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ABSTRACTS

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Dedicated to the 125th birthday anniversary of Professor Andrea Razmadze

12.08.1889 – 02.10.1929

The well-known Georgian mathematician, one of the founders of Tbilisi State University, A. Razmadze was born on August 12, 1889 in the village of Chkhenishi (Samtredia region, Georgia). In 1906 he finished the real school in Kutaisi, and in 1910 he graduated from the Moscow State University. The first A. Razmadze's scientific work dealing with the basic problem of variational calculus with free ends was published in 1914 in the journal *"Matematische Annalen"*. The lemma belonging to A. Razmadze allows one to get easily Euler's differential equation; the same lemma results, as particular cases, in lemmas due to Dubois–Reymond and Lagrange. The work *"Sur les exstremales discontinues dans le calcul des variations"* published in 1925 won him wide recognition and popularity. Later A. Razmadze used this work for his Doctoral Thesis which he defended in Sorbonne University (France). A. Razmadze's theory of discontinuous extremals has found reflection and further development in the works of many scientists. In 1924, A. Razmadze participated in the work of the International Congress of Mathematicians (Toronto, Canada). Up to his dying day, A. Razmadze kept close scientific contacts with C. Carathéodory and L. Tonelli. The last A. Razmadze's work *"Sur les solutions périodiques et les extrémales fermées du calcul de variations"* was estimated by C. Carathéodory as follows: *"The theory of closed extremals on a plane is brought by A. Razmadze to perfection"*. Later, the results obtained by A. Razmadze in the abovementioned work, C. Carathéodory included in one of the sections of his book *"Variationrechnung und partielle differentialgleichungen, Lpz.-B.,* 1935" under the title *"The Hadamard and Razmadze Theory"*. Text-books in mathematics in Georgian language for the first time were published under A. Razmadze's authorship.

A. Razmadze passed away in 1929 in the age of 40. The Georgian Mathematical Society announced with deep regret the untimely death of professor A. Razmadze, founder and president of the Society, first Georgian mathematician, founder of higher mathematical education in Georgia. Italian mathematician L. Tonelli wrote: *"Mathematics in the person of A. Razmadze has lost one of its outstanding scientists, researcher with lucid mentality and keen intellect"*.

Dynamics of Integro-Differential Cellular Neural Network Model

G. Agranovich

Department of Electrical and Electronic Engineering, Ariel University Center of Samaria, Ariel, Israel

E-mail: agr@ariel.ac.il

E. Litsyn

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel

E-mail: elena.litsyn@weizmann.ac.il

A. Slavova

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria

E-mail: slavova@math.bas.bg

1 Introduction

The main aim of this presentation is to study the dynamics of integro-differential Cellular Neural Network model from both theoretical and numerical points of view.

Let us first consider the following integro-differential problem:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} - \beta \int\limits_0^t e^{-\gamma(t-s)}u(s)\,ds + f(u), \quad t \in (0,T], \tag{1}$$

where γ and β are positive constants, f is a nonlinear function depending on u. If we take it in the form $f(u) = u(1-u)(u-\alpha)$, α being positive constant $0 < \alpha < 1$, this model describes the nerve impulse transmission and is known as FitzHugh–Nagumo equation [1]. For the equation (1) stability results are established in [1]. Finite difference method is proposed in order to solve (1) numerically. The application of this method to the above integro-differential problem needs a great storage of information in each time level. For this reason in this paper we shall propose Cellular Neural Networks (CNN) appraoch in order to study such kind of problems in real time due to the parallelism of the proposed architecture.

The model we shall consider is a more general form of the famous Hodgkin–Huxley model for propagation of the voltage pulse trough a nerve axon [3]:

$$u_t - D\nabla^2 u = \sigma u(u-\alpha)(1-u) - \beta \int^t g(u(s,x))\,ds, \tag{2}$$

where $0 < x,t < 1$, $0 < \alpha < 1$, σ, $\beta > 0$, D-the diffusion coefficients, g is a nonlinear function depending on u. The proposed equation (2) is a nonlinear parabolic integro-differential equation, $u(x,t)$ is a membrane in a nerve axon, the steady state $u = 0$ represents the resting state of the nerve. For (2) travelling wave solutions have been constructed in [5]. In this paper we shall study the dynamics of (2). We shall construct CNN architecture for integro-differential equation (2) in the nex section.

2 Integro-Differential CNN Model and its Dynamics

Cellular Neural Networks (CNNs) [2] are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable. Namely speaking, this phenomenon arises as the loss of stability and the birth of a limit cycles.

It is known that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs) [2]. The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. Consider the following well-known PDE, generally referred to us in the literature as a reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u,$$

where $u \in \mathbf{R}^N$, $f \in \mathbf{R}^N$, D is a matrix with the diffusion coefficients, and $\nabla^2 u$ is the Laplacian operator in $\mathbf{R}^2$. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A-template. An one-dimensional discretized Laplacian template will be in the following form:

$$A_1 = (1, -2, 1).$$

This is the two-dimensional discretized Laplacian A_2 template:

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For the integro-differential equation (2), CNN model will be the following:

$$\frac{du_{ij}}{dt} - DA_2 * u_{ij} = \sigma u_{ij}(1 - u_{ij})(u_{ij} - \alpha) - \beta \int_0^t g(u_{ij}(s))\, ds, \quad 1 \le i, j \le M. \tag{3}$$

We shall use an approximative method for studying the dynamics of integro-differential CNN (ID-CNN) model (3), based on a special Fourier transform. The idea of using Fourier expansion for finding the solutions of PDEs is well known in physics [4]. This special spectral technique is related to Harmonic Balance Method [4] well known in control theory and in the study of electronic oscillators as describing function method. The method is based on the fact that all cells in CNN are identical [2]. It is usually applied for discovering the existence and characteristics of periodic solutions.

In our case we apply the following double Fourier transform:

$$F(s, z_1, z_2) = \sum_{i=-\infty}^{\infty} z_1^{-i} \sum_{j=-\infty}^{\infty} z_2^{-j} \int_{-\infty}^{\infty} f_{ij}(t) \exp(-st)\, dt. \tag{4}$$

We apply this transform to ID-CNN model (3) and we obtain the following transfer function [4]:

$$H(s, z_1, z_2) = \frac{s}{s^2 - s(z_2^{-1} + z_2 - 4 + z_1^{-1} + z_1) + \beta}\,. \tag{5}$$

In the above transfer function $s = i\omega_0$, $z_1 \exp(i\Omega_1)$, $z_2 = \exp(i\Omega_2$, where ω_0, Ω_1, Ω_2 are temporal and two spatial frequencies, respectively, $i = \sqrt{-1}$.

We are looking for the periodic solutions of (3) of the form:

$$u_{ij}(t) = U_{m_0} \sin(\omega_0 t + i\Omega_1 + j\Omega_2), \tag{6}$$

where the temporal frequency is $\omega_0 = \frac{2\pi}{T_0}$, $T_0 > 0$ is the minimal period. If we take boundary conditions for ID-CNN model (3) which will make the array circular, we obtain:

$$\Omega_1 + \Omega_2 = \frac{2K\pi}{n}\,, \quad 0 \le K \le n-1, \quad n = M.M. \tag{7}$$

Applying describing function technique we obtain the following system for unknowns U_{m_0}, ω_0, Ω_1 and Ω_2:

$$\begin{gathered}
\Omega_1 + \Omega_2 = \frac{2K\pi}{n}\,, \quad 0 \le K \le n-1, \\
1 + \Big(\sigma\alpha + \frac{3}{4}\,\sigma U_{m_0}^2\Big)\,\frac{\omega_0 A}{(\beta - \omega_0^2)^2 + A\omega_0^2} = 0, \\
1 + \Big(\sigma\alpha + \frac{3}{4}\,\sigma U_{m_0}^2\Big)\,\frac{\omega_0(\beta - \omega_0^2)}{(\beta - \omega_0^2)^2 + A\omega_0^2} = 0,
\end{gathered} \tag{8}$$

where $A = 4 - 2\cos\Omega_1 - 2\Omega_2$.

Then the following proposition holds.

Proposition 1. *ID-CNN model* (3) *with circular array of M cells has periodic solutions with period $T_0 = \frac{2\pi}{\omega_0}$ and amplitude U_{m_0} for all $\Omega_1 + \Omega_2 = \frac{2K\pi}{n}$, $0 \le K \le n-1$.*

We obtain the following computer simulations of the solutions of ID-CNN model (3):

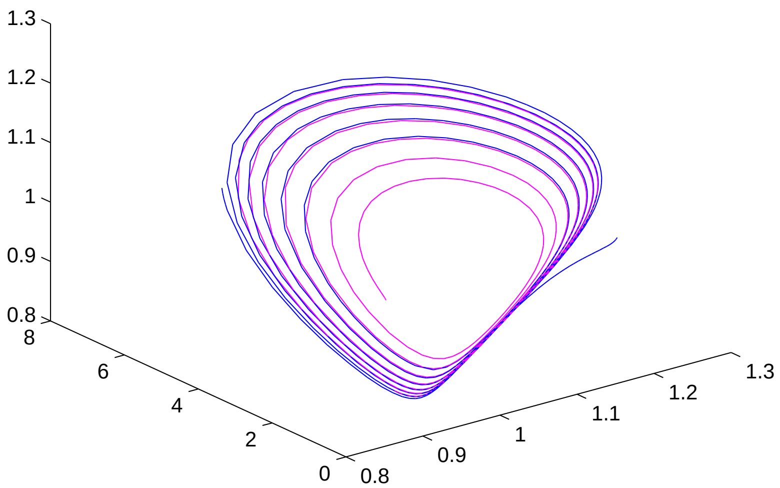

Figure 1. Computer simulations of the periodic solutions of ID-CNN model (3).

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On the Well-Possed Question of the Antiperiodic Problem for Systems of Linear Generalized Differential Equations

Malkhaz Ashordia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Sukhumi State University, Tbilisi, Georgia
E-mail: ashord@rmi.ge

We consider the well-possed question for the ω-antiperiodic problem for linear generalized ordinary differential equations of the form

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in \mathbb{R} \tag{1}$$

under the ω-antiperiodic condition

$$x(t+\omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{2}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R} \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with bounded variation components on every closed interval from $\mathbb{R}$, and ω is a fixed positive number.

Let the system (1) have the unique ω-antiperiodic solution x_0.

Along with the system (1) consider the sequence of the systems

$$dx(t) = dA_k(t) \cdot x(t) + df_k(t) \ \ (k = 1, 2, \dots), \tag{1_k}$$

where $A_k : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f_k : \mathbb{R} \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with bounded variation components on every closed interval from $\mathbb{R}$.

We give the necessary and sufficient condition for a sequence of ω-antiperiodic problems $(1_k), (2)$ $(k = 1, 2, \dots)$ to have a unique solution x_k for sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \ \text{ uniformly on } \ \mathbb{R}. \tag{3}$$

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, [1], [2], [4], [5] and references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil in connection with investigation the well-possed problem for the Cauchy problem for ordinary differential equations.

The use will be made of the following notation and definitions.

$\mathbb{R}$ is the real axis. $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices. $O_{n\times m}$ is the zero $n \times m$ matrix. I_n is the identity $n \times n$-matrix. $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors.

If $X : [a, b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a, b]$ of its components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \bigvee\limits_a^t(x_{ij})$ for $a < t \le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t.

$\mathrm{BV}([a, b], \mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ (i.e. $\bigvee\limits_a^b(X) < \infty$) with the norm $\|X\|_s = \sup\{\|X(t)\| : \ t \in [a, b]\}$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the space of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}^+_\omega(\mathbb{R},\mathbb{R}^{n\times m})$ and $\mathrm{BV}^-_\omega(\mathbb{R},\mathbb{R}^{n\times m})$ are the sets of all matrix-functions $G:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$ and there exists a constant matrix $C\in\mathbb{R}^{n\times m}$ such that, respectively, $G(t+\omega)\equiv G(t)+C$ and $G(t+\omega)\equiv G(t)+C$.

$s_j:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ $(j=0,1,2)$ are the operators defined, respectively, by $s_1(x)(a)=s_2(x)(a)=0$, $s_1(x)(t)=\sum\limits_{a<\tau\le t}d_1x(\tau)$ and $s_2(x)(t)=\sum\limits_{a\le\tau<t}d_2x(\tau)$ for $a<t\le b$, and $s_0(x)(t)\equiv x(t)-s_1(x)(t)-s_2(x)(t)$.

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $s<t$, then $\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum\limits_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum\limits_{s\le\tau<t}x(\tau)d_2g(\tau)$, where $\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$. So that $\int\limits_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see, [3]–[5]).

We use the operators. If $X\in\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ and $Y\in\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$, then $\mathcal{B}(X,Y)(t)=X(t)Y(t)-X(0)Y(0)-\int\limits_0^t dX(\tau)\cdot Y(\tau)$; if, in addition, $\det(X(t))\ne0$ for $t\in\mathbb{R}$, then $\mathcal{I}(X,Y)(t)=\int\limits_0^t d(X(\tau)+\mathcal{B}(X,Y)(\tau))\cdot X^{-1}(\tau)$.

A vector-function $\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be solution of the system (1) if $x(t)-x(s)=\int\limits_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)$ for $s<t$; $s,t\in\mathbb{R}$.

We will assume that $A,A_k\in\mathrm{BV}^+_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ and $f,f_k\in\mathrm{BV}^-_\omega(\mathbb{R},\mathbb{R}^n)$ $(k=1,2,\dots)$, i.e. $A(t+\omega)=A(t)+C$, $A_k(t+\omega)\equiv A_k(t)+C_k$ and $f(t+\omega)\equiv-f(t)+c$, $f_k(t+\omega)=-f_k(t)+c_k$ $(k=1,2,\dots)$ where $C,C_k\in\mathbb{R}^{n\times n}$ $(k=1,2,\dots)$ and $c,c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ are, respectively, some constant matrices and vectors. In addition, without loss of generality we assume that $A(0)=A_k(0)=O_{n\times n}$ and $f(0)=f_k(0)=0$ $(k=1,2,\dots)$. Moreover, we assume $\det(I_n+(-1)^jd_jA(t))\ne0$ for $t\in\mathbb{R}$ $(j=1,2)$.

Definition 1. We say that a sequence (A_k,f_k) $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A,f)$ if the ω-antiperiodic problem $(1_k),(2)$ has a unique solution x_k for any sufficiently large k and the condition (3) holds.

Statement 1. *The following statements are valid:*

(a) *if x is a solution of the system* (1)*, then the function $y(t)=-x(t+\omega)$ $(t\in\mathbb{R})$ is a solution of the system* (1)*, as well;*

(b) *the problem* (1), (2) *is solvable if and only if the system* (1) *on the closed interval $[0,\omega]$ has a solution satisfying the boundary condition*

$$x(0)=-x(\omega). \tag{4}$$

More than, the set of restrictions of the solutions of the problem (1), (2) *on $[0,\omega]$ coincides with the set of solutions of the problem* (1), (4).

Theorem 1. *The inclusion*

$$\big((A_k,f_k)\big)_{k=1}^{+\infty}\in\mathcal{S}(A,f) \tag{5}$$

is valid if and only if there exists a sequence of matrix-functions $H,H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\sup\bigvee_0^\omega\big(H_k+\mathcal{B}(H_k,A_k)\big)<+\infty, \tag{6}$$

$$\inf\big\{\big|\det(H(t))\big|:\ t\in[0,\omega]\big\}>0, \tag{7}$$

and the conditions

$$\lim_{k\to+\infty}H_k(t)=H(t), \tag{8}$$

$$\lim_{k\to+\infty}\mathcal{B}(H_k,A_k)(t)=\mathcal{B}(H,A)(t), \tag{9}$$

$$\lim_{k\to+\infty}\mathcal{B}(H_k,f_k)(t)=\mathcal{B}(H,f)(t)$$

are fulfilled uniformly on $[0,\omega]$.

Theorem 2. *Let* $A_*\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f_*\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$ *be such that*

$$\det\big(I_n+(-1)^jd_jA_*(t)\big)\neq 0\ \ \textit{for}\ \ t\in[0,\omega]\ \ (j=1,2)$$

and the system

$$dx(t)=dA_*(t)\cdot x(t)+df_*(t)$$

has a unique ω-antiperiodic solution x_*. Let, moreover, there exist sequences of matrix-and vector-functions $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *and* $h_k\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$ $(k=1,2\dots)$, *respectively, such that* $h_k(0)=-h_k(\omega)$ $(k=1,2,\dots)$, $\inf\big\{\big|\det(H_k(t))\big|:\ t\in[0,\omega]\big\}>0$ $(k=1,2,\dots)$, $\lim\limits_{k\to+\infty}\sup\bigvee\limits_a^b A_{*k}<+\infty$, *and the conditions* $\lim\limits_{k\to+\infty}A_{*k}(t)=A_*(t)$ *and* $\lim\limits_{k\to+\infty}f_{*k}(t)=f_*(t)$ *are fulfilled uniformly on* $[0,\omega]$, *where* $A_{*k}(t)\equiv\mathcal{I}_k(H_k,A_k)(t)$ $(k=1,2,\dots)$ *and*

$$f_{*k}(t)\equiv h_k(t)-h_k(0)+\mathcal{B}_k(H_k,f_k)(t)-\int\limits_0^t dA_{*k}(\tau)\cdot h_k(t)\ \ (k=1,2,\dots).$$

Then the system (1_k) *has the unique ω-antiperiodic solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty}\|H_kx_k+h_k-x_*\|_s=0.$$

Corollary 1. *Let the conditions* (6) *and* (7) *hold, and let the conditions* (8), (9) *and*

$$\lim_{k\to+\infty}\bigg(\mathcal{B}(H_k,f_k-\varphi_k(t)+\int\limits_0^t d\mathcal{B}(H_k,A_k)(s)\cdot\varphi_k(s)\bigg)=\mathcal{B}(H,f)(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H,H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Then the system* (1_k) *has a unique ω-antiperiodic solution x_k for any sufficiently large k and* $\lim\limits_{k\to+\infty}\|x_k-\varphi_k-x_*\|_s=0$.

Corollary 2. *Let the conditions* (6) *and* (7) *hold, and let the conditions* (8),

$$\lim_{k\to+\infty}\int\limits_0^t H_k(s)\,dA_k(s)=\int\limits_0^t H(s)\,dA(s),\ \ \lim_{k\to+\infty}\int\limits_0^t H_k(s)\,df_k(s)=\int\limits_0^t H(s)\,df(s),$$

$$\lim_{k\to+\infty}d_jA_k(t)=d_jA(t)\ \ (j=1,2)\ \ \textit{and}\ \ \lim_{k\to+\infty}d_jf_k(t)=d_jf(t)\ \ (j=1,2)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H,H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Let, moreover, either*

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\big(\|d_jA_k(t)\|+\|d_jf_k(t)\|\big)<+\infty\ \ (j=1,2),$$

or

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\|d_jH_k(t)\|<+\infty\ \ (j=1,2).$$

Then the inclusion (5) *holds.*

Corollary 3. *Let the conditions* (6) *and* (7) *hold, and let the conditions* (8),

$$\lim_{k\to+\infty} A_k(t) = A(t), \tag{10}$$

$$\lim_{k\to+\infty} f_k(t) = f(t)), \tag{11}$$

$$\lim_{k\to+\infty} \int_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot A_k(s) = A_*(t), \quad \lim_{k\to+\infty} \int_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot f_k(s) = f_*(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k, A_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, *and* $f_* \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$. *Let, moreover, the system*

$$dx(t) = d\big(A(t) - A_*(t)\big)\cdot x(t) + d\big(f(t) - f_*(t)\big)$$

has the unique ω*-antiperiodic solution. Then* $((A_k, f_k))_{k=1}^{+\infty} \in \mathcal{S}(A - A_*, f - f_*)$.

Corollary 4. *Let there exist a natural number* m *and matrix-functions* $B_j \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(j = 0,\dots,m-1)$ *such that* $\lim\limits_{k\to+\infty} \sup \mathop{\vee}\limits_0^\omega (A_{km}) < +\infty$, *and the conditions* $\lim\limits_{k\to+\infty} (A_{km}(t) - A_{km}(0) = A(t)$ *and* $\lim\limits_{k\to+\infty} (f_{km}(t) - f_{km}(0)) = f(t)$, *be fulfilled uniformly on* $[0,\omega]$, *where*

$$H_{k0}(t) \equiv I_n, \;\; H_{k\,j+1\,0}(t) \equiv \prod_{j+1}^{1} \big(I_n - A_{kl}(t) + A_{kl}(0) + B_l(t) - B_l(0)\big),$$

$$A_{k\,j+1} \equiv H_{kj}(t) + \mathcal{B}(H_{kj}, A_k)(t), \;\; f_{k\,j+1} \equiv \mathcal{B}(H_{kj}, f_k)(t).$$

Then the inclusion (5) *holds.*

If $m = 1$, then Corollary 4 has the following form.

Corollary 5. *Let* $\lim\limits_{k\to+\infty} \sup \mathop{\vee}\limits_0^\omega (A_k) < +\infty$, *and the conditions* (10) *and* (11) *be fulfilled uniformly on* $[0,\omega]$. *Then the inclusion* (5) *holds.*

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On Existence of Quasi-Periodic Solutions to a Nonlinear Singular Higher-Order Differential Equation and Asymptotic Classification of Its Solutions for the Forth Order

I. Astashova

Lomonosov Moscow State University, Moscow State University of Economics, Statistics and Informatics, Moscow, Russia
E-mail: `ast@diffiety.ac.ru`

1 Introduction

The paper is devoted to the existence of oscillatory quasi-periodic, in some sense, solutions to the higher-order singular Emden–Fowler type differential equation

$$y^{(n)} + p_0\,|y|^k \operatorname{sgn} y = 0, \ \ n > 2, \ \ k \in \mathbb{R}, \ \ 0 < k < 1, \ \ p_0 \neq 0, \tag{1}$$

and to the asymptotic classification of solutions to this equation with $n = 4$.

A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1] and [4]. Results on the existence of solutions with special asymptotic behavior are contained in [2, 3, 5–8]. Results on asymptotic classification of solutions to (1) with $n = 3$, $k > 0$, $k \neq 1$, and $n = 4$, $k > 1$, are given in [4] and [9].

2 On Existence of Quasi-Periodic Oscillatory Solutions

Put

$$\alpha = \frac{n}{k-1}.$$

Theorem 2.1. *For any integer $n > 2$ and real positive $k < 1$ there exists a non-constant oscillatory periodic function h such that for any p_0 with $(-1)^n p_0 > 0$ and any real x^* the function*

$$y(x) = |p_0|^{-\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \ \ -\infty < x < x^*, \tag{2}$$

is a solution to equation (1).

Remark. Note that the same result for equation (1) with $n \geq 2$ and $k > 1$ was obtained earlier in [6–8]. A result on the existence of a positive solution similar to (2) with positive periodic function h for $n = 12, 13, 14$ and $k > 1$ is proved in [5].

3 On Asymptotic Classification of Solutions to Emden–Fowler Singular Equations of the Forth Order

The asymptotic classification of all possible solutions to the forth-order Emden–Fowler type differential equations

$$y^{\mathrm{IV}}(x) + p_0|y|^k \operatorname{sgn} y = 0, \ \ 0 < k < 1, \ \ p_0 > 0, \tag{3}$$

and

$$y^{\mathrm{IV}}(x) - p_0|y|^k \operatorname{sgn} y = 0, \ \ 0 < k < 1, \ \ p_0 > 0, \tag{4}$$

is given.

3.1 Definitions and Preliminary Results

In the case of regular nonlinearity $k > 1$, only maximally extended solutions are considered because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then special behavior can occur also near internal points of the domains. This is why a notion of *maximally uniquely extended (MUE)* solutions is introduced.

Definition. A solution $u : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MUE-solution* if the following conditions hold:

(i) the equation has no solution equal to u on some subinterval of (a, b) and not equal to u at some point of (a, b);

(ii) either there is no solution defined on another interval containing (a, b) and equal to u on (a, b) or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of (a, b).

In this article all MUE-solutions to equation (3) and (4) are classified according to their behavior near the boundaries of their domains. All maximally extended solution can be classified through investigation of possible ways to join several MUE-solutions.

Consider the equation

$$y^{(n)} + p(x, y, y', \dots, y^{(n-1)})\,|y|^k \operatorname{sgn} y = 0, \quad n \ge 2, \quad k \in \mathbb{R}, \quad 0 < k < 1,$$

with positive $p(x, y_0, \dots, y_{n-1})$.

Note that, because of the condition $0 < k < 1$, the classical theorem on the uniqueness of solutions cannot be applied to Cauchy problems with $y(x_0) = 0$. Nevertheless, the following assertion holds (see [4, 7.3]).

Theorem 3.1. *Let the function $p(x, y_0, \dots, y_{n-1})$ be continuous in x and Lipschitz continuous in $y_0, \dots, y_{n-1}$. Then for any tuple of numbers $x_0, y_0^0, \dots, y_{n-1}^0$ with not all y_i^0 equal to zero, the corresponding Cauchy problem $y(x_0) = y_0^0, \dots, y^{(n-1)}(x_0) = y_{n-1}^0$ has a unique solution.*

Remark. While the uniqueness conditions hold, the property of continuous dependence of solution on initial data fulfils (see [10, V, Theorem 2.1]).

3.2 Main Results. Asymptotic classification of solutions to equations (3) and (4)

Theorem 3.2. *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MUE-solutions to equation (3) are divided into the following three types according to their asymptotic behavior (see Figure 1).*

1. *Oscillatory solutions defined on semi-axes $(-\infty, b)$. The distance between their neighboring zeros infinitely increases near $-\infty$ and tends to zero near b. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$ and $\overline{\lim\limits_{x\to-\infty}}\,|y^{(j)}(x)| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum the following estimates hold:*

$$C_1|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2|x-b|^{-\frac{4}{k-1}} \tag{5}$$

with positive constants C_1 and C_2 depending only on k and p_0.

2. *Oscillatory solutions defined on semi-axes $(b, +\infty)$. The distance between their neighboring zeros tends to zero near b and infinitely increases near $+\infty$. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$ and $\overline{\lim\limits_{x\to+\infty}}\,|y^{(j)}(x)| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum estimates (5) hold with positive constants C_1 and C_2 depending only on k and p_0.*

3. *Oscillatory solutions defined on* $(-\infty,+\infty)$. *All their derivatives* $y^{(j)}$ *with* $j=0,1,2,3,4$ *satisfy*
$$\overline{\lim_{x\to-\infty}}\,|y^{(j)}(x)|=\overline{\lim_{x\to+\infty}}\,|y^{(j)}(x)|=\infty.$$
At the points of local extremum the estimates
$$C_1|x|^{-\frac{4}{k-1}}\le|y(x)|\le C_2|x|^{-\frac{4}{k-1}} \tag{6}$$
hold near $-\infty$ *or* $+\infty$ *with positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.

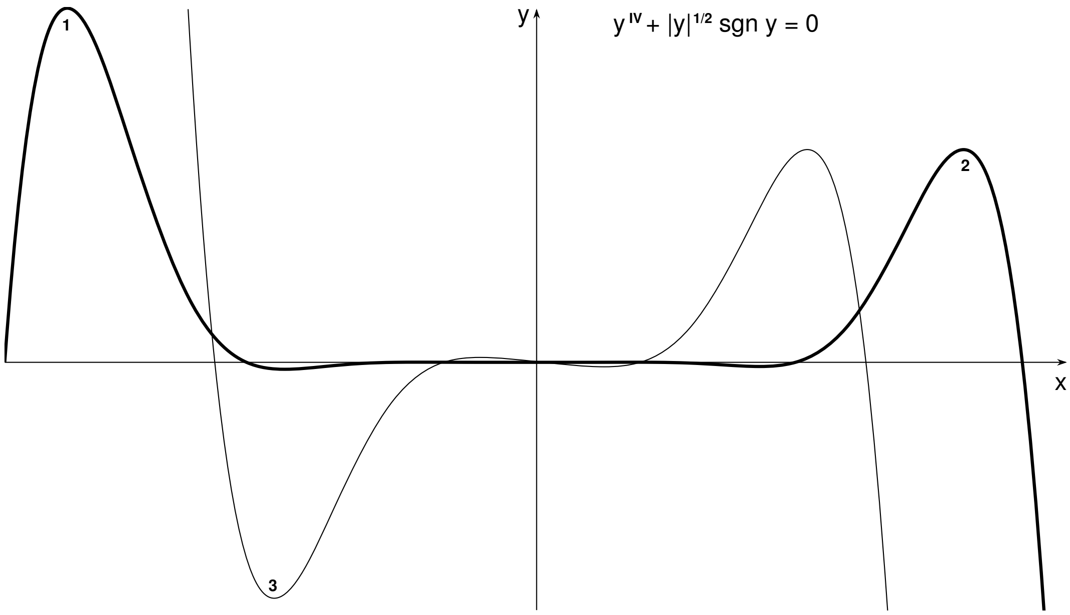

Figure 1.

Theorem 3.3. *Suppose* $0<k<1$ *and* $p_0>0$. *Then all MUE-solutions to equation* (4) *are divided into the following thirteen types according to their asymptotic behavior* (*see Figure* 2).

1–2. *Defined on semi-axes* $(b,+\infty)$ *solutions with the power asymptotic behavior near the boundaries of the domain* (*with the relative signs* $\pm$)*:*
$$y(x)\sim\pm C_{4k}(x-b)^{-\frac{4}{k-1}},\ \ x\to b+0,$$
$$y(x)\sim\pm C_{4k}x^{-\frac{4}{k-1}},\ \ x\to+\infty,$$
where
$$C_{4k}=\Big(\frac{4(k+3)(2k+2)(3k+1)}{p_0\,(k-1)^4}\Big)^{\frac{1}{k-1}}.$$

3–4. *Defined on semi-axes* $(-\infty,b)$ *solutions with the power asymptotic behavior near the boundaries of the domain* (*with the relative signs* $\pm$)*:*
$$y(x)\sim\pm C_{4k}|x|^{-\frac{4}{k-1}},\ \ x\to-\infty,$$
$$y(x)\sim\pm C_{4k}(b-x)^{-\frac{4}{k-1}},\ \ x\to b-0.$$

5. *Defined on the whole axis periodic oscillatory solutions. All of them can be received from one, say $z(x)$, by the relation*
$$y(x) = \lambda^4 z(\lambda^{k-1} x + x_0)$$
with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–7. *Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to -\infty$ and have the power asymptotic behavior near $+\infty$ (with the relative signs $\pm$):*
$$y(x) \sim \pm C_{4k} x^{-\frac{4}{k-1}}, \quad x \to +\infty.$$
For each solution a finite limit of the absolute values of its local extrema exists as $x \to -\infty$.

8–9. *Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to +\infty$ and have the power asymptotic behavior near $-\infty$ (with the relative signs $\pm$):*
$$y(x) \sim \pm C_{4k} |x|^{-\frac{4}{k-1}}, \quad x \to -\infty.$$
For each solution a finite limit of the absolute values of its local extrema exists as $x \to +\infty$.

10–13. *Defined on $(-\infty, +\infty)$ solutions having the power asymptotic behavior both near $-\infty$ and $+\infty$ (with the relative pairs of signs $\pm$):*
$$y(x) \sim \pm C_{4k}(p(b)) |x|^{-\frac{4}{k-1}}, \quad x \to \pm\infty.$$

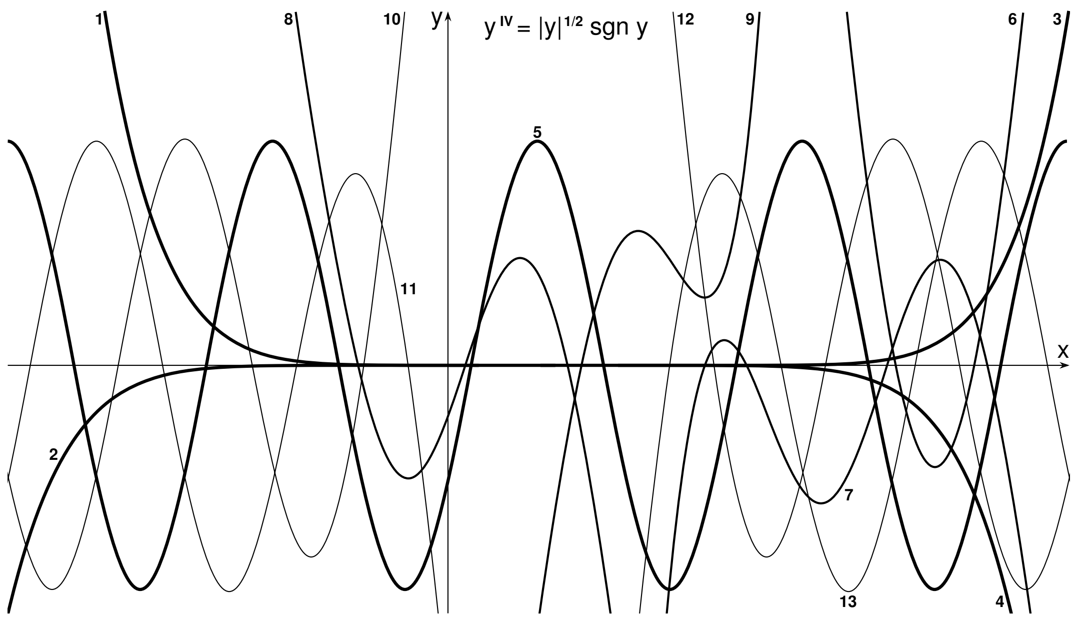

Figure 2.

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The Solution of the Erugin Problem on the Existence of Irregular Solutions of the Linear System in the Case of Triangular Periodic Matrix

M. S. Belokursky

Department of Differential Equations and Theory of the Functions, Gomel State University of F. Scorina, Gomel, Belarus
E-mail: drakonsm@ya.ru

A. K. Demenchuk

Department of Differential Equations, Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: demenchuk@im.bas-net.by

As we know, periodic differential systems under certain conditions can have periodic solutions whose period incommensurable with the period of the system [1]–[6]. These periodic solutions are inherent in a fairly wide class of differential systems, and called strongly irregular. We also note that oscillatory processes forms at the natural frequency of oscillations of systems, generally incommensurate with the frequency of external force in a number of applied problems [7].

In the monograph [3] N. P. Erugin considered the linear system of the form

$$\dot{x} = (AP(t) + B)x, \quad t \in R, \quad x \in R^n, \quad n \geqslant 2, \tag{1}$$

where A, B – constants $(n \times n)$-matrix, $P(t)$ – continuous ω-periodic $(n \times n)$-matrix. In the system (1) the matrix A and $P(t)$ will be called the stationary and periodic coefficients, respectively. For the system (1) with diagonal periodic coefficient $P(t)$ Erugin studied the problems of existence of periodic strongly irregular solutions. In particular, it was proved that if the matrix A is nonsingular, the desired solutions of the system (1) do not exist. The case of nondiagonal matrix $P(t)$ remained unexplored.

It should be noted that the system of type (1) is considered in solving control problems: control of asymptotic invariants, including Lyapunov exponents of stationary control systems by means of periodic controls [8], [9], problems of stabilization of linear control systems periodic feedback, including Brockett problem [10], [11].

In this paper, we consider the existence problem of strongly irregular periodic solutions of the system (1) with an upper triangular periodic coefficient

$$p_{ij}(t) \equiv 0, \quad i > j \quad (i, j = 1, \dots, n), \tag{2}$$

where $p_{ij}(t)$ – the elements of the matrix $P(t)$.

First we consider the case where the stationary coefficient is nonsingular, that is

$$\det A \neq 0. \tag{3}$$

Let $x(t)$ be a Ω-periodic solution to (1), it is considered that at least one of its components is different from the constant and the ratio ω/Ω is an irrational number. By [5], the vector $x(t)$ satisfies the system

$$\dot{x} = (A\widehat{P} + B)x, \quad (AP(t) - A\widehat{P})x = 0,$$

where $\widehat{P} = \frac{1}{\omega}\int\limits_0^{\omega} P(\tau)\,d\tau$ is an average ω-periodic coefficient. By condition (3) last system takes the form

$$\dot{x} = (A\widehat{P} + B)x, \quad \widetilde{P}(t)x = 0, \quad \widetilde{P}(t) = P(t) - \widehat{P}, \tag{4}$$

with

$$\operatorname{rank}_{\text{col}} \widetilde{P} = r < n. \tag{5}$$

By (5) there is a constant nonsingular $(r \times r)$-matrix Q such that the first $d = n - r$ columns of the matrix $\widetilde{P}(t)Q$ will be zero, but the remaining columns are linearly independent. Next, we replace $x = Qy$, which brings the system (4) to the system

$$\dot{y} = Fy, \quad \widetilde{P}_1(t)y = 0 \quad \big(F = Q^{-1}(A\widehat{P} + B)Q, \ \ \widetilde{P}_1(t) = \widetilde{P}(t)Q\big). \tag{6}$$

The system (6) has the following structure

$$\dot{y}^{[d]} = F_{d,d}y^{[d]}, \quad F_{r,d}y^{[d]} = 0, \quad y_{[r]} = 0,$$

where $F_{d,d}$, $F_{r,d}$ – the left upper and lower blocks of the matrix F (subscript indicates the dimension). Among eigenvalues of the coefficient matrix $F_{d,d}$ will be the numbers

$$\pm i\lambda_j \ \ \big(j = 1, \dots, d'; \ \ d' \leqslant [d/2]\big), \tag{7}$$

where $\lambda_j = 2k_j\pi/\Omega$, $k_j \in N$. Let l_j – the number of groups of elementary divisors corresponding to the eigenvalues $\pm i\lambda_j$ $(j = 1, \dots, d'; \ l_1 + \dots + l_{d'} = l)$. This means that $y^{[d]}(t)$ presented by trigonometric polynomial of the form $y^{[d]}(t) = \sum\limits_{j=1}^{d'} a_j \cos\lambda_j t + b_j \sin\lambda_j t$, where the coefficients a_j, b_j depend on $2l$ arbitrary real constants, for which we have the identity

$$F_{r,d}\sum_{j=1}^{d'} a_j \cos\lambda_j t + b_j \sin\lambda_j t \equiv 0. \tag{8}$$

Then the system (1) has a strongly irregular periodic solution

$$x(t) = Q \operatorname{col}\big(y^{[d]}(t), 0, \dots, 0\big). \tag{9}$$

Theorem 1. *Let for the system* (1) *conditions* (2) *and* (3) *be satisfied.*

If the system (1) *has a strongly irregular periodic solution, then this solution will be a trigonometric polynomial of the form* (9)*. The conditions* (5)*,* (7) *and* (8) *are necessary and sufficient for the function* (9) *to be the solution of the system* (1)*.*

Corollary 1. *If all diagonal elements of the upper triangular periodic coefficients are nonstationary, then the system* (1) *does not have strongly irregular periodic solutions.*

Now consider the case of stationary singular coefficient

$$\operatorname{rank} A = r < n. \tag{10}$$

Let $x(t)$ be a Ω-periodic solution to (1), it is considered that at least one of its components is nonconstant and the ratio ω/Ω is an irrational number. According [5], the vector $x(t)$ satisfies the system

$$A\widetilde{P}(t)x = 0. \tag{11}$$

By (10) there is a constant nonsingular $(n \times n)$-matrix S such that the system (11) takes the form

$$C\widetilde{P}(t)x = 0, \quad C = SA. \tag{12}$$

We denote the trapezoidal $(r \times n)$-matrix formed by the first r rows of the matrix C, through C_1, where $\operatorname{rank} C_1 = r$. Thus system (12) takes the form

$$C_1 \widetilde{P}(t)x = \widetilde{P}_1(t)x = 0. \tag{13}$$

Then there exist constants k, linearly independent vectors $\alpha^{(1)}, \dots, \alpha^{(k)}$ such that $(\alpha^{(j)}, \widetilde{P}_1^{(j)}(t)) \equiv 0$ $(j = 1, \dots, k)$. The system (13) has k linearly independent irregular periodic solutions of the form $x^{(j)}(t) = \alpha^{(j)} \varphi_j(t)$, where $\varphi_1(t), \dots, \varphi_k(t)$ are some Ω-periodic functions. Denote by Λ and $X(t)$ – $(n \times k)$-matrix whose columns are vectors $\alpha^{(1)}, \dots, \alpha^{(k)}$ and $x^{(1)}(t), \dots, x^{(k)}(t)$, respectively. We write the last equation in the matrix form $X(t) = \Lambda \Phi$, where Φ is a diagonal matrix with functions $\varphi_1(t), \dots, \varphi_k(t)$ on the main diagonal. In view of the linear independence of the vectors $\alpha^{(1)}, \dots, \alpha^{(k)}$ the matrix Λ has a nonzero minor of order k. Let this minor is located in the rows with numbers $i_1, \dots, i_k$, Λ_1 – the corresponding matrix and Λ_2 – $(n-k) \times k$-matrix composed of the remaining rows of Λ. Then the resulting matrix equality splits into $X'(t) = \Lambda_1 \Phi$, $X''(t) = \Lambda_2 \Phi$, where matrix $X'(t)$ formed by the rows with numbers $i_1, \dots, i_k$, and $X''(t)$ formed by remaining rows.

Take in account the notation from [5], the vector $x(t)$ satisfies the system

$$\begin{pmatrix} \dot{x}' \\ \dot{x}'' \end{pmatrix} = \begin{pmatrix} A'\widehat{P}' + B_1' & A'\widehat{P}'' + B_1'' \\ A''\widehat{P}' + B_2' & A''\widehat{P}'' + B_2'' \end{pmatrix} \begin{pmatrix} x' \\ x'' \end{pmatrix},$$

where the blocks B_1' and B_1'' formed by the first k rows of the matrix B' and B'', and blocks B_2', B_2'' formed by remaining $n-k$ rows of these matrices. Then the matrix $H = A'\widehat{P}' + B_1' + (A'\widehat{P}'' + B_1'')\Lambda_2 \Lambda_1^{-1}$ has purely imaginary eigenvalues

$$\pm i\lambda_s \ \ \big(s = 1, \dots, k'; \ \ 1 \leqslant k' \leqslant [k/2] \big), \tag{14}$$

where $\lambda_s = 2k_s \pi / \Omega$, $k_s \in N$. Let p_s – the number of groups of elementary divisors corresponding to the eigenvalues $\pm i\lambda_s$ $(s = 1, \dots, k';\ p_1 + \dots + p_{k'} = p)$. This means that $x'(t)$ presented by trigonometric polynomial of the form

$$x'(t) = \sum_{s=1}^{k'} \alpha_s \cos \lambda_s t + \beta_s \sin \lambda_s t, \tag{15}$$

where the coefficients α_s, β_s depend on $2l$ arbitrary real constants. We have the identity

$$\Big(\Lambda_2 \Lambda_1^{-1} \big(A'P' + B_1' + (A'P'' + B_1'') \Lambda_2 \Lambda_1^{-1} \big) - A''P' - B_2' - (A''P'' + B_2'') \Lambda_2 \Lambda_1^{-1} \Big) x'(t) \equiv 0. \tag{16}$$

Then the system (1) has a strongly irregular periodic solution

$$x(t) = \operatorname{ord} \Big\{ x_{i_1}(t), \dots, x_{i_k}(t), x_{i_{k+1}}(t), \dots, x_{i_n}(t) \Big\} = \operatorname{ord} \big\{ \operatorname{col} (x'(t), x''(t)) \big\}, \tag{17}$$

where $\operatorname{ord}\{\cdot\}$ means ordering vector components $\{\cdot\}$ in ascending order of their indices.

Theorem 2. *Let for the system* (1) *conditions* (2) *and* (10) *be satisfied.*

If the system (1) *has a strongly irregular periodic solution, then this solution will be a trigonometric polynomial of the form* (15), (17).

The vector (17) *is the solution of the system* (1) *if the system (13) has* $0 < k < n$ *linearly independent stationary solutions and conditions* (14), (16) *are satisfied.*

Remark. A similar result holds in the case of lower triangular periodic coefficient.

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On a Weak Solvability of One Nonlocal Boundary-Value Problem in Weighted Sobolev Space

Givi Berikelashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Department of Mathematics of Georgian Technical University, Tbilisi, Georgia
E-mail: bergi@rmi.ge; berikela@yahoo.com

Nodar Khomeriki

Department of Mathematics of Georgian Technical University, Tbilisi, Georgia
E-mail: n.khomeriki@gtu.ge

Let $\Omega = \{x \in \mathbb{R}^n : \ 0 < x_k < 1\}$ be the open unit cube in $\mathbb{R}^n$ with boundary Γ, and $\Gamma_0 = \Gamma \setminus \Gamma_*$, $\Gamma_* = \{(0, x_2, \dots, x_n) : \ 0 < x_k < 1, \ k = 2, \dots, n\}$.

Consider the nonlocal boundary-value problem

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} \Big(a_k \frac{\partial u}{\partial x_k} \Big) = -f(x), \quad x \in \Omega, \tag{1}$$

$$u(x) = 0, \ x \in \Gamma_0, \tag{2}$$

$$\ell u := \int\limits_0^1 \beta(x_1) u(x) \, dx_1 = 0, \quad 0 \le x_k \le 1, \quad k = 2, \dots, n, \tag{3}$$

where $\beta(t) = \varepsilon t^{\varepsilon - 1}$, $\varepsilon \in (0; 1)$.

Define the operator

$$Gv = v - Hv,$$

where H is the weighted Hardy operator associated to conditions (3):

$$Hv = \frac{1}{\rho(x_1)} \int\limits_0^{x_1} \beta(t) v(t, x_2, \dots, x_n) \, dt, \quad \rho(x_1) = \int\limits_0^{x_1} \beta(t) \, dt = x_1^{\varepsilon}.$$

By $L_2(\Omega, \rho)$ we denote the weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u, v)_\rho = \int\limits_\Omega \rho(x_1) u(x) v(x) \, dx, \quad \|u\|_\rho = (u, u)_\rho^{1/2}.$$

The weighted Sobolev space $W_2^1(\Omega, \rho)$ is usually defined as a linear set of all functions $u(x) \in L_2(\Omega, \rho)$, whose distributional derivatives $\partial u / \partial x_k$, $k = 1, 2, \dots, n$ are in $L_2(\Omega, \rho)$. It is a normed linear space if equipped with the norm

$$\|u\|_{W_2^1(\Omega,\rho)} = \big(\|u\|_\rho^2 + |u|_{W_2^1(\Omega,\rho)}^2 \big)^{1/2}, \quad |u|_{W_2^1(\Omega,\rho)}^2 = \sum_{k=1}^{n} \Big\| \frac{\partial u}{\partial x_k} \Big\|_\rho^2.$$

Define the subspace of space $W_2^1(\Omega,\rho)$ which can be obtained by closing the set

$$\overset{\circ}{C}{}^{\infty}(\overline{\Omega})=\Big\{u\in C^{\infty}(\overline{\Omega}):\ \operatorname{supp}u\cap\Gamma_0=\varnothing,\ \ell u=0,\ 0<x_k<1,\ k=2,\dots,n\Big\}$$

with the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$. Denote it by $\overset{\circ}{W}{}_2^1(\Omega,\rho)$.

Let the right-hand side $f(x)$ in equation (1) be a linear continuous functional on $\overset{\circ}{W}{}_2^1(\Omega,\rho)$ which can be represented as

$$f=f_0+\sum_{k=1}^{n}\frac{\partial f_k}{\partial x_k}\,,\ \ f_k(x)\in L_2(\Omega,\rho),\ \ k=0,1,2,\dots,n. \tag{4}$$

Assume

$$\begin{gathered}\nu\le a_k(x)\le\mu\ \ (k=1,\dots,n),\\ \nu,\mu=const>0,\ \ 0\le\frac{\partial}{\partial x_1}\,(a_kx_1^{1-\varepsilon})\in L_\infty(\Omega),\ \ (k=2,\dots,n).\end{gathered} \tag{5}$$

Definition. We say that the function $u\in\overset{\circ}{W}{}_2^1(\Omega,\rho)$ is a weak solution of problem (1)–(5) if the relation

$$a(u,v)=\langle f,v\rangle,\ \ \forall\,v\in\overset{\circ}{W}{}_2^1(\Omega,\rho) \tag{6}$$

holds, where

$$\begin{gathered}a(u,v)=\Big(a_1\,\frac{\partial u}{\partial x_1}\,,\frac{\partial v}{\partial x_1}\Big)_\rho+\sum_{k=2}^{n}\Big(a_k\,\frac{\partial u}{\partial x_k}\,,G\,\frac{\partial v}{\partial x_k}\Big)_\rho,\\ \langle f,v\rangle=(f_0,Gv)_\rho-\sum_{k=1}^{n}\Big(f_k,\frac{\partial}{\partial x_k}\,Gv\Big)_\rho.\end{gathered}$$

Equality (6) formally is obtained from $(\Delta u+f,Gv)_\rho=0$ by integration by parts, taking into account that

$$\begin{aligned}\Big(\frac{\partial v}{\partial x_1}\,,Gu\Big)_\rho&=-\Big(v,\frac{\partial u}{\partial x_1}\Big)_\rho,\\ \Big(\frac{\partial v}{\partial x_k}\,,Gu\Big)_\rho&=-\Big(v,G\,\frac{\partial u}{\partial x_k}\Big)_\rho,\ \ k=2,\dots,n.\end{aligned}$$

Theorem. *The problem* (1)–(5) *has a unique weak solution from* $\overset{\circ}{W}{}_2^1(\Omega,\rho)$.

On Conditions for the Solvability of the Periodic Problem for Second Order Linear Functional Differential Equations

Eugene Bravyi

Perm National Research Polytechnic University, Perm, Russia
E-mail: bravyi@perm.ru

Consider the periodic boundary value problem

$$\begin{cases} \ddot{x}(t) = \lambda(Tx)(t) + f(t) & \text{for almost all } t \in [a,b], \\ x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b), \end{cases} \tag{1}$$

where λ is a real number, $T : \mathbf{C}[a,b] \to \mathbf{L}[a,b]$ is a linear bounded operator, $f \in \mathbf{L}[a,b]$, a solution $x : [a,b] \to \mathbb{R}$ has an absolutely continuous derivative, $\mathbf{C}[a,b]$ is the space of continuous functions $x : [a,b] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{C}} = \max\limits_{t\in[a,b]} |x(t)|$, $\mathbf{L}[a,b]$ is the space of integrable functions $z : [a,b] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{L}} = \int\limits_a^b |z(t)|\,dt$.

Suppose we investigate this problem under some uncertainty: we know only some sign properties of the operator T and a result of the action of T on some function, for example, we know $T\mathbf{1}$. For such family of operators, we will find necessary and sufficient conditions for problems (1) to have solutions for all functional differential equations with such functional operators. Thus, unimprovable sufficient conditions for the unique solvability of the periodic boundary value problem will be obtained. Conditions for the solvability with integral restrictions on functional operators can be found in works by I. Kiguradze, R. Hakl, A. Lomtatidze, S. Mukhigulashvili, A. Ronto, J. Sremr and others [1, 2, 3, 4].

Here we determine the best constants in the solvability conditions for a kind of point-wise restrictions.

Let a function $p \in \mathbf{L}[a,b]$ be given. Define the piecewise linear functions

$$q_{t_1,t_2}(t) \equiv \begin{cases} \dfrac{(t-a)(t_2-t_1)}{b-a}, & t \in [a,t_1), \\ t_2 - t - \dfrac{(b-t)(t_2-t_1)}{b-a}, & t \in [t_1,t_2), \\ -\dfrac{(b-t)(t_2-t_1)}{b-a}, & t \in [t_2,b], \end{cases}$$

$$q_{t_1,t_2,p}(t) \equiv q_{t_1,t_2}(t) - \int\limits_a^b p(s) q_{t_1,t_2}(s)\,ds, \quad t \in [a,b].$$

For every $z : [a,b] \to \mathbb{R}$, denote $z^+(t) \equiv \frac{z(t)+|z(t)|}{2}$, $z^-(t) \equiv \frac{z(t)-|z(t)|}{2}$.

Theorem 1. *Let $T\mathbf{1} = p$, $\int\limits_a^b p(t)\,dt = 1$, the functionals $x \mapsto (Tx)(t)$ be monotone for almost all $t \in [a,b]$, and*

$$\lambda \neq 0, \quad |\lambda| < \frac{1}{\max\limits_{a\le t_1<t_2\le b} \int\limits_a^b (p^+(t) q^+_{t_1,t_2,p}(t) + p^-(t) q^-_{t_1,t_2,p}(t))\,dt}\,. \tag{2}$$

Then periodic problem (1) *has a unique solution.*

Note that the function p may change its sign and the constant in the right-hand side of (2) is exact.

Corollary 1. *Let* $p \in \mathbf{L}[a,b]$ *be given,* $\mathcal{P} \equiv |\int\limits_a^b p(t)\,dt| \neq 0$*,* $f \in \mathbf{L}[a,b]$*. The periodic boundary value problem*

$$\begin{gathered}\ddot{x}(t) = p(t)x(h(t)) + f(t) \ \text{ for a.a. } t \in [a,b],\\ x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b),\end{gathered}$$

has a unique solution for all measurable functions $h : [a,b] \to [a,b]$ *if*

$$\max_{a \le t_1 < t_2 \le b} \int\limits_a^b \Big(p^+(t) q^+_{t_1,t_2,p/\mathcal{P}}(t) + p^-(t) q^-_{t_1,t_2,p/\mathcal{P}}(t)\Big)\,dt < 1. \tag{4}$$

Theorem 2. *Let non-negative* q*,* $r \in \mathbf{L}[a,b]$ *be given,* $\int\limits_a^b (q(t) - r(t))\,dt = 1$*,* λ *is a real number. The periodic boundary value problem*

$$\begin{gathered}\ddot{x}(t) = \lambda\big((T^+x)(t) - (T^-x)(t)\big) + f(t) \ \text{ for a.a. } t \in [0,1],\\ x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b),\end{gathered}$$

has a unique solution for all linear positive operators T^+*,* $T^- : \mathbf{C}[a,b] \to \mathbf{L}[a,b]$ *such that* $T^+\mathbf{1} = q$*,* $T^-\mathbf{1} = r$ *if*

$$\lambda \neq 0, \quad |\lambda| < \frac{1}{\max\limits_{a \le t_1 < t_2 \le b} \int\limits_a^b (q(t) g^+_{t_1,t_2,q-r}(t) + r(t) g^-_{t_1,t_2,q-r}(t))\,dt}. \tag{3}$$

Hypothesis 1. *Let* $p \in \mathbf{L}[0,1]$*,* $p(t) \ge 0$*,* $t \in [0,1]$*, be given,* $p(t) = p(1-t)$ *for all* $t \in [0,1]$*, and* $p(t) = p(1/2 - t)$ *for all* $t \in [0,1/2]$*,* $f \in \mathbf{L}[0,1]$*. Then the periodic boundary value problem*

$$\begin{gathered}\ddot{x}(t) = \lambda\, p(t)x(h(t)) + f(t) \ \text{ for a.a. } t \in [0,1],\\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1),\end{gathered}$$

has a unique solution for all measurable functions $h : [0,1] \to [0,1]$ *if*

$$0 < |\lambda| < \frac{1}{\max\Big\{ \int\limits_0^{1/4} t\,p(t)\,dt,\ \int\limits_0^{1/4} (1/4 - t)\,p(t)\,dt \Big\}}. \tag{5}$$

The best constants in the solvability conditions (2)–(5) for some functions p, q, r can be calculated in the explicit form.

First consider the problem

$$\begin{gathered}\ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1],\\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1),\end{gathered} \tag{6}$$

where $T : \mathbf{C}[a,b] \to \mathbf{L}[0,1]$ is a linear bounded operator such that

$$(T\mathbf{1})(t) = \begin{cases} p, & t \in \Big[0, \dfrac{1}{2}\Big],\\ q, & t \in \Big(\dfrac{1}{2}, 1\Big], \end{cases}$$

and the functionals $x \mapsto (Tx)(t)$ are monotone for almost all $t \in [0,1]$. Let

$$P \equiv \max\big\{|p|, |q|\big\}, \quad Q \equiv \operatorname{sign}(pq) \min\big\{|p|, |q|\big\}.$$

Theorem 3. *Problem* (6) *has a unique solution if*

$$P \in (0,32], \quad \frac{-16P}{16+P} < Q \tag{7}$$

or

$$P \in (32,64), \quad \frac{P}{128-P}\big(\sqrt{512P} - P - 128\big) < Q < \sqrt{512P} - 3P. \tag{8}$$

Let

$$r(t) = \begin{cases} p, & t \in \big[0, \frac{1}{2}\big], \\ q, & t \in \big(\frac{1}{2}, 1\big], \end{cases}$$

Corollary 2. *The periodic problem*

$$\begin{gathered} \ddot{x}(t) = r(t)x(h(t)) + f(t) \ \textit{ for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{gathered}$$

has a unique solution for every measurable function $h : [0,1] \to [0,1]$ *if one of conditions* (7), (8) *holds.*

Theorem 4. *The periodic boundary value problem*

$$\begin{gathered} \ddot{x}(t) = \lambda\, 2tx(h(t)) + f(t) \ \textit{ for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{gathered}$$

has a unique solution for all measurable functions $h : [0,1] \to [0,1]$ *if and only if*

$$0 \neq |\lambda| < \frac{1}{\max\limits_{k\in[0,1/2],\, s\in[k,1-k]} g_1(k,s)g_2(k,s)} \equiv \lambda^* \in (29.328, 29.329),$$

where

$$g_1(k,s) \equiv \Big(\frac{-1+3k+k^2-3s+3s^2}{9(1-2k)}\Big)^2,$$

$$g_2(k,s) \equiv -2k\big(1+2k-7k^2+4k^3-6s+6sk-3s^2+12s^2k\big).$$

Theorem 5. *The periodic boundary value problem*

$$\begin{gathered} \ddot{x}(t) = \lambda\,(Tx)(t) + f(t) \ \textit{ for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{gathered}$$

has a unique solution for all linear positive (*or negative*) *operators* $T : \mathbf{C} \to \mathbf{L}[0,1]$ *with* $(T\mathbf{1})(t) \equiv 6\,t\,(1-t)$ (*or* $(T\mathbf{1})(t) \equiv -6\,t\,(1-t)$) *if and only if*

$$0 \neq |\lambda| < \frac{16}{\max\limits_{t\in[0,1/2]} t(1-2t)(-4t^2+6t+3)} \in (29.737, 29.738).$$

Theorem 6. *Let* $n \in \mathbb{N}$. *The periodic boundary value problem*

$$\begin{gathered} \ddot{x}(t) = \lambda\big(1-\cos^3(4n\pi t)\big)x(h(t)) + f(t) \ \textit{ for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{gathered}$$

has a unique solution for all measurable functions $h : [0,1] \to [0,1]$ *if and only if*

$$0 \neq |\lambda| < \begin{cases} \dfrac{288\,n^2\,\pi^2}{28+9n^2\pi^2} & \textit{for odd } n, \\ 32 & \textit{for even } n. \end{cases}$$

Theorem 7. *Let $n \in \mathbb{N}$. The periodic boundary value problem*

$$\ddot{x}(t) = \lambda\big(1 - \cos(4n\pi t)\big)x(h(t)) + f(t) \ \text{ for a.a. } t \in [0,1],$$
$$x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1),$$

has a unique solution for all measurable $h : [0,1] \to [0,1]$ if and only if

$$0 \neq |\lambda| < \begin{cases} \dfrac{32n^2\pi^2}{4 + n^2\pi^2} & \text{for odd } n, \\ 32 & \text{for even } n. \end{cases}$$

Theorem 8. *The periodic boundary value problem*

$$\ddot{x}(t) = \lambda\big(p(t) = 1 - \cos^4(2n\pi t)\big)x(h(t)) + f(t) \ \text{ for a.a. } t \in [0,1],$$
$$x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1),$$

has a unique solution for all measurable functons $h : [0,1] \to [0,1]$ if and only if

$$0 \neq |\lambda| < \begin{cases} \dfrac{160n^2\pi^2}{16 + 5n^2\pi^2} & \text{if } n \text{ is odd}, \\ 32 & \text{if } n \text{ is even}. \end{cases}$$

Let $n \in \mathbb{N}$ and

$$p(t) = \begin{cases} t^n & \text{if } t \in \Big[0, \dfrac{1}{4}\Big], \\ \Big|\dfrac{1}{2} - t\Big|^n & \text{if } t \in \Big(\dfrac{1}{4}, \dfrac{3}{4}\Big], \\ (1-t)^n & \text{if } t \in \Big(\dfrac{3}{4}, 1\Big]. \end{cases}$$

Theorem 9. *The periodic boundary value problem*

$$\ddot{x}(t) = \lambda\, p(t)x(h(t)) + f(t) \ \text{ for a.a. } t \in [0,1],$$
$$x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1),$$

has a unique solution for all measurable functions $h : [0,1] \to [0,1]$ if and only if

$$0 \neq |\lambda| < (n+2)\, 4^{n+2}.$$

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Existence, Uniqueness and Continuous Dependence on Control of Localized Solutions to Neural Field Equations

Evgenii Burlakov, John Wyller

Norwegian University of Life Sciences, 1432 Ås, Norway
e-mail: evgenii.burlakov@nmbu.no; john.wyller@nmbu.no

Neural activity models which incorporate brain electrical stimulation effects have recently been used in the study of different type of neural disorders like epilepsy and Parkinson's disease [1–4]. The starting point for the analysis in these papers is a discretized version of the nonlocal neural field model known as the Amari model [5–8], extended with a term describing the stimulation effects. The latter term is considered as a control variable and corresponding optimization problems are discussed. There is a need for investigating the well-posedness of such models as well as justifying the numerical optimization procedure used in [9].

This serves as a motivation for the present research: We consider the model

$$w(t,x)=\int\limits_a^t\int\limits_{R^m} f\big(t,s,x,y,w(s,y),u(s,y)\big)\,dy\,ds,\quad t\in[a,\infty),\quad x\in R^m, \tag{1}$$

involving the control variable $u:[a,\infty)\times R^m\to R^k$ which is assumed to be Lebesgue measurable. This model generalizes a number of nonlocal models which have extensively been used in neural field theory [5–8]. Our aim is to study well-posedness of this model.

To this end, let us introduce the following notations:

- R^m is the m-dimensional real vector space with the norm $|\cdot|$;
- Λ is some metric space;
- for any $\mathfrak{S}\subset\Lambda$, $r>0$, we denote $B_\Lambda(\mathfrak{S},r)=\bigcup\limits_{\mathfrak{s}\in\mathfrak{S}}\{\lambda\in\Lambda \mid \ \rho_\Lambda(\lambda,\mathfrak{s})<r\}$;
- μ is the Lebesgue measure;
- $M([a,\infty)\times R^m,\mu,R^k)$ is a metric space of measurable functions $\mathfrak{m}:[a,\infty)\times R^m\to R^k$ with the distance
$$\rho_M(\mathfrak{m}_1,\mathfrak{m}_2)=\operatorname*{vrai\,sup}_{(t,x)\in[a,\infty)\times R^m}\big|\mathfrak{m}_1(t,x)-\mathfrak{m}_2(t,x)\big|;$$
- $L(\Omega,\mu,R^n)$ is the space of all measurable integrable functions $\chi:\Omega\to R^n$ with the norm
$$\|\chi\|_{L(\Omega,\mu,R^n)}=\int\limits_\Omega|\chi(s)|\,ds,\quad 1\le p<\infty;$$
- $C_0(\Omega,R^n)$ is the space of all continuous functions $\vartheta:\Omega\to R^n$ satisfying the additional condition $\lim\limits_{|x|\to\infty}|\vartheta(x)|=0$ in the case if Ω is unbounded, with the norm
$$\|\vartheta\|_{C_0(\Omega,R^n)}=\max_{x\in\Omega}|\vartheta(x)|;$$

- $C([a,b],C_0(\Omega,R^n))$ is the space of all continuous functions $\nu:[a,b]\to C_0(\Omega,R^n)$, with the norm
$$\|\nu\|_{C([a,b],C_0(\Omega,R^n))}=\max_{t\in[a,b]}\|\nu(t)\|_{C_0(\Omega,R^n)}.$$

We assume that for some $r_0>0$, the following conditions on the model (1) are fulfilled:

(i) For any $t\in[a,\infty)$, $w\in R^n$, $x\in R^m$ and any ball $u\in R^k$, the function $f(t,\,\cdot\,,x,\,\cdot\,,w,u):[a,\infty)\times R^m\to R^n$ is measurable.

(ii) For almost all $(s,y)\in[a,\infty)\times R^m$ and any $u\in R^k$, the function $f(\,\cdot\,,s,\,\cdot\,,y,\,\cdot\,,u)$ is continuous.

(iii) For any $b\in(a,\infty)$ and any $r>0$, it holds true that
$$\lim_{\mathfrak{r}\to\infty}\sup_{t\in[a,b],\,x\in R^m\setminus B_{R^m}(0,\mathfrak{r})}\bigg|\int\limits_a^t\int\limits_{R^m}f(t,s,x,y,w,u)\,dy\,ds\bigg|=0$$
uniformly for all $w\in B_{R^n}(0,r)$, $u\in B_{R^k}(u_0(s,y),r_0)$ $((s,y)\in[a,\infty)\times R^m)$.

(iv) For any $b\in(a,\infty)$ and any $r>0$, there exists such $g_{(b,r)}\in L([a,b]\times R^m,\mu,R)$ that $|f(t,s,x,y,w,u)|\le g_{(b,r)}(s,y)$ for all $x\in R^m$, $w\in B_{R^n}(0,r)$, $t\in[a,b]$, $u\in B_{R^k}(\{u_0(t,x)\},r_0)$.

Definition. Choose an arbitrary $u\in B_M(u_0,r_0)$. We define a *local solution* to eq. (1), defined on $[a,a+\gamma]\times R^m$, $\gamma\in(0,\infty)$, to be a function $w_\gamma\in C([a,a+\gamma],C_0(R^m,R^n))$, that satisfies the equation (1) on $[a,a+\gamma]\times R^m$. We define a *maximally extended solution* to eq. (1), defined on $[a,a+\eta)\times R^m$, $\eta\in(0,\infty)$, to be a function $w_\eta:[a,a+\eta)\times R^m\to R^n$, whose restriction w_γ on $[a,a+\gamma]\times R^m$ with any $\gamma<\eta$ is its local solution and $\lim\limits_{\gamma\to\eta-0}\|w_\gamma\|_{C([a,a+\gamma],C_0(R^m,R^n))}=\infty$. We define a *global solution* to eq. (1) to be a function $w:[a,\infty)\times R^m\to R^n$, whose restriction w_γ on $[a,a+\gamma]\times R^m$ with any $\gamma\in(0,\infty)$ is its local solution.

We are now in position to formulate the main result in the present study:

Theorem. *Let the assumptions* (i)–(iv) *hold true. Assume that the following conditions are satisfied:*

1) *For the given $r_0>0$ and any $r>0$ there exists $\widetilde{f}_r(s,y)\in L([a,\infty)\times R^m,\mu,R)$ (independent of u) such that $|f(t,s,x,y,w_1,u)-f(t,s,x,y,w_2,u)|\le\widetilde{f}_r(s,y)|w_1-w_2|$ for all $w_1,w_2\in B_{R^n}(0,r)$, $u\in B_{R^k}(u_0(s,y),r_0)$, $t\in[a,\infty)$, $x\in B_{R^m}(0,r)$.*

2) *For any $w\in R^n$, $t\in[a,\infty)$, $x\in R^m$, $\Delta\to0$ it holds true that:*
$$\Big|f(t,\,\cdot\,,x,\,\cdot\,,w,u_0(\,\cdot\,,\,\cdot\,)+\Delta)-f(t,\,\cdot\,,x,\,\cdot\,,w,u_0)\Big|\to0$$
in measure on $[a,\infty)\times R^m$.

Then for each $u\in B_M(u_0,r_0)$, eq. (1) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution. Moreover, if at $u=u_0$ eq.* (1) *has a local solution $w_{0\gamma}$ defined on $[a,a+\gamma]\times R^m$, then for any $\{u_i\}\subset M([a,\infty)\times R^m,\mu,R^k)$, $\rho_M(u_i,u_0)\to0$ one can find number I such that for all $i>I$ eq.* (1) *has a local solution $w_\gamma=w_\gamma(u_i)$ defined on $[a,a+\gamma]\times\Omega$ and $\|w_\gamma(u_i)-w_{0\gamma}\|_{C([a,a+\gamma],C_0(R^m,R^n))}\to0$.*

Proof. We are going to use Theorem 2.1 in [10] on well-posedness of parameterized operator Volterra equations, so we represent (1) in terms of operator equation in the following way:
$$w=F(w,u),\quad (F(w,u))(t,x)=\int\limits_a^t\int\limits_{R^m}f\big(t,s,x,y,w(s,y),u(s,y)\big)\,dy\,ds.$$

Here, for each $u \in B_M(u_0, r_0)$, $F : C([a,b], C_0(R^m, R^n)) \to C([a,b], C_0(R^m, R^n))$ provided that the conditions (i)–(iv) are fulfilled.

According to Theorem 2.1 in [10], we need to check the following two conditions:

1. There exists $q < 1$ such that for any $r > 0$ one can find $\delta > 0$ such that the following two conditions are satisfied for all $w_1, w_2 \in C([a,b], C_0(R^m, R^n))$, such that $\|w_1\|_{C([a,b],C_0(R^m,R^n))} \le r$, $\|w_2\|_{C([a,b],C_0(R^m,R^n))} \le r$:

$$\mathfrak{q}_1)\quad \sup_{t\in[a,a+\delta],\, x\in R^m} \bigg| \int_a^t \int_{R^m} f\big(t,s,x,y,w_1(s,y),u(s,y)\big)\,dy\,ds - \int_a^t \int_{R^m} f\big(t,s,x,y,w_2(s,y),u(s,y)\big)\,dy\,ds \bigg| \le$$

$$\le q \sup_{t\in[a,a+\delta],\, x\in R^m} \big|w_1(s,y) - w_2(s,y)\big|,$$

$\mathfrak{q}_2)$ for any $\gamma \in (0, b-a-\delta]$, the condition $w_1(t,\,\cdot\,) = w_2(t,\,\cdot\,)$ implies that

$$\sup_{t\in[a,a+\gamma+\delta],\, x\in R^m} \bigg| \int_a^t \int_{R^m} f\big(t,s,x,y,w_1(s,y),u(s,y)\big)\,dy\,ds - \int_a^t \int_{R^m} f\big(t,s,x,y,w_2(s,y),u(s,y)\big)\,dy\,ds \bigg| \le$$

$$\le q \sup_{t\in[a,a+\gamma+\delta],\, x\in R^m} \big|w_1(s,y) - w_2(s,y)\big|.$$

2. For an arbitrary $w \in C([a,b], C_0(R^m, R^n))$, the operator $F : C([a,b], C_0(R^m, R^n)) \to C([a,b], C_0(R^m, R^n))$ is continuous at (w, u_0).

We now check the validity of $\mathfrak{q}_2$). Choose an arbitrary $b \in (a, \infty)$, $q_0 < 1$, $r > 0$. Let $\gamma \in (0, b-a)$ and $w_1(t,\,\cdot\,) = w_2(t,\,\cdot\,)$, $t \in [a, a+\gamma]$, where $w_1, w_2 \in B_{C([a,b],C_0(R^m,R^n))}(0,r)$. Using assumptions (i)–(iv) and condition 1) of Theorem 1, we get the following estimates

$$\sup_{t\in[a,a+\gamma+\delta],\, x\in R^m} \bigg| \int_a^t \int_{R^m} f\big(t,s,x,y,w_1(s,y),u(s,y)\big)\,dy\,ds -$$

$$- \int_a^t \int_{R^m} f\big(t,s,x,y,w_2(s,y),u(s,y)\big)\,dy\,ds \bigg| \le$$

$$\le \frac{\varepsilon}{2} + \sup_{t\in[a,a+\gamma+\delta],\, x\in B_{R^m}(0,r_\varepsilon)} - \int_{a+\gamma}^{a+\gamma+\delta} \int_{R^m} \Big| f\big(t,s,x,y,w_1(s,y),u(s,y)\big)\,dy\,ds -$$

$$- f\big(t,s,x,y,w_2(s,y),u(s,y)\big) \Big|\,dy\,ds \le$$

$$\le \frac{\varepsilon}{2} + \sup_{t\in[a,a+\gamma+\delta],\, x\in B_{R^m}(0,r_\varepsilon)} \bigg| \int_{a+\gamma}^{a+\gamma+\delta} \int_{R^m} \widetilde{f}_r(s,y) \|w_1 - w_2\|_{C([a,b],BC(R^m,R^n))}\,dy\,ds \bigg| \le \varepsilon.$$

Here, $r_\varepsilon > 0$, $\delta > 0$ can be chosen in such a way that $\varepsilon < q_0$. Thus, we checked that condition $\mathfrak{q}_2$ is satisfied. The verification of condition $\mathfrak{q}_1$ is analogous.

In order to prove validity of **2**, we take arbitrary $\varepsilon > 0$, $\widehat{w} \in C([a,b], C_0(R^m, R^n))$, $w_i \subset C([a,b], C_0(R^m, R^n))$, $u_i \subset M([a,\infty) \times R^m, \mu, R^k)$, $\|\widehat{w} - w_i\|_{C([a,b],C_0(R^m,R^n))}$, $\rho_M(u_i, u_0) \to 0$

$(i\to\infty)$, and the estimate

$$\big\|F(\widehat{w},u_0)-F(w_i,u_i)\big\|_{C([a,b],C_0(R^m,R^n))}=$$
$$=\sup_{t\in[a,b],\,x\in R^m}\bigg|\int_a^t\int_{R^m} f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)\,dy\,ds-\int_a^t\int_{R^m} f\big(t,s,x,y,w_i(s,y),u_i(s,y)\big)\,dy\,ds\bigg|\le$$
$$\le\frac{\varepsilon}{3}+\sup_{t\in[a,b],\,x\in B_{R^m}(0,r_\varepsilon)}\bigg|\int_a^t\int_{R^m} f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)\,dy\,ds-$$
$$-\int_a^t\int_{R^m} f\big(t,s,x,y,w_i(s,y),u_i(s,y)\big)\,dy\,ds\bigg|=$$
$$=\frac{\varepsilon}{3}+\sup_{t\in[a,b],\,x\in B_{R^m}(0,r_\varepsilon)}\int_a^t\int_{R^m}\Big(\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)-f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)\Big|+$$
$$+\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)-f\big(t,s,x,y,w_i(s,y),u_i(s,y)\big)\Big|\Big)\,dy\,ds.$$

Estimating the first summand of the integrand, we get

$$\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)-f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)\Big|\le$$
$$\le\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)-f\big(\overline{t}^{\,\varepsilon},s,\overline{x}^{\,\varepsilon},y,\overline{\widehat{w}^{\varepsilon}},u_0(s,y)\big)\Big|+$$
$$+\Big|f\big(\overline{t}^{\,\varepsilon},s,\overline{t}^{\,\varepsilon},y,\overline{\widehat{w}^{\varepsilon}},u_0(s,y)\big)-f\big(\overline{t}^{\,\varepsilon},s,\overline{x}^{\,\varepsilon},y,\overline{\widehat{w}^{\varepsilon}},u_i(s,y)\big)\Big|+$$
$$+\Big|f\big(\overline{t}^{\,\varepsilon},s,\overline{x}^{\,\varepsilon},y,\overline{\widehat{w}^{\varepsilon}},u_i(s,y)\big)-f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)\Big|.$$

Here $\overline{t}^{\,\varepsilon}$, $\overline{x}^{\,\varepsilon}$, $\overline{\widehat{w}^{\varepsilon}}$ are approximations of t, x, $\widehat{w}(s,y)$, taking finite number of values (on their compact ranges of definition). Thus, using the condition 2) of Theorem 1 and the assumptions (i)–(iv), the first and third summands on the right-hand side of the inequality go to 0 uniformly with respect to $(s,y)\in[a,b]\times R^m$ and the second summand go to 0 in measure on $[a,b]\times R^m$.

Next, estimation of $|f(t,s,x,y,\widehat{w}(s,y),u_i(s,y))-f(t,s,x,y,w_i(s,y),u_i(s,y))|$ using the condition 1) of Theorem 1, gives uniform convergence of this expression to 0 on $[a,b]\times R^m$.

Thus, we can find such I that for any $i>I$, we get

$$\sup_{t\in[a,b],\,x\in B_{R^m}(0,r_\varepsilon)}\int_a^t\int_{R^m}\Big(\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_0(s,y)\big)-f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)\Big|+$$
$$+\Big|f\big(t,s,x,y,\widehat{w}(s,y),u_i(s,y)\big)-f\big(t,s,x,y,w_i(s,y),u_i(s,y)\big)\Big|\Big)\,dy\,ds\le 2\,\frac{\varepsilon}{3}\,,$$

which concludes the verification of conditions of Theorem 2.1 in [10] and completes the proof. □

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The Relation Between the Existence of Bounded Solutions of Differential Equations and the Corresponding Difference Equations

V. Danilov

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail: danilov_vy@ukr.net

O. Karpenko

National Aviation University, Kyiv, Ukraine
E-mail: olyakare@gmail.com

V. Kravets

State Agrotechnical University of Militopol, Militopol, Ukraine
E-mail: v_i_kravets@list.ru

We study the connection between the existence of bounded (on real axis) solutions of differential equations and the corresponding difference equations. We obtain the conditions under which the existence of bounded solutions of differential equations implies the existence of bounded solution of difference equation, and vice versa.

Throughout this work, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^d$ is the Euclidian space of d – dimensional vectors, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of integers. The euclidian norm in $\mathbb{R}^d$ is denoted through $|\cdot|$, and $\|\cdot\|$ is the matrix norm in the same space.

Consider the following system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{1}$$

$t \in \mathbb{R}$, $x \in D$ for $D \subset \mathbb{R}^d$, and the corresponding system of difference equations

$$x^h(t+h) = x^h(t) + hX\big(t, x^h(t)\big), \tag{2}$$

where $h > 0$ is the step of difference equation. We assume that the function $X(t,x)$ is continuously differentiable and bounded together with its partial derivatives, i.e. $\exists\, C > 0$ such that

$$|X(t,x)| + \left|\frac{\partial X(t,x)}{\partial t}\right| + \left\|\frac{\partial X(t,x)}{\partial x}\right\| \le C \tag{3}$$

for $t \in R$, $x \in D$, where $\frac{\partial X}{\partial x}$ is the corresponding Jacobi matrix.

In this paper we study the connection between the existence of globally bounded solutions of (1) and of the corresponding system (2).

Here are some necessary statements and definition used later.

Consider the system (2) for $t = t_0 + kh$, where t_0 is fixed. We have

$$x^h_{k+1} = x^h_k + hX(t_0 + kh, x^h_k), \tag{4}$$

where $k \in \mathbb{Z}$, $h > 0$, $x^h_k = x^h(t_0 + kh)$, $x^h(t_0) = x_0$. The following results are used throughout the work.

Lemma 1. *Let $x(t)$ and x_k^h be the solutions of* (1) *and* (4) *on the interval* $[t_0, t_0 + T]$ *such that* $x(t_0) = x_0^h = x_0$, $x \in D$. *Then, if the inequality* (3) *holds, we have the estimate*

$$\big|x(t_0 + kh) - x_k^h\big| < he^{CT}[1 + KT], \tag{5}$$

for $kh < T$, where $K = C + C^2$.

Lemma 2. *If the inequality* (3) *holds, any solution of* (4) x_k^h *continuously depends on the initial data x_0, until it reaches the boundary of D.*

Definition 1. We say that a solution $x^h(t)$ of system (2), defined for $t \in \mathbb{R}$, is exponentially stable uniformly in t_0 if there exist $\delta > 0$, $N > 0$ and $\alpha > 0$ such that for any solution $y^h(t)$ of the system (2) such that $|y^h(t_0) - x^h(t_0)| < \delta$ for $t \geq t_0$ we have the inequality

$$\big|x^h(t) - y^h(t)\big| \leq Ne^{-\alpha(t-t_0)}\big|x^h(t_0) - y^h(t_0)\big|, \tag{6}$$

where constants δ, N and α do not depend on t_0.

Consider the system (4) for $t_0 = 0$:

$$x_{k+1}^h = x_k^h + hX(kh, x_k^h), \tag{7}$$

Definition 2. A solution x_k^h of (7) is called *exponentially stable* uniformly in k_0 if it satisfies the conditions in 1 with t_0 replaced with k_0 and t replaced with kh.

Our main results are the following theorems.

Theