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M-1}\big[(u^k_{\overline{x},\ell})^2+(v^k_{\overline{x},\ell})^2\big]\Big)^p u^{j+1}_{\overline{x}x,i}=f^j_{1,i},\\ \frac{v_i^{j+1}-v_i^j}{\tau}-\Big(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M-1}\big[(u^k_{\overline{x},\ell})^2+(v^k_{\overline{x},\ell})^2\big]\Big)^p v^{j+1}_{\overline{x}x,i}=f^j_{2,i},\\ i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,\end{gathered}\tag{9}$$
$$u_0^j=v_0^j=u^j_{\overline{x}\,M}=v^j_{\overline{x}\,M}=0,\ \ j=0,1,\dots,N,\tag{10}$$
$$u_i^0=U_{0,i},\quad v_i^0=V_{0,i},\ \ i=0,1,\dots,M.\tag{11}$$

Multiplying equations in (9) scalarly by $u_i^{j+1}$ and $v_i^{j+1}$, respectively, it is not difficult to get the inequalities:

$$\|u^n\|^2+\sum_{j=1}^n\|u^j_{\overline{x}}]|^2\tau<C,\quad \|v^n\|^2+\sum_{j=1}^n\|v^j_{\overline{x}}]|^2\tau<C,\ \ n=1,2,\dots,N.\tag{12}$$

Here and below $C$ is a positive constant independent from $\tau$ and $h$.

The a priori estimates (12) guarantee the stability of scheme (9)–(11). Note that the uniqueness of a solution of scheme (9)–(11) can be proved too.

The main statement of this note can be stated as follows.

**Theorem 2.** *If problem* (6)–(8) *has a sufficiently smooth solution* $(U(x,t),V(x,t))$, *then the solution* $u^j=(u_1^j,u_2^j,\dots,u_M^j)$, $v^j=(v_1^j,v_2^j,\dots,v_M^j)$, $j=1,2,\dots,N$ *of the difference scheme* (9)–(11) *tends to the solution of the continuous problem* (6)–(8) $U^j=(U_1^j,U_2^j,\dots,U_M^j)$, $V^j=(V_1^j,V_2^j,\dots,V_M^j)$, $j=1,2,\dots,N$ *as* $\tau\to0$, $h\to0$ *and the following estimates are true:*

$$\|u^j-U^j\|\le C(\tau+h),\quad \|v^j-V^j\|\le C(\tau+h).$$

We have carried out numerous numerical experiments for problem (6)–(8) with different kind of right hand sides and initial-boundary conditions.

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# Asymptotic Behaviour of Solutions of $n$-Order Differential Equations with Regularly Varying Nonlinearities

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Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[ \to ]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, the $\varphi_j : \Delta Y_j \to ]0; +\infty[$ are continuous functions regularly varying as $y^{(j)} \to Y_j$ of order $\sigma_j$, $j = \overline{0, n-1}$, $\Delta Y_j$ is a one-sided neighborhood of the point $Y_j$, $Y_j \in \{0, \pm\infty\}$[1].

The equation (1) is a particular case of the equation, comprehensively studied by V. M. Evtukhov and A. M. Klopot [3]

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}),$$

where $n \geq 2$, $\alpha_k \in \{-1, 1\}$ $(k = \overline{1, m})$, the $p_k : [a, \omega[ \to ]0, +\infty[$ $(k = \overline{1, m})$ are continuous functions, $-\infty < a < \omega \leq +\infty$, the $\varphi_{kj} : \Delta Y_j \to ]0; +\infty[$ $(k = \overline{1, m},\ j = \overline{0, n-1})$ are continuous functions regularly varying as $y^{(j)} \to Y_j$ of order $\sigma_j$, $\Delta Y_j$ is a one-sided neighborhood of the point $Y_j$, $Y_j$ is equal to either 0 or $\pm\infty$.

From mentioned results necessary and sufficient existence conditions of the so-called $\mathcal{P}_{+\infty}(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of equation (1) can be obtained for all $\lambda_0$ $(-\infty \leq \lambda_0 \leq +\infty)$. Moreover, asymptotic representations as $t \to +\infty$ of such solutions and their derivatives of order up to $n-1$ can be established.

It follows from the definition of these solutions that

$$\lim_{t \to +\infty} y^{(j)}(t) = Y_j \in \{0, \pm\infty\} \ \ (j = \overline{0, n-1}), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0.$$

However, the set of the monotonous solutions of equation (1), defined in some neighborhood of $+\infty$, also can have the solutions such that for each of them there exists a number $k \in \{1, \dots, n\}$ so that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{2}$$

When $k = 1, 2$ or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to positive constants, as $y^{(i)} \to Y_i$, a question on the existence of the solutions of type (2) of equation (1) can be solved without any assumption on the limits. Otherwise, we can not get asymptotic formulas of these solutions and their derivatives of order up to $n-1$ directly from equation (1).

Some results concerning existence of solutions of type (2) were obtained in corollary 8.2 of the monograph by I. T. Kiguradze and T. A. Chanturia [2, Ch. II, § 8, p. 207] for general type equations. But these results provide for considerably strict restriction on the $(n-k+1)$-st derivative of solution.

[1]For $Y_j = \pm\infty$ here and in the following all numbers in the neighborhood $\Delta Y_j$ are assumed to have constant sign.

In order to receive new results with less strict restrictions on behaviour of this and following derivatives of order $\leq n-1$ in case, when $k \in \{3, \dots, n\}$ and not all $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to positive constant as $y^{(i)} \to Y_i$, let us introduce the following definition.

**Definition.** The solution $y$ of the differential equation (1) is referred (for $k \in \{3, \dots, n\}$) to as a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, +\infty[ \subset [a, +\infty[$ and satisfies the conditions

$$\lim_{t\to+\infty} y^{(n-k)}(t) = c \ \ (c \neq 0), \quad \lim_{t\to+\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t)y^{(n)}(t)} = \lambda_0. \tag{3}$$

It is obvious that by virtue of the first relative (3) for these solutions the following representations hold

$$y^{(l-1)}(t) = \frac{ct^{n-l-k+1}}{(n-l-k+1)!}\,[1+o(1)] \ \ (l = \overline{1, n-k}) \ \text{ as } \ t \to +\infty \tag{4}$$

and $c \in \Delta Y_{n-k}$.

It readily follows from the form of equation (1) that $y^{(n)}(t)$ has constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l = \overline{1, k-1})$ are strictly monotone functions in neighborhood of $+\infty$ and by virtue of (2) can tend only to zero as $t \to +\infty$. Therefore it is necessary that

$$Y_{j-1} = 0 \ \text{ for } \ j = \overline{n-k+2, n}. \tag{5}$$

Let us introduce the numbers $\mu_j$ $(j = \overline{0, n-1})$

$$\mu_j = \begin{cases} 1 & \text{if } \ Y_j = +\infty, \\ & \text{or } \ Y_j = 0, \ \text{ and } \ \Delta Y_j \ \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } \ Y_j = -\infty, \\ & \text{or } \ Y_j = 0 \ \text{ and } \ \Delta Y_j \ \text{ is a left neighborhood of the point } 0, \end{cases}$$

such that

$$\mu_j\mu_{j+1} > 0 \ \text{ for } \ j = \overline{0, n-k-1}, \quad \mu_j\mu_{j+1} < 0 \ \text{ for } \ j = \overline{n-k+1, n-2}. \tag{6}$$

Besides, note that in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \ \ (j = \overline{0, n-1}), \ \ \operatorname{sign} y^{(n)}(t) = \alpha. \tag{7}$$

In this case along with (6) the following inequality hold

$$\alpha\mu_{n-1} < 0. \tag{8}$$

Moreover, it follows from (4) that

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \ \mu_{n-k} > 0, \\ -\infty & \text{if } \ \mu_{n-k} < 0, \end{cases} \quad \text{for } \ j = \overline{1, n-k}. \tag{9}$$

These conditions on $\mu_j$ $(j = \overline{0, n-1})$ and $\alpha$ are necessary for existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1).

The aim of the present paper is to obtain necessary and sufficient existence conditions of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions $(k \in \{3, \dots, n\})$ of equation (1) for $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$, and establish asymptotic as $t \to +\infty$ formulas of their derivatives of order $\leq n-1$. Moreover, the question on the quantity of studied solutions will be solved.

It is significant that by virtue of the obtained results by V. M. Evtukhov [1] studied solutions of equation (1) hold the following a priori asymptotic conditions.

**Lemma 1.** *Let $k \in \{3,\dots,n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then for each $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation* (1) *the following asymptotic as $t \to +\infty$ relations hold*

$$y^{(l-1)}(t) \sim \frac{[(\lambda_0-1)t]^{n-l}}{\prod\limits_{i=l}^{n-1} a_{0i}}\, y^{(n-1)}(t) \ \ (l = \overline{n-k+2, n-1}), \tag{10}$$

*where $y : [t_0, +\infty[ \to \mathbb{R}$ is an arbitrary $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation* (1), *$a_{0i} = (n-i)\lambda_0 - (n-i-1)$ $(i = \overline{1, n-1})$.*

We say that a continuous function $L : \Delta Y_0 \to ]0, +\infty[$ slowly varying as $y \to Y_0$ satisfies condition $S_0$ if

$$L(\mu e^{[1+o(1)]\ln|y|}) = L(y)[1+o(1)] \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta Y_0),$$

where $\mu = \operatorname{sign} y$.

Condition $S_0$ is necessarily satisfied for functions $L$ that have a nonzero finite limit as $y \to Y_0$, for functions of the form

$$L(y) = |\ln|y||^{\gamma_1}, \quad L(y) = |\ln|y||^{\gamma_1} \big|\ln|\ln|y||\big|^{\gamma_2},$$

where $\gamma_1, \gamma_2 \neq 0$, and for many other functions.

We need the following auxiliary notations:

$$\gamma = 1 - \sum_{j=n-k+1}^{n-1} \sigma_j, \quad \nu = \sum_{j=n-k+1}^{n-2} \sigma_j (n-j-1),$$

$$a_{0j} = (n-j)\lambda_0 - (n-j-1) \ \ (j = \overline{1, n-1}), \quad C = \prod_{j=n-k+1}^{n-2} \left| \frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_j},$$

$$I(t) = \varphi_{n-k}(c) M(c) \int_A^t p(\tau)\tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau,$$

where

$$A = \begin{cases} a_1 & \text{if } \displaystyle\int_{a_1}^{+\infty} p(\tau)\tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau = \pm\infty, \\ +\infty & \text{if } \displaystyle\int_{a_1}^{+\infty} p(\tau)\tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau < +\infty, \end{cases}$$

$a_1 \geq a$ such that $\mu_{j-1} t^{n-k-j+1} \in \Delta Y_{j-1}$ $(j = \overline{1, n-k})$ for $t \geq a_1$,

$$M(c) = \prod_{j=1}^{n-k} \left| \frac{c}{(n-j-k+1)!} \right|^{\sigma_{j-1}}.$$

**Theorem 1.** *Let $\gamma \neq 0$, $k \in \{3,\dots,n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then for existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation* (1), *it is necessary that $c \in \Delta Y_{n-k}$, along with* (5), (6), (8), (9) *inequalities*

$$\lambda_0 < 1, \quad a_{0j+1} > 0 \ \ (j = \overline{n-k+1, n-2}) \tag{11}$$

*hold and the following condition be satisfied:*

$$\lim_{t\to+\infty}\frac{tI'(t)}{I(t)}=\frac{\gamma}{\lambda_0-1}\,. \tag{12}$$

*Moreover, each solution of that kind admits along with* (2) *and* (4) *the asymptotic representations* (10) *as* $t\to+\infty$ *and*

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{L_{n-1}(y^{(n-1)}(t))\prod\limits_{j=n-k+1}^{n-2}L_j\Big(\frac{[(\lambda_0-1)t]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}y^{(n-1)}(t)\Big)}=\alpha\mu_{n-1}\gamma CI(t)[1+o(1)].$$

**Theorem 2.** *Let* $\gamma\neq0$, $k\in\{3,\dots,n\}$, $\lambda_0\in\mathbb{R}\setminus\{0,\frac12,\dots,\frac{k-3}{k-2},1\}$ *and functions* $L_j$ $(j=\overline{n-k+1,n-1})$ *slowly varying as* $y^{(j)}\to Y_j$ *satisfy condition* $S_0$. *In addition, let* $c\in\Delta Y_{n-k}$ *and conditions* (5), (6), (8), (9), (11) *and* (12) *hold. Then, in case of existence of* $\mathcal{P}^k_{+\infty}(\lambda_0)$*-solutions of equation* (1),

$$\int\limits_{a_2}^{+\infty}\tau^{k-2}\Big|I(\tau)L_{n-1}(\mu_{n-1}\tau^{\frac{1}{\lambda_0-1}})\prod_{j=n-k+1}^{n-2}L_j(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}})\Big|^{\frac{1}{\gamma}}\,d\tau<+\infty, \tag{13}$$

*where* $a_2\geq a_1$ *such that* $\mu_{j-1}t^{\frac{a_{0j}}{\lambda_0-1}}\in\Delta Y_{j-1}$ $(j=\overline{n-k+2,n-1})$, $\mu_{n-1}t^{\frac{1}{\lambda_0-1}}\in\Delta Y_{n-1}$ *for* $t\geq a_2$, *and each solution of that kind admits along with* (4) *the following asymptotic representations as* $t\to+\infty$

$$\begin{aligned}
y^{(n-k)}(t)&=c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+o(1)],\\
y^{(l-1)}(t)&=\frac{\mu_{n-1}(\lambda_0-1)^{n-l}t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1}a_{0i}}\,W'(t)[1+o(1)]\ \ (l=\overline{n-k+2,n-1}),\\
y^{(n-1)}(t)&=\mu_{n-1}\,\frac{W'(t)}{t^{k-2}}\,[1+o(1)],
\end{aligned} \tag{14}$$

*where*

$$W(t)=\int\limits_{+\infty}^{t}\tau^{k-2}\Big|\gamma CI(\tau)L_{n-1}(\mu_{n-1}\tau^{\frac{1}{\lambda_0-1}})\prod_{j=n-k+1}^{n-2}L_j(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}})\Big|^{\frac{1}{\gamma}}\,d\tau.$$

*If the inequality* $\sigma_{n-1}\neq1$ *holds along with mentioned conditions, then equation* (1) *has at least one* $\mathcal{P}^k_{+\infty}(\lambda_0)$*-solution that admits such representations. Moreover, for each* $c\in\Delta Y_{n-k}$ *in case* $\lambda_0\in]-\infty,\frac{k-2}{k-1}[\setminus\{0,\frac12,\dots,\frac{k-3}{k-2}\}$ $(\lambda_0\in[\frac{k-2}{k-1};1[)$ *there exists an* $(n-k+1)$*-parameter* ($(n-k)$*-parameter, respectively*) *family of solutions with such representations if* $\sigma_{n-1}>1$, *and* $(n-k)$*-parameter* ($(n-k-1)$*-parameter*) *if* $\sigma_{n-1}<1$.

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# Controllability Linear Differential Systems with Many Inputs by Means of Differential-Algebraic Regulator

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Consider the control system

$$\dot{x} = Ax + Bu, \quad t \geq 0, \tag{1}$$

with the initial condition $x(0) = x_o$, where $x \in R^n$, and $u \in R^r$, $A$, $B$ are constant matrices of appropriate sizes, $x_0 \in R^n$.

**Definition 1.** System (1) is said to be controllable if for each initial condition $x_0$, there exists a time $t_1$, $0 < t_1 < +\infty$, and piecewise continuous control $u(t)$, $0 \leq t \leq t_1$, such that the solution $x(t)$, $t \geq 0$, of system (1) satisfies the condition $x(t_1) = 0$.

It is known [3] that for the controllability of system (1) it is necessary and sufficient that

$$\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n. \tag{2}$$

According to the controllability (by Kalman [3]) the input is chosen from the class of piecewise continuous functions. At the same time it is interesting the possibility to choose the control from restricted class.

Let the control be constructed by the input

$$u(t) = Cy(t) \tag{3}$$

of the differential-algebraic system

$$D_0\dot{y}(t) = Dy(t), \ y(0) = y_0, \tag{4}$$

where $y, y_0 \in R^n$, $C$ – $r \times n$-matrix, $D_0D$ – $n \times n$-matrices.

We say that system (4) is the dynamical regulator for system (1).

**Definition 2.** System (1) is said to be controllable by dynamical regulator (3) if for each initial condition $x_0$, there exists a time $t_1$, $0 < t_1 < +\infty$, and initial condition $y_0$ of the regulator (4) such that $x(t_1) = 0$.

**Theorem.** *System* (1) *is controllable by dynamical regulator* (4) *if and only if*

$$\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n$$

*and*

$$\operatorname{rank}(CD_0^dD_0, CD_0^dKD_0, \dots, CD_0^dK^{n-1}D_0) = n,$$

*where* $D_0^d$ *– Drazin inverse of* $D_0$, $K = DD_0^d$.

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# On Asymptotic Behavior of Singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-Solutions of Second-Order Differential Equations

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Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[ \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is continuous function, $-\infty < a < \omega \le +\infty$, $\Delta_{Y_i}$ $(i \in \{0,1\})$ is a one-side neighborhood of $Y_i$ and $Y_i$ $(i \in \{0,1\})$ is either 0 or $\pm\infty$. We assume that the numbers $\mu_i$ $(i = 0,1)$ given by the formula

$$\mu_i = \begin{cases} 1 & \text{if either } Y_i = +\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is right neighborhood of the point 0,} \\ -1 & \text{if either } Y_i = -\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is left neighborhood of the point 0,} \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \text{ for } Y_0 = \pm\infty \text{ and } \mu_0\mu_1 < 0 \text{ for } Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in a left neighborhood of $\omega$ and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \text{ for } t \in [t_0, \omega[, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0,1).$$

In monograph [1] definitions of singular solutions of first and second kinds are introduced. Here we study Eq. (1) on class singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solutions, that are defined as follows.

**Definition 1.** Let $t_* < \omega$. A solution $y$ of Eq. (1) on interval $[t_0, t_*[ \subset [a, \omega[$ is called singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it satisfies the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \text{ for } t \in [t_0, t_*[, \quad \lim_{t\uparrow t_*} y^{(i)}(t) = Y_i \ \ (i = 0,1), \quad \lim_{t\uparrow t_*} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Note that the singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solution of Eq. (1) is noncontinuable to the right solution. Depending on the values of $\lambda_0$ the set of all such solutions of Eq.(1) is disconnected into 4 disjoint subsets respective to the values of $\lambda_0$: $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$. Here we'll formulate the properties of singular $P_{t_*}(Y_0, Y_1, \lambda_0)$-solutions that correspond to the value $\lambda_0 = \pm\infty$. With this aim, we impose a restriction on the function $f$.

**Definition 2.** We say that a function $f$ satisfies condition $(\mathrm{RN})^*_\infty$ if there exists a number $\alpha_0 \in \{-1, 1\}$, a positive number $A_*$ and continuous functions $\varphi_i : \Delta_{Y_i} \to ]0, +\infty[$ $(i = 0,1)$ of orders $\sigma_i$ $(i = 0,1)$ regular varying [1] as $z \to Y_i$ $(i = 0,1)$ such that for arbitrary continuously differentiable functions $z_i : [a, \omega[\ \Delta_{Y_i}$ $(i = 0,1)$ satisfying the conditions

$$\lim_{t\uparrow t_*} z_i(t) = Y_i \ \ (i = 0,1),$$

$$\lim_{t\uparrow t_*} \frac{(t - t_*)z_0'(t)}{z_0(t)} = 1, \quad \lim_{t\uparrow t_*} \frac{(t - t_*)z_1'(t)}{z_1(t)} = 0,$$

[1] Definition of regular varying function see in [2].

one has representation

$$f(t,z_0(t),z_1(t))=\alpha_0A_*\varphi_0(z_0(t))\varphi_1(z_1(t))[1+o(1)] \ \text{ as } \ t\uparrow t_*.$$

For each singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solution assuming that the function $f$ satisfies condition $(\mathrm{RN})^*_\infty$ with condition (2) we have

$$\alpha_0\mu_1>0 \ \text{ for } \ Y_1=\pm\infty \ \text{ and } \ \alpha_0\mu_1<0 \ \text{ for } \ Y_1=0. \tag{3}$$

**Definition 3.** We say that a slowly varying as $z\to Y_i$ $(z\in\Delta_{Y_i})$ $(i\in\{0,1\})$ function $L:\Delta_{Y_i}\to\,]0;+\infty[$ satisfies the condition $S$ if for any continuous differentiable function $l:\Delta_{Y_i}\to\,]0;+\infty[$, such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{zl'(z)}{l(z)}=0,$$

the following condition takes place

$$L(zl(z))=L(z)(1+o(1)) \ \text{ as } \ z\to Y_i \ \ (z\in\Delta_{Y_i}).$$

We introduce an auxiliary function $\overline{I_\infty}$ by the formula

$$\overline{I_\infty}(t)=\int\limits_{A_\infty}^{t}(t_*-\tau)^{-1}L_0(\mu_0(t_*-\tau))\,d\tau,$$

where the integration limit $A_\infty\in\{a_\infty;t_*\}$ $(a_\infty>a)$ is chosen so as the integrals $\overline{I_\infty}$ tends either to zero or to $\pm\infty$ as $t\uparrow t_*$, $L_0(z)=\varphi_0(z)|z|^{-\sigma_0}$.

**Theorem 1.** [1] *Let the function $f$ satisfy condition $(\mathrm{RN})_{\lambda_0}$, the function $\varphi_0$ satisfy condition $S$. Moreover, let the orders $\sigma_i$ $(i=0,1)$ of the functions $\varphi_i$ $(i=0,1)$ regularly varying as $y^{(i)}\to Y_i$ $(i=0,1)$ satisfy the condition $\sigma_0+\sigma_1\neq1$. Then, for the existence of singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions of the differential equation* (1)*, it is necessary and sufficient that together with conditions* (2)*,* (3) *the conditions*

$$\sigma_0=-1,\quad \sigma_1\neq2,\quad Y_0=0,\quad Y_1=\mu_1\lim_{t\uparrow t_*}|\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}},$$
$$\mu_0\mu_1<0,\quad \alpha_0\mu_1(2-\sigma_1)\overline{I_\infty}(t)>0 \ \ \text{as} \ \ t\in\,]a_\infty,t_*[$$

*hold. Moreover, each solution of this kind admits the asymptotic representations*

$$\frac{y'(t)^2}{\varphi_1(y'(t))}=\alpha_0\mu_1(2-\sigma_1)A_*\overline{I_\infty}(t)[1+o(1)],\quad \frac{y'(t)}{y(t)}=\frac{(1+o(1))}{(t-t_*)} \ \ \text{as} \ \ t\uparrow\omega$$

*and such solutions form a one-parameter family if $\alpha_0\mu_1(2-\sigma_1)>0$.*

**Theorem 2.** *Let the function $f$ satisfy condition $(\mathrm{RN})_{\lambda_0}$, the functions $\varphi_0$, $\varphi_1$ satisfy condition $S$, $\sigma_0+\sigma_1\neq1$. Then each singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions (in case of the existence) of the differential equation* (1) *admits the asymptotic representations*

$$y(t)=\mu_0(t_*-t)\Big(|2-\sigma_1|A_*|\overline{I_\infty}(t)|L_1\big(\mu_1A_*|\,|\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}}\big)\Big)^{\frac{1}{2-\sigma_1}}(1+o(1)),$$
$$y'(t)=\mu_1\Big(|2-\sigma_1|A_*|\overline{I_\infty}(t)|L_1\big(\mu_1A_*||\overline{I_\infty}(t)|^{\frac{1}{2-\sigma_1}}\big)\Big)^{\frac{1}{2-\sigma_1}}(1+o(1)) \ \ \text{as} \ \ t\uparrow t_*.$$

[1]Theorem 1, Theorem 2 are obtained as corollaries from theorems of [3].

To illustrate Theorem 1, we give the result of Eq. (1) of special form

$$y'' = \frac{\sum\limits_{k=1}^{m} \alpha_k A_{*k} \varphi_{k0}(y)\varphi_{k1}(y')}{\sum\limits_{k=m+1}^{m+n} \alpha_k A_{*k} \varphi_{k0}(y)\varphi_{k1}(y')}\,, \tag{4}$$

where $\alpha_k \in \{-1,1\}$ $(k=1,\dots,m{+}n)$, $A_{*k} = const > 0$ $(k=1,\dots,m{+}n)$ and $\varphi_{ki} : \Delta_{Y_i} \to ]0,+\infty[$ $(k=1,\dots,n+m;\ i=0,1)$ are regular varying as $z \to Y_i$ continuous functions of $\sigma_{ki}$-th orders.

**Theorem 3.** *Let for any $i \in \{1,\dots,m\}$, $j \in \{m+1,\dots,m+n\}$ inequalities*

$$\sigma_{i0} - \sigma_{j0} + \sigma_{i1} - \sigma_{j1} \neq 1, \quad \sigma_{i0} - \sigma_{k0} < 0 \ \text{ for } \ k \in \{1,\dots,m\} \setminus \{i\},$$
$$\sigma_{j0} - \sigma_{k0} < 0 \ \text{ for } \ k \in \{m+1,\dots,m+n\} \setminus \{j\}$$

*hold and function $\frac{\varphi_{i0}}{\varphi_{j0}}$ satisfy condition S. Then, for the existence of singular $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions of the differential equation (4), it is necessary and sufficient that together with conditions (2), (3) the conditions*

$$\mu_0\mu_1 < 0, \quad \alpha_i\alpha_j\mu_1(2-\sigma_{i1}-\sigma_{j1})\overline{I_{\infty ij}}(t) > 0 \ \text{ as } \ t \in ]a_\infty, t_*[\,,$$
$$\sigma_{i0} - \sigma_{j0} = -1, \quad \sigma_{i1} - \sigma_{j1} \neq 2, \quad Y_0 = 0, \quad Y_1 = \mu_1 \lim_{t \uparrow t_*} |\overline{I_{\infty ij}(t)}|^{\frac{1}{2-\sigma_{i1}-\sigma_{j1}}},$$

*where*

$$\overline{I_{\infty ij}}(t) = \int\limits_{A_\infty}^{t} (t_* - \tau)^{-1}\, \frac{L_{0i}(\mu_0(t_*-\tau))}{L_{0j}(\mu_0(t_*-\tau))}\, d\tau, \quad L_{0k}z = \varphi_{0k}(z)|z|^{\sigma_{0k}}, \ \ k = i,j,$$

*hold. Moreover, each solution of this kind admits the asymptotic representations*

$$\frac{y'(t)^2\varphi_{j1}(y'(t))}{\varphi_{i1}(y'(t))} = \alpha_i\alpha_j\mu_1(2-\sigma_{i1}+\sigma_{j1})\,\frac{A_{*i}}{A_{*j}}\overline{I_{\infty ij}}(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{(1+o(1))}{(t-t_*)} \ \text{ as } \ t \uparrow \omega$$

*and such solutions form a one-parameter family if $\alpha_i\alpha_j\mu_1(2-\sigma_{i1}+\sigma_{j1}) > 0$.*

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# On Non-Negative Periodic Solutions of Second-Order Differential Equations with Mixed Sub-Linear and Super-Linear Non-Linearities

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Consider the periodic problem

$$\boxed{u'' = p(t)u + q(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),} \tag{1}$$

where $p \in L([0,\omega])$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under a *solution* of problem (1), as usually, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere and verifies periodic conditions.

We are interested in the existence and uniqueness of a **non-trivial non-negative** solution of problem (1) in the case when the function $q$ may contain both sub-linear and super-linear non-linearities. In particular, it follows from Corollary 4 stated below that for an arbitrary $p \in L([0,\omega])$, the problem

$$u'' = p(t)u + \sqrt[3]{u} - u^3; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has at least one non-trivial non-negative solution.

**Definition 1.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \leq 0 \quad \text{for } t \in [0,\omega]\big)$$

is fulfilled.

**Definition 2.** We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a nontrivial sign-constant solution.

Introduce the hypothesis

$$\left.\begin{array}{l} q(t,x)\leq q_0(t,x) \quad \text{for a.e. } t\in[0,\omega] \text{ and all } x\geq x_0,\\ x_0\geq 0, \quad q_0\colon [0,\omega]\times[x_0,+\infty[\,\to[0,+\infty[ \text{ is a Carathéodory function,}\\ q_0(t,\,\cdot\,)\colon [x_0,+\infty[\,\to[0,+\infty[ \text{ is non-decreasing for a.e. } t\in[0,\omega],\\ \lim\limits_{x\to+\infty}\dfrac{1}{x}\displaystyle\int_0^\omega q_0(s,x)\,\mathrm{d}s=0. \end{array}\right\} \qquad (H_1)$$

A general existence result reads as follows.

**Theorem 1.** *Let hypothesis $(H_1)$ be fulfilled and*

$$q(t,0)\leq 0 \quad \textit{for a.e. } t\in[0,\omega]. \tag{2}$$

*Let, moreover, there exist functions $\alpha,\beta\in AC^1([0,\omega])$ satisfying*

$$\begin{gathered} \alpha(t)>0, \quad \beta(t)>0 \quad \textit{for } t\in[0,\omega],\\ \alpha''(t)\geq p(t)\alpha(t)+q(t,\alpha(t)) \quad \textit{for a.e. } t\in[0,\omega], \quad \alpha(0)=\alpha(\omega), \quad \alpha'(0)\geq\alpha'(\omega),\\ \beta''(t)\leq p(t)\beta(t)+q(t,\beta(t)) \quad \textit{for a.e. } t\in[0,\omega], \quad \beta(0)=\beta(\omega), \quad \beta'(0)\leq\beta'(\omega). \end{gathered}$$

*Then problem* (1) *has at least one solution $u$ such that*

$$u(t)\geq 0 \quad \textit{for } t\in[0,\omega], \quad u\not\equiv 0, \tag{3}$$

*and*

$$\min\big\{\alpha(t_u),\beta(t_u)\big\}\leq u(t_u)\leq\max\big\{\alpha(t_u),\beta(t_u)\big\} \quad \textit{for some } t_u\in[0,\omega]. \tag{4}$$

**Corollary 1.** *Let inequality* (2) *hold, hypothesis $(H_1)$ be fulfilled,*

$$\left.\begin{array}{l} q(t,x)\leq -xg(t,x) \quad \textit{for a.e. } t\in[0,\omega] \textit{ and all } x>\kappa,\\ \kappa\geq 0, \quad g\colon [0,\omega]\times\,]\kappa,+\infty[\,\to\mathbb{R} \textit{ is a locally Carathéodory function,}\\ g(t,\,\cdot\,)\colon\, ]\kappa,+\infty[\,\to\mathbb{R} \textit{ is non-decreasing for a.e. } t\in[0,\omega], \end{array}\right\} \qquad (H_2)$$

*and*

$$\left.\begin{array}{l} q(t,x)\geq xg_1(t,x)-g_2(t,x) \quad \textit{for a.e. } t\in[0,\omega] \textit{ and all } x\in\,]0,\delta],\\ \delta>0, \quad g_1,g_2\colon [0,\omega]\times\,]0,\delta]\to\mathbb{R} \textit{ are locally Carathéodory functions,}\\ g_1(t,\,\cdot\,)\colon\, ]0,\delta]\to\mathbb{R} \textit{ is non-increasing for a.e. } t\in[0,\omega],\\ g_2(t,\,\cdot\,)\colon\, ]0,\delta]\to\mathbb{R} \textit{ is non-decreasing for a.e. } t\in[0,\omega],\\ \lim\limits_{x\to 0+}\dfrac{1}{x}\displaystyle\int_0^\omega |g_2(s,x)|\,\mathrm{d}s=0. \end{array}\right\} \qquad (H_3)$$

*Let, moreover, there exist a non-negative function $\ell\in L([0,\omega])$ and numbers $r_1\in\,]0,\delta]$, $r_2>\kappa$ such that*

$$p+g_1(\,\cdot\,,r_1)\in\mathcal{V}^-(\omega), \quad p+\ell-g(\,\cdot\,,r_2)\in\operatorname{Int}\mathcal{V}^+(\omega).$$

*Then problem* (1) *has at least one solution $u$ satisfying condition* (3).

Now we provide efficient conditions guaranteeing the existence of a non-trivial non-negative solution of problem (1).

**Corollary 2.** *Let inequality* (2) *hold, hypotheses* ($H_1$), ($H_2$), *and* ($H_3$) *be fulfilled, and*

$$\lim_{x\to\kappa+} g(t,x) \le 0 \quad \textit{for a.e. } t\in[0,\omega], \qquad \lim_{x\to+\infty}\int_0^\omega g(s,x)\,\mathrm{d}s = +\infty. \tag{5}$$

*Let, moreover, at least one of the following conditions be satisfied:*

(a) $p\in\mathcal{V}^-(\omega)$ *and*

$$g_1(t,\delta)\ge 0 \quad \textit{for a.e. } t\in[0,\omega]; \tag{6}$$

(b) $p\in\mathcal{V}_0(\omega)$, *inequality* (6) *holds, and* $g_1(\,\cdot\,,\delta)\not\equiv 0$;

(c) $p\in\mathcal{V}^+(\omega)$, *inequality* (6) *holds, and*

$$\lim_{x\to 0+}\int_0^\omega g_1(s,x)\,\mathrm{d}s = +\infty; \tag{7}$$

(d)

$$\lim_{x\to 0+}\int_E g_1(s,x)\,\mathrm{d}s = +\infty \quad \textit{for every } E\subseteq[0,\omega], \ \ \operatorname{meas} E > 0. \tag{8}$$

*Then problem* (1) *has at least one solution u satisfying condition* (3).

Further, we present some consequences of the general results for the following particular cases of (1):

$$\boxed{u'' = p(t)u + h(t)\ln(1+|u|) - f(t)\ln(1+|u|)u; \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega)} \tag{9}$$

and

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u - f(t)|u|^\mu \operatorname{sgn} u; \quad u(0)=u(\omega), \ \ u'(0)=u'(\omega),} \tag{10}$$

where $h,f\in L([0,\omega])$ and $\lambda,\mu>0$, $(1-\lambda)(\mu-1)>0$.

**Corollary 3.** *Let*

$$f(t)\ge 0 \quad \textit{for a.e. } t\in[0,\omega], \ \ f\not\equiv 0, \tag{11}$$

*and*

$$h(t)\ge 0 \quad \textit{for a.e. } t\in[0,\omega].$$

*Then problem* (9) *has a positive solution if and only if* $p+h\in\mathcal{V}^-(\omega)$.

Concerning problem (10), we first recall a known result in the case, when $0<\mu<1<\lambda$.

**Proposition 1.** *Let* $0<\mu<1<\lambda$ *and*

$$h(t)\ge 0, \quad f(t)\ge 0 \quad \textit{for a.e. } t\in[0,\omega], \ \ h\not\equiv 0, \ \ f\not\equiv 0. \tag{12}$$

*If, moreover,* $p\in\mathcal{V}^-(\omega)$, *then problem* (10) *has a positive solution.*

**Definition 3.** We say that the function $p\in L([0,\omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = \widetilde{p}(t)u; \quad u(a)=0, \ \ u(b)=0$$

has no non-trivial solution for any $a,b\in\mathbb{R}$ satisfying $0<b-a<\omega$, where $\widetilde{p}$ is the $\omega$-periodic extension of the function $p$ to the whole real axis.

For the case, when $0 < \lambda < 1 < \mu$, we get the following statement.

**Corollary 4.** *Let $0 < \lambda < 1 < \mu$, relation* (11) *hold, and one of the following conditions be satisfied:*

(1) $h(t) > 0$ *for a.e.* $t \in [0,\omega]$*;*

(2) $h(t) \geq 0$ *for a.e.* $t \in [0,\omega]$*,* $h \not\equiv 0$*, and* $p \in \mathcal{D}(\omega)$*.*

*Then problem* (10) *has at least one non-trivial non-negative solution.*

Finally, we discuss the question of the positivity of solutions of problem (10), where $0 < \lambda < 1 < \mu$. We start with the following proposition, which provides a sufficient condition guaranteeing that any non-trivial sign-constant solution of problem (10) has no zero, i. e., it is either positive or negative.

**Proposition 2.** *Let $p \in \operatorname{Int}\mathcal{D}(\omega)$. Then there exists $\varrho > 0$ such that for any $\lambda \in ]0,1[$, $\mu > 1$, and $h, f \in L([0,\omega])$ satisfying conditions* (12) *and*

$$\Big(\frac{\omega}{4}\Big)^{\frac{\mu-1}{1-\lambda}} \mathrm{e}^{\frac{\omega(\mu-1)}{8(1-\lambda)}\,\|[p]_+\|_L}\, \|h\|_L^{\frac{\mu-1}{1-\lambda}} \|f\|_L \leq \varrho, \tag{13}$$

*any non-trivial non-negative solution of problem* (10) *is positive.*

In some particular cases, the number $\varrho$ appearing in Proposition 2 can be estimated from below. For example, the following statement holds.

**Corollary 5.** *Let $0 < \lambda < 1 < \mu$, condition* (12) *hold, and*

$$\|[p]_-\|_L < \frac{4}{\omega}\,,$$
$$\Big(\frac{\omega}{4}\Big)^{\frac{\mu-1}{1-\lambda}} \mathrm{e}^{\frac{\omega(\mu-1)}{8(1-\lambda)}\,\|[p]_+\|_L}\, \|h\|_L^{\frac{\mu-1}{1-\lambda}} \|f\|_L \leq \frac{4}{\omega} - \|[p]_-\|_L\,. \tag{14}$$

*Then problem* (10) *has at least one positive solution. Moreover, every non-trivial non-negative solution of problem* (10) *is positive.*

The assertion of the previous corollary remains true if $p \in \mathcal{V}^+(\omega)$ and the point-wise condition (15) is satisfied instead of (14).

**Corollary 6.** *Let $0 < \lambda < 1 < \mu$, $p \in \mathcal{V}^+(\omega)$, condition* (12) *hold, and*

$$\Big(\frac{\omega}{4}\,\|h\|_L\, \mathrm{e}^{\frac{\omega}{8}\,\|[p]_+\|_L}\Big)^{\frac{\mu-\lambda}{1-\lambda}} f(t) \leq h(t) \quad \text{for a.e. } t \in [0,\omega]. \tag{15}$$

*Then problem* (10) *has at least one positive solution. Moreover, every non-trivial non-negative solution of problem* (10) *is positive.*

**Remark 1.** The inclusion $p \in \mathcal{V}^+(\omega)$ holds, for example, if

$$\|[p]_+\|_L \leq \|[p]_-\|_L \leq \frac{4}{\omega}\,, \quad p \not\equiv 0.$$

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# Some Properties of Minimal Malkin Estimates

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Consider the linear differential system

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \tag{1}$$

with a bounded piecewise continuous coefficient matrix $A$ and the Cauchy matrix $X_A$. Suppose that $\|A(t)\| \leq a < +\infty$ for all $t \geq 0$. In [8], see also [9, p. 379] and [1, p. 236], I. G. Malkin has used estimations of the form

$$\|X_A(t,s)\| \leq D \exp(\alpha(t-s) + \beta s), \;\; t \geq s \geq 0, \;\; D > 0, \;\; \alpha, \beta \in \mathbb{R}, \tag{2}$$

in order to investigate asymptotic stability of the trivial solution to a system

$$\dot{y} = A(t)y + f(t,y), \;\; y \in \mathbb{R}^n, \;\; t \geq 0,$$

with a nonlinear perturbation $f(t,y)$ of a higher order. An ordered pair $(\alpha, \beta) \in \mathbb{R}^2$ is called a Malkin estimation for system (1) if there exists a number $D = D(\alpha, \beta) > 0$ such that (2) holds. We denote the set of all Malkin estimations for system (1) by $E(A)$.

A pair $(\alpha, \beta) \in \mathbb{R}^2$ is said to be a minimal Malkin estimation [7] if $(\alpha + \xi, \beta + \eta) \in E(A)$ for all $\xi > 0$, $\eta > 0$, and $(\alpha + \xi, \beta + \eta) \notin E(A)$ for all $\xi \leq 0$, $\eta \leq 0$, $\xi^2 + \eta^2 \neq 0$. Note that a minimal Malkin estimation is not necessarily an element of $E(A)$ by definition; an example is given below. On the other hand, if $(\alpha, \beta) \in E(A)$ and numbers $\xi$ and $\eta$ are nonnegative, then the pair $(\alpha + \xi, \beta + \eta)$ satisfies inequality (2) with the same $D = D(\alpha, \beta)$ since $t \geq s \geq 0$, i.e. the inclusion $(\alpha + \xi, \beta + \eta) \in E(A)$ is now valid.

We denote the set of all minimal Malkin estimations for system (1) by $M(A)$.

It can be easily seen that the set of minimal Malkin estimations for system (1) coincides with the set of Grudo characteristic vectors [2] for the function $\|X_A(t,s)\|$ with respect to the cone $C = \{(t,s) \in \mathbb{R}^2 : \; t \geq s \geq 0\}$. Using this fact and the results of [2] we can give [7] another description for the set $M(A)$. Let $K = \{(\alpha, \beta) \in \mathbb{R}^2 : \; \alpha > 0, \; \beta > 0\}$ be the positive cone of $\mathbb{R}^2$ and $\preccurlyeq$ be the partial order in $\mathbb{R}^2$ corresponding to $K$. Then $M(A)$ coincides with the set of all minimal with respect to $\preccurlyeq$ elements of $\operatorname{cl} E(A)$, where cl is the operator of closure.

The invariant uniform exponent $\iota[x]$ of a nonzero solution $x$ to system (1) is the number $\sup N(x)$, where the set $N(x)$ consists of all numbers

$$\varlimsup_{k \to +\infty} \frac{1}{(t_k - s_k)} \ln \frac{\|x(t_k)\|}{\|x(s_k)\|}$$

such that the sequence of pairs $\tau_k = (t_k, s_k) \in \mathbb{R}^2$, $t_k \geq s_k \geq 0$, $k \in \mathbb{N}$, satisfy the condition $\inf\limits_k s_k^{-1} t_k > 1$ and $t_k - s_k \to +\infty$ as $k \to +\infty$.

The invariant general exponent $\mathrm{I}_0(A)$ for system (1) is the number

$$\mathrm{I}_0(A) = \sup_{\theta > 0} \varlimsup_{s \to +\infty} \frac{1}{(\theta - 1)s} \ln \|X_A(\theta s, s)\|. \tag{3}$$

These two exponents are invariant with respect to generalized Lyapunov transformations [3], whereas the analogous Bohl uniform and general exponents are not invariant.

There exists an alternative characterization for $\mathrm{I}_0(A)$ given in [7]. Namely, $\mathrm{I}_0(A)$ is the first component of a unique pair $(\alpha, 0) \in M(A)$. It should be stressed that the pair $(\mathrm{I}_0(A), 0)$ is always in $M(A)$, but the inclusion $(\mathrm{I}_0(A), 0) \in E(A)$ is not valid in general. Indeed, according to [1, p. 109], [4, p. 68], and [5, p. 63] for any $\varepsilon > 0$ we have

$$\|X_A(t,s)\| \le D_\varepsilon \exp\big((\Omega_0(A)+\varepsilon)(t-s)\big) \tag{4}$$

with some $D_\varepsilon > 0$, where

$$\Omega_0(A) = \lim_{T\to+\infty} \overline{\lim_{k\to\infty}}\, T^{-1} \ln \big\|X_A(kT, kT-T)\big\| \tag{5}$$

is the general exponent of system (1). A similar estimation

$$\|X_A(t,s)\| \le D_\varepsilon \exp(\alpha(t-s)) \tag{6}$$

with $\alpha < \Omega_0(A)$ is not possible at all. Thus, $(\Omega_0(A)+\varepsilon, 0) \in E(A)$ for each $\varepsilon > 0$ and there are no pairs $(\alpha, 0) \in E(A)$ with $\alpha < \Omega_0(A)$. On the other hand, from (3) and (5) we can assert that the inequality $\Omega_0(A) \ge \mathrm{I}_0(A)$ is always valid and that $\Omega_0(A) > \mathrm{I}_0(A)$ in general. Thereby $(\mathrm{I}_0(A), 0) \notin E(A)$ in general too.

It was proved in [7] that the invariant general exponent $\mathrm{I}_0(A)$ is the attainable upper bound for invariant uniform exponents under exponentially small perturbations. Our aim is to obtain some similar interpretation for all elements of $M(A)$. To this end, we first obtain some alternative formulas for $\mathrm{I}_0(A)$ and $\iota[x]$.

**Proposition 1.** *For any system* (1) *the equalities*

$$\mathrm{I}_0(A) = \lim_{\theta\to 1+0} \overline{\lim_{s\to+\infty}} \frac{1}{(\theta-1)s} \ln\|X_A(\theta s, s)\| = \lim_{\theta\to 1+0} \overline{\lim_{k\to\infty}} \frac{1}{(\theta-1)\theta^k} \ln\big\|X_A(\theta^{k+1}, \theta^k)\big\|$$

*hold.*

*Proof.* Let

$$R(\theta,s) = \frac{1}{(\theta-1)s} \ln\|X_A(\theta s, s)\|, \quad R(\theta) = \overline{\lim_{k\to\infty}} R(\theta, \theta^k), \quad \underline{\mathrm{I}} = \underline{\lim}_{\theta\to 1+0} R(\theta).$$

Take any $\varepsilon > 0$, $\theta > 1$ and put $\vartheta = 1 + \varepsilon a^{-1}(\theta-1)/(\theta+1)$. By definition of lower limit, for any $\varepsilon > 0$ and $\vartheta > 1$ there exists a number $\theta_\varepsilon \in ]1, \vartheta]$ such that the inequality $R(\theta_\varepsilon) < \underline{\mathrm{I}} + \varepsilon$ holds. Then by definition of upper limit, for the same $\varepsilon > 0$ there exists a number $N_\varepsilon \in \mathbb{N}$ such that the inequality

$$R(\theta_\varepsilon, \theta_\varepsilon^j) < \overline{\lim_{j\to\infty}} R(\theta_\varepsilon, \theta_\varepsilon^j) + \varepsilon < \underline{\mathrm{I}} + 2\varepsilon$$

is valid for each $j > N_\varepsilon$.

Take any $s > \theta_\varepsilon^{N_\varepsilon}$ and find numbers $p, q \in \mathbb{N}$ such that $s \in [\theta_\varepsilon^p, \theta_\varepsilon^{p-1}[$ and $\theta s \in [\theta_\varepsilon^{q+2}, \theta_\varepsilon^{q+1}[$. Then we have

$$\theta_\varepsilon^p - s \le \theta_\varepsilon^p - \theta_\varepsilon^{p-1} = \theta_\varepsilon^{p-1}(\theta_\varepsilon - 1) \le (\theta_\varepsilon - 1)s,$$
$$\theta s - \theta_\varepsilon^{q+1} \le \theta_\varepsilon^{q+2} - \theta_\varepsilon^{q+1} = \theta_\varepsilon^{q+1}(\theta_\varepsilon - 1) \le (\theta_\varepsilon - 1)\theta s,$$

and

$$(\theta-1)sR(\theta,s)\le \ln\|X(\theta s,\theta_\varepsilon^{q+1})\|+\ln\|X(\theta_\varepsilon^p,s)\|+\sum_{j=p}^{q}\ln\|X(\theta_\varepsilon^{j+1},\theta_\varepsilon^j)\|$$
$$\le a(\theta s-\theta_\varepsilon^{q+1}+\theta_\varepsilon^p-s)+\sum_{j=p}^{q}(\theta_\varepsilon^{j+1}-\theta_\varepsilon^j)R(\theta_\varepsilon,\theta_\varepsilon^j)$$
$$\le as(\theta+1)(\theta_\varepsilon-1)+(\theta_\varepsilon^{q+1}-\theta_\varepsilon^p)\max_{q\le j\le p}R(\theta_\varepsilon,\theta_\varepsilon^j)\le as(\theta+1)(\vartheta-1)+(\theta-1)s\max_{q\le j\le p}R(\theta_\varepsilon,\theta_\varepsilon^j).$$

By the above assumptions we have

$$R(\theta,s)\le a(\theta+1)(\vartheta-1)/(\theta-1)+\max_{q\le j\le p}R(\theta_\varepsilon,\theta_\varepsilon^j)\le \max_{j\ge N_\varepsilon}R(\theta_\varepsilon,\theta_\varepsilon^j)+\varepsilon\le \underline{\mathrm{I}}+3\varepsilon,$$

for all $\varepsilon>0$ and $\theta>1$ and all sufficiently large $s$. Hence, the relation $\widetilde{R}(\theta):=\overline{\lim\limits_{s\to\infty}}R(\theta,s)\le\underline{\mathrm{I}}$ is valid for each $\theta>1$. Now, we obtain

$$\mathrm{I}_0:=\sup_{\theta>1}\widetilde{R}(\theta)\le\underline{\mathrm{I}}\ \text{ and }\ \overline{\lim_{\theta\to1+0}}\widetilde{R}(\theta)\le\underline{\mathrm{I}}.$$

On the other hand, $\underline{\lim\limits_{\theta\to1+0}}\widetilde{R}(\theta)\ge\overline{\lim\limits_{\theta\to1+0}}R(\theta)=\underline{\mathrm{I}}$, since $\widetilde{R}(\theta)\ge R(\theta)$. Thus,

$$\underline{\lim_{\theta\to1+0}}\widetilde{R}(\theta)\ge\underline{\mathrm{I}}\ge\overline{\lim_{\theta\to1+0}}\widetilde{R}(\theta)$$

and therefore the limit $\lim\limits_{\theta\to1+0}\widetilde{R}(\theta)=\underline{\mathrm{I}}\ge\mathrm{I}_0$ exists. Since the last inequality is possible only as an equality, we have the required assertion. □

**Remark.** The above proof essentially follows from the well-known scheme of the similar proof for general exponent, see [1, p. 110], [4, p. 67], or [5, p. 61].

**Proposition 2.** *For any nonzero solution $x$ to system* (1) *the following equalities*

$$\iota[x]=\sup_{\theta>0}\overline{\lim_{s\to+\infty}}\frac{1}{(\theta-1)s}\ln\frac{\|x(\theta s)\|}{\|x(s)\|}=\lim_{\theta\to1+0}\overline{\lim_{s\to+\infty}}\frac{1}{(\theta-1)s}\ln\frac{\|x(\theta s)\|}{\|x(s)\|}$$
$$=\lim_{\theta\to1+0}\overline{\lim_{k\to\infty}}\frac{1}{(\theta-1)\theta^k}\ln\frac{\|x(\theta^{k+1})\|}{\|x(\theta^k)\|}$$

*are valid.*

To prove Proposition 2, we use some theorems from [11] concerning the growth of $x$ instead of standard estimates for the Cauchy matrix used in the proof of Proposition 1, but the rest of the proof is rather analogous to previous one.

**Definition.** The number

$$\iota_\theta[x]:=\overline{\lim_{s\to+\infty}}\frac{1}{(\theta-1)s}\ln\frac{\|x(\theta s)\|}{\|x(s)\|}$$

is called the $\theta$-uniform exponent of a nonzero solution $x$ to system (1).

Together with original system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \;\; y \in \mathbb{R}^n, \;\; t \geq 0, \tag{7}$$

with piecewise continuous bounded perturbation matrix $Q$. Let $\mathfrak{R}_\sigma$ be the set of all piecewise continuous bounded perturbations $Q$ such that

$$\lambda[Q] = \varlimsup_{t \to +\infty} t^{-1} \ln \|Q(t)\| < -\sigma, \;\; \sigma \in \mathbb{R}.$$

Put

$$\mathrm{i}_\theta(A+Q) = \sup_y \iota_\theta[y],$$

where the supremum is taken over all non-trivial solutions of system (7).

**Theorem.** *For any* $(\alpha, \beta) \in M(A)$*, there exists a number* $\theta > 1$ *such that*

$$\alpha = \sup \big\{ \mathrm{i}_\theta(A+Q) : \; Q \in \mathfrak{R}_\beta \big\}.$$

The proof is based on Millionshchikov's rotation method [10], [3], [5, p. 75].

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# An Estimate for Solutions to a Uniformly Charged Functional Differential Equation with Full Memory

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## 1 Introduction

Here we consider a class of functional differential systems that arises under attempts to reduce functional differential systems with continuous and discrete times [3] to equations with only continuous time having in mind to apply some results from the theory of functional differential equations [2]. First we recall the description of a class of continuous-discrete functional differential equations with linear Volterra operators and appropriate spaces where those are considered. Then a continuous-discrete system is reduced to a continuous system that turns out to be a charged functional differential system with a full memory. For this system, an estimate of solutions, which can be useful for analysis of their properties, is obtained.

## 2 Preliminaries

To describe the continuous subsystem, let us introduce the linear operator $\mathcal{L}$ :

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int_0^t K(t,s)\dot{x}(s)\,ds + A(t)x(0), \quad t \in [0,T]. \tag{1}$$

Here the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $\{(t,s) :\ 0 \leqslant s \leqslant t < \infty\}$ and such that

$$|k_{ij}(t,s)| \leqslant \kappa(t), \quad i,j = 1,\dots,n,$$

where function $\kappa$ is summable on $[0,T]$ for any finite $T > 0$, the elements $(n \times n)$-matrix $A$ are summable on $[0,T]$ for any finite $T > 0$. By $AC^n[0,T]$ we denote the space of absolutely continuous functions $x : [0,T] \to R^n$, $L^n[0,T]$ denotes the space of functions Lebesgue summable on $z : [0,T] \to R^n$,

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}, \quad \|z\|_{L^n} = \int_0^T |z(t)|\,dt,$$

where $|\alpha| = \max\limits_{i=1,\dots,n} |\alpha_i|$ for $\alpha = col(\alpha_1,\dots,\alpha_n) \in R^n$ (we reserve $\|\cdot\|$ for the corresponding norm in $R^n$). The operator $\mathcal{L} : AC^n[0,T] \to L^n[0,T]$ is bounded. The theory of equation $\mathcal{L}x = f$ is thorouhgly treated in [2, 6]. The equation $\mathcal{L}x = f$ covers differential equations with concentrated and/or distributed delay and integrodifferential Volterra equations. The Cauchy problem

$$\mathcal{L}x = f, \quad x(0) = \alpha$$

is uniquely solvable for any $f \in L^n[0,T]$ and $\alpha \in R^n$ and its solution has the representation

$$x(t) = X(t)\alpha + \int\limits_0^t C_1(t,s)f(s)\,ds,$$

where $X(\,\cdot\,)$ is the fundamental matrix, $C_1(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix [5].

For description of the discrete subsystem, we introduce the operator $\Lambda$:

$$(\Lambda y)(t_i) = y(t_i) - \sum_{j<i} B_{ij} y(t_j), \ \ i = 1,2,\dots,\mu, \ \ 0 = t_0 < t_1 < \dots < t_\mu = T.$$

Here $B_{ij}$ are constant $(\nu\times\nu)$-matrices. Denote $J = \{t_0, t_1, \dots, t_\mu\}$, $FD^\nu(\mu)$ is the space of functions $y : J \to R^\nu$ normed by $\|y\|_{FD^\nu(\mu)} = \sum\limits_{i=0}^{\mu} |y(t_i)|$. Recall some facts on equation $\Lambda y = g$ (see, for instance, [1]). The Cauchy problem

$$\Lambda y = g, \quad y(0) = \beta$$

is uniquely solvable for any $g \in FD^\nu(\mu)$ $\beta \in R^\nu$ and its solution has the form

$$y(t_i) = Y(t_i)\beta + \sum_{j\leqslant i} C_2(i,j)g(t_j), \ \ i = 1,2,\dots,\mu, \tag{2}$$

where $Y(\,\cdot\,)$ is the fundamental matrix, $C_2(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

Consider the system

$$(\mathcal{L}x)(t) = \sum_{j:\ t_j<t} U_j(t)y(t_j) + f(t), \ \ t \in [0,T], \tag{3}$$

$$(\Lambda y)(t_i) = \sum_{j: t_j<t_i} A_{ij}x(t_j) + g(t_i), \ \ i = 1,2,\dots,\mu, \tag{4}$$

that consists of subsystem (3) with continuous time and subsystem (4) with discrete time. Here $A_{ij}$ are constant matrices of dimension $\nu \times n$, $U_j$ are $(n \times \nu)$-matrices with summable elements. The subsystems are connected between each other with respect their states.

## 3 A charged functional differential system

To reduce system (3), (4) to an equation with respect to $x(\,\cdot\,)$, we solve (4) with respect to $y(\,\cdot\,)$ by means of (2):

$$y(t_i) = Y(t_i)y(t_0) + \sum_{j\leqslant i} C_2(i,j)\Big(\sum_{j:\ t_\ell<t_j} A_{j\ell}x(t_\ell)\Big) + \sum_{j\leqslant i} C_2(i,j)g(t_j), \ \ i = 1,2,\dots,\mu,$$

and then substitute the right-hand side of the latter into (3). After immediate calculations subsystem (3) can be rewritten in the form of a charged (by the terms $V_j(t)x(t_j)$) functional differential equation

$$(\mathcal{L}x)(t) = \sum_{j: t_j<t} V_j(t)x(t_j) + r(t), \ \ t \in [0,T].$$

In the sequel, we consider this equation in the case $t_j = j$ and assume that $T$ is as great as we wish:

$$(\mathcal{L}x)(t) = \sum_{j<t} V_j(t)x(j) + r(t), \ \ t \in [0,\infty). \tag{5}$$

Our aim is to obtain an estimate of solutions to (5). We derive this estimate on the base of the following Lemma that is a kind of the Gronwall-Bellman inequality.

**Lemma.** *Let $p(j)$, $q(j)$, $\upsilon(j)$, $z(j)$, $j=0,1,2,\dots$ be nonnegative sequences such that*

$$z(j)\le \upsilon(j)+p(j)\sum_{k=0}^{j-1}q(k)z(k),\quad k=1,2,\dots,\quad z(0)\le \upsilon(0). \tag{6}$$

*Then the estimate*

$$z(j)\le \upsilon(j)+p(j)\sum_{\ell=0}^{j-1}M_{j\ell}\,q(\ell)\upsilon(l),\quad j=1,2,\dots, \tag{7}$$

*where*

$$M_{j\ell}=\exp\Big(\sum_{i=\ell}^{j-1}p(i)q(i)\Big),$$

*holds.*

**Remark.** Let us note that, as to compare with the traditional version of (6), where $\upsilon(j)=cp(j)$, $c>0$ and the estimate has the form

$$z(j)\le cp(j)\prod_{\ell=0}^{j-1}\big(1+p(\ell)q(\ell)\big) \tag{8}$$

(see, for instance, Corollary of Lemma 1.1 [4]), the estimate (7) can be much more sharp. Really, put $\upsilon(j)=1+1/(1+j)$; $p(j)=1/(1+j)$; $q(j)=1/(1+j)^2$. By means of (7) we obtain $z(100)\le 1.1$, whereas (8) gives $z(100)\le 6.5$.

Denote

$$d_j=X(j)x_0+\int_0^j C_1(j,s)r(s)\,ds,\quad D_{jk}=\int_k^j C_1(j,s)V_k(s)\,ds.$$

**Theorem.** *Let the following inequalities take place:*

$$|d_j|\le \upsilon(j),\quad \|D_{jk}\|\le p(j)q(k),\quad j,k=1,2,\dots,\quad k\le j,$$

*where $\upsilon(j)$, $p(j)$, $q(j)$, $j=1,2,\dots$ are nonnegative sequences. Then the estimate* (7) *holds for $z(j)=|x(j)|$.*

*Proof.* First we use the representation of solutions to (1) as applied to (5):

$$x(t)=X(t)x_0+\int_0^t C_1(t,s)r(s)\,ds+\int_0^t C_1(t,s)\sum_{k<s}V_k(s)x(k)\,ds,\quad t\in[0,T].$$

Thus, for sections $x(j)$, we have the system

$$x(j)=X(j)x_0+\int_0^j C_1(j,s)r(s)\,ds+\int_0^j C_1(j,s)\sum_{k<s}V_k(s)x(k)\,ds. \tag{9}$$

Next note that the expression

$$\int_0^j C_1(j,s)\sum_{k<s}V_k(s)x(k)\,ds$$

can be written in the form

$$\sum_{k<j} D_{jk}x(k).$$

This follows from the immediate calculations. Denote

$$w(j) = X(j)x_0 + \int_0^j C_1(j,s)r(s)\,ds$$

and rewrite (9) in the form

$$x(t_j) = w(t_j) + \sum_{k<j} D_{jk}x(k). \tag{10}$$

To complete the proof, it remains to apply Lemma to the inequality

$$|x(j)| \le |w(j)| + \sum_{k<j} \|D_{jk}\|\,|x(k)|,$$

which follows from (10). □

This Theorem makes it possible to take into account asymptotic properties of the Cauchy matrix, the coefficients $V_j(t)$ as weights of the charges $x(j)$, and the free term $r(t)$ in (5) to answer questions about asymptotic behaviour of solutions. Here we restrict ourselves by the following example.

**Example.** Consider the linear charged differential equation

$$\dot{x}(t) + 2tx(t) = \sum_{j<t} v_j(t)x(j) + r(t), \quad t \in [0,\infty),$$

where $|v_j(t)| \le c\,\frac{1}{(1+j)^2}$ . For this equation, the solution $x(t)$ with the initial condition $x(0) = x_0$ is bounded on $[0,\infty)$ for any $r(t)$ such that the inequality $|r(t)| \le d(1+t)$ holds with a $d > 0$ almost everywhere on $[0,\infty)$, and the estimate

$$|x(j)| \le \Big(e^{-j^2} + \frac{11}{10}\,\frac{ce^{\frac{11}{5}c}}{\frac{e}{4}+j}\Big)|x_0| + \frac{3}{2}\Big(1 + \frac{2ce^{\frac{11}{5}c}}{\frac{e}{4}+j}\Big)d, \quad j = 1,2,\dots$$

holds.

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# Approximation of the Optimal Control Problem on an Interval with a Family of Optimization Problems on time Scales

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This work is devoted to the study of the limiting behavior of the optimal control problem for dynamic equations defined on a family of time scales $\mathbb{T}_\lambda$, in the regime when the graininess function $\mu_\lambda$ converges to zero as $\lambda \to 0$. At the same time the segment of the time scale $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ approaches $[t_0, t_1]$ e.g. in the Hausdorff metric. The natural question that arises is how the optimal control problem on the time scale is related to the corresponding control problem on the interval $[t_0, t_1]$.

The time scales theory was introduced by S. Hilger [6] (1988) as a unified theory for both discrete and continuous analysis. For reader's convenience, we present several notions from this theory which are used in this paper.

Time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$, $A_\mathbb{T} := A \cap \mathbb{T}$ for $A \subset \mathbb{R}$, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$ is the forward jump operator, $\rho : \mathbb{T} \to \mathbb{T}$, $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$ is the backward jump operator (here $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$), $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$ is called the graininess function. A point $t \in \mathbb{T}$ is called left-dense (LD) (left-scattered (LS), right-dense (RD) or right-scattered (RR)) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$), $\mathbb{T}^k := \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum $M$, $\mathbb{T}^k := \mathbb{T}$ otherwise.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is called $\Delta$-differentiable at $t \in \mathbb{T}^k$ if the limit

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$.

Let $\Lambda \subset \mathbb{R}$, such that 0 is a limit point of $\Lambda$, be the set of indices. Consider the family of time scales $\mathbb{T}_\lambda, \lambda \in \Lambda$ such that $\sup \mathbb{T}_\lambda = \infty$. For any $t_0, t_1 \in \mathbb{T}_\lambda$ denote $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ and $\mu_\lambda = \sup_{t \in [t_0, t_1]_{\mathbb{T}_\lambda}} \mu(t)$. Assume

$$\mu_\lambda(t) \to 0 \ \text{ as } \ \lambda \to 0. \tag{1}$$

For every $\mathbb{T}_\lambda$ consider the optimal control problem on the time scale $[t_0, t_1]_{\mathbb{T}_\lambda}$:

$$\begin{cases} x^\Delta = f(t, x, u), \\ x(t_0) = x, \\ J_\lambda(u) = \displaystyle\int\limits_{[t_0, t_1)_{\mathbb{T}_\lambda}} L(t, x(t), u(t))\, \Delta t + \Psi(x(t_1)) \longrightarrow \inf, \ \ u \in \mathcal{U}(t_0). \end{cases} \tag{2}$$

Along with (2), consider the corresponding continuous optimal control problem on the interval $[t_0, t_1]$:

$$\begin{cases} \dfrac{dx(t)}{dt} = f\big(t, x(t), u(t)\big), \\ x(t_0) = x, \\ J(u) = \displaystyle\int\limits_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt + \Psi(x(t_1)) \longrightarrow \inf, \quad u \in \mathcal{U}(t_0), \end{cases} \tag{3}$$

where $x \in \mathbb{R}^d$, $u \in U \subset \mathbb{R}^m$, $U$ – compact set, $\mathcal{U}(t_0) := L^\infty([t_0, t_1]_\mathbb{T}, U)$, i.e. the set of bounded, $\Delta$ – measurable functions [2, Chapter 5.7] defined on $[t_0, t_1]_\mathbb{T}$ and taking values in $U$ for each $t \in [t_0, t_1]_\mathbb{T}$, is called the set of admissible controls.

Assume that $f$, $L$ and $\Psi$ satisfy

(i) $f : [t_0, t_1]_\mathbb{T} \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $L : [t_0, t_1]_\mathbb{T} \times \mathbb{R}^d \times U \to \mathbb{R}^1$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^1$;

(ii) $f$ is continuous and globally Lipschitz in $x$ with the Lipschitz constant $K$;

(iii) $L$ and $\Psi$ are continuous and globally Lipschitz in $x$ with the Lipschitz constant $K$.

The Bellman function in this case is

$$V(t_0, x) := \inf_{u(\,\cdot\,) \in \mathcal{U}(t_0)} J(t_0, x, u). \tag{4}$$

Denote by $V_\lambda(t_0, x)$ and $V(t_0, x)$ the corresponding Bellman functions for these problems, given by (4). Our main result is the following theorem.

**Theorem 1.** *Let $\mathbb{T}_\lambda$ be such that (1) holds. In addition, assume that*

1) *The functions $f$, $f_x$ and $L$ are continuous on $[t_0, t_1] \times \mathbb{R}^d \times U$;*

2) *$f$ and $L$ are globally Lipschitz in $x$, with Lipschitz constant $K > 0$.*

*Then*

$$V_\lambda(t_0, \,\cdot\,) \to V(t_0, \,\cdot\,) \ \ in \ \ C_{loc}(\mathbb{R}^d), \ \ \lambda \to 0.$$

The proof of the main result will heavily rely on two lemmas.

Without loss of generality, we assume that $t_0 = 0$ and $t_1 = 1$. Consider an arbitrary time scale $\mathbb{T}_\lambda$ and an arbitrary admissible control $u_\lambda(t)$ on it. Let $x_\lambda(t)$ be a corresponding admissible trajectory. Denote by $\widetilde{u}_\lambda(t)$ the extension of $u_\lambda(t)$ to the entire interval $[0, 1]$:

$$\widetilde{u}_\lambda(t) := \begin{cases} u_\lambda(t), & t \in [0, 1]_{\mathbb{T}_\lambda}, \\ u_\lambda(r), & t \in [r, \sigma(r)), \ \ r \in \mathrm{RS}. \end{cases} \tag{5}$$

This control is admissible for the problem (3).

**Lemma 1.** *Let $x(t)$ be a solution of*

$$\begin{cases} \dfrac{dx}{dt} = f\big(t, x, \widetilde{u}_\lambda(t)\big), \\ x(0) = x_0. \end{cases}$$

*Then*

$$\Bigg| \int\limits_{[0,1)_{\mathbb{T}_\lambda}} L\big(t, x_\lambda(t), u_\lambda(t)\big)\, \Delta t - \int\limits_0^1 L\big(t, x(t), \widetilde{u}_\lambda(t)\big)\, dt \Bigg| \longrightarrow 0, \ \ \lambda \to 0.$$

Let $u^\lambda_{ts}(\,\cdot\,)$ be an arbitrary admissible control for the problem (2) and $x^\lambda_{ts}(\,\cdot\,)$ be the corresponding trajectory. Similarly, let $x(\,\cdot\,)$ be an admissible trajectory of the problem (3) which corresponds to the admissible control $u(\,\cdot\,)$.

**Lemma 2.** *For any admissible control $u(\,\cdot\,)$ for the problem* (3) *and for every time scale $\mathbb{T}_\lambda$, there is an admissible control $u^\lambda_{ts}(\,\cdot\,)$ for the problem* (2) *such that*

$$\big|J(u) - J_\lambda(u^\lambda_{ts})\big| \longrightarrow 0, \quad \lambda \to 0.$$

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# Non-Oscillation Criteria for Two-Dimensional System of Nonlinear Ordinary Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$ , we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $p, g : \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions such that

$$g(t) \geq 0 \quad \text{for a.e. } t \geq 0. \tag{2}$$

By a solution of system (1) on the interval $J \subseteq [0, +\infty[$ we understand a pair $(u, v)$ of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in $J$ and satisfy equalities (1) almost everywhere in $J$.

**Definition 1.** A solution $(u, v)$ of system (1) is called *non-trivial* if $|u(t)| + |v(t)| \neq 0$ for $t \geq 0$. We say that a non-trivial solution $(u, v)$ of system (1) is *non-oscillatory* if at least one of its component does not have a sequence of zeros tending to infinity.

**Remark 2.** It was proved by Mirzov in [11] that all non-extendable solutions of system (1) are defined on the whole interval $[0, +\infty[$ . Therefore, when we are speaking about a solution of system (1), we assume that it is defined on $[0, +\infty[$ . Moreover, in [11, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1) if the function $g$ is nonnegative. Especially, under assumption (2), if system (1) has a non-oscillatory solution, then any other its non-trivial solution is also non-oscillatory. Consequently, it is possible to introduce the following definition.

**Definition 3.** We say that system (1) is *non-oscillatory* if all its non-trivial solutions are non-oscillatory.

Oscillation and non-oscillation theory for ordinary differential equations and their systems is a widely studied topic of the qualitative theory of differential equation. Below presented results are closely related to those which are established in [1, 2, 4–10, 12, 13]. Some criteria stated in these papers are generalized below.

Indeed, one can see that system (1) is a generalization of the equation

$$u'' + \frac{1}{\alpha}\, p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u = 0, \tag{3}$$

where $\alpha \in ]0, 1]$ and $p : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. This equation is studied in the existing literature and some oscillation and non-oscillation criteria for equation (3) can be found, e.g., in [5, 8].

Moreover, many results (see, e.g., survey given in [2]) are known in the non-oscillation theory for the so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0, \tag{4}$$

where $q > 1$, $p, r : [0, +\infty[\, \to \mathbb{R}$ are continuous and $r$ is positive. It is clear that (4) is a particular case of system (1). Indeed, if the function $u$, with the properties $u \in C^1$ and $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$, is a solution of equation (4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution of system (1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q - 1$.

However, there are some restrictions on functions $p$ and $g$ in the above-mentioned papers. It is usually assumed that $p(t) \geq 0$ or $\int\limits_0^t p(s)\,ds > 0$ for large $t$. Moreover, the coefficient $g(t) := r^{\frac{1}{1-q}}(t)$ of the half-linear equation (4) cannot have zero points in any neighbourhood of infinity. Below we formulate criteria without these additional assumptions.

We consider two different cases, when the coefficient $g$ is non-integrable and integrable on the half-line.

## a) The case $\int\limits_0^{+\infty} g(s)\,ds = +\infty$

At first, we assume that

$$\int\limits_0^{+\infty} g(s)\,ds = +\infty, \tag{5}$$

and we put

$$f(t) := \int\limits_0^t g(t)\,ds \quad \text{for } \ t \geq 0.$$

In view of assumptions (2) and (5), there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in the behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \quad \text{for } \ t > 0$$

and, moreover,

$$\lim_{t \to +\infty} f(t) = +\infty.$$

We put

$$c_\alpha(t) := \frac{\alpha}{f^\alpha(t)} \int\limits_0^t \frac{g(s)}{f^{1-\alpha}(s)} \bigg( \int\limits_0^s p(\xi)\,d\xi \bigg)\,ds \quad \text{for } \ t > 0.$$

It is known (see [3, Corollary 2.5 (with $\nu = 1 - \alpha$)]) that if a finite limit of the function $c_\alpha(t)$ does not exist and $\liminf\limits_{t \to +\infty} c_\alpha(t) > -\infty$, then system (1) is oscillatory. Consequetly, in what follows it is natural to assume that

$$\lim_{t \to +\infty} c_\alpha(t) =: c_\alpha^* \in \mathbb{R}. \tag{6}$$

We put

$$Q(t;\alpha) := f^\alpha(t)\bigg(c_\alpha^* - \int\limits_0^t p(s)\,ds\bigg) \quad \text{for } \ t > 0,$$

where the number $c^*_\alpha$ is given by (6). Moreover, we denote lower and upper limits of the function $Q(\,\cdot\,;\alpha)$ as follows

$$Q_*(\alpha) := \liminf_{t\to+\infty} Q(t;\alpha), \quad Q^*(\alpha) := \limsup_{t\to+\infty} Q(t;\alpha).$$

**Theorem 4.** *Let* (6) *hold. Let, moreover, the inequalities*

$$-\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha} < Q_*(\alpha) \quad \text{and} \quad Q^*(\alpha) < \frac{1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

*be satisfied. Then system* (1) *is nonoscillatory.*

We denote by $B(\xi)$ the greatest root of the equation

$$|x|^{\frac{\alpha}{\alpha+1}} + x + \xi = 0,$$

where $\xi \leq 0$. Now we can formulate the next theorem which complements the previous one in a certain sense.

**Theorem 5.** *Let* (6) *hold. Let, moreover, the inequalities*

$$-\infty < Q_*(\alpha) \leq -\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

*and*

$$Q^*(\alpha) < [B(Q_*(\alpha))]^{\frac{\alpha}{\alpha+1}} - B(Q_*(\alpha))$$

*be satisfied. Then system* (1) *is nonoscillatory.*

### b) The case $\int\limits_0^{+\infty} g(s)\,ds < +\infty$

Now we assume that the coefficient $g$ is integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,ds < +\infty.$$

Let

$$\widetilde{f}(t) := \int\limits_t^{+\infty} g(t)\,ds \quad \text{for } t \geq 0.$$

In view of assumptions (2) and (5), we have

$$\lim_{t\to+\infty} \widetilde{f}(t) = 0$$

and

$$\widetilde{f}(t) > 0 \quad \text{for } t \geq 0.$$

We put

$$\widetilde{c}_\alpha(t) := \widetilde{f}(t) \int\limits_0^t \frac{g(s)}{\widetilde{f}^2(s)} \bigg( \int\limits_0^s \widetilde{f}^{\alpha+1}(\xi) p(\xi)\,d\xi \bigg)\,ds \quad \text{for } t \geq 0.$$

According to [3, Corollary 2.11 (with $\nu = 1 - \alpha$)], the system (1) is oscillatory if function $\widetilde{c}_\alpha(t)$ does not have a finite limit and $\liminf\limits_{t\to+\infty} \widetilde{c}_\alpha(t) > -\infty$. Consequently, we assume that there exists a finite limit of the function $\widetilde{c}_\alpha$, i.e.,

$$\lim_{t\to+\infty} \widetilde{c}_\alpha(t) =: \widetilde{c}_\alpha^* \in \mathbb{R}.$$

We denote

$$\widetilde{Q}(t;\alpha) := \frac{1}{\widetilde{f}(t)} \bigg( \widetilde{c}_\alpha^* - \int_0^t \widetilde{f}^{\alpha+1}(s) p(s)\, ds \bigg) \quad \text{for } t > 0.$$

Moreover, we denote lower and upper limits of the functions $\widetilde{Q}(\,\cdot\,;\alpha)$ as follows

$$\widetilde{Q}_*(\alpha) := \liminf_{t\to+\infty} \widetilde{Q}(t;\alpha), \quad \widetilde{Q}^*(\alpha) := \limsup_{t\to+\infty} \widetilde{Q}(t;\alpha).$$

Now we formulate next nonoscilation criteria by using lower and upper limits of the function $\widetilde{Q}(t;\alpha)$. We denote by $\widetilde{A}(\nu)$ and $\widetilde{B}(\nu)$ the smallest and the greatest root of the equation

$$\alpha|x|^{\frac{\alpha+1}{\alpha}} + (\alpha+1)x + \nu = 0.$$

**Theorem 6.** *Let the inequalities*

$$\widetilde{A}(\nu) + \nu < \widetilde{Q}_*(\alpha) \quad \textit{and} \quad \widetilde{Q}^*(\alpha) < \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}$$

*be fulfilled with* $\nu = \frac{2\alpha+1}{\alpha+1}\big(\frac{\alpha}{1+\alpha}\big)^{1+\alpha}$. *Then system* (1) *is nonoscillatory.*

The following theorem complements previous one in a certain sense. Before we formulate it, we denote by $\widehat{B}(\eta)$ the greatest root of the equation

$$\alpha|x|^{\frac{\alpha+1}{\alpha}} - \alpha x + \eta = 0,$$

where $\eta < \big(\frac{\alpha}{\alpha+1}\big)^{\alpha+1}$.

**Theorem 7.** *Let the inequalities*

$$-\infty < \widetilde{Q}_*(\alpha) \le \widetilde{A}(\nu) + \nu$$

*with* $\nu = \frac{2\alpha+1}{\alpha+1}\big(\frac{\alpha}{1+\alpha}\big)^{1+\alpha}$, *and*

$$\widetilde{Q}^*(\alpha) < \widetilde{Q}_*(\alpha) + \widehat{B}(\widetilde{Q}_*(\alpha)) + \widetilde{B}\Big(\widetilde{Q}_*(\alpha) + \widehat{B}(\widetilde{Q}_*(\alpha))\Big)$$

*be satisfied. Then system* (1) *is nonoscillatory.*

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# The Cauchy–Nicoletti Weighted Problem for Nonlinear Singular Functional Differential Systems

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Let $-\infty < a < b < +\infty$, and let $J \subset [a,b]$ be the measurable set such that

$$J \neq [a,b], \ \ \operatorname{mes} J = b-a.$$

Consider the functional differential system with deviating arguments

$$\frac{du_i(t)}{dt} = f_i\big(t, u_1(t), \dots, u_n(t), u_1(\tau_1(t)), \dots, u_n(\tau_n(t))\big) \ \ (i=1,\dots,n) \tag{1}$$

with the weighted boundary conditions

$$\limsup_{t\to t_i} \frac{|u_i(t)|}{\varphi_i(t)} < +\infty \ \ (i=1,\dots,n). \tag{2}$$

Here $f_i : J \times \mathbb{R}^{2n} \to \mathbb{R}$ $(i=1,\dots,n)$ are measurable in the first and continuous in the last $2n$ arguments function,

$$t_i \in [a,b] \setminus J \ \ (i=1,\dots,n),$$

while $\varphi_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ and $\tau_i : J \to [a,b]$ $(i=1,\dots,n)$ are, respectively, absolutely continuous and continuous functions such that

$$\varphi_i(t) > 0 \quad \text{for } \ t \neq t_i \ \ (i=1,\dots,n),$$
$$\varphi_i'(t)(t-t_i) \geq 0, \quad \tau_i(t) \neq t_i \quad \text{for } \ t \in J \ \ (i=1,\dots,n).$$

A vector function $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ with absolutely continuous components $u_1,\dots,u_n$ is said to be **a solution of system** (1) if it satisfies that system almost everywhere on $J$. The solution $(u_i)_{i=1}^n$ of system (1) is said to be **a solution of problem** (1), (2) if it satisfies conditions (2).

Note that the boundary conditions

$$u_i(t_i) = 0 \ \ (i=1,\dots,n) \tag{3}$$

are called Cauchy–Nicoletti conditions, and problem (1), (3) is said to be a Cauchy–Nicoletti problem (see, e.g., [1–3, 5–8], where the Cauchy–Nicoletti problem is investigated both for differential and functional differential systems). Thus it is natural to call the boundary conditions (2) and problem (1), (2) the Cauchy–Nicoletti weighted conditions and the Cauchy–Nicoletti weighted problem, respectively.

We are interested in study of problem (1), (2) in the case where system (1) has non-integrable singularities in the time variable, i.e., where

$$\int_a^b \Big( \sum_{i=1}^n \big| f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \big| \Big)\, dt = +\infty \quad \text{if } \ \sum_{i=1}^n (|x_i| + |y_i|) > 0.$$

For singular systems of ordinary differential equations, the unimprovable conditions for the solvability and unique solvability of the Cauchy–Nicoletti weighted problem are established by I. Kiguradze [2, 4]. In this paper, analogous results are obtained for the singular problem (1), (2).

Below everywhere we use the following notation.

- $I_k = [a,b] \setminus \{t_k\}$ $(k = 1,\dots,n)$.
- $\chi_k(t,\delta,\lambda) = \begin{cases} 0 & \text{if } t \in [t_k-\delta, t_k+\delta], \\ \lambda & \text{if } t \notin [t_k-\delta, t_k+\delta]. \end{cases}$
- $L_{loc}(I_k;\mathbb{R})$ is the set of Lebesgue integrable on each closed interval contained in $I_k$ functions $v : I_k \to \mathbb{R}$.
- $X = (x_{ik})_{i,k=1}^n$ is the $n \times n$ matrix with the components $x_{ik}$ $(i,k = 1,\dots,n)$.
- $r(X)$ is the spectral radius of the matrix $X$.

Moreover, below everywhere it is assumed that

$$f^*_{\rho,k} \in L_{loc}(I_k;\mathbb{R}) \quad \text{for every} \ \ \rho > 0 \ \ (k = 1,\dots,n),$$

where

$$f^*_{\rho,k}(t) = \max\bigg\{\Big|f_k\Big(t, \varphi_1(t)x_1,\dots,\varphi_n(t)x_n, \varphi_1(\tau_1(t))y_1,\dots,\varphi_n(\tau_n(t))y_n\Big)\Big| : \ \sum_{i=1}^n (|x_i| + |y_i|) \le \rho\bigg\}.$$

Along with (1) we consider the functional differential system

$$\frac{du_i(t)}{dt} = \chi_i(t,\delta,\lambda) f_i\big(t, u_1(t),\dots,u_n(t), u_1(\tau_1(t)),\dots,u_n(\tau_n(t))\big) \ \ (i = 1,\dots,n), \tag{4}$$

depended on parameters $\lambda \in [0,1]$ and $\delta \in ]0,1[$ .

**Theorem 1** (A principle of a priori boundedness)**.** *Let there exist a positive constant $\rho$ such that for every $\delta \in ]0,1[$ and $\lambda \in [0,1]$ any solution $(u_i)_{i=1}^n$ of problem (4), (2) admits the estimates*

$$|u_i(t)| \le \rho\varphi_i(t) \quad \textit{for} \ \ t \in [a,b] \ \ (i = 1,\dots,n).$$

*Then problem (1), (2) has at least one solution.*

**Theorem 2.** *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t,x_1,\dots,x_n,y_1,\dots,y_n)\operatorname{sgn}[(t-t_i)x_i] \\ \le |\varphi_i'(t)|\Big[\sum_{k=1}^n \big(p_{1ik}\frac{|x_k|}{\varphi_k(t)} + p_{2ik}\frac{|y_k|}{\varphi_k(\tau_k(t))}\big) + q\Big] \ \ (i = 1,\dots,n)$$

*be fulfilled, where $p_{1ik}$, $p_{2ik}$ $(i,k = 1,\dots,n)$ and $q$ are nonnegative constants, at that the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies the inequality*

$$r(\mathcal{P}) < 1. \tag{5}$$

*Then problem (1), (2) has at least one solution.*

**Remark 1.** Under the conditions of Theorem 2, each function $f_i$ may have the singularity of arbitrary order at the point $t_i$. Indeed, if $\varphi_i(t) = |t - t_i|$ $(i = 1,\dots,n)$, then the conditions of the above-mentioned theorem are satisfied, for example, by the functions

$$f_i(t,x_1,\dots,x_n,y_1,\dots,y_n) = \exp\Big(\frac{1 + |x_1| + \dots + |x_n| + |y_1| + \dots + |y_n|}{|t-t_i|}\Big)(t_i - t)x_i \\ + \sum_{k=1}^n \Big(p_{1ik}\,\frac{|x_k|}{|t-t_k|} + p_{2ik}\,\frac{|y_k|}{|\tau_k(t) - t_k|}\Big) + q \ \ (i = 1,\dots,n).$$

Condition (5) in Theorem 2 is unimprovable and it cannot be replaced by the condition

$$r(\mathcal{P}) \leq 1.$$

What is more, the following theorem is valid.

**Theorem 3.** *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \operatorname{sgn}(t - t_i) \geq |\varphi_i'(t)| \Big[ \sum_{k=1}^{n} \Big( p_{1ik} \, \frac{|x_k|}{\varphi_k(t)} + p_{2ik} \, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) + q \Big] \ \ (i = 1, \dots, n)$$

*be fulfilled, where $p_{1ik} \geq 0$, $p_{2ik} \geq 0$ $(i, k = 1, \dots, n)$, $q > 0$, and the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies the inequality*

$$r(\mathcal{P}) \geq 1.$$

*Then problem* $(1), (2)$ *has no solution.*

Along with $(1), (2)$ let us consider the perturbed problem

$$\frac{dv_i(t)}{dt} = f_i\big(t, v_1(t), \dots, v_n(t), v_1(\tau_1(t)), \dots, v_n(\tau_n(t))\big) + h_i(t) \ \ (i = 1, \dots, n), \tag{6}$$

$$\limsup_{t \to t_i} \frac{|v_i(t)|}{\varphi_i(t)} < +\infty \ \ (i = 1, \dots, n), \tag{7}$$

and introduce

**Definition.** Problem $(1), (2)$ is said to be **well-posed** if:

(i) it has a unique solution $(u_i)_{i=1}^n$;

(ii) there exists a positive constant $\rho$ such that for arbitrary integrable functions $h_k : J \to \mathbb{R}$ $(k = 1, \dots, n)$, satisfying the conditions

$$\nu_k(h_k) = \sup \bigg\{ \frac{1}{\varphi_k(t)} \bigg| \int_{t_k}^{t} |h_k(s)| \, ds \bigg| : \ t \in I_k \bigg\} < +\infty \ \ (k = 1, \dots, n),$$

problem $(6), (7)$ is solvable and its every solution satisfies the inequalities

$$|v_i(t) - u_i(t)| \leq \rho \Big[ \sum_{k=1}^{n} \nu_k(h_k) \Big] \varphi_i(t) \quad \text{for } \ t \in [a, b] \ \ (i = 1, \dots, n).$$

**Theorem 4.** *Let on the set $J \times \mathbb{R}^{2n}$ the inequalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \operatorname{sgn}[(t - t_i) x_i] \leq |\varphi_i'(t)| \sum_{k=1}^{n} \Big( p_{1ik} \, \frac{|x_k|}{\varphi_k(t)} + p_{2ik} \, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) \ \ (i = 1, \dots, n)$$

*be fulfilled, where $p_{1ik}$, $p_{2ik}$ $(i, k = 1, \dots, n)$ are nonnegative constants, and the matrix $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^n$ satisfies inequality* $(5)$*. Then problem* $(1), (2)$ *is well-posed.*

Theorems 3 and 4 yield the following result.

**Corollary 1.** *Let on the set* $J \times \mathbb{R}^{2n}$ *the equalities*

$$f_i(t, x_1, \dots, x_n, y_1, \dots, y_n) = \varphi_i'(t) \sum_{k=1}^{n} \Big( p_{1ik} \, \frac{|x_k|}{\varphi_k(t)} + p_{2ik} \, \frac{|y_k|}{\varphi_k(\tau_k(t))} \Big) \ \ (i = 1, \dots, n)$$

*hold, where* $p_{1ik}$, $p_{2ik}$ $(i, k = 1, \dots, n)$ *are nonnegative constants. Then for problem* (1), (2) *to be well-posed it is necessary and sufficient that the matrix* $\mathcal{P} = (p_{1ik} + p_{2ik})_{i,k=1}^{n}$ *to satisfy inequality* (5)*.*

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# Global Attractor of Impulsive Parabolic System Without Uniqueness

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An autonomous evolution system is called impulsive dynamical system (impulsive DS) if its trajectories have jumps at moments of intersection with certain surface of the phase space. These systems are an important subclass of systems with impulsive perturbations at fixed moments of time whose qualitative theory was developed in [6]. In this paper, using the theory of global attractors for multi-valued semiflows [3], we describe the dynamics of infinite-dimensional impulsive systems without uniqueness of solution of the Cauchy problem. We consider global attractor as a minimal uniformly attracting set for corresponding multi-valued semiflow [4]. Using the results of [1, 2], we construct abstract theory of multi-valued impulsive dynamical systems and apply obtained results to weakly non-linear impulsive parabolic system.

Let $(X,\rho)$ be a metric space, $P(X)$ $(\beta(X))$ be a set of all non-empty (non-empty bounded) subset of $X$.

**Definition 1** ([3])**.** A multi-valued map $G : R_+ \times X \to P(X)$ is called multi-valued dynamical system (MDS) if

$$\forall\, x \in X \ \ G(0,x) = x \ \text{ and } \ \forall\, t,s \geq 0 \ \ G(t+s,x) \subseteq G(t, G(s,x)).$$

**Definition 2** ([4])**.** A non-empty subset $\Theta \subset X$ is called a global attractor of MDS $G$ if

1) $\Theta$ is a compact set;
2) $\Theta$ is uniformly attracting set, i.e., $\forall\, B \in \beta(X)$ $\operatorname{dist}(G(t,B),\Theta) \longrightarrow 0$, $t \to \infty$;
3) $\Theta$ is minimal among all closed uniformly attracting sets.

**Lemma 1.** *Assume that MDS $G$ satisfies dissipativity condition:*

$$\exists\, B_0 \in \beta(X), \ \ \forall\, B \in \beta(X), \ \ \exists\, T = T(B) > 0, \ \ \forall\, t \geq T \ \ G(t,B) \subset B_0. \tag{1}$$

*Then the following conditions are equivalent:*

1) *MDS $G$ has a global attractor $\Theta$;*
2) *MDS $G$ is asymptotically compact, i.e.,*

$$\forall\, t_n \nearrow \infty \ \ \forall\, B \in \beta(X), \ \ \forall\, \xi_n \in G(t_n, B) \ \ \text{sequence } \{\xi_n\} \ \text{is precompact in } X.$$

Impulsive MDS $G$ consists of a given non-empty closed *impulsive* set $M \subset X$, compact-valued *impulsive* map $I : M \to P(X)$ and a given family $K$ of continuous maps $\varphi : [0,+\infty) \to X$ satisfying the following properties:

K1) $\forall\, x \in X, \exists\, \varphi \in K : \varphi(0) = x$;

K2) $\forall\, \varphi \in K, \forall\, s \geq 0 \ \varphi(\,\cdot\, + s) \in K$.

We denote

$$K_x = \{\varphi \in K \mid \varphi(0) = x\}.$$

Impulsive MDS describes the following behaviour: a phase point moves along trajectories of $K$ and when it meets the impulsive set $M$, it jumps onto a new position from the set of *impulsive points* $IM$.

For "well-posedness" of impulsive problem we assume the following conditions:

$$\begin{gathered} M \cap IM = \varnothing; \\ \forall\, x \in M, \ \ \forall\, \varphi \in K_x, \ \ \exists\, \tau = \tau(\varphi) > 0, \ \ \forall\, t \in (0, \tau) \ \ \varphi(t) \notin M. \end{gathered} \tag{2}$$

We denote

$$\forall\, \varphi \in K \ M^+(\varphi) = \bigcup_{t>0} \varphi(t) \cap M.$$

If $M^+(\varphi) \neq \varnothing$, then there exists a moment of time $s := s(\varphi) > 0$ such as

$$\forall\, t \in (0, s) \ \ \varphi(t) \notin M, \ \ \varphi(s) \in M. \tag{3}$$

Hence, we can define the following function : $K \to (0, +\infty]$ :

$$s(\varphi) = \begin{cases} s, & \text{if } \ M^+(\varphi) \neq \varnothing, \\ +\infty, & \text{if } \ M^+(\varphi) = \varnothing. \end{cases} \tag{4}$$

Impulsive trajectory $\widetilde{\varphi}$, starting from the point $x \in X$, is a right continuous function

$$\widetilde{\varphi}(t) = \begin{cases} \varphi_n(t - t_n), & \text{if } \ t \in [t_n, t_{n+1}), \\ x^+_{n+1}, & \text{if } \ t = t_{n+1}, \end{cases} \tag{5}$$

where $\{x^+_n\}_{n \geq 1} \subset IM$ are impulsive points, $\{s_n\}_{n \geq 0} \subset (0, +\infty)$ are the corresponding moments of time, $\{\varphi_n\}_{n \geq 0} \subset K$, $\varphi_0(0) = x$ and $\forall\, n \geq 0$ $t_0 := 0$, $t_{n+1} := \sum\limits_{k=0}^{n} s_k$, $n \geq 0$.

By $\widetilde{K}_x$ we denote the set of all impulsive trajectories starting from $x \in X$.

We assume that every impulsive trajectory is defined on $[0, +\infty)$, i.e.,

$$\forall\, x \in X \ \ \text{every} \ \ \widetilde{\varphi} \in \widetilde{K}_x \ \ \text{is defined on} \ \ [0, +\infty). \tag{6}$$

**Definition 3.** A multi-valued map $G : \mathbb{R}_+ \times X \to P(X)$

$$G(t, x) = \{\widetilde{\varphi}(t) \mid \widetilde{\varphi} \in \widetilde{K}_x\} \tag{7}$$

is called impulsive MDS.

**Lemma 2.** *Let conditions* K1), K2), (2), (6) *be satisfied. Then* (7) *defines the MDS* $G$.

To state further results concerning invariance property of the global attractor we have to impose additional constraints on the parameters of our impulsive problem:

K3) $\forall\, x_n \to x$, $\forall\, \varphi_n \in K_{x_n}$, $\exists\, \varphi \in K_x$ such that on some subsequence

$$\forall\, t \geq 0 \ \ \varphi_n(t) \to \varphi(t);$$

I) the compact-valued map $I : M \to P(X)$ is upper-semicontinuous [3];

S1) if for $x \in X \setminus M$, $x_n \to x$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ we have $\forall\, t \geq 0$ $\varphi_n(t) \to \varphi(t)$, then

$$\begin{cases} s(\varphi) = \infty, & \text{if } s(\varphi_n) = \infty \text{ for infinitely many } n \geq 1, \\ s(\varphi_n) \to s(\varphi), & \text{otherwise.} \end{cases}$$

**Lemma 3.** *Assume that impulsive MDS $G$ satisfies* K1), K2), (2), (6), K3), I), S1) *and $\Theta$ is a global attractor of $G$. Then the following property holds:*

$$\forall\, t \geq 0, \ \ \forall\, \xi \in \Theta \setminus M \ \ G(t,\xi) \cap (\Theta \setminus M) \neq \varnothing. \tag{8}$$

*If, additionally, $G$ is single-valued, then*

$$\forall\, t \geq 0 \ \ G(t, \Theta \setminus M) \subseteq \Theta \setminus M. \tag{9}$$

*In order to prove the inverse embedding in* (9), *it is necessary to impose the following additional assumptions on $K$, $M$, $I$:*

K4) *$\forall\, x_n \to x$, $\forall\, \varphi_n \in K_{x_n}$, $\exists\, \varphi \in K_x$ such that on some subsequence*

$$\varphi_n \to \varphi \ \ \text{uniformly on every } [a,b] \subset [0,\infty), \tag{10}$$

S2) *if for $\forall\, x_n \notin M$ $x_n \to x \in M$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ we have $\forall\, t \geq 0$ $\varphi_n(t) \to \varphi(t)$, then either $s(\varphi_n) = \infty$ for an infinite number $n \geq 1$,*

$$\text{or } \ s(\varphi_n) \to 0, \ \ n \to \infty.$$

**Lemma 4.** *Assume that impulsive MDS $G$ satisfies* K1), K2), (2), (6), K3), I), S1), K4), S2), *and $\Theta$ is a global attractor of $G$. Then*

$$\forall\, t \geq 0 \ \ \Theta \setminus M \subseteq G(t, \Theta \setminus M). \tag{11}$$

*If $\forall\, x \in X$, $\forall\, t,s \geq 0$ $G(t+s,x) = G(t, G(s,x))$, then in* (11) *equality takes place.*

We apply obtained results for impulsive weakly non-linear parabolic problem. Let $\Omega \subset R^n$ be a bounded domain. For unknown functions $u(t,x)$, $v(t,x)$ on $(0,+\infty) \times \Omega$ we consider the following weakly non-linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \Delta u + \varepsilon f_1(u,v), \\ \dfrac{\partial v}{\partial t} = a_2 \Delta v + b \Delta u + \varepsilon f_2(u,v), \end{cases} \tag{12}$$

where $\varepsilon > 0$ is a small parameter, $a_1, a_2 > 0$, $|b| < 2\sqrt{a_1 a_2}$. Continuous non-linear functions $f_i : R^2 \longmapsto R$, $i = 1,2$ satisfy the following condition:

$$\exists\, C > 0 \ \ \forall\, u,v \in R \ \ |f_1(u,v)| + |f_2(u,v)| \leq C. \tag{13}$$

It is known that under such conditions for every $\varepsilon > 0$, $z_0 \in X$ there exists at least one solution $\varphi(\,\cdot\,) = \begin{pmatrix} u(\,\cdot\,) \\ v(\,\cdot\,) \end{pmatrix} \in C([0,+\infty), X)$ of the problem (12) with $\varphi(0) = z_0$, where $X = L_2(\Omega) \times L_2(\Omega)$ is a phase space.

Thus the problem (12) generates the family of continuous maps:

$$K^{\varepsilon} = \big\{\varphi : [0,+\infty) \to X \mid \ \varphi \text{ is a solution of (12)}\big\},$$

which satisfies conditions K1), K2). For fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$ we consider the following impulsive perturbation:

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid \ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1, \ |(u,\psi_1)| \le \gamma \right\}, \tag{14}$$

$$I : M \to P(X) \ \text{ such that for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M,$$

$$Iz \subseteq \left\{ \begin{pmatrix} c_1' \\ d_1' \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \mid \ |c_1'| \le \gamma, \ \alpha c_1' + \beta d_1' = 1 + \mu \right\}, \tag{15}$$

where $\{\psi_i\}_{i=1}^{\infty}$ are eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$.

**Theorem.** *For sufficiently small $\varepsilon > 0$ impulsive problem* (12), (14), (15) *generates an impulsive MDS $G_\varepsilon : R_+ \times X \longmapsto P(X)$, which has a global attractor $\Theta_\varepsilon$ and*

$$\operatorname{dist}(\Theta_\varepsilon, \Theta) \longrightarrow 0, \ \ \varepsilon \to 0, \tag{16}$$

*where $\Theta$ is global attractor of impulsive system* (12), (14), (15) *with $\varepsilon = 0$.*

*Moreover, if $I : M \longmapsto P(X)$ is upper semicontinuous map, then*

$$\forall\, t \ge 0 \ \ G_\varepsilon(t, \Theta_\varepsilon \setminus M) = \Theta_\varepsilon \setminus M. \tag{17}$$

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# An $m$-Dimensional Linear Pfaff Equation with Arbitrary Characteristic Sets

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Consider the linear Pfaff system

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \;\; x \in R^n, \;\; t = (t_1, t_2, \dots, t_m) \in R^m_+, \;\; i = \overline{1, m}, \tag{1}$$

with bounded coefficient matrices $A_i(t)$ continuously differentiable in $R^m_+ = \{t \in R^m : \; t \geq 0\}$ and satisfying the condition of complete integrability [1, pp. 14–24], [2, pp. 16–26]. The characteristic vector [1, p. 83], [3], $\lambda[x] = \lambda$ and the lower characteristic vector [4] $p[x] = p$ of a nontrivial solution $x : R^m_+ \to R^n \setminus \{0\}$ of system (1) is defined by the conditions

$$L_x(\lambda) \equiv \overline{\lim_{t\to\infty}} \frac{\ln \|x(t)\| - (\lambda, t)}{\|t\|} = 0, \;\; L_x(\lambda - \varepsilon e_i) > 0, \;\; \forall\, \varepsilon > 0, \;\; i = 1, \dots, m, \tag{2}$$

$$l_x(p) \equiv \underline{\lim_{t\to\infty}} \frac{\ln \|x(t)\| - (p, t)}{\|t\|} = 0, \;\; l_x(p + \varepsilon e_i) < 0, \;\; \forall\, \varepsilon > 0, \;\; i = 1, \dots, m, \tag{3}$$

where $e_i = (\underbrace{0, \dots, 0, 1}_{i}, 0, \dots, 0) \in R^m_+$ is a unit coordinate vector. The characteristic set $\Lambda_x$ [3] and the lower characteristic set $P_x$ [4] of a nontrivial solution $x : R^m_+ \to R^n \setminus \{0\}$ of system (1) is defined as the unions of all characteristic vectors $\Lambda_x = \cup \lambda[x]$ and all lower characteristic vectors $P_x = \cup p[x]$ of that solution. The sets [3], [4] $\Lambda(A) = \bigcup\limits_{x \neq 0} \Lambda_x$ and $P(A) = \bigcup\limits_{x \neq 0} P_x$ referred respectively to as the characteristic and the lower characteristic sets of system (1).

We generalize the statement on joint implementation of the characteristic and the lower characteristic sets of the linear Pfaff system (1) with two-dimensional time ($m = 2$) [6] on the system (1) with $m$-dimensional time $t$.

**Definition 1** ([9])**.** A set $D \subset R^m$ is said to be *bounded above* (respectively, *below*) if there exists an $r \in R^m$ such that $d \leq r$ (respectively, $d \geq r$) for all $d \in D$ ($d \leq r$ is equivalent to the inequalities $d_i \leq r_i$, $i = \overline{1, m}$).

We introduce an analog of notions of least upper bound and greatest lower bound of a one-dimensional set for a bounded set $D \subset R^m$ [10, p. 11], [7, p. 32] without considering these bounds as elements of an ordered set of subsets of the space $R^m$. To this end, to each point $r \in R^m$, we assign the sets

$$\overline{K}(r) = \{p \in R^m : \; p \geq r\}, \quad \underline{K}(r) = \{p \in R^m : \; p \leq r\},$$

which are referred to as the *upper and lower direct $m$-dimensional angles*, respectively, *with vertex at the point $r$*.

**Definition 2** ([9])**.** The *least upper* (respectively, *greatest lower*) *bound* of a set $D \subset R^m$ bounded above (respectively, below) is defined as the set $\sup D$ (respectively, $\inf D$) of vertices of all upper direct $m$-dimensional angles $\overline{K}(r)$ (respectively, lower direct $m$-dimensional angles $\underline{K}(r)$), each of which has the unique common point, the angle vertex, with the set $\overline{D}$,

$$\sup D \equiv \big\{r \in R^m : \ \overline{D} \cap \overline{K}(r) = \{r\}\big\} \quad \Big(\text{respectively, } \ \inf D \equiv \big\{r \in R^m : \ \overline{D} \cap \underline{K}(r) = \{r\}\big\}\Big).$$

**Definition 3** ([9])**.** A set $D \subset R^m$ is said to be *upper closed* (respectively, *lower closed*) if it contains the least upper bound (respectively, the greatest lower bound) of itself.

Let the set $D \subset R^m$ be a connected upper and lower closed convex set. Note that the sets are its least upper bound $\sup D$ and greatest lower bound $\inf D$ have the properties of characteristic and lower characteristic sets, respectively.

**Theorem.** *Let sets $P \subset R^m$ and $\Lambda \subset R^m$ be defined, respectively, convex function $p_m = f_P(p_1, \dots, p_{m-1}) : R^{m-1} \to R$ and concave function $\lambda_m = f_\Lambda(\lambda_1, \dots, \lambda_{m-1}) : R^{m-1} \to R$ continuous monotonically decreasing in their convex closed bounded domain, and satisfy*

$$\sup\big\{p_i : \ (p_1, p_2, \dots, p_m) \in P\big\} \le \inf\big\{\lambda_i : \ (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda\big\}, \quad i = \overline{1, m}.$$

*Then there exists a completely integrable Pfaff equation*

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \ \ x \in R, \ \ t \in R^m_+, \ \ i = \overline{1, m}, \tag{$1_2$}$$

*with bounded infinitely differentiable coefficient $A_i(t)$ with characteristic set $\Lambda(A) = \Lambda$ and lower characteristic set $P(A) = P$.*

**Sketch of the proof.** Without loss of generality, one can assume (to within a shift) that the set $P \subset R^m$ lies in the $m$-dimensional cube $[d_1, d_2] \times \dots \times [d_1, d_2] \subset R^m_-$, and the set $\Lambda \subset R^m$ lies in the cube $[|d_2|, |d_1|] \times \dots \times [|d_2|, |d_1|] \subset R^m_+$, where $d_1 < d_2 \le 0$.

## I. Preliminary construction

Let us assume that the sets $P$ and $\Lambda$, determines the functions $f_P$ and $f_\Lambda$, admit the following parametric representation

$$P: \ \ p = H(\alpha) \ \text{ and } \ \Lambda: \ \ p = G(\alpha), \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m-1}), \ \ \alpha_i \in [0, 1].$$

By the assumptions of the theorem, for each point of the sets $P$ and $\Lambda \subset R^m$, there exists a tangent hyperplane, and if several tangent hyperplanes exist at some point of that set, then we take a hyperplane whose normal has coordinates of one sign. In addition, any of those tangent hyperplanes $\mu$ at the set $P \subset R^m$ lies not below that set, and any of those tangent hyperplanes $\nu$ at the set $\Lambda \subset R^m$ lies not above that set $\Lambda$. It means that for each $s \in P$, there exists $M_s \in \mu$ such that $s \le M_s$, and for each $s \in \Lambda$, there exists $M_s \in \nu$ such that $s \ge M_s$. Let the tangent hyperplane $\mu$ of the set $P$ at the point $H(\alpha)$ and the tangent hyperplane $\nu$ of the set $\Lambda$ at the point $G(\alpha)$ be defined by the points $q^{(i)}(\alpha) \in R^m$ and $r^{(i)}(\alpha) \in R^m$, $i = \overline{1, m}$, respectively,

$$\begin{aligned}
\mu(\alpha, \zeta) &= q^{(1)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)(1 - \zeta_1) + q^{(2)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)\zeta_1 + \cdots \\
&\quad + q^{(m-1)}(\alpha) \cdot (1 - \zeta_{m-1})\zeta_{m-2} + q^{(m)}(\alpha) \cdot \zeta_{m-1}, \ \ \zeta = (\zeta_1, \zeta_2, \dots, \zeta_{m-1}), \ \ \zeta_i \in [0, 1], \\
\nu(\alpha, \zeta) &= r^{(1)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)(1 - \zeta_1) + r^{(2)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)\zeta_1 + \cdots \\
&\quad + r^{(m-1)}(\alpha) \cdot (1 - \zeta_{m-1})\zeta_{m-2} + r^{(m)}(\alpha) \cdot \zeta_{m-1}, \ \ \zeta = (\zeta_1, \zeta_2, \dots, \zeta_{m-1}), \ \ \zeta_i \in [0, 1].
\end{aligned}$$

In this case, we set $q^{(1)}(\alpha) = H(\alpha)$, $r^{(1)}(\alpha) = G(\alpha)$ and require that the projections of those tangents $\mu(\alpha,\zeta)$ and $\nu(\alpha,\zeta)$ to the coordinate axes lies inside the corresponding projections of the sets $P$ and $\Lambda$, respectively.

We construct the sequence $\{\tau_n^{(j)}(h)\}$, $h = (h_1, h_2, \dots, h_{m-1})$, where $j$ for any fixed $n \in N$ ranges over the values $1, 2$, and $h_i$ for fixed values of $n$, $j$, $h_1, \dots, h_{i-1}$ ranges over the values $1, \dots, 2^n$. We set the first element $\tau_1^{(1)}(1, \dots, 1)$ of that sequence to unity, and other elements obtained by multiplying by two the previous element of this sequence.

As a result, we obtain

$$\tau_n^{(j)}(h) = 2^{2^{\sum\limits_{l=1}^{n}(2^{(l-1)})^{m-1} + (j-1)(2^n)^{m-1} + (h_1-1)(2^n)^{m-2} + \dots + (h_{m-3}-1)(2^n)^2 + (h_{m-2}-1)2^n + h_{m-1} - 1}}$$
$$\leq \tau_{n+1}^{(1)}(1, \dots, 1) = 2^{2^{\sum\limits_{l=1}^{n+1}(2^{(l-1)})^{m-1}}} \equiv 2^{\sigma_m(n)}.$$

We set $\tau_t = t_1 + t_2 + \dots + t_m$. We divide the subset $R_+^m = \{t = (t_1, t_2, \dots, t_m):\ t_i \geq 0\}$ of the space $R^m$ by the planes $\tau_t = 2^k$, $k \in N$, into the layers $\{t \in R_+^m:\ 2^k \leq \tau_t < 2^{k+1}\}$, with the closed "lower" face and the open "upper" face. By $\Pi_0^{(1)}(1, \dots, 1)$ we denote the initial layer $\{t \in R_+^m:\ 0 \leq \tau_t < \tau_1^{(1)}(1, \dots, 1)\}$. Next successively denote the layers by $\Pi_n^{(j)}(h)$, where $j$ takes the values 1, 2 for a fixed $n \in N$, and $h_i$ takes the values $1, \dots, 2^n$ for a fixed $n$, $j$, $h_1, \dots, h_{i-1}$. The lower part of the layer $\Pi_n^{(j)}(h)$ is defined as the layer

$$\widetilde{\Pi}_n^{(j)}(h) = \Big\{t \in \Pi_n^{(j)}(h):\ \tau_n^{(j)}(h) \leq \tau_t < \overline{\tau}_n^{(j)}(h)\Big\},$$

where

$$\overline{\tau}_n^{(j)}(h) \equiv \tau_n^{(j)}(h)\sqrt{2}\,,$$

and the top part is defined as the layer

$$\widetilde{\widetilde{\Pi}}_n^{(j)}(h) = \Big\{t \in \Pi_n^{(j)}(h):\ \overline{\tau}_n^{(j)}(h) \leq \tau_t < \overline{\tau}_n^{(j)}(h)\sqrt{2}\Big\}.$$

Following [4], [9], on the segment $\Delta_0^{(1)} = [0, 1]$ we construct perfect set

$$P_0 = \bigcap_{n=1}^{+\infty} \bigcup_{k=1}^{2^n} \Delta_n^{(k)},$$

similar to the Cantor perfect set [8, p. 50] with a nonzero Lebesgue measure and modified step functions $\Theta(\alpha)$ [8, p. 200]. Wherein the length of the $n$st rank segments $\Delta_n^{(k)}$ will be assumed equal $\varepsilon_n = \exp(d_1 \cdot 2^{\sigma_m(n)})$, and the middle of these segments will be denoted $\alpha_n^{(k)}$. Next on the segment $\Delta_0^{(1)} = [0, 1]$ we define continuous nondecreasing Cantor step function $\Theta(\alpha):\ \Delta_0^{(1)} \to [0, 1] = \{\Theta(\alpha):\ \alpha \in P_0\}$ with intervals $\delta_n^{(k)} = \Delta_n^{(k)} \setminus (\Delta_{n+1}^{(2k-1)} \cup \Delta_{n+1}^{(2k)})$ of constant values.

Note that by the definition of $P_0$ for all the $n \in N$ there exists a number $k = k^{(n)}(\alpha) \in \{1, \dots, 2^n\}$, for which the inequality $|\alpha_n^{(k)} - \alpha| \leq \varepsilon_n/2$, $k = k^{(n)}(\alpha)$, $n \in N$. Therefore we have $\Theta\big(\alpha_n^{(k_n(\alpha))}\big) \to \alpha$ if $n \to \infty$. We introduce the notation $\Theta(\alpha, h) \equiv (\Theta(\alpha_n^{(h_1)}), \dots, \Theta(\alpha_n^{(h_{m-1})}))$, $n \in N$.

## II. Construction of the equation

For further constructions, we use the following functions infinitely differentiable on the interval $[\tau_1, \tau_2]$:

$$e_{01}(\tau,\tau_1,\tau_2)=\begin{cases}\exp\big\{-[\tau-\tau_1]^{-2}\exp\big(-[\tau-\tau_2]^{-2}\big)\big\} & \text{if } \tau_1<\tau<\tau_2,\\ i-1 & \text{if } \tau=\tau_i,\ \ i=1,2,\end{cases}$$

$$e_{00}(\tau,\tau_1,\tau_2)=\begin{cases}\exp\big(2^4(\tau_2-\tau_1)^{-4}-(\tau-\tau_1)^{-2}(\tau-\tau_2)^{-2}\big) & \text{if } \tau_1<\tau<\tau_2,\\ 0 & \text{if } \tau=\tau_i,\ \ i=1,2,\end{cases}$$

these are analogs of standard functions infinitely differentiable on the segment $[0,1]$. Note that the function $e_{00}(\tau,\tau_1,\tau_2)$ attains its maximum value unity in the middle of the segment $[\tau_1,\tau_2]$. On the sets

$$\Pi^{(1)}\equiv\bigcup_{n=1}^{+\infty}\bigcup_{h_1=1}^{2^n}\cdots\bigcup_{h_{m-1}=1}^{2^n}\Pi_n^{(1)}(h)$$

and

$$\Pi^{(2)}\equiv\bigcup_{n=1}^{+\infty}\bigcup_{h_1=1}^{2^n}\cdots\bigcup_{h_{m-1}=1}^{2^n}\Pi_n^{(2)}(h),$$

we introduce the vector functions

$$\mathcal{Q}^{(i)}(\tau_t)=\begin{cases}0 & \text{if } t\in\widetilde{\Pi}_n^{(j)}(h),\\ q^{(i)}(\Theta(\alpha,h))e_{00}\Big(\dfrac{\tau_t}{\overline{\overline{\tau}}_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\widetilde{\Pi}}_n^{(j)}(h),\end{cases}\qquad i=\overline{1,m},$$

$$\mathcal{R}^{(i)}(\tau_t)=\begin{cases}0 & \text{if } t\in\widetilde{\Pi}_n^{(j)}(h),\\ r^{(i)}(\Theta(\alpha,h))e_{00}\Big(\dfrac{\tau_t}{\overline{\overline{\tau}}_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\widetilde{\Pi}}_n^{(j)}(h),\end{cases}\qquad i=\overline{1,m}.$$

We introduce the functions

$$\mathcal{E}(t)=e^{(\mathcal{Q}^{(1)}(\tau_t),t)}+e^{(\mathcal{Q}^{(2)}(\tau_t),t)}+\cdots+e^{(\mathcal{Q}^{(m)}(\tau_t),t)}\ \ \text{if } t\in\Pi^{(1)},$$
$$E(t)=\big[e^{-(\mathcal{R}^{(1)}(\tau_t),t)}+e^{-(\mathcal{R}^{(2)}(\tau_t),t)}+\cdots+e^{(\mathcal{R}^{(m)}(\tau_t),t)}\big]^{-1}\ \ \text{if } t\in\Pi^{(2)}.$$

Obviously, the function $\mathcal{E}(t)$ takes a value equal to $m$ if $t\in\widetilde{\Pi}_n^{(1)}(h)$, and the function $E(t)$ takes a value equal to $m^{-1}$ if $t\in\widetilde{\Pi}_n^{(2)}(h)$. We construct the function $x(t)$, $t\in R_+^m$, by the following rule

$$x(t)=\begin{cases}m^{-1}+[m-m^{-1}]e_{01}\Big(\dfrac{\tau_t}{\tau_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\Pi}_n^{(1)}(1,1,\dots,1),\\ \mathcal{E}(t) & \text{if } t\in\Pi^{(1)}\setminus\widetilde{\Pi}_n^{(1)}(1,1,\dots,1),\\ m+[m^{-1}-m]e_{01}\Big(\dfrac{\tau_t}{\tau_n^{(j)}}(1,1,\dots,1),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\Pi}_n^{(2)}(1,1,\dots,1),\\ E(t) & \text{if } t\in\Pi^{(2)}\setminus\widetilde{\Pi}_n^{(2)}(1,1,\dots,1).\end{cases}$$

This function is infinitely differentiable and is a solution of the Pfaff equation $(1_2)$ with bounded infinitely differentiable on $R_+^m$ coefficients

$$A_i(t)=x^{-1}(t)\,\frac{\partial x(t)}{\partial t_i}\,.$$

The infinite differentiability of $A_i(t)$ follows from the similar property of the functions, through which they are defined. Boundedness of coefficients $A_i(t)$ easy to show with the help of estimates given in [5] for functions $\frac{de_{01}(\tau,\tau_1,\tau_2)}{d\tau}$ and $\frac{de_{00}(\tau,\tau_1,\tau_2)}{d\tau}$, defined on any interval $[\tau_1,\tau_2]$ of length $\tau_2-\tau_1\le 1/2$.

# III. Computation of the characteristic sets

Using conditions (2) and (3), the definition of the characteristic and the lower characteristic vectors, and the obvious estimates

$$\ln \mathcal{E}(t) > \max_{i\in\{1,2,\dots,m\}} \big\{(\mathcal{Q}^{(i)}(\tau_t),t)\big\}, \quad \ln E(t) < \min_{i\in\{1,2,\dots,m\}} \big\{(\mathcal{R}^{(i)}(\tau_t),t)\big\},$$

can be shown that the characteristic set of functions $x(t)$ is the set $\Lambda = \Lambda_E$, and the lower characteristic set of functions $x(t)$ is the set $P = P_{\mathcal{E}}$.

## Comment

The result for equation $(1_2)$ is easy to transfer on system (1).

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# Existence and Asymptotic Properties of Kneser Solutions to Singular Differential Problems

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## 1 Formulation of the problem

Analytical results presented here are based on a common research with Jana Burkotová and they are contained in the paper [1] where in addition numerical simulations are discussed. In particular, here we study the existence and asymptotic behaviour of Kneser solutions to the nonlinear second order ODE,

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \ \ t \in [0, \infty), \tag{1}$$

satisfying

$$u(0) = u_0 \in (0, L), \ \ 0 \le u(t) \le L \ \text{ for } \ t \in [0, \infty), \tag{2}$$

or

$$u(0) = u_0 \in (L_0, 0), \ \ L_0 \le u(t) \le 0 \ \text{ for } \ t \in [0, \infty), \tag{3}$$

where the interval $[L_0, L]$ is specified in the following way:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0.$$

Note that equation (1) is singular because we assume that $p(0) = 0$ (see (6)), and therefore there is a time singularity at $t = 0$.

A function $u$ is called *a solution to equation* (1) *on* $[0, \infty)$ if $u \in C^1[0, \infty)$, $pu' \in C^1[0, \infty)$, and $u$ satisfies equation (1) for all $t \in [0, \infty)$. The solution $u$ to equation (1) on $[0, \infty)$ is called *a solution to problem* (1), (2) or problem (1), (3) if $u$ additionally satisfies condition (2) or (3), respectively. A solution $u$ to equation (1) on $[0, \infty)$ is called a *Kneser solution* if there exists $t_0 > 0$ such that

$$u(t)u'(t) < 0 \ \text{ for } \ t \in [t_0, \infty). \tag{4}$$

## 2 Existence of Kneser solutions to singular equation (1)

In this section, the existence of Kneser solutions to problems (1), (2) and (1), (3) is discussed under the assumptions that $f$ is continuous on $[L_0, L]$, $p$ is continuous on $[0, \infty)$ and $p \equiv q$. For more details see [1] and [5]. For the existence of other types of solutions and a deeper study of this problems see also [2], [3], [4].

**Theorem 1.** *Let us assume that*

$$f \in \mathrm{Lip}_{loc}(0, L], \quad f(x) > 0 \ \textit{ for } \ x \in (0, L), \tag{5}$$

$$p \in C^1(0, \infty), \ \ p(0) = 0, \ \ p' > 0 \ \textit{ on } \ (0, \infty), \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \tag{6}$$

$$\frac{p'(t)\int\limits_0^t p(s)\,\mathrm{d}s}{p^2(t)} \geq c, \quad t \in (0,\infty), \tag{7}$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}\,, \quad x \in (0, A_0], \tag{8}$$

*hold for some $c > \frac{1}{2}$ and $A_0 \in (0, L)$, where $F(x) = \int\limits_0^x f(z)\,\mathrm{d}z$.*

*Then, for each $u_0 \in (0, A_0]$ there exists a unique Kneser solution $u$ to problem* (1), (2) *with $p \equiv q$. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \;\; u'(0) = 0, \;\; u'(t) < 0, \;\; t \in (0,\infty).$$

A dual statement for an initial condition $u_0$ from a negative neighbourhood of zero is given in the following theorem.

**Theorem 2.** *Let us assume that* (6) *and* (7) *with a constant $c > \frac{1}{2}$ hold, and let*

$$f \in \mathrm{Lip}_{loc}[L_0, 0), \;\; f(x) < 0 \;\text{ for }\; x \in (L_0, 0). \tag{9}$$

*Further, assume that there exists $B_0 \in (L_0, 0)$ such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}\,, \quad x \in [B_0, 0), \tag{10}$$

*is satisfied.*

*Then, for each $u_0 \in [B_0, 0)$, there exists a unique Kneser solution $u$ to problem* (1), (3) *with $p \equiv q$. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \;\; u'(t) > 0, \;\; t \in (0,\infty).$$

To our knowledge, the existence of Kneser solutions to singular problems (1), (2) and (1), (3) with $p(0) = 0$ and $p \neq q$ remains an open problem. Let us note, that the condition $u'(0) = 0$ is necessary for the smoothness of the solution in the case where $p \equiv q$ is an increasing function. To see this, let us consider a solution $u$ to (1), (2) or (1), (3). Since $u \in C^1[0,\infty)$, the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. Since $f$ is continuous on $[L_0, L]$ and $u(0) \in (L_0, L)$, there exist $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. We now integrate (1) and use the monotonicity of $p$ to obtain

$$|u'(t)| = \left|\frac{1}{p(t)}\int\limits_0^t p(s)f(u(s))\,\mathrm{d}s\right| \leq \frac{M}{p(t)}\int\limits_0^t p(s)\,\mathrm{d}s \leq Mt, \;\; t \in (0,\delta).$$

Consequently, $u'(0) = 0$ holds.

# 1 Asymptotic properties of Kneser solutions

This section focuses on properties of Kneser solutions to problems (1), (2) and (1), (3) in the neighbourhood of infinity. Asymptotic formulas for the solutions and for their first derivatives are provided. In the following analysis, we assume that the data functions $p$ and $q$ are regularly varying at infinity and

$$f \in C[L_0, L], \;\; xf(x) > 0 \;\text{ for }\; x \in (L_0, 0) \cup (0, L). \tag{11}$$

A function $g$, which is positive and measurable on $[\tau_0,\infty)$, $\tau_0>0$, is called *regularly varying of index* $\alpha\in\mathbb{R}$ if for each $\lambda>0$

$$\lim_{t\to\infty}\frac{g(\lambda t)}{g(t)}=\lambda^{\alpha}.$$

The set of all regularly varying functions of index $\alpha$ is denoted by $RV(\alpha)$.

Our proofs are based on

**Karamata Integration Theorem.** *Let $L(t)\in SV$, $c>0$.*

(i) *If $\alpha>-1$, then*

$$\int_c^t s^{\alpha}L(s)\,ds\sim\frac{1}{\alpha+1}\,t^{\alpha+1}L(t)\ \ \text{as}\ \ t\to\infty.$$

(ii) *If $\alpha<-1$, then*

$$\int_t^{\infty} s^{\alpha}L(s)\,ds\sim-\frac{1}{\alpha+1}t^{\alpha+1}L(t)\ \ \text{as}\ \ t\to\infty.$$

(iii) *If $\alpha=-1$, then*

$$l(t)=\int_c^t\frac{L(s)}{s}\,ds\in SV\ \ \text{and}\ \ \lim_{t\to\infty}\frac{L(t)}{l(t)}=0.$$

Note, that if

$$p\in C[0,\infty),\ \ p>0\ \text{ on }\ (0,\infty),\ \ p(0)=0, \tag{12}$$

$$q\in C[0,\infty),\ \ q>0\ \text{ on }\ (0,\infty), \tag{13}$$

then problems $(1),(2)$ and $(1),(3)$ have no Kneser solutions in case that

$$\int_1^{\infty}\frac{\mathrm{d}s}{p(s)}=\infty. \tag{14}$$

This follows from (12), (13), (11) and the following arguments: Let $u$ be a solution to $(1),(2)$. Then, $pu'$ is decreasing for $t>0$. Assume that $pu'\le0$ for $t\ge t_1>0$. By integrating inequality $p(t)u'(t)<p(t_1)u'(t_1)=K<0$, we obtain

$$u(t)\le u(t_1)+K\int_{t_1}^t\frac{\mathrm{d}s}{p(s)}\,.$$

Therefore, as $t$ tends to infinity, $\lim\limits_{t\to\infty}u(t)\le-\infty$ contradicting (2). This means that $u'>0$ on $[t_0,\infty)$. Hence, any solution of $(1),(2)$ is increasing and there exists no Kneser solution to $(1),(2)$. Similar arguments can be given for problem $(1),(3)$. According to the Karamata Integration Theorem, condition (14) is satisfied when $p\in RV(\alpha)$ with $\alpha<1$. For $\alpha=1$, the integral may be convergent (or may not) and hence Kneser solutions to the problem could exist. Therefore, in the following asymptotic analysis, we restrict our attention to the case $\alpha\ge1$. We first formulate the asymptotic properties of Kneser solutions to problem $(1),(2)$, or $(1),(3)$.

**Theorem 3.** *Assume that* (11) *holds and that* $p \in RV(\alpha) \cap C[0,\infty)$, $q \in RV(\beta) \cap C[0,\infty)$, $\alpha \geq 1$, $\beta > 0$, $\beta - \alpha > -1$. *Let* $u$ *be a Kneser solution to problem* (1), (2) *or* (1), (3). *Then,*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{15}$$

We finally focus our attention to the first derivatives of Kneser solutions.

**Theorem 4.** *Assume that* (11) *holds and that* $p \in RV(\alpha) \cap C[0,\infty)$, $\alpha \geq 1$, $q \in RV(\beta) \cap C[0,\infty)$, $\beta > 0$, $\beta - \alpha > -1$, *and in addition*

$$\exists\, r > 1 : \liminf_{x\to 0} \frac{|f(x)|}{|x|^r} > 0, \quad \limsup_{x\to 0} \frac{|f(x)|}{|x|^r} < \infty. \tag{16}$$

*Let* $u$ *be a Kneser solution to problem* (1), (2) *or* (1), (3). *Then, for any* $\varepsilon > 0$

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1}-\varepsilon} |u(t)| = 0. \tag{17}$$

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# On Existence of Solutions with Prescribed Number of Zeros to Third Order Emden–Fowler Equations with Singular Nonlinearity and Variable Coefficient

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## 1 Introduction

The problem of the existence of solutions to Emden–Fowler type equations with prescribed number of zeros on a given domain is investigated.

Consider the equation

$$y''' = -p(t, y, y', y'')|y|_+^k, \text{ where } k \in (0, 1), \quad 0 < m \leqslant p(t, y_0, y_1, y_2) \leqslant M < \infty, \tag{1.1}$$

function $p(t, y_1, y_2, y_3)$ is continuous and it is Lipschitz continuous in $(y_1, y_2, y_3)$. By $|y|_+^k$ we denote $|y|^k \operatorname{sgn} y$.

The equations similar to (1.1) were considered in the previous papers. The existence of solutions with a given number of zeros on the prescribed interval was proved. In [4] equations of the third- and the fourth- order with constant coefficient and $k \in (0, 1) \cup (1, +\infty)$ was investigated. In [6] we provide our results regarding high-order Emden–Fowler type equation with constant coefficient and regular nonlinearity ($k > 1$). This result was proved using a theorem obtained by I. Astashova in [2]. The work [7] contains theorems regarding equation (1.1) with $k > 1$. Now we generalize the result obtained in [7] to the case of singular nonlinearity $k \in (0, 1)$.

## 2 Main result

**Theorem 2.1.** *For any $k \in (0, 1)$, $-\infty < a < b < +\infty$, and integer $j \geqslant 2$, equation* (1.1) *has a solution defined on the segment $[a, b]$, vanishing at its endpoints, and having exactly $j$ zeros on $[a, b]$.*

The idea of the proof is as follows. In [1] it was proved that any solution $y(t)$ is oscillatory if the conditions $y(a) = 0$, $y'(a) > 0$, $y''(a) > 0$ hold. We cannot rely on the Continuous Dependence On Parameters theorem, because its conditions do not fulfill. Nevertheless, solutions to (1.1) (in some extent) are continuous, and we prove this fact. After that we prove that the location of the $N$-th zero of solution $y(t)$ depends continuously on its initial data. Then we can make upper and lower estimates of that location. Finally, we prove that there exist initial data such that the $N$-th zero of the related solution $y(t)$ is exactly at the point $b$.

### 2.1 Continuous Dependence of Solutions on Initial Data

**Lemma 2.1.** *Let $y(t)$ be a solution to equation* (1.1) *defined on $[t_0, I^*]$ and satisfying $y(t_0) = y_0$, $y'(t_0) = y_1 \neq 0$, $y''(t_0) = y_2$. Then there exists $I \in (t_0, I^*)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $(z_0, z_1, z_2)$ belonging to the $\delta$-neighborhood of $(y_0, y_1, y_2)$ and any continuous* *in* $(t,x_0,x_1,x_2)$ *and Lipschitz continuous in* $(x_0,x_1,x_2)$ *function* $q(t,x_0,x_1,x_2)$ *satisfying for all* $(t,\xi_1,\xi_2,\xi_3)$ *the inequality*

$$\big|p(t,\xi_1,\xi_2,\xi_3)-q(t,\xi_1,\xi_2,\xi_3)\big|<\delta,$$

*the solution* $z(t)$ *to the Cauchy problem*

$$\begin{cases} z'''=-q(t,z,z',z'')|z|_+^k,\\ z(t_0)=z_0,\\ z'(t_0)=z_1,\\ z''(t_0)=z_2 \end{cases} \tag{2.1}$$

*is extensible onto* $[t_0,I]$ *and satisfies on this segment the inequalities*

$$|y(t)-z(t)|<\varepsilon,\quad |y'(t)-z'(t)|<\varepsilon,\quad |y''(t)-z''(t)|<\varepsilon.$$

By integrating equation (1.1) three times and taking into account the initial data, we can obtain that the solution $y(t)$ satisfies

$$y(t)=y_0+y_1(t-t_0)+y_2\,\frac{(t-t_0)^2}{2}-\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}p(\xi,y,y',y'')|y|_+^k\,d\xi\,d\tau\,d\eta.$$

From this we can obtain the following estimate:

$$\begin{aligned}|z(t)-y(t)|\leqslant |y_0-z_0|+|z_1-y_1|\,|t-t_0|+|z_2-y_2|\,\frac{|t-t_0|^2}{2}\\ \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}\big|(p(\xi,y,y',y'')-q(\xi,z,z',z''))|y|_+^k\big|\,d\xi\,d\tau\,d\eta\\ +\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|q(\xi,z,z',z'')|\,\big||y|_+^k-|z|_+^k\big|\,d\xi\,d\tau\,d\eta.\end{aligned} \tag{2.2}$$

Our goal is to prove that if the difference of the initial data is small, then the difference of the solutions is small too. For example, take a look at the last term of (2.2)

$$\begin{aligned}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|q(\xi,z,z',z'')|\,\big||y|_+^k-|z|_+^k\big|\,d\xi\,d\tau\,d\eta\leqslant M\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}\big||y|_+^k-|z|_+^k\big|\,d\xi\,d\tau\,d\eta\\ =M\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|y|^k\Big|1-\Big|\frac{z}{y}\Big|_+^k\Big|\,d\xi\,d\tau\,d\eta\leqslant M\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|y|^k\,\frac{1}{k}\,|1-\frac{z}{y}\,|\,d\xi\,d\tau\,d\eta\\ =\frac{M}{k}\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|y|^{k-1}|y-z|\,d\xi\,d\tau\,d\eta\leqslant\frac{M}{k}\,\max_{[t_0,I]}|y-z|\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|y|^{k-1}\,\frac{y'}{y'}\,d\xi\,d\tau\,d\eta\\ \leqslant\max_{[t_0,I]}|y-z|\,\frac{M}{k}\,\max_{[t_0,I]}\Big|\frac{1}{y'}\Big|\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta}|y|^{k-1}y'\,d\xi\,d\tau\,d\eta\end{aligned}$$

$$\leqslant \max_{[t_0,I]} |y-z| \, \frac{M}{k} \, \max_{[t_0,I]} \Big|\frac{1}{y'}\Big| \int_{t_0}^{t} \int_{t_0}^{\tau} \frac{1}{k} \, \big||y(I)|^k - |y(t_0)|^k\big| \, d\tau \, d\eta$$

$$= \max_{[t_0,I]} |y-z| L_3 (t-t_0)^2 \big||y(I)|^k - |y(t_0)|^k\big|. \quad (2.3)$$

Here $L_3$ depends only on $k$, $q(t, y_0, y_1, y_2)$ and $y(t)$. From (2.2) we can obtain the following inequality:

$$|y-z| \leqslant L_1 \max\big\{|z_0-y_0|, |z_1-y_1|, |z_2-y_2|\big\}$$
$$+ L_2 (t-t_0)^3 \Big( \max_{[t_0,I]} |y-z| + \max_{[t_0,I]} |y'-z'| + \max_{[t_0,I]} |y''-z''| + \max |p-q| \Big)$$
$$+ \Big( L_3 (t-t_0)^2 \big||y(I)|^k - |y(t_0)|^k\big| \Big) \max_{[t_0,I]} |y-z|. \quad (2.4)$$

Similarly, we can integrate equations twice or once and obtain estimates for $|y'-z'|$ and $|y''-z''|$, respectively. Adding all the estimates obtained together, we get the evaluation

$$\Big( \max_{[t_0,I]} |y-z| + \max_{[t_0,I]} |y'-z'| + \max_{[t_0,I]} |y''-z''| \Big)$$
$$\leqslant \frac{K_1}{1 - K_2[(I-t_0)]} \Big( \max\{|z_0-y_0|, |z_1-y_1|, |z_2-y_2|, \max |p-q|\} \Big),$$

where $K_1 > 0$, $K_1$ and $K_2[x]$ do not depend on $\varepsilon$, and $K_2[x] > 0$ is a function tending to zero as $x \to 0$. It is possible to choose $I$ such that $1 - K_2[(I-t_0)] > 0$.

The evaluation shows that if $\max\{|z_0-y_0|, |z_1-y_1|, |z_2-y_2|, \max |p-q|\}$ is sufficiently small, then

$$\max_{[t_0,I]} |y-z| + \max_{[t_0,I]} |y'-z'| + \max_{[t_0,I]} |y''-z''| < \varepsilon.$$

This proves the theorem.

**Theorem 2.2.** *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0) = y_0 \geqslant 0$, $y'(t_0) = y_1 > 0$, $y''(t_0) = y_2 \geqslant 0$. Suppose $y(t)$ is defined on a segment $[t_0, I]$ and has a finite number of zeros on it. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $z(t)$ is a solution to* (1.1) *with initial data $z(t_0) = z_0$, $z'(t_0) = z_1$, $z''(t_0) = z_2$, and $(z_0, z_1, z_2)$ belongs to the $\delta$-neighborhood of $(y_0, y_1, y_2)$, then $z(t)$ is extensible onto $[t_0, I]$ and satisfies on it the inequalities $|y(t) - z(t)| < \varepsilon$, $|y'(t) - z'(t)| < \varepsilon$, $|y''(t) - z''(t)| < \varepsilon$.*

Using Lemma 2.1, we put segments of fixed length on every zero of $y(t)$. In such segments continuous dependency on initial data is proven by Lemma 2.1. Between those segments either $y(t) > a > 0$ or $y(t) < b < 0$, and therefore the Continuous Dependence On Parameters theorem holds (because the right-hand side of (1.1) is not Lipschitz continuous only near $y = 0$). Combining all the segments, we prove Theorem 2.2.

## 2.2 Continuous Dependence of Zeros on Initial Data

**Theorem 2.3** (see [7])**.** *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0) = 0$, $y'(t_0) = y_1 > 0$, $y''(t_0) = y_2 > 0$. We denote by $T(y_1, y_2)$ the location of the first zero of $y(t)$ after $t_0$. Then $T(y_1, y_2)$ is a continuous function.*

**Theorem 2.4.** *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0) = 0$, $y'(t_0) = y_1 > 0$, $y''(t_0) = y_2 > 0$. We denote by $t_n(y_1, y_2)$ the location of the $n$-th zero of $y(t)$ after $t_0$. Then $t_n(y_1, y_2)$ is a continuous function, and $|t_n(y_1, y_2) - t_0|$ runs over all positive numbers.*

Now we can prove the main theorem. If we want a solution $y(t)$ to have exactly $j$ zeros on the segment $[a,b]$, we can find suitable initial data for this. Let $y(a)=0$, $y'(a)=c_1>0$, $y'(a)=c_2>0$. Denote by $t_j(c_1,c_2)$ the location of the $(j-1)$-th zero of $y(t)$ after $a$. It follows from Theorem 2.4 that $|t_j(c_1,c_2)-a|$ is a continuous function, and this function runs over all positive numbers. Therefore, $|t_j(c_1,c_2)-a|=b$ has a solution $(c_1^*,c_2^*)$. If $y(a)=0$, $y'(a)=c_1^*>0$, $y'(a)=c_2^*>0$, then $y(t)$ has exactly $j$ zeros on $[a,b]$, and this proves the theorem.

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# Investigation of Carathéodory Functional Boundary Value Problems by Division into Subintervals

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We study the problem

$$\frac{du(t)}{dt} = f(t, u(t)), \ \ t \in [a, b], \ \ \Phi(u) = d, \tag{1}$$

where $\Phi : C([a, b], \mathbb{R}^n)$ is a vector functional (possibly non-linear), $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the Carathéeodory conditions in a certain bounded set, which will be specified below, and $d$ is a given vector.

Note that investigation of solutions of problem (1) in the paper [4] is based on reduction it to a certain simpler parametrized "model-type" problem

$$\frac{du(t)}{dt} = f(t, u(t)), \ \ t \in [a, b], \ \ u(a) = z, \ \ u(b) = \eta, \tag{2}$$

where $z := col(z_1, \dots, z_n)$, $\eta := col(\eta_1, \dots, \eta_n)$ are unknown parameters. Investigation of solutions of problems (2) was connected with the properties of the special sequence of functions $\{u_m(t, z, \eta)\}_{m=0}^{\infty}$ well posed on the interval $t \in [a, b]$. We note that the sufficient condition for the uniform convergence of sequence $\{u_m(t, z, \eta)\}_{m=0}^{\infty}$ consists in the assumption that the maximal in modulus eigenvalue of the matrix $Q = \frac{3(b-a)}{10} K$ is smaller than one, $r(Q) < 1$, where $|f(t, u_1) - f(t, u_2)| \le K|u_1 - u_2|$, a.e. $t \in [a, b]$, $u_1, u_2 \in D$, $D$ is some closed bounded set. To improve twice this sufficient convergence condition, in [1–3, 6] a special interval halving and parametrization technique were suggested.

Following to the idea used in numerical methods for approximate solution of initial value problems for ordinary differential equations, let us fix a natural $N$ and choose $N+1$ grid points

$$t_k = t_{k-1} + h_k, \ \ k = 1, \dots, N, \ \ t_0 = a, \ \ t_N = b, \tag{3}$$

where $h_k$, $k = 1, \dots, N$, are the corresponding step sizes. Thus, $[a, b]$ is divided into $N$ subintervals $[t_0, t_1], [t_1, t_2], [t_2, t_3], \dots, [t_{N-1}, t_{1N}]$.

The aim of this note is to show that by using an $N$ subintervals divisions of type (3) and an appropriate parametrization technique one can $N$ times improve the sufficient convergence condition. It seems that in the case of boundary value problems interval division for approximations in analytic form was first used in [5].

Let us fix certain closed bounded sets $D^k \subset \mathbb{R}^n$, $k = 0, 1, 2, \dots, N$, and focus on the absolutely continuous solutions $u$ of problem (1) whose values at the nodes (3) lie in the corresponding sets $D^k$, i.e. $u(t_k) \in D^k$, $k = 0, 1, 2, \dots, N$.

Based on $D^k$ we introduce the sets

$$D_{k-1,k} := (1-\theta)z^{(k-1)} + \theta z^{(k)}, \quad z^{(k-1)} \in D^{k-1}, \ \ z^{(k)} \in D^k, \ \ \theta \in [0,1], \ \ k = 1, 2, \dots, N,$$

and its some componentwise $\rho^{(k)}$-vector neighbourhoods $D^{[k]} := B(D_{k-1,k}, \rho^{(k)})$, $k = 1, 2, \dots, N$, where $B(D_{k-1,k}, \rho^{(k)}) := \bigcup\limits_{\xi \in D_{k-1,k}} B(\xi, \rho^{(k)})$ and $B(\xi, \rho^{(k)}) := \{\nu \in \mathbb{R}^n : \ |\nu - \xi| \le \rho^{(k)}\}$. Recall that $D_{k-1,k}$ is the set of all possible straight line segments joining points of $D^{k-1}$ with points of $D^k$.

Let us "freeze" the values of $u$ at the nodes (3) by formally putting

$$u(t_k) = z^{(k)} = col(z_1^{(k)}, z_2^{(k)}, \dots, z_n^{(k)}), \ \ k = 0, 1, 2, \dots, N,$$

and consider the restrictions of equation (1) to each of the subintervals of the division (3).

Instead of (1) we introduce $N$ "model-type" problems

$$\frac{dx^{(k)}}{dt} = f(t, x^{(k)}), \ \ t \in [t_{k-1}, t_k], \ \ x(t_{k-1}) = z^{(k-1)}, \ \ x(t_k) = z^{(k)}, \ \ k = 1, 2, \dots, N, \tag{4}$$

where the vectors $z^{(0)}$, $z^{(1)}, \dots, z^{(N)} \in \mathbb{R}^n$ will be regarded as unknown parameters whose values are to be determined. Note that the length of the intervals in problems (4), which will be studied independently, are equal to step-size $h_k$ in opposition to $b-a$ in the case of the original BVP (1).

To study the solutions of (4) we will use the special parametrized successive approximations $x_m^{(k)}(t, z^{(k-1)}, z^{(k)})$ constructed in analytic form and well defined on the intervals $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$, respectively.

**Assumption 1.** *There exist non-negative vectors $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(N)}$ such that*

$$\rho^{(k)} \ge \frac{h_k}{2}\, \delta_{[t_{k-1},t_k], D^{[k]}}(f) \ \ \text{for all} \ \ k = 1, 2, \dots, N,$$

*where*

$$\delta_{[t_{k-1},t_k], D^{[k]}}(f) := \frac{1}{2}\Big[ \operatorname*{ess\,sup}_{(t,x) \in [t_{k-1},t_k] \times D^{[k]}} f(t,x) - \operatorname*{ess\,inf}_{(t,x) \in [t_{k-1},t_k] \times D^{[k]}} f(t,x) \Big]. \tag{5}$$

**Assumption 2.** *There exist non-negative matrices $K_1, K_2, \dots, K_N$ such that*

$$\big| f(t, u_1) - f(t, u_2) \big| \le K_k |u_1 - u_2|, \ \ a.e. \ \ t \in [t_{k-1}, t_k], \ \ u_1, u_2 \in D^{[k]}. \tag{6}$$

**Assumption 3.** *The maximal in modulus eigenvalue of the matrix $Q_k = \frac{3h_k}{10} K_k$, $k = 1, 2, \dots, N$, is smaller than one, $r(Q_k) < 1$.*

Let us define for problems (4) the recurrence parametrized sequences of functions

$$x_0^{(k)}(t, z^{(k-1)}, z^{(k)}) := z^{(k-1)} + \frac{(t - t_{k-1})}{h_k}\,\big[z^{(k)} - z^{(k-1)}\big] = \Big[1 - \frac{t - t_{k-1}}{h_k}\Big] z + \frac{t - t_{k-1}}{h_k}\, z^{(k)}, \tag{7}$$

$$t \in [t_{k-1}, t_k], \ \ k = 1, 2, \dots, N,$$

$$x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) := z^{(k-1)} + \int\limits_{t_{k-1}}^{t} f\big(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds$$

$$- \frac{t - t_{k-1}}{h_k} \int\limits_{t_{k-1}}^{t_k} f\big(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds + \frac{t - t_{k-1}}{h_k}\,\big[z^{(k)} - z^{(k-1)}\big], \tag{8}$$

for all $m = 1, 2, \dots, z^{(k-1)} \in \mathbb{R}^n$, $z^{(k)} \in \mathbb{R}^n$ and $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$.

**Theorem 1.** *Let Assumptions 1–3 hold. Then, for any fixed vectors* $(z^{(0)}, z^{(1)}, \dots, z^{(N)}) \in D^0 \times D^1 \times \dots \times D^N$ *and* $k = 1, 2, \dots, N$*:*

1. *The limit:* $\lim\limits_{m\to\infty} x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) = x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$*, exists uniformly in* $t \in [t_{k-1}, t_k]$*.*
2. *The limit function satisfies the conditions*
$$x_\infty^{(k)}(t_{k-1}, z^{(k-1)}, z^{(k)}) = z^{(k-1)}, \quad x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = z^{(k)}.$$
3. *The function* $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$ *is the unique absolutely continuous solution of the integral equation*
$$x^{(k)}(t) = z^{(k-1)} + \int\limits_{t_{k-1}}^{t} f(s, x^{(k)}(s))\,ds - \frac{t - t_{k-1}}{h_k} \int\limits_{t_{k-1}}^{t_k} f(s, x^{(k)}(s))\,ds + \frac{t - t_{k-1}}{h_k} \big[z^{(k)} - z^{(k-1)}\big], \ \ t \in [t_{k-1}, t_k],$$
*in the domain* $D^{[k]}$*.*

*In other words,* $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$ *is the unique solution of the following Cauchy problem for the modified system of integro-differential equations:*
$$\frac{dx^{(k)}}{dt} = f(t, x^{(k)}) + \frac{1}{h_k}\,\Delta^{(k)}(z^{(k-1)}, z^{(k)}), \ \ t \in [t_{k-1}, t_k], \ \ x(t_{k-1}) = z^{(k-1)},$$
*where* $\Delta^{(k)}(z^{(k-1)}, z^{(k)}) : D^{k-1} \times D^k \to \mathbb{R}^n$ *are the mapping given by formula*
$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f(s, x^{(k)}(s))\,ds.$$

4. *The following estimates hold for* $m \geq 0$*:*
$$\big|x_\infty^{(k)}(\,\cdot\,, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(\,\cdot\,, z^{(k-1)}, z^{(k)})\big| \leqslant \frac{10}{9}\,\alpha_1(t, t_{k-1}, h_k) Q_k^m (1_n - Q_k)^{-1} \delta_{[t_{k-1}, t_k], D^{[k]}}(f), \ \ t \in [t_{k-1}, t_k],$$
*where* $\delta_{[t_{k-1}, t_k], D^{[k]}}(f))$ *is given in* (5) *and*
$$|\alpha_1(t, t_{k-1}, h_k)| \leq \frac{h_k}{2}\,, \ \ t \in [t_{k-1}, t_k].$$

Theorem 1 guarantees that under the assumed conditions, the functions $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)}) : [t_{k-1}, t_k] \to \mathbb{R}^n$, $k = 1, 2, \dots, N$, are well defined for all $(z^{(k-1)}, z^{(k)}) \in D^{k-1} \times D^k$. Therefore, by putting
$$u_\infty(t, z^{(0)}, z^{(1)}, \dots, z^{(N)}) := \begin{cases} x_\infty^{(1)}(t, z^{(0)}, z^{(1)}), & \text{if } t \in [t_0, t_1], \\ x_\infty^{(2)}(t, z^{(1)}, z^{(2)}), & \text{if } t \in [t_1, t_2], \\ \dots\dots\dots\dots\dots\dots & \\ x_\infty^{(N)}(t, z^{(N-1)}, z^{(N)}), & \text{if } t \in [t_{N-1}, t_N] \end{cases} \tag{9}$$

we obtain a function $u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)}) : [a,b] \to \mathbb{R}^n$, which is well defined for the values $z^{(k)} \in D^k$, $k = 0,1,2,\dots,N$. This function is obviously continuous since at the points $t = t_k$ we have

$$x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = x_\infty^{(k)}(t_k, z^{(k)}, z^{(k+1)}), \ \ k = 1,2,\dots,N.$$

**Theorem 2.** *Let the conditions of Theorem* 1 *hold. Then:*

1. *The function* $u_\infty(t, z^{(k-1)}, z^{(k)}) : [a,b] \to \mathbb{R}^n$ *defined by* (9) *is an absolutely continuous solution of problem* (1) *if and only if the vectors* $z^{(k)}$, $k = 0,1,2,\dots,N$, *satisfy the system of* $n(N+1)$ *numerical equations*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f\big(s, x_\infty^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\,ds = 0, \ \ k = 1,2,\dots,N,$$
$$\Delta^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) = \Phi\big(u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})\big) - d = 0. \tag{10}$$

2. *For every solution* $U(\,\cdot\,)$ *of problem* (1) *with* $U(t_k) \in D^k$, $k = 0,1,2,\dots,N$, *there exist vectors* $z^{(k)}$, $k = 0,1,\dots,N$, *such that* $U(\,\cdot\,) = u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})$, *where the function* $u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})$ *is given in* (9).

Although Theorem 2 provides a theoretical answer to the question on the construction of a solution of the BVP (1), its application faces difficulties due to the fact that the explicit form of $x_\infty^{(j)}(s, z^{(j-1)}, z^{(j)})$ and the functions $\Delta^{(k)}(z^{(k-1)}, z^{(k)}) : D^{k-1} \times D^k \to \mathbb{R}^n$, $k = 1,2,\dots,N$, $\Delta^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) : D^0 \times D^1 \times \cdots \times D^N \to \mathbb{R}^n$, appearing in (10) is usually unknown. This complication can be overcome by using $x_m^{(k)}(s, z^{(k-1)}, z^{(k)})$ of form (8) for a fixed $m$, which will lead one to the so-called *approximate determining equations:*

$$\Delta_m^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f\big(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\,ds = 0, \ \ k = 1,2,\dots,N,$$
$$\Delta_m^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) = \Phi\big(u_m(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})\big) - d = 0. \tag{11}$$

Note that, unlike system (10), the $m$-th approximate determining system (11) contains only terms involving the functions $x_m^{(j)}(\,\cdot\,, z^{(j-1)}, z^{(j)})$ which are explicitly known.

It is natural to expect that approximations to the unknown solution of problem (1) can be obtained by using the function

$$u_m(t, \widetilde{z}^{(0)}, \widetilde{z}^{(1)}, \dots, \widetilde{z}^{(N)}) := \begin{cases} x_m^{(1)}(t, \widetilde{z}^{(0)}, \widetilde{z}^{(1)}), & \text{if } t \in [t_0, t_1], \\ x_m^{(2)}(t, \widetilde{z}^{(1)}, \widetilde{z}^{(2)}), & \text{if } t \in [t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_m^{(N)}(t, \widetilde{z}^{(N-1)}, \widetilde{z}^{(N)}), & \text{if } t \in [t_{N-1}, t_N] \end{cases}$$

where $\widetilde{z}^{(k)} \in D^k$, $k = 0,1,2,\dots,N$, are solutions of the numerical system (11).

The constructivity of a suggested technique is shown on the following example with four absolute continuous solutions:

$$\frac{du_1(t)}{dt} = \begin{cases} u_1u_2 - \frac{48}{25}\,t^3 + \frac{44}{25}\,t^2 - \frac{17}{100}\,t - \frac{7}{10}\,, & t \in \Big[0, \frac{1}{4}\Big], \\ u_1u_2 + \frac{48}{25}\,t^3 - \frac{28}{25}\,t^2 - \frac{131}{20}\,t + \frac{483}{200}\,, & t \in \Big[\frac{1}{4}, \frac{1}{2}\Big], \end{cases}$$

$$\frac{du_2(t)}{dt} = \begin{cases} t(u_1 - u_2) - \frac{16}{5}\,t^3 + \frac{7}{5}\,t^2 - \frac{131}{20}\,t + \frac{4}{5}\,, & t \in \Big[0, \frac{1}{4}\Big], \\ t(u_1 - u_2) + \frac{16}{5}\,t^3 - \frac{9}{5}\,t^2 + \frac{1}{4}\,t + \frac{3}{5}\,, & t \in \Big[\frac{1}{4}, \frac{1}{2}\Big], \end{cases}$$

$$\begin{cases} \displaystyle\int_0^{\frac{1}{2}} u_1^2(s)\,ds = \frac{47}{1000}\,, \\ \displaystyle\int_0^{\frac{1}{2}} u_2^2(s)\,ds = \frac{47}{1000}\,. \end{cases}$$

For $N = 2$, $t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $m = 5$ these four solutions are defined by the approximate values of parameters $z^{(0)}$, $z^{(1)}$, $z^{(2)}$ given in table.

| | 1-solution | 2-solution | 3-solution | 4-solution |
|---|---|---|---|---|
| $\widetilde{z}_1^{(0)}$ | 0.3999999998 | 0.4469892219 | −0.1615332331 | −0.2084976508 |
| $\widetilde{z}_2^{(0)}$ | 0.25 | −0.3803603881 | 0.2769448823 | −0.3583253898 |
| $\widetilde{z}_1^{(1)}$ | 0.2499999998 | 0.2446667248 | −0.3540518758 | −0.3583375962 |
| $\widetilde{z}_2^{(1)}$ | 0.2500000001 | −0.3606725966 | 0.2579658912 | −0.3583910008 |
| $\widetilde{z}_1^{(2)}$ | 0.2499999998 | 0.2046115983 | −0.4035821965 | −0.3584724797 |
| $\widetilde{z}_2^{(2)}$ | 0.4000000003 | −0.1585615166 | 0.3508654384 | −0.2082301147 |

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# The Plane Rotatability Indicators of a Differential System

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In a Euclidean space $\mathbb{R}^n$ with $n>1$, consider the set $\mathcal{M}^n$ of linear systems

$$\dot{x}=A(t)x, \quad x\in\mathbb{R}^n, \ \ t\in\mathbb{R}^+\equiv[0,\infty), \tag{1}$$

with continuous operator-functions $A:\mathbb{R}^+\to\operatorname{End}\mathbb{R}^n$, identified with the systems themselves. Developing the ideas from the papers [1–7], we study the Lyapunov type indicators which are responsible for the oscillation of solutions: in this case, for their rotatability in a specially chosen planes in which it is the most significant.

Let $\mathcal{S}(A)$ be the set of all solutions of system (1), and let $\mathcal{G}^k(A)$ be the set of all its $k$-dimensional subspaces. The asterisk as subscript of a linear space denotes the set with the zero removed.

**Definition 1.** For a given linearly independent solutions $x,y\in\mathcal{S}_*(A)$ of the system $A\in\mathcal{M}^n$ and for a moment $t\in\mathbb{R}^+$ define *the angle of rotation* of function $x$ in direction of function $y$ and, respectively, the *trace variation* of function $x$ in the time from 0 to $t$ by the following formulas

$$\Psi(x,y,t)\equiv\bigg|\int\limits_0^t\big(\dot{e}_{x(\tau)},R_{y(\tau)}e_{x(\tau)}\big)\,d\tau\bigg|, \quad \mathrm{P}(x,t)\equiv\int\limits_0^t|\dot{e}_{x(\tau)}|\,d\tau, \tag{2}$$

where $e_a\equiv a/|a|$ is a normalized vector $a$, and $R_ba$ is the result of rotation of the vector $a$ by the angle $\pi/2$ to the half-plane which contains the vector $b$ (linearly independent of $a$).

**Definition 2.** For each *plane* (two-dimensional subspace) $G\in\mathcal{G}^2(A)$ of solutions of the system $A\in\mathcal{M}^n$ define the *weak* and, respectively, *strong rotatability indicators* of the plane $G$: *the lower one*

$$\check{\psi}^\circ(G)\equiv\varliminf_{t\to\infty}\inf_{L\in\operatorname{Aut}\mathbb{R}^n}\frac{1}{t}\,\Psi(Lx,Ly,t), \quad \check{\psi}^\bullet(G)\equiv\inf_{L\in\operatorname{Aut}\mathbb{R}^n}\varliminf_{t\to\infty}\frac{1}{t}\,\Psi(Lx,Ly,t) \tag{3}$$

and *the upper one*

$$\hat{\psi}^\circ(G)\equiv\varlimsup_{t\to\infty}\inf_{L\in\operatorname{Aut}\mathbb{R}^n}\frac{1}{t}\,\Psi(Lx,Ly,t), \quad \hat{\psi}^\bullet(G)\equiv\inf_{L\in\operatorname{Aut}\mathbb{R}^n}\varlimsup_{t\to\infty}\frac{1}{t}\,\Psi(Lx,Ly,t), \tag{4}$$

where $x$ and $y$ form a basis in $G$.

**Remark 1.** If one replaces in formulas (3) and (4) for each $t\in\mathbb{R}^+$ the angle of rotation $\Psi(Lx,Ly,t)$ of the function $Lx$ in direction of the function $Ly$ in time from 0 to $t$ by the trace variation $\mathrm{P}(Lx,t)$ of the function $Lx$ in the same time (see eq. (2)), then the resulting formulas will give corresponding *wandering indicators* $\hat{\rho}^\circ(x)$, $\hat{\rho}^\bullet(x)$, $\check{\rho}^\circ(x)$, $\check{\rho}^\bullet(x)$ of the solution $x\in\mathcal{S}_*(A)$ of the system $A\in\mathcal{M}^n$ (see [3] in somewhat different notation).

**Definition 3.** For each solution $x\in\mathcal{S}_*(A)$ of the system $A\in\mathcal{M}^n$ define *weak* and, respectively, *strong plain rotatability indicators* of the solution $x$: *the lower one*

$$\check{\psi}^\circ(x,A)\equiv\sup_{x\in G\in\mathcal{G}^2(A)}\check{\psi}^\circ(G), \quad \check{\psi}^\bullet(x,A)\equiv\sup_{x\in G\in\mathcal{G}^2(A)}\check{\psi}^\bullet(G) \tag{5}$$

and *the upper one*

$$\hat{\psi}^{\circ}(x,A) \equiv \sup_{x\in G\in\mathcal{G}^2(A)} \hat{\psi}^{\circ}(G), \quad \hat{\psi}^{\bullet}(x,A) \equiv \sup_{x\in G\in\mathcal{G}^2(A)} \hat{\psi}^{\bullet}(G). \tag{6}$$

**Definition 4.** If the upper indicator in Definitions 2 and 3 coincides with the similar lower one, then it is called *exact* and its accent (check or hat) is removed, and in case of coincidence of weak indicator with the similar strong one it is called *absolute* and its circle (empty or full) is omitted.

**Definition 5.** For each system $A \in \mathcal{M}^n$, by *the spectrum* of an indicator defined on the set $\mathcal{S}_*(A)$ or $\mathcal{G}^2(A)$ (or perhaps only on a part of these) we mean the set of all its values on that set.

**Remark 2.** The case $n = 2$ is special in that the plane $G \in \mathcal{G}^2(A)$ of solutions of the system $A \in \mathcal{M}^2$ coincides with the whole space $\mathcal{S}(A)$, and hence, indicators (3) and (4) coincide with the corresponding *oriented rotatability indicators* $\check{\theta}^{\circ}(x) = \check{\theta}^{\bullet}(x)$ and $\hat{\theta}^{\circ}(x) = \hat{\theta}^{\bullet}(x)$ of some solution $x \in G_*$ (actually, of any one; see [7] in other notation), and they are the absolute lower $\check{\psi}(G)$ and upper $\hat{\psi}(G)$ rotatability indicators of the plane $G = \mathcal{S}(A)$, respectively, and have one-point spectrum.

The apparent incorrectness of Definition 2, in the part of its possible dependence on the choice of linearly independent solutions $x, y$ in $G$ and of a scalar product in $\mathbb{R}^n$, is eliminated by

**Theorem 1.** *The rotatability indicators of a plane $G \in \mathcal{G}^2(A)$ of solutions of any system $A \in \mathcal{M}^n$, defined by formulas* (3) *and* (4)*, are invariant under the choice of a basis $x, y \in G_*$ and the choice of a Euclidean structure in $\mathbb{R}^n$.*

The proof of Theorem 1 is provided by

**Lemma 1.** *For any plane $G \in \mathcal{G}^2(A)$ of any system $A \in \mathcal{M}^n$, there are a system $B \in \mathcal{M}^2$ and a continuously differentiable family of orthogonal transformations*

$$U(t) : G(t) \to G(0) \equiv \mathbb{R}^2, \quad t \in \mathbb{R}^+, \ \ U(0) = I,$$

*sending any linearly independent solutions $x, y \in G_*$ into solutions $u, v \in \mathcal{S}(B)$ such that*

$$u \equiv Ux, \quad v \equiv Uy, \quad \Psi(x,y,t) = \Psi(u,v,t), \ \ t \in \mathbb{R}^+.$$

According to the notation given in Definition 3 for the plane rotatability indicator of a solution of a system, it is not uniquely determined by that solution alone and may depend on the other solutions of the system, which is justified by

**Theorem 2.** *There exist an autonomous system $A \in \mathcal{M}^3$ and a non-autonomous system $B \in \mathcal{M}^3$, having a common solution $x \in \mathcal{S}_*(A) \cap \mathcal{S}_*(B)$ with exact, absolute, but different plane rotatability indicators*

$$\psi(x,A) > \psi(x,B).$$

There exists a usual order in the set of plane indicators [3]: the lower indicators do not exceed the upper ones and the weak indicators do not exceed the strong ones. In addition, the seminorm

$$\|A\|_I \equiv \varlimsup_{t\to\infty} \frac{1}{t} \int_0^t \|A(\tau)\|\, d\tau < \infty, \quad \|A(\tau)\| \equiv \sup_{|e|=1} |A(\tau)e|, \tag{7}$$

in the space $\mathcal{M}^n$ gives the upper bound for all the wandering indicators and hence for all the indicators introduced in Definitions 2 and 3, since the following assertion holds.

**Theorem 3.** *For any solution $x \in G_*$ from any plane $G \in \mathcal{G}^2(A)$ of solutions of any system $A \in \mathcal{M}^n$ the following estimates hold*

$$0 \le \check{\psi}^\circ(G) \le \check{\psi}^\circ(x,A) \le \check{\rho}^\circ(x), \quad \check{\psi}^\bullet(G) \le \check{\psi}^\bullet(x,A) \le \check{\rho}^\bullet(x),$$
$$\hat{\psi}^\circ(G) \le \hat{\psi}^\circ(x,A) \le \hat{\rho}^\circ(x), \quad \hat{\psi}^\bullet(G) \le \hat{\psi}^\bullet(x,A) \le \hat{\rho}^\bullet(x) \le \|A\|_I.$$

The inequalities in Theorem 3 between the plane rotatability indicators and the wandering indicators are not equalities in general, already for solutions of two-dimensional systems (but non-autonomous, according to Theorem 10 below) as shown by

**Theorem 4.** *There exists a system $A \in \mathcal{M}^2$ such that the plane rotatability indicators of all solutions $x \in \mathcal{S}_*(A)$ are exact, absolute, and the same but do not coincide with the wandering indicators, which are also exact, absolute, and the same:*

$$\psi(x,A) < \rho(x).$$

If in Definition 2 instead of the exact lower bounds over all automorphisms of the phase space the upper bounds are taken, then so defined indicators are upper estimated neither by the seminorm (7) nor by anything else, as shown by

**Theorem 5.** *For any $\varepsilon > 0$ there exists a system $A \in \mathcal{M}^3$ satisfying the conditions*

$$\|A(t)\| \le \begin{cases} \varepsilon, & t \in [0,1], \\ 0, & t \ge 1, \end{cases} \qquad \|A\|_I = 0,$$

*such that all the indicators of some plane $G \in \mathcal{G}^2(A)$ obtained from formulas* (3) *and* (4) *by replacement of all the exact lower bounds by the upper ones equal $\infty$.*

If in Definition 3 instead of the exact upper bounds over all planes of solution space (containing the given solution) the lower bounds are taken, then so defined indicators are too less informative, already for three-dimensional autonomous systems as shown by

**Theorem 6.** *All the indicators of all solutions $x \in \mathcal{S}_*(A)$ of any autonomous $A \in \mathcal{M}^3$ obtained from formulas* (5) *and* (6), *with the exact upper bounds replaced by the lower ones, equal* 0.

In the case of an autonomous system $A \in \mathcal{M}^n$ all the spectra of various indicators from Definitions 2–4 are closely related to the *spectrum* $|\operatorname{Im}\operatorname{Sp}(A)|$ – the set of *absolute values of imaginary parts* of the eigenvalues of the operator $A \in \operatorname{End}\mathbb{R}^n$. This relationship is described by the next three theorems.

**Theorem 7.** *For any autonomous system $A \in \mathcal{M}^n$ the spectrum of the exact absolute rotatability indicator of a plane includes the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

**Theorem 8.** *There exists an autonomous system $A \in \mathcal{M}^n$ with the spectrum of the exact absolute rotatability indicator of a plane not included in the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

**Theorem 9.** *For any autonomous system $A \in \mathcal{M}^n$ the spectrum of the exact weak, as well as strong, plane rotatability indicator of a solution coincides with the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.*

As an example confirming the validity of Theorem 8, it suffices to take a four-dimensional autonomous system with eigenvalues $\pm i, \pm 2i$: its exact absolute rotatability indicators for at least one of planes equal zero. The proof of Theorem 9 is provided by

**Theorem 10.** *For each solution $x \in \mathcal{S}_*(A)$ of any autonomous system $A \in \mathcal{M}^n$ the weak and strong plane rotatability indicators are exact and coincide with the similar wandering indicators*

$$\psi^\circ(x, A) = \rho^\circ(x), \quad \psi^\bullet(x, A) = \rho^\bullet(x). \tag{8}$$

To prove Theorem 10 it is enough, in its turn, to make sure that the next assertion is true.

**Lemma 2.** *For each solution $x \in \mathcal{S}_*(A)$ of any autonomous system $A \in \mathcal{M}^n$ there exists a linearly independent with $x$ solution $y \in \mathcal{S}_*(A)$ satisfying the condition*

$$\Psi(Lx, Ly, t) = \mathrm{P}(Lx, t), \quad L \in \operatorname{Aut}\mathbb{R}^n, \ \ t \in \mathbb{R}^+.$$

In Lemma 2, in the case when the initial value $x(0)$ of a solution $x$ is an eigenvector for $A \in \operatorname{End}\mathbb{R}^n$ corresponding to a real eigenvalue, any nonzero solution is suitable as a solution $y$ related to the solution $x$, otherwise there is a suitable one, for example, the function $y = Ax$.

**Remark 3.** Applying Theorem 10 and the results of the papers [3, 4] to each of the indicators (8), we can describe the distribution of its values over the space $\mathcal{S}_*(A)$, namely, on the steps of some flag of subspaces in $\mathcal{S}(A)$ it takes constant values ranging in some special order over all the numbers of the spectrum $|\operatorname{Im}\operatorname{Sp}(A)|$.

Theorems 9 and 10 justify the introduction of the plain rotatability indicators of a solution in Definition 3. But equalities (8) do not extend to non-autonomous systems $A \in \mathcal{M}^n$: by Theorem 4 already for $n = 2$ and by Theorem 2 even when the function $x$ is a solution of some autonomous system.

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# Asymptotic Behavior of Solutions of Third Order Nonlinear Differential Equations Close to Linear Ones

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The differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^{\sigma} \tag{1}$$

is considered, where $\alpha_0 \in \{-1; 1\}$, $\sigma \in \mathbb{R}$, $p : [a, w) \to (0, +\infty)$ is a continuous function; $a < w \leq +\infty$.

Asymptotic properties of solutions of equation (1) when $\sigma = 0$ were investigated in detail in the work by I. T. Kiguradze [6, § 6]. For second order equations of the form (1) the asymptotic of solutions of this class was studied in the works by V. M. Evtukhov and Mousa Jaber Abu Elshour [1, 3].

In this work the equation of the third order equation (1) is investigated using the methodology proposed by V. M. Evtukhov for differential equations of $n$-th order in [2] and further developed in the works [4, 5, 9]. Some results for equation (1) we published in [7, 8].

The solution $y$ of equation (1), defined on the interval $[t_y, w) \subset [a, w)$ is called $P_w(\lambda_0)$ solution if it satisfies the following conditions:

$$\lim_{t \to w} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty, \end{cases} \qquad (k = 0, 1, 2), \qquad \lim_{t \to w} \frac{(y''(t))^2}{y'''(t) y'(t)} = \lambda_0.$$

Necessary and sufficient conditions for the existence of $P_w(\lambda_0)$ solutions of equation (1) are stated. The asymptotic representation of such solutions and their derivatives up to second order when $t \to w$ were received.

Let us introduce the necessary notation.

$$\pi_w(t) = \begin{cases} t & \text{if } \ w = +\infty, \\ t - w & \text{if } \ w < +\infty, \end{cases} \quad I_A(t) = \int_A^t \pi_w^2(\tau) p(\tau)\, d\tau, \quad I_B(t) = \int_B^t p^{\frac{1}{3}}(\tau)\, d\tau,$$

$$A = \begin{cases} a & \text{if } \displaystyle\int_a^w |\pi_w(\tau)|^2 p(\tau)\, d\tau = +\infty, \\ w & \text{if } \displaystyle\int_a^w |\pi_w(\tau)|^2 p(\tau)\, d\tau < +\infty, \end{cases} \qquad B = \begin{cases} a & \text{if } \displaystyle\int_a^w p^{\frac{1}{3}}(\tau)\, d\tau = +\infty, \\ w & \text{if } \displaystyle\int_a^w p^{\frac{1}{3}}(\tau) d\tau < +\infty, \end{cases}$$

$$q(t) = p(t) \pi_\omega^3(t) \big| \ln \pi_\omega^2(t) \big|^{\sigma}, \quad Q(t) = \int_a^t p(\tau) \pi_\omega^2(\tau) \big| \ln \pi_\omega^2(t) \big|^{\sigma}\, d\tau.$$

Let us formulate the main theorem on the existence of $P_w(\lambda_0)$ solutions of equation (1).

**Theorem 1.** *Let $\sigma \neq 1$. Then for the existence of $P_w(\lambda_0)$ solutions of equation (1), where $\lambda_0 \in \mathbb{R}\backslash\{0;1;\frac{1}{2}\}$, it is necessary, and if the function $p:[a,w)\to(0,+\infty)$ is continuous and differentiable and*

$$\lambda_0 \neq \frac{-(2+\sigma)\pm\sqrt{(2+\sigma)^2+8}}{4}\,, \quad \lambda_0 \neq \frac{-1\pm\sqrt{3}}{2}\,, \quad \lambda_0 \neq \frac{-(2-\sigma)\pm\sqrt{(2+\sigma)^2+8}}{4}\,,$$

*then it is also sufficient that*

$$\alpha_0\lambda_0(2\lambda_0-1)(\lambda_0-1)\pi_w(t)>0, \quad \lim_{t\to w}\frac{p(t)\pi_w^3(t)}{\big|\frac{(1-\sigma)(1-\lambda_0)^2}{\lambda_0}\,I_A(t)\big|^{\frac{\sigma}{\sigma-1}}} = \alpha_0\frac{|\lambda_0|\,|2\lambda_0-1|}{|\lambda_0-1|^3}\,. \tag{2}$$

*Moreover, for each of such solutions there are asymptotic representation as $t\to w$*

$$\ln|y(t)| = \nu\Big((1-\sigma)\frac{(\lambda_0-1)^2}{\lambda_0}I_A(t)\Big)^{\frac{1}{1-\sigma}}(1+O(1)),$$
$$\frac{y'(t)}{y(t)} = \frac{(2\lambda_0-1)}{(\lambda_0-1)\pi_w(t)}\,(1+O(1)), \quad \frac{y''(t)}{y'(t)} = \frac{\lambda_0}{(\lambda_0-1)\pi_w(t)}\,(1+O(1)),$$

*where $\nu = \operatorname{sign}(\alpha_0(\lambda_0-1)(1-\sigma)I_A(t))$.*

**Theorem 2.** *Let $\sigma\neq 3$. Then for the existence of $P_w(1)$ solutions of equation (1) it is necessary, and if $p:[a,w)\to(0,+\infty)$ is continuous and differentiable and such that there is a finite or equal $\pm\infty$*

$$\lim_{t\to w}\frac{(p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{3-\sigma}})'}{p^{\frac13}(t)|I_B(t)|^{\frac{3\sigma}{3-\sigma}}}\,,$$

*then it is also sufficient that*

$$\lim_{t\to w}\pi_w(t)p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{3-\sigma}} = \infty.$$

*Moreover, for each of such solutions the are asymptotic representation as $t\to w$*

$$\ln|y(t)| = \mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{3-\sigma}{3}}(1+O(1)),$$
$$\frac{y'(t)}{y(t)} = p^{\frac13}\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}(1+O(1)), \quad \frac{y''(t)}{y(t)} = p^{\frac13}\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}(1+O(1)),$$

*where $\mu = \operatorname{sign}(\frac{3-\sigma}{3}\,I_B(t))$.*

**Theorem 3.** *For the existence of $P_w(\pm\infty)$ solution of equation (1), necessary and sufficient conditions are:*

$$\lim_{t\to w}q(t)=0, \quad \lim_{t\to w}Q(t)=\infty.$$

*Moreover, for each of such solutions there are asymptotic representation as $t\to w$*

$$\ln|y(t)| = \ln\pi_w^2(t) + \frac{\alpha_0 Q(t)}{2}\,(1+O(1)),$$
$$\ln|y'(t)| = \ln|\pi_w(t)| + \frac{\alpha_0 Q(t)}{2}\,(1+O(1)), \quad \ln|y''(t)| = \frac{\alpha_0 Q(t)}{2}\,(1+O(1)).$$

The asymptotic of solutions in Theorems 1–4 is written in implicit form. The conditions for the existence of solutions of equation (1) of the specified type were obtained in which their asymptotic performance, as well as derivatives of first and second order can be written in explicit form.

**Theorem 4.** *Let* $\sigma(1-\sigma)\neq 0$ *and conditions* (2) *take place. Let, in addition* $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac{1}{2}\}$, $\lambda_0\neq -1\pm\sqrt{3}$ *and the functions*

$$h_1(t)=\frac{p(t)\pi_\omega^3(t)}{\big|\frac{(1-\sigma)(1-\lambda_0)^2}{\lambda_0}I_A(t)\big|^{\frac{\sigma}{\sigma-1}}}-\frac{\alpha_0|\lambda_0|\,|2\lambda_0-1|}{|\lambda_0-1|^3},\quad h_2(t)=\Big|(1-\sigma)\,\frac{(\lambda_0-1)^2}{\lambda_0}\,I_A(t)\Big|^{\frac{1}{\sigma-1}},$$

*such that*

$$\lim_{t\to\omega}\frac{h_1(t)}{h_2(t)}=0.$$

*Then the differential equation* (1) *has* $P_w(\lambda_0)$ *solution, which allows asymptotic representation as* $t\to w$

$$y(t)=(\pm1+o(1))e^{\nu\left|(1-\sigma)\,\frac{(\lambda_0-1)^2}{\lambda_0}I_A(t)\right|^{\frac{1}{1-\sigma}}},$$

$$y'(t)=\frac{(2\lambda_0-1)}{(\lambda_0-1)\pi_w(t)}\,\big(\pm1+o(1)\big)\,e^{\nu\left|(1-\sigma)\,\frac{(\lambda_0-1)^2}{\lambda_0}\,I_A(t)\right|^{\frac{1}{1-\sigma}}},$$

$$y''(t)=\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2\pi_w^2(t)}\,\big(-1\pm o(1)\big)\,e^{\nu\left|(1-\sigma)\,\frac{(\lambda_0-1)^2}{\lambda_0}\,I_A(t)\right|^{\frac{1}{1-\sigma}}}.$$

Here is a consequence of this theorem, if $\sigma=0$, i.e. for the linear differential equation

$$y'''=\alpha_0p(t)y, \tag{3}$$

where $\alpha_0\in\{-1;1\}$, $\sigma\in\mathbb{R}$, $p:[a,w)\to(0,+\infty)$ is a continuous function; $a<w\leq+\infty$.

**Corollary.** *Let for the differential equation* (3),

$$\lim_{t\to\omega}p(t)\pi_\omega^3(t)=c_0>0\ \ and\ \ \int\limits_a^\omega\Big|\frac{p(t)\pi_\omega^3(t)-c_0}{\pi_\omega(t)}\Big|\,dt<+\infty.$$

*Then, if*

$$-\frac{16}{36}<\frac{c_0}{\alpha_0}<\frac{1}{3}$$

*and*

$$\bigg(32\Big(\frac{\alpha_0}{c_0}\Big)^3+36\Big(\frac{\alpha_0}{c_0}\Big)^2-2\,\frac{\alpha_0}{c_0}+6\bigg)^2-\bigg(32\Big(\frac{\alpha_0}{c_0}\Big)^3-2\Big(\frac{\alpha_0}{c_0}\Big)^2+24\,\frac{\alpha_0}{c_0}\bigg)^2\Big(1+\frac{36c_0}{16\alpha_0}\Big)<0,$$

*the differential equation* (3) *has a fundamental system of solutions* $y_i$ $(i=1,2,3)$, *admitting asymptotic representation as* $t\to\omega$

$$y_i(t)=(1+o(1))e^{\left[\alpha_0\,\frac{(\lambda_i-1)^2}{\lambda_i}\,I_A(t)\right]},$$

$$y'_i(t)=\frac{(2\lambda_i-1)}{(\lambda_i-1)\pi_w(t)}\,(1+o(1))e^{\left[\alpha_i\,\frac{(\lambda_i-1)^2}{\lambda_i}\,I_A(t)\right]},$$

$$y''_i(t)=\frac{\lambda_i(2\lambda_i-1)}{(\lambda_i-1)^2\pi_w^2(t)}\,(1+o(1))e^{\left[\alpha_0\frac{(\lambda_i-1)^2}{\lambda_i}\,I_A(t)\right]},$$

*where* $\lambda_i$ $(i=1,2,3)$ *– the roots of the algebraic equation*

$$\lambda^3-\lambda^2\Big(3+2\,\frac{\alpha_0}{c_0}\Big)+\lambda\Big(3+\frac{\alpha_0}{c_0}\Big)-1=0.$$

The obtained asymptotics are consistent with the already known results for linear differential equations.

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# Variation Formulas of Solution for One Class of Controlled Functional Differential Equation with Several Delays and the Continuous Initial Condition

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Let $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ be open sets. Let $\theta_{i2} > \theta_{i1} > 0$, $i = \overline{1,s}$ be given numbers and $n$-dimensional function $f(t,x,x_1,\dots,x_s,u)$ satisfy the following conditions: for almost all fixed $t \in I = [a,b]$ the function $f(t,\cdot\,) : O^{1+s} \times U_0 \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x,x_1,\dots,x_s,u) \in O^{1+s} \times U_0$ the functions $f(t,x,x_1,\dots,x_s,u)$, $f_x(t,\cdot\,)$ and $f_{x_i}(t,\cdot\,)$, $i = \overline{1,s}$, $f_u(t,\cdot\,)$ are measurable on $I$; for compact sets $K \subset O$, $U \subset U_0$ there exist a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that

$$\big|f(t,x,x_1,\dots,x_s,u)\big| + |f_x(t,\cdot\,)| + \sum_{i=1}^{s} |f_{x_i}(t,\cdot\,)| + |f_u(t,\cdot\,)| \le m_{K,U}(t)$$

for all $(x,x_1,\dots,x_s,u) \in K^{1+s} \times U$ and for almost all $t \in I$. Furthermore, $\Phi$ is the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, $\widehat{\tau} = a - \max\{\theta_{12},\dots,\theta_{s2}\}$, and $\Omega$ is the set of measurable control functions $u : I \to U$ with $clu(I)$ is a compact set and $clu(I) \subset U$.

To each element

$$\mu = (t_0,\tau_1,\dots,\tau_s,\varphi,u) \in \Lambda = [a,b) \times [\theta_{11},\theta_{12}] \times \cdots [\theta_{s1},\theta_{s2}] \times \Phi \times \Omega$$

we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t)\big) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0]. \tag{2}$$

Condition (2) is said to be the continuous initial condition since always $x(t_0) = \varphi(t_0)$.

**Definition.** Let $\mu = (t_0,\tau_1,\dots,\tau_s,\varphi,u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0,b]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to $\mu$ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let us introduce the set of variation:

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta\varphi, \delta u) : \; |\delta t_0| \le \alpha, \; |\delta\tau_i| \le \alpha, \; i = \overline{1,s},$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \; \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \; |\lambda_i| \le \alpha, \; i = \overline{1,k}\Big\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta u_i \in \Omega - u_0$, $i = \overline{1,k}$. Here $\varphi_0 \in \Phi, u_0 \in \Omega$ are fixed functions and $\alpha > 0$ is a fixed number.

Let $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$. Let $x_0(t)$ be the solution corresponding to $\mu_0$.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it (see [2, Theorem 1.3]).

By the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Now we introduce the increment of the solution $x_0(t) = x(t;\mu_0)$ :

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \;\; (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V.$$

**Theorem 1.** *Let the following conditions hold:*

1) *the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t)$, $t \in I_1$, is bounded;*

2) *function $f(w,u)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{1+s}$ is bounded on $I \times O^{s+1} \times U_0$*

3) *there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \quad \lim_{w \to w_0} f(w, u_0(t)) = f^-, \;\; w \in (a, t_{00}] \times O^{1+s},$$

*where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$, we have*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \tag{3}$$

*Here*

$$\delta x(t;\delta\mu) = Y(t_{00}; t)\big[\dot{\varphi}_0^- - f^-\big]\delta t_0 + \beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu) = Y(t_{00};t)\delta\varphi(t_{00}) - \sum_{i=1}^{s}\bigg[\int_{t_{00}}^{t} Y(\xi;t) f_{x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\,d\xi\bigg]\delta\tau_i$$
$$+ \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t} Y(\xi + \tau_{i0}; t) f_{x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\,d\xi + \int_{t_{00}}^{t} Y(\xi;t) f_u[\xi]\delta u(\xi)\,d\xi,$$

*where $Y(\xi;t)$ is the $n \times n$-matrix function satisfying the equation*

$$Y_\xi(\xi;t) = -Y(\xi;t) f_x[\xi] - \sum_{i=1}^{s} Y(\xi + \tau_{i0}; t) f_{x_i}[\xi + \tau_{i0}], \;\; \xi \in [t_{00}, t]$$

*and the condition*

$$Y(\xi;t) = \begin{cases} H & for \;\; \xi = t, \\ \Theta & for \;\; \xi > t, \end{cases}$$

*$H$ is the identity matrix and $\Theta$ is the zero matrix;*

$$f_{x_i}[\xi] = f_{x_i}\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \dots, x_0(\xi - \tau_{s0}), u_0(\xi)\big).$$

The expression (4) is called the variation formula of solution. The addend

$$-\sum_{i=1}^{s}\bigg[\int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$

in the formula (4) is the effects of perturbations of the delays $\tau_{i0}$, $i=\overline{1,s}$.

The expression

$$Y(t_{00};t)\Big\{\delta\varphi(t_{00})+\big[\dot{\varphi}_0^- - f^-\big]\delta t_0\Big\}+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t} Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

is the effect of the continuous initial condition and perturbation of the initial moment $t_{00}$ and the initial function $\varphi_0(t)$.

The expression

$$\int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

is the effect of perturbation of the control function $u_0(t)$.

In [4] variation formulas of solution were proved for the equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau),u(t)\big)$$

with the condition (2) in the case when the initial moment and delay variations have the same signs.

In the present paper, the equation with several delays is considered and variation formulas of solution are obtained with respect to wide classes of variations (see $V^-$ and $V^+$).

Variation formulas of solution for various classes of controlled delay functional differential equations, without perturbations of delays, are proved in [1,3].

**Theorem 2.** *Let the conditions* 1) *and* 2) *of the Theorem* 1 *hold. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0} f(w,u_0(t))=f^+,\ \ w\in[t_{00},b).$$

*Then for any* $\hat{t}\in(t_{00},t_{10})$ *there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[\hat{t},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$ *the formula* (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)\big[\dot{\varphi}_0^+ - f^+\big]\delta t_0+\beta(t;\delta\mu).$$

**Theorem 3.** *Let the conditions* 1) *and* 2) *of the Theorem* 1 *and the condition* 6) *hold. Moreover,*

$$\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ := \hat{f}.$$

*Then for any* $\hat{t}\in(t_{00},t_{10})$ *there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[\hat{t},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$ *the formula* (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)\hat{f}\delta t_0+\beta(t;\delta\mu).$$

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# On Some Special Classes of Solutions of the Countable Block-Diagonal Differential System

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in[0,\varepsilon_0],\ \varepsilon_0\in\mathbf{R}^+\big\}.$$

**Definition 1.** We say that a function $p(t,\varepsilon)$, in general a complex-valued, belongs to the class $S(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$ if

1) $p:G(\varepsilon_0)\to\mathbf{C}$;
2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to $t$;
3) $\dfrac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k^*(t,\varepsilon)$ $(0\le k\le m)$,

$$\|p\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^m\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

**Definition 2.** We say that a function $f(t,\varepsilon,\theta)$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$ if this function can be represented as

$$f(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon)\in S(m;\varepsilon_0)$ $(n\in\mathbf{Z})$;
2) $\|f\|_{F(m;\varepsilon_0,\theta)}\overset{def}{=}\sum\limits_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty$;
3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

The set of functions of the class $F(m;\varepsilon_0;\theta)$ forms a linear space, that turns into a complete normed space by introducing norms $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. The chain of next inclusions are true: $F(0;\varepsilon_0;\theta)\supset F(1;\varepsilon_0;\theta)\supset\cdots\supset F(m;\varepsilon_0;\theta)$.

Suppose we have two functions of the class $F(m;\varepsilon_0;\theta)$,

$$u(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}u_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),\quad v(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}v_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

The product of these functions we define by the formula:

$$(uv)(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}\Big(\sum_{s=-\infty}^{\infty}u_{n-s}(t,\varepsilon)v_s(t,\varepsilon)\Big)\exp(in\theta(t,\varepsilon)).$$

Obviously, $uv\in F(m;\varepsilon_0;\theta)$.

We formulate some properties of the norm $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. Let $u,v\in F(m;\varepsilon_0;\theta)$, $k=const$. Then

1) $\|ku\|_{F(m;\varepsilon_0;\theta)}=|k|\cdot\|u\|_{F(m;\varepsilon_0;\theta)}$;
2) $\|u+v\|_{F(m;\varepsilon_0;\theta)}\leq\|u\|_{F(m;\varepsilon_0;\theta)}+\|v\|_{F(m;\varepsilon_0;\theta)}$;
3) $\|u\|_{F(m;\varepsilon_0;\theta)}=\sum\limits_{k=0}^{m}\Big\|\dfrac{1}{\varepsilon^k}\,\dfrac{\partial^k u}{\partial t^k}\Big\|_{F(0;\varepsilon_0;\theta)}$;
4) $\|uv\|_{F(m;\varepsilon_0;\theta)}\leq 2^m\|u\|_{F(m;\varepsilon_0;\theta)}\cdot\|v\|_{F(m;\varepsilon_0;\theta)}$.

**Definition 3.** We say that the infinite vector $x(t,\varepsilon)=\operatorname{col}(x_1(t,\varepsilon),x_2(t,\varepsilon),\dots)$ belongs to the class $S_1(m;\varepsilon_0)$ if $x_j\in S(m;\varepsilon_0)$ $(j=1,2,\dots)$ and

$$\|x\|_{S_1(m;\varepsilon_0)}\overset{def}{=}\sup_j\|x_j\|_{S(m;\varepsilon_0)}<+\infty.$$

**Definition 4.** We say that the infinite matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=1,2,\dots}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}\in S(m;\varepsilon_0)$, and

$$\|A\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{S(m;\varepsilon_0)}<+\infty.$$

**Definition 5.** We say that the infinite vector $x(t,\varepsilon,\theta)=\operatorname{col}(x_1(t,\varepsilon,\theta),x_2(t,\varepsilon,\theta),\dots)$ belongs to the class $F_1(m;\varepsilon_0;\theta)$ if $x_j\in F(m;\varepsilon_0)$ $(j=1,2,\dots)$, and

$$\|x\|_{F_1(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\|x_j\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

**Definition 6.** We say that the infinite matrix $A(t,\varepsilon,\theta)=(a_{jk}(t,\varepsilon,\theta)_{j,k=1,2,\dots}$ belongs to the class $F_2(m;\varepsilon_0,\theta)$ if $a_{jk}\in F(m;\varepsilon_0,\theta)$, and

$$\|A\|_{F_2(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Consider the countable system of differential equations:

$$\frac{dx}{dt}=A(t,\varepsilon)x+f(t,\varepsilon,\theta)+\mu X(t,\varepsilon,\theta,x),\tag{1}$$

where $t,\varepsilon\in G(\varepsilon_0)$, $x=\operatorname{col}(x_1,x_2,\dots)\in D\subset l_1$ ($l_1$ – the space of boundary numerical sequences), $f=\operatorname{col}(f_1,f_2,\dots)\in F_1(m;\varepsilon_0;\theta)$, $A=\operatorname{diag}[A_1,A_2,\dots]$, $A_j=A_j(t,\varepsilon)=(a_{j,\alpha\beta})_{\alpha,\beta=1,2}$ $(j=1,2,\dots)$, $a_{j,\alpha\beta}\in S(m;\varepsilon_0)$ $(j=1,2,\dots;\ \alpha,\beta=1,2)$, eigenvalues of matrix $A_j(t,\varepsilon)$ have a kind $\pm i\omega_j(t,\varepsilon)$, $\omega_j\in\mathbf{R}^+$ $(j=1,2,\dots)$; infinite vector-function $X=\operatorname{col}(X_1,X_2,\dots)\in F_1(m;\varepsilon_0;\theta)$ with respect to $t$, $\varepsilon$, $\theta$ and continuous with respect to $x\in D$; parameter $\mu\in(0,\mu_0)\subset\mathbf{R}^+$.

The purpose of the article is to establish conditions under which the system (1) has a particular solution $x(t,\varepsilon,\theta,\mu)\in F_1(m_1;\varepsilon_1;\theta)$ $(0\leq m_1\leq m;\ 0<\varepsilon_1\leq\varepsilon_0)$.

We assume the next conditions.

$1^0$. $\inf\limits_{G(\varepsilon_0)} |a_{j,12}(t,\varepsilon)| = a_0 > 0$ $(j = 1, 2, \dots)$.

$2^0$. $\sup\limits_{j} \sup\limits_{G(\varepsilon_0)} \omega_j(t,\varepsilon) = \omega < +\infty$.

$3^0$. $\forall\, n \in \mathbf{Z}$: $|n| \le (2\omega + 1)\varphi_0^{-1}$:

$$\inf_{G(\varepsilon_0)} \big|k\omega_j(t,\varepsilon) - n\varphi(t,\varepsilon)\big| \ge \gamma > 0 \ \ (k = 1, 2; \ \ j = 1, 2, \dots).$$

$4^0$. The functions $X_j$ $(j = 1, 2, \dots)$ have in $D$ continuous particular derivations with respect to $x_1, x_2, \dots$ up to order $2q+1$ $(q \in \mathbf{N})$, and if $x_1, x_2, \dots \in F(m;\varepsilon_0;\theta)$, then all these derivations belong to the class $F(m;\varepsilon_0;\theta)$ also, and

$$\sup_j \Big\| \frac{\partial^{2q+1} X_j(x_1, x_2, \dots)}{\partial x_{k_1}^{q_1} \partial x_{k_2}^{q_2} \cdots \partial x_{k_s}^{q_s}} \Big\|_{F(m;\varepsilon_0;\theta)} < +\infty$$

$(q_1 + q_2 + \dots + q_s = 2q + 1;\ k_1, k_2, \dots, k_s \in \mathbf{N})$.

**Lemma 1.** *Let the countable system of the differential equations*

$$\frac{dx}{dt} = \Big(\Lambda(t,\varepsilon) + \sum_{l=1}^{q} B_l(t,\varepsilon,\theta)\mu^l\Big)x, \tag{2}$$

*where* $x = \operatorname{col}(x_1, x_2, \dots)$, $\Lambda(t,\varepsilon) = \operatorname{diag}(\lambda_1(t,\varepsilon), \lambda_2(t,\varepsilon), \dots)$, $\lambda_j \in S(m;\varepsilon_0)$, $B_l(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$ $(l = 1, \dots, q)$, $\mu \in (0, \mu_0) \subset \mathbf{R}^+$, *satisfy the condition:* $\forall\, n \in \mathbf{Z}$, $j \neq k$:

$$\inf_{G(\varepsilon_0)} \big|\lambda_j(t,\varepsilon) - \lambda_k(t,\varepsilon) - in\varphi(t,\varepsilon)\big| \ge \gamma_1 > 0,$$

*where* $\varphi(t,\varepsilon)$ *– the function is involved in the definition of the class* $F(m;\varepsilon_0;\theta)$. *Then there exists* $\mu_1 \in (0, \mu_0)$ *such that* $\forall\, \mu \in (0, \mu_1)$ *there exists a non-degenerate transformation*

$$x = \Big(E + \sum_{l=1}^{q} \Phi_l(t,\varepsilon,\theta)\mu^l\Big)y,$$

*where* $\Phi_l \in F_2(m;\varepsilon_0;\theta)$ $(l = 1, \dots, q)$, *which leads the system* (2) *to the kind:*

$$\frac{dy}{dt} = \Big(\Lambda(t,\varepsilon) + \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l + \varepsilon \sum_{l=1}^{q} V_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1} W(t,\varepsilon,\theta,\mu)\Big)y,$$

*where* $U_l(t,\varepsilon)$ *– infinite diagonal matrices whose elements belong to the class* $S(m;\varepsilon_0)$, $V_l, W \in F_2(m-1;\varepsilon_0;\theta)$ $(l = 1, \dots, q)$.

**Lemma 2.** *Let the system* (1) *satisfy conditions* $1^0$–$4^0$. *Then there exists* $\mu_2 \in (0, \mu_0)$ *such that* $\forall\, \mu \in (0, \mu_2)$ *there exists a transformation of kind*

$$x = \xi(t,\varepsilon,\theta,\mu) + \Psi(t,\varepsilon,\theta,\mu)y, \tag{3}$$

*where* $\xi(t,\varepsilon,\theta,\mu) \in F_1(m;\varepsilon_0;\theta)$, $\Psi(t,\varepsilon,\theta,\mu) \in F_2(m;\varepsilon_0;\theta)$, *which leads the system* (1) *to the kind:*

$$\begin{aligned} \frac{dy}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)y + \varepsilon h(t,\varepsilon,\theta,\mu) + \mu^{2q} r(t,\varepsilon,\theta,\mu) \\ + \varepsilon C(t,\varepsilon,\theta,\mu)y + \mu^{q+1} P(t,\varepsilon,\theta,\mu)y + \mu Y(t,\varepsilon,\theta,y,\mu), \end{aligned} \tag{4}$$

*where* $\widetilde{\Lambda}(t,\varepsilon) = \operatorname{diag}[\Lambda_1(t,\varepsilon),\Lambda_2(t,\varepsilon),\dots]$, $\Lambda_j(t,\varepsilon) = \operatorname{diag}(-i\omega_j(t,\varepsilon), i\omega_j(t,\varepsilon))$ $(j = 1,2,\dots)$, $K_l(t,\varepsilon) = \operatorname{diag}(k_{l,1}(t,\varepsilon), k_{l,2}(t,\varepsilon),\dots) \in S_2(m;\varepsilon_0)$, $h, r \in F_1(m-1;\varepsilon_0;\theta)$, $C, P \in F_2(m-1;\varepsilon_0;\theta)$. *Vector-function* $Y$ *belongs to the class* $F_1(m;\varepsilon_0;\theta)$ *with respect to* $(t,\varepsilon,\theta)$ *and contains the terms not lower than the second order with respect to the components of vector* $y$.

**Theorem 1.** *Let the system* (4) *satisfy the condition: there exists* $q_0 \in \mathbf{N}$ *such that* $|\operatorname{Re} k_{q_0,j}(t,\varepsilon)| \ge \gamma_0 > 0$, *and for all* $l = 1,\dots,q_0-1$ *(if* $q_0 > 1$*):* $\operatorname{Re} k_{l,j}(t,\varepsilon) \equiv 0$ $(j = 1,2,\dots)$. *Then there exists* $\mu_3 \in (0,\mu_0)$, $\varepsilon_1(\mu) \in (0,\varepsilon_0)$ *such that for all* $\mu \in (0,\mu_3)$, $\varepsilon \in (0,\varepsilon_1(\mu))$ *the system* (4) *has a particular solution* $y(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_1(\mu))$.

*Proof.* We make in the system (4) the substitution:

$$y = \frac{\varepsilon + \mu^{2q}}{\mu^{q_0}}\,\widetilde{y},$$

where $\widetilde{y}$ is a new unknown vector. We obtain:

$$\frac{d\widetilde{y}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu)$$
$$+\,\varepsilon C(t,\varepsilon,\theta,\mu)\widetilde{y} + \mu^{q+1}P(t,\varepsilon,\theta,\mu)\widetilde{y} + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\,\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y},\mu). \quad (5)$$

Consider the appropriate linear homogeneous and diagonal system:

$$\frac{d\widetilde{y}^{(0)}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y}^{(0)} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu). \quad (6)$$

In the paper [2] it has been found that the conditions of the theorem guarantee the existence of a particular solution $\widetilde{y}^{(0)}(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_0;\theta)$ of the system (6), and there exists $M \in (0,+\infty)$ such that

$$\|\widetilde{y}^{(0)}\|_{F_1(m-1;\varepsilon_0;\theta)} \le \frac{M}{\gamma_0\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,\|h\|_{F_1(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,\|r\|_{F_1(m-1;\varepsilon_0;\theta)}\Big)$$
$$< \frac{M}{\gamma_0}\big(\|h\|_{F_1(m-1;\varepsilon_0;\theta)} + \|r\|_{F_1(m-1;\varepsilon_0;\theta)}\big).$$

We seek the solution belonging to the class $F_1(m-1;\varepsilon_1(\mu);\theta)$ of the system (5) by the method of succesive approximations, defining the initial approximations $\widetilde{y}^{(0)}$ and the subsequents approximations defining as solutions, belonging to the class $F_1(m-1;\varepsilon_0;\theta)$ of the countable linear, homogeneous and diagonal systems:

$$\frac{d\widetilde{y}^{(s+1)}}{dt} = \Big(\widetilde{\Lambda}(t,\varepsilon) + \sum_{l=1}^{q} K_l(t,\varepsilon)\mu^l\Big)\widetilde{y}^{(s+1)} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,h(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,r(t,\varepsilon,\theta,\mu)$$
$$+\,\varepsilon C(t,\varepsilon,\theta,\mu)\widetilde{y}^{(s)} + \mu^{q+1}P(t,\varepsilon,\theta,\mu)\widetilde{y}^{(s)} + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\,\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y}^{(s)},\mu), \quad s = 0,1,2,\dots. \quad (7)$$

Let

$$\Omega = \Big\{\widetilde{y} \in F_1(m-1;\varepsilon_0;\theta) :\ \|\widetilde{y} - \widetilde{y}^{(0)}\|_{F_1(m-1;\varepsilon_0;\theta)} \le d\Big\}.$$

By virtue of the condition $4^0$, there exists $L(d) \in (0,+\infty)$ such that $\forall\,\widetilde{y},\widetilde{z} \in \Omega$:

$$\big\|\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y},\mu) - \widetilde{Y}(t,\varepsilon,\theta,\widetilde{z},\mu)\big\|_{F_1(m-1;\varepsilon_0;\theta)} \le L(d)\|\widetilde{y} - \widetilde{z}\|_{F_1(m-1;\varepsilon_0;\theta)}.$$

Using the ordinary technique of the contraction mapping principle [1], it is easy to show that there exists $\mu_3 \in (0,\mu_0)$, $N_1 \in (0,+\infty)$ such that $\forall\,\mu \in (0,\mu_0)$, $\forall\,\varepsilon \in (0,\varepsilon_1(\mu))$, where $\varepsilon_1(\mu) = N_1\mu^{2q_0-1}$, the process (7) converges to the solution $\widetilde{y}(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_1(\mu);\theta)$ of the system (5). □

Lemma 2 and Theorem 1 immediately yield the following theorem.

**Theorem 2.** *Let the system* (1) *satisfy conditions* $1^0$–$4^0$*, and the system* (4)*, which is obtained from the system* (1) *by the transformation* (3)*, satisfy the conditions of Theorem* 1*. Then there exists* $\mu_4 \in (0,\mu_0)$*,* $\varepsilon_2(\mu) \in (0,\varepsilon_0)$ *such that* $\forall\,\mu \in (0,\mu_4)$*,* $\varepsilon \in (0,\varepsilon_2(\mu))$ *the system* (1) *has a particular solution* $x(t,\varepsilon,\theta,\mu) \in F_1(m-1;\varepsilon_2(\mu);\theta)$*.*

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# On Fractional Boundary Value Problems with Positive and Increasing Solutions

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Let $J=[0,1]$ and $\mathbb{R}_0=[0,\infty)$.

We consider the fractional boundary value problem

$$ {}^c\!D^\alpha u(t)=q\big(t,u(t),u'(t)\big)\,{}^c\!D^\beta u(t)+f\big(t,u(t),u'(t)\big), \tag{1}$$

$$u(0)=ku'(0),\quad u(1)=ku'(1),\ \ k\geq \frac{1}{\alpha-1}\,, \tag{2}$$

where $1<\beta<\alpha\leq 2$, ${}^c\!D$ denotes the Caputo fractional derivative and

$(H_1)$ $f,q\in C(J\times\mathbb{R}_0^2)$ and

$$0\leq f(t,x,y),\ \ 0\leq q(t,x,y)\leq W<\infty\ \text{ for }\ (t,x,y)\in J\times\mathbb{R}_0^2. \tag{3}$$

The further conditions on $f$ will be specified later.

We recall that *the Riemann–Liouville fractional integral* $I^\gamma x$ of order $\gamma>0$ of a function $x:J\to\mathbb{R}$ is defined as [1,2]

$$I^\gamma x(t)=\int\limits_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s$$

and *the Caputo fractional derivative* ${}^c\!D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is given as

$${}^c\!D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\Big(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}\,s^k\Big)\,\mathrm{d}s,$$

provided that the right-hand sides exist. Here, $\Gamma$ is the Euler gamma function and $n=[\gamma]+1$, $[\gamma]$ means the integral part of the fractional number $\gamma$. $\Lambda^0$ is the identical operator and if $n\in\mathbb{N}$, then ${}^c\!D^n x(t)=x^{(n)}(t)$.

In particular,

$${}^c\!D^\gamma x(t)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\limits_0^t \frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)}\,\big(x(s)-x(0)-x'(0)s\big)\,\mathrm{d}s,\ \ \gamma\in(1,2).$$

**Definition.** We say that $u$ is *a solution of equation* (1) if $u\in C^1(J)$, ${}^c\!D^\alpha u\in C(J)$ and (1) holds for $t\in J$. A solution $u$ of (1) satisfying the boundary condition (2) is called *a solution of problem* (1), (2). We say that $u$ is *a positive and increasing solution of problem* (1), (2) if $u>0$ and $u'>0$ on $J$.

The special case of problem (1), (2) is the problem

$$u''(t) = q\big(t, u(t), u'(t)\big)\,{}^c\!D^{\beta} u(t) + f\big(t, u(t), u'(t)\big), \tag{4}$$

$$u(0) = ku'(0), \quad u(1) = ku'(1), \;\; k \geq 1. \tag{5}$$

Equation (4) is called the generalized Bagley–Torvik fractional differential equation (see [2–6]).

We are interested in the existence of positive and increasing solutions to problem (1), (2). To this end for $a \in C(J)$ introduce an operator $\Lambda_a : C(J) \to C(J)$ as

$$\Lambda_a x(t) = a(t) I^{\alpha-\beta} x(t).$$

For $n \in \mathbb{N}$, let $\Lambda_a^n = \underbrace{\Lambda_a \circ \Lambda_a \circ \cdots \circ \Lambda_a}_{n}$ be $n$th iteration of $\Lambda_a$ and $\mathcal{B}_a$ be an operator acting on $C(J)$ defined by the formula

$$\mathcal{B}_a x(t) = \sum_{n=0}^{\infty} \Lambda_a^n x(t).$$

For $\gamma > 0$, let $E_\gamma$ be the classical Mittag–Leffler functions [1, 2]

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)}\,, \quad z \in \mathbb{R}.$$

In the following result, solutions of the auxiliary linear fractional differential equation

$${}^c\!D^{\alpha} u(t) = a(t)\,{}^c\!D^{\beta} u(t) + r(t), \tag{6}$$

satisfying (2), are given by the operator $\mathcal{B}_a$.

**Lemma 1.** *Let $a, r \in C(J)$. Then the function*

$$u(t) = I^{\alpha}\mathcal{B}_a r(t) + (t + k)\Big(kI^{\alpha-1}\mathcal{B}_a r(t)\big|_{t=1} - I^{\alpha}\mathcal{B}_a r(t)\big|_{t=1}\Big), \;\; t \in J,$$

*is the unique solution to problem* (6), (2)*.*

Let

$$\mathcal{S} = \big\{x \in C^1(J) : \;\; x(t) \geq 0, \;\; x'(t) \geq 0 \;\text{ for }\; t \in J\big\}$$

and, under condition $(H_1)$, introduce the Nemytskii operators $\mathcal{Q}, \mathcal{F} : \mathcal{S} \to C(J)$,

$$\mathcal{Q}x(t) = q\big(t, x(t), x'(t)\big), \quad \mathcal{F}x(t) = f\big(t, x(t), x'(t)\big),$$

where $q$ and $f$ are from (1). It is clear that $\mathcal{S}$ is a cone in $C^1(J)$. Note that, by the definition,

$$\Lambda_{\mathcal{Q}x} y(t) = q\big(t, x(t), x'(t)\big) I^{\alpha-\beta} y(t).$$

Keeping in mind, Lemma 1 define an operator $\mathcal{K}$ acting on $\mathcal{S}$ by the formula

$$\mathcal{K}x(t) = I^{\alpha}\mathcal{L}_{\mathcal{Q}x} x(t) + (t + k)\Big(kI^{\alpha-1}\mathcal{L}_{\mathcal{Q}x} x(t)\big|_{t=1} - I^{\alpha}\mathcal{L}_{\mathcal{Q}x} x(t)\big|_{t=1}\Big),$$

where

$$\mathcal{L}_{\mathcal{Q}x} x(t) = \mathcal{B}_{\mathcal{Q}x}\mathcal{F}x(t)$$

and $k \geq 1/(\alpha - 1)$ is from (2).

The properties of $\mathcal{K}$ are summarized in the following lemma.

**Lemma 2.** *Let* $(H_1)$ *hold. Then* $\mathcal{K}:\mathcal{S}\to\mathcal{S}$, $\mathcal{K}$ *is a completely continuous operator and if* $u$ *is a fixed point of* $\mathcal{K}$, *then* $u$ *is a solution to problem* (1), (2).

In view of Lemma 2, we need to prove that the operator $\mathcal{K}$ admits a fixed point. The existene of a fixed point of $\mathcal{K}$ is proved in Theorem 1 by the Schauder fixed point theorem, while in Theorem 2 by the Guo–Krasnoselskii fixed point theorem on cones. We work with the following growth condition on the function $f$.

$(H_2)$ For $t\in J$ and $x,y\in\mathbb{R}_0$, the estimate

$$f(t,x,y)\le\varphi(x+y)$$

holds, where $\varphi\in C(\mathbb{R}_0)$, $\varphi$ is positive, nondecreasing and there exists $M>0$ such that

$$\varphi(M)\le\frac{M\Gamma(\alpha+1)}{(1+k)(\alpha k+\alpha-1)E_{\alpha-\beta}(W)}\,, \tag{7}$$

where $W$ is from $(H_1)$.

**Theorem 1.** *Let* $(H_1)$ *and* $(H_2)$ *hold. Let* $f(t_0,0,0)>0$ *for some* $t_0\in J$. *Then there exists at least one positive and increasing solution to problem* (1), (2).

If $f(t,0,0)=0$ on $J$, we can't apply Theorem 1 to problem (1), (2). In this case $u=0$ is a solution of this problem.

**Example 1.** Let $\rho,\mu\in(0,1)$, $a,p\in C(J)$ and $p(t_0)\neq 0$ for some $t_0\in J$. Theorem 1 guarantees that the equation

$${}^cD^\alpha u=\big|a(t)+\cos(x-y)\big|{}^cD^\beta u+|p(t)|+u^\rho+(u')^\mu$$

has at least one positive and increasing solution satisfying condition (2).

**Corollary 1.** *Let* $(H_1)$ *and* $(H_2)$ *with* (7) *replaced by*

$$\varphi(M)\le\frac{2M}{(1+k)(2k+1)E_{2-\beta}(W)}$$

*hold. Let* $f(t_0,0,0)>0$ *for some* $t_0\in J$. *Then there exists at least one positive and increasing solution to problem* (4), (5).

**Theorem 2.** *Let* $(H_1)$ *and* $(H_2)$ *hold. Let*

$$\lim_{x,y\in\mathbb{R}_0,\,x+y\to 0}\frac{f(t,x,y)}{x+y}>\frac{\Gamma(\alpha+1}{2(k\alpha-1)}\ \ \textit{uniformly on } J.$$

*Then problem* (1), (2) *has at least one positive and increasing solution.*

**Example 2.** Let $a,p\in C(J)$ and $p>\frac{\Gamma(\alpha+1)}{2(k\alpha-1)}$. Theorem 2 guarenrees that there exists a positive and increasing solution of the equation

$${}^cD^\alpha u=\big|a(t)+e^{-u}\sin u'\big|{}^cD^\beta u+p(t)(u+u')e^{-u-u'}, \tag{8}$$

satisfying condition (2). Note that $u=0$ is also a solution to problem (8), (2).

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# On Existence, Uniqueness and Continuous Dependence from Initial Datum of Mild Solution for Neutral Stochastic Differential Equation of Reaction-Diffusion Type in Hilbert Space

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## 1 Introduction

The following initial-value problem is considered

$$d\bigg(u(t,x)+\int\limits_{\mathbb{R}^d} b(t,x,u(\alpha(t),\xi),\xi)\,d\xi\bigg)$$

$$=\big(\Delta_x u(t,x)+f(t,u(\alpha(t),x),x)\big)\,dt+\sigma\big(t,u(\alpha(t),x),x\big)\,dW(t,x),\ \ 0<t\le T,\ \ x\in\mathbb{R}^d, \tag{1}$$

$$u(t,x)=\phi(t,x),\ \ -r\le t\le 0,\ \ x\in\mathbb{R}^d,\ \ r>0, \tag{2}$$

where $\Delta_x\equiv\sum\limits_{i=1}^{d}\partial_{x_i}^2$ is $d$-measurable operator of Laplace, $\partial_{x_i}^2\equiv\frac{\partial^2}{\partial x_i^2}$, $i\in\{1,\dots,d\}$, $W(t)=W(t,\,\cdot\,)$ is $L_2(\mathbb{R}^d)$-valued $Q$-Wiener process, $\{f,\sigma\}\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are some given functions to be specified later, $\phi\colon[-r,0]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is an initial-datum function and $\alpha\colon[0,T]\to[-r,\infty)$ is a delay-function.

Differential equations with delay have appeared as mathematical models of real processes, evolution of which depends on previous states. Number of works are devoted to investigation qualitative theory of stochastic differential equations with delay in finite-dimensional spaces. With regard to such equations in infinite-dimensional spaces, let us remark the work [3], where theorem on existence and uniqueness of mild solution to neutral stochastic differential equation in Hilbert space has been proved. But conditions of this theorem are formulated in an abstract form, therefore it is difficult to check them directly for concrete equations in specific spaces, e.g., for stochastic partial differential equations of reaction-diffusion type. For such equations abstract mappings are generated by real-valued functions as operator of Nemytskii. Thus our expectations to receive conditions in terms of coefficients of these equations, i.e. in terms of real-valued functions, are natural. If such conditions are found, it will be possible to check them easily while solving concrete applied problems. Equation (1), considered in our work, is special case of equation from the work [3]. It has an applied importance: it models behavior of various dynamical systems in physics and mathematical biology. Equations of such type are well known in literature and have a wide range of applications. The presence of an integral term in (1) turns this equation into nonlocal neutral stochastic equation of reaction-diffusion type.

## 2 Preliminaries

Throughout the article $L_2(\mathbb{R}^d)$ will denote real Hilbert space with an inner product $(f,g)_{L_2(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} f(x)g(x)\,dx$ and the corresponding norm $\|f\|_{L_2(\mathbb{R}^d)} = \sqrt{\int\limits_{\mathbb{R}^d} f^2(x)\,dx}$. Let $\{e_n(x), n\in\{1,2,\dots\}\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}} \|e_n\|_{L_\infty(\mathbb{R}^d)} \le 1$. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space. We now define $Q$-Wiener $L_2(\mathbb{R}^d)$-valued process $W(t)=W(t,\,\cdot\,)$ as follows

$$W(t,x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, e_n(x)\beta_n(t), \quad t\ge 0, \quad x\in\mathbb{R}^d, \tag{3}$$

where $\{\beta_n(t), n\in\{1,2,\dots\}\}\subset\mathbb{R}$ are independent standard one-dimensional Wiener processes on $t\ge 0$, $\{\lambda_n, n\in\{1,2,\dots\}\}$ is a sequence of positive numbers such that $\sum\limits_{n=1}^{\infty}\lambda_n < \infty$. Let $\{\mathcal{F}_t(dW), t\ge 0\}$ be normal filtration, generated by (3). It means that $\mathcal{F}_t(dW)$ is the least $\sigma$-algebra such that increments $W(t)-W(s)$ are measurable with respect to this $\sigma$-algebra for $0\le s\le t$. It is clear that $W(t)-W(s)$, $s\le t$, are independent from $\mathcal{F}_s(dW)$.

In what fellows, we will need some facts on the Cauchy problem for heat-equation

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad t>0, \quad x\in\mathbb{R}^d,\\ u(0,x) &= g(x), \quad x\in\mathbb{R}^d. \end{aligned} \tag{4}$$

Let us denote

$$\mathscr{K}(t,x) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac d2}}\exp\Big\{-\dfrac{|x|^2}{4t}\Big\}, & t>0,\\ 0, & t<0, \end{cases} \quad \text{– heat-kernel.}$$

**Proposition 2.1** ([1, p. 47])**.** *If $g$ in (4) belongs to $L_2(\mathbb{R}^d)$, then it's solution will be represented by the following formula*

$$u(t,x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t,x-\xi)g(\xi)\,d\xi,$$

*an besides $u\in C^\infty((0,\infty)\times\mathbb{R}^d)$.*

**Proposition 2.2** ([1, pp. 242–244])**.** *Operators $S(t)\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$, generating solution of the Cauchy problem (4) by the rule*

$$u(t,x) = (S(t)g(\,\cdot\,))(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t,x-\xi)g(\xi)\,d\xi,$$

*form an analytic contractive $(C_0$-$)$semi-group of operators, i.e. the following estimate is valid*

$$\big\|(S(t)g(\,\cdot\,))(x)\big\|^2_{L_2(\mathbb{R}^d)} \le \|g(x)\|^2_{L_2(\mathbb{R}^d)},$$

*and besides Laplacian $\Delta_x$ is an infinitesimal generator of this semi-group.*

**Proposition 2.3** ([2, p. 274])**.** *For partial derivatives of $\mathscr{K}$ the following estimate is true*

$$\big|\partial_t^r\partial_x^s\mathscr{K}(t,x)\big| \le c_{r,s}t^{-\frac d2 - r - \frac s2}\exp\Big\{-\frac{c_0|x|^2}{t}\Big\}, \quad c_{r,s}>0, \quad c_0<\frac14\,. \tag{5}$$

**Proposition 2.4.** *If $g$ in* (4) *belongs to $L_1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, then solution of this problem will satisfy the following limit relations*

$$\lim_{|x|\to\infty} u(t,x)=0, \quad \lim_{|x|\to\infty} \partial_t u(t,x)=0. \tag{6}$$

◀ The proof follows from standard theorems on possibility to limit transition in Lebesgue integral and differentiability of integral by parameter via using estimate (5). ▶

From Propositions 2.1 and 2.4 we have the following result.

**Proposition 2.5** ([2, p. 360])**.** *If relations* (6) *are valid, then for some $C_T>0$, depending only on $T$, solution of* (4) *will satisfy*

$$\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}(\Delta_x u(t,x))^2\,dx=\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\big\|D_x^2u(t,x)\big\|_d^2\,dx\le C_T\int\limits_{\mathbb{R}^d}\|D^2g(x)\|_d^2\,dx,$$

*where* $\nabla_x\equiv(\partial_{x_1}\cdots\partial_{x_d})^\top$, $D_x^2\equiv\begin{pmatrix}\partial_{x_1}^2 & \cdots & \partial_{x_1x_d}\\ \vdots & \ddots & \vdots\\ \partial_{x_dx_1} & \cdots & \partial_{x_d}^2\end{pmatrix}$ *is Hesse-operator, $\|\cdot\|_d$ is the corresponding norm of matrix.*

# 3 Formulation of the problem

The following assumptions are the main, assumed in the article.

**3.1)** $\alpha\colon[0,T]\to[-r,\infty)$ is function from $C^1([0,T])$ such that $0<\alpha'\le 1$ (observe that there exist a constant $c>0$ and a unique point $0\le t^*\le T$ such that $\frac{1}{\alpha'}\le c$, $\alpha(t^*)=0$);

**3.2)** $\{f,\sigma\}\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are measurable with respect to all of their variables functions, and $b$ is continuous by its first argument;

**3.3)** initial-datum function $\phi(t,\cdot,\omega)\colon[-r,0]\times\Omega\to L_2(\mathbb{R}^d)$ is $\mathcal{F}_0$-measurable random variable, independent from $W$, with almost surely continuous paths and such that

$$\mathbf{E}\phi^2(t)<\infty,\ \ -r\le t\le 0,$$
$$\mathbf{E}\sup_{-r\le t\le 0}\|\phi(t)\|^p_{L_2(\mathbb{R}^d)}<\infty,\ \ p>2;$$

**3.4)** for $\{f,\sigma\}$, there exist a constant $L>0$ and a function $\chi\colon[0,T]\times\mathbb{R}^d\to[0,\infty)$ such that

$$\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\chi^2(t,x)\,dx<\infty$$

and the following conditions of linear-growth and Lipschitz are valid

$$|f(t,u,x)|\le\chi(t,x)+L|u|,\ \ 0\le t\le T,\ \ u\in\mathbb{R},\ \ x\in\mathbb{R}^d,$$
$$\big|f(t,u,x)-f(t,v,x)\big|\le L|u-v|,\ \ 0\le t\le T,\ \ \{u,v\}\subset\mathbb{R},\ \ x\in\mathbb{R}^d,$$
$$|\sigma(t,u,x)|\le L(1+|u|),\ \ 0\le t\le T,\ \ u\in\mathbb{R},\ \ x\in\mathbb{R}^d,$$
$$\big|\sigma(t,u,x)-\sigma(t,v,x)\big|\le L|u-v|,\ \ 0\le t\le T,\ \ \{u,v\}\subset\mathbb{R},\ \ x\in\mathbb{R}^d;$$

**3.5)** $|b(t,x,0,\xi)| \le b_1(x,\xi)$, $0 \le t \le T$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, where function $b_1 \colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ satisfies conditions

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} b_1(x,\zeta)\,d\zeta\,dx < \infty, \quad \int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} b_1(x,\zeta)\,d\zeta\Big)^2\,dx < \infty;$$

**3.6)** there exists a function $l \colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ such that

$$\big|b(t,x,u,\xi) - b(t,x,v,\xi)\big| \le l(x,\xi)|u-v|, \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ \{u,v\} \subset \mathbb{R},$$

and $l$ satisfies the following conditions

$$\int\limits_{\mathbb{R}^d}\sqrt{\int\limits_{\mathbb{R}^d} l^2(x,\zeta)\,d\zeta}\,dx < \infty, \quad \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(x,\zeta)\,d\zeta\,dx < \infty;$$

**3.7)** for each $x \in \mathbb{R}^d$, there exist partial derivatives $\partial_{x_i} b$, $\partial_{x_i x_j} b$, $\{i,j\} \subset \{1,\dots,d\}$, and for gradient-vector $\nabla_x b$ and Hesse-matrix $D_x^2 b$ the following condition of linear-growth by the third argument is true

$$|\nabla_x b(t,x,u,\xi)| + \|D_x^2 b(t,x,u,\xi)\|_d \le \psi(t,x,\xi)(1+|u|), \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ u \in \mathbb{R},$$

and for $D_x^2 b$ – Lipschitz condition

$$\big\|D_x^2 b(t,x,u,\xi) - D_x^2 b(t,x,v,\xi)\big\|_d \le \psi(t,x,\xi)|u-v|, \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ \{u,v\} \subset \mathbb{R},$$

where function $\psi \colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} \psi(t,x,\xi)\,d\xi\Big)^2\,dx < \infty, \quad \sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \psi^2(t,x,\xi)\,d\xi\,dx < \infty,$$

and besides for each point $x_0 \in \mathbb{R}^d$, there exist its vicinity $B_\delta(x_0)$ and a nonnegative function $\varphi$ such that

$$\sup_{0 \le t \le T} \varphi(t,\,\cdot\,,x_0,\delta) \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d), \ \ \delta > 0,$$
$$\big|\psi(t,x,\xi) - \psi(t,x_0,\xi)\big| \le \varphi(t,\xi,x_0,\delta)|x-x_0|, \ \ 0 \le t \le T, \ \ |x-x_0| < \delta, \ \ \xi \in \mathbb{R}^d.$$

**Definition 3.1.** Continuous random process $u(t,\,\cdot\,,\omega) : [-r,T] \times \Omega \to L_2(\mathbb{R}^d)$ is called mild solution of (1), (2) if it

1) is $\mathcal{F}_t$-measurable for almost all $-r \le t \le T$;

2) satisfies the following integral equation

$$u(t,\,\cdot\,) = S(t)\Big(\phi(0,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b\big(0,\,\cdot\,,\phi(-r,\zeta),\zeta\big)\,d\zeta\Big) - \int\limits_{\mathbb{R}^d} b\big(t,\,\cdot\,,u(\alpha(t),\xi),\xi\big)\,d\xi$$
$$- \int\limits_0^t \Delta_{(\,\cdot\,)}\Big(S(t-s)\int\limits_{\mathbb{R}^d} b\big(s,\,\cdot\,,u(\alpha(s),\zeta),\zeta\big)\,d\zeta\Big)\,ds$$

$$
\begin{aligned}
&\quad + \int\limits_0^t S(t-s) f\big(s, u(\alpha(s), \cdot), \cdot\big)\, ds \\
&\quad + \int\limits_0^t S(t-s) \sigma\big(s, u(\alpha(s), \cdot), \cdot\big)\, dW(s, \cdot), \ \ 0 \le t \le T, \\
&u(t, \cdot) = \phi(t, \cdot), \ \ -r \le t \le 0, \ \ r > 0.
\end{aligned}
$$

**Remark 3.1.** It is assumed in the definition above that all integrals make sense.

Our first result is concerned with existence and uniqueness of solution to (1), (2).

**Theorem 3.1** (existence and uniqueness)**.** *Suppose that assumptions* 3.1–3.7 *are satisfied. Then, if*

$$
\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} l^2(x, \xi)\, d\xi\, dx < \frac{1}{4}\,,
$$

*the Cauchy problem* (1), (2) *has a unique for* $0 \le t \le T$ *mild solution.*

**Remark 3.2.** If we replace an initial range $[-r, 0]$ from (2) with $[s-r, s]$ for arbitrary $s \ge 0$, it will be possible to guarantee existence and uniqueness of mild solution to (1), (2) for $0 \le s \le t$.

Concerning continuation of mild solution to (1), (2) on the whole semi-axis $t \ge 0$, the following corollary is true.

**Corollary 3.1.** *If in Theorem* 3.1 *conditions* 3.4–3.7 *are valid for* $t \ge 0$*, then the Cauchy problem* (1), (2) *has a unique mild solution for* $t \ge 0$*.*

The next result is concerned with continuous dependence of $u$ from the corresponding initial-datum function $\phi$.

**Theorem 3.2** (continuous dependence)**.** *Under the conditions of Theorem* 3.1*, there exists* $C(T) > 0$ *such that for arbitrary admissible initial-datum functions* $\phi$ *and* $\phi_1$ *the following estimates hold*

$$
\mathbf{E} \sup_{0 \le t \le T} \big\| u(t, \phi) - u(t, \phi_1) \big\|^p_{L_2(\mathbb{R}^d)} \le C(T) \mathbf{E} \sup_{-r \le t \le 0} \| \phi(t) - \phi_1(t) \|^p_{L_2(\mathbb{R}^d)}, \ \ p > 2,
$$

*where* $u(t, \phi)$ *denotes solution* $u(t, x)$ *of* (1) *that satisfies* (2)*.*

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# Effects of Several Delays Perturbations in the Variation Formulas of Solution for a Functional Differential Equation with the Discontinuous Initial Condition

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Let $\theta_{i2} > \theta_{i1} > 0$, $i = \overline{1,s}$, be given numbers and $O \subset \mathbb{R}^n$ be an open set. Let $E_f$ be the set of functions $f : I \times O^{1+s} \to \mathbb{R}^n$, $I = [a,b]$, satisfying the following conditions: for almost all fixed $t \in I$ the function $f(t,\,\cdot\,) : O^{1+s} \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x, x_1, \dots, x_s) \in O^{1+s}$ the functions $f(t, x, x_1, \dots, x_s)$, $f_x(t,\,\cdot\,)$ and $f_{x_i}(t,\,\cdot\,)$, $i = \overline{1,s}$, are measurable on $I$; for any $f \in E_f$ and compact set $K \subset O$ there exists a function $m_{f,K}(t) \in L_1(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$, such that

$$\big|f(t,x,x_1,\dots,x_s)\big| + |f_x(t,\,\cdot\,)| + \sum_{i=1}^{s} \big|f_{x_i}(t,\,\cdot\,)\big| \le m_{f,K}(t)$$

for all $(x, x_1, \dots, x_s) \in K^{1+s}$ and for almost all $t \in I$.

Let $\Phi$ be the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, where $\widehat{\tau} = a - \max\{\theta_{12}, \dots, \theta_{s2}\}$. To each element $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda = [a,b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f$ we set in correspondence the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x(t_0) = x_0. \tag{2}$$

The condition (2) is said to be the discontinuous initial condition since, in general, $x(t_0) \neq \varphi(t_0)$.

**Definition.** Let $\mu = (t_0, \tau_1, \dots, \tau_s, x_0, \varphi, f) \in \Lambda$. A function $x(t) = x(t;\mu) \in O, t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element $\mu$ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let us introduce the set of variation:

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau_1, \dots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) : \; |\delta t_0| \le \alpha, \; |\delta\tau_i| \le \alpha, \; i = \overline{1,s},$$
$$|\delta x_0| \le \alpha, \; \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \; \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \; |\lambda_i| \le \alpha, \; i = \overline{1,k}\Big\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta f_i \in E_f - f_0$, $i = \overline{1,k}$, $\varphi_0 \in \Phi$, $f_0 \in E_f$ are fixed functions; $\alpha > 0$ is a fixed number.

Let

$$\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda \tag{3}$$

be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1,s}$. Let $x_0(t)$ be the solution corresponding to $\mu_0$. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it (see [4, Theorem 1.2]). By the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ to the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau}, t_{10} + \delta_1]$. Now we introduce the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \quad (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V.$$

**Theorem 1.** *Let the following conditions hold:*

1) $\tau_{10} < \dots < \tau_{s0}$ *(see* (3)*) and* $t_{00} + \tau_{s0} < t_{10}$*;*
2) *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$, $t \in I_1$, *is bounded;*
3) *the function* $f_0(w)$, $w = (t, x, x_1, \dots, x_s) \in I \times O^{1+s}$, *is bounded;*
4) *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \quad w \in (a, t_{00}] \times O^{1+s},$$

*where* $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}))$*;*

5) *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}, \quad w_{1i}, w_{2i} \in (a,b) \times O^{1+s}, \quad i = \overline{1,s},$$

*where*

$$\begin{aligned} w_{1i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}), \\ &x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big), \\ w_{2i}^0 = \Big(&t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}), \\ &\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \dots, x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big). \end{aligned}$$

*Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, $t_{00} + \tau_{s0} < t_{10} - \delta_2$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^-$, *where* $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$, *we have*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \tag{4}$$

*Here*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0^- \delta t_0 + \beta(t; \delta\mu), \tag{5}$$

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta x_0 - \Big[\sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}\Big] \delta t_0$$

$$-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int\limits_{t_{00}}^{t_{00}+\tau_{i0}}Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi$$

$$+\int\limits_{t_{00}+\tau_{i0}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$

$$+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi,$$

*where $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation*

$$Y_\xi(\xi;t)=-Y(\xi;t)f_{0x}[\xi]-\sum_{i=1}^{s}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t]$$

*and the condition:*

$$Y(\xi;t)=H\ \textit{ for }\ \xi=t,\quad Y(\xi;t)=\Theta\ \textit{ for }\ \xi>t;$$

*$H$ is the identity matrix and $\Theta$ is the zero matrix;*

$$f_{0x_i}[\xi]=f_{0x_i}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big),$$
$$\delta f[\xi]=\delta f\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0})\big).$$

The expression (5) is called the variation formula of solution. The addend

$$-\sum_{i=1}^{s}\bigg[Y(t_{00}+\tau_{i0};t)f_{0i}+\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i$$

in the formula (5) is the effects of perturbations of the delays $\tau_{i0}$, $i=\overline{1,s}$. For the ordinary differential equation the variation formula of solution has been proved in the monograph R. V. Gamkrelidze [1]. In [3] variation formulas of solution were proved for the equation $\dot{x}(t)=f(t,x(t),x(t-\tau))$ with the condition (2) in the case when the initial moment and delay variations have the same signs. In the present paper, the equation with several delays is considered and variation formulas of solution are obtained with respect to wide classes of variations (see $V^-$ and $V^+$). Variation formulas of solution for various classes of delay functional differential equations, without perturbations of delays, are proved in [2].

**Theorem 2.** *Let the conditions* 1)–3) *and* 5) *of the Theorem* 1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{1+s}. \tag{6}$$

*Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\delta t_0\le 0\}$, the formula* (4) *holds. Here*

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f_0^+\delta t_0+\beta(t;\delta\mu).$$

**Theorem 3.** *Let the conditions* 1)–4) *of the Theorem* 1 *hold. Moreover, there exists the finite limits:*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}\big[f_0(w_{1i})-f_0(w_{2i})\big]=f_{0i}^-,\ \ w_{1i},w_{2i}\in(a,t_{00}+\tau_{i0}]\times O^{1+s},\ \ i=\overline{1,s},$$

*Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_1^-$, *where* $V_1^- = \{\delta\mu \in V : \delta t_0 \leq 0, \delta\tau_i \leq 0, \ i = \overline{1, s}\}$ *the formula* (4) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^- + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^-\Big] \delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^-\big] \delta\tau_i + \beta_1(t; \delta\mu),$$

*where*

$$\begin{aligned}
\beta_1(t; \delta\mu) &= Y(t_{00}; t)\delta x_0 \\
&+ \sum_{i=1}^{s} \Bigg[ \int_{t_{00}}^{t_{00}+\tau_{i0}} Y(\xi; t) f_{0x_i}[\xi] \dot{\varphi}_0(\xi - \tau_{i0})\, d\xi + \int_{t_{00}+\tau_{i0}}^{t} Y(\xi; t) f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0})\, d\xi \Bigg] \delta\tau_i \\
&+ \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi)\, d\xi + \int_{t_{00}}^{t} Y(\xi; t) \delta f[\xi]\, d\xi.
\end{aligned}$$

**Theorem 4.** *Let the conditions* 1)–3) *of the Theorem* 1 *and the condition* (6) *hold. Moreover, there exists the finite limits:*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}^+, \ \ w_{1i}, w_{2i} \in [t_{00} + \tau_{i0}, b) \times O^{1+s}, \ \ i = \overline{1, s},$$

*Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_1^+$, *where* $V_1^+ = \{\delta\mu \in V : \delta t_0 \geq 0, \ \delta\tau_i \geq 0, \ i = \overline{1, s}\}$ *the formula* (4) *holds. Here*

$$\delta x(t; \delta\mu) = -\Big[Y(t_{00}; t) f_0^+ + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t) f_{0i}^+\Big] \delta t_0 - \sum_{i=1}^{s} \big[Y(t_{00} + \tau_{i0}; t) f_{0i}^+\big] \delta\tau_i + \beta_1(t; \delta\mu).$$

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# On Exponential Equivalence of Solutions to Nonlinear Differential Equations

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## 1 Introduction

The equations

$$y^{(n)} + \frac{a}{x^2}\,y + p(x)y|y|^{k-1} = f(x), \tag{1.1}$$

$$z^{(n)} + \frac{a}{x^2}\,z + p(x)z|z|^{k-1} = 0 \tag{1.2}$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$ are considered. Functions $p(x)$ and $f(x)$ are assumed to be continuous as $x > x_0 > 0$, $p(x) \not\equiv 0$. Exponential equivalence of solutions to equations (1.1), (1.2) is proved under some assumptions on the function $f(x)$. If $a = 0$, equation (1.2) is well-known Emden–Fowler equation:

$$z^{(n)} + p(x)z|z|^{k-1} = 0.$$

A lot of results on the asymptotic behaviour of solutions to this equation and its generalizations were obtained in [1,2,4–6]. Note that equation (1.2) with $a \neq 0$ can't be reduced to Emden–Fowler differential equation by any substitution of dependent or independent variables.

## 2 Exponential equivalence of solutions to nonlinear differential equations

Consider the differential equations

$$y^{(n)} + \frac{a}{x^2}\,y + p(x)y|y|^{k-1} = e^{-\alpha x}f(x), \tag{2.1}$$

$$z^{(n)} + \frac{a}{x^2}\,z + p(x)z|z|^{k-1} = e^{-\alpha x}g(x). \tag{2.2}$$

with $n \geq 2$, $k > 1$, $a \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$.

**Lemma 2.1** ([3])**.** *If function $y(x)$ and its $n$-th derivative $y^{(n)}(x)$ tend to zero as $x \to +\infty$, then the same holds for $y^{(j)}(x)$, $0 < j < n$.*

**Lemma 2.2.** *Let $y(x)$ be a solution to equation* (2.1) *such that $y(x)$ tends to zero as $x \to +\infty$. Then it holds*

$$y(x) = \mathbf{J^n}\Big[e^{-\alpha x}f(x) - \frac{a}{x^2}\,y(x) - p(x)[y(x)]^k_{\pm}\Big]$$

*with $[y(x)]^k_{\pm} = |y|^{k-1}y$.* $\mathbf{J}$ *is the operator that maps tending to zero as $x \to +\infty$ function $\varphi(x)$ to its antiderivative:*

$$\mathbf{J}[\varphi](x) = -\int\limits_x^{+\infty} \varphi(t)\,dt.$$

**Theorem 2.1.** *Let $p(x)$, $f(x)$, $g(x)$ be continuous bounded functions defined as $x > x_0 > 0$, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation* (2.1) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (2.2) *such that*

$$|z(x) - y(x)| = O(e^{-\alpha x}), \;\; x \to +\infty.$$

**Remark 2.1.** Obviously, equations (2.1) and (2.2) in Theorem 2.1 can be swapped.

Back to equations (1.1), (1.2):

$$y^{(n)} + \frac{a}{x^2}\, y + p(x) y |y|^{k-1} = f(x),$$
$$z^{(n)} + \frac{a}{x^2}\, z + p(x) z |z|^{k-1} = 0$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$.

**Corollary 2.1.1.** *Suppose continuous function $f(x)$ satisfies the following condition*

$$f(x) = O(e^{-\alpha x}), \;\; \alpha > 0.$$

*Let function $p(x)$ be a continuous bounded function, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation* (1.1) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (1.2) *such that*

$$|y(x) - z(x)| = O(e^{-\alpha x}), \;\; x \to +\infty.$$

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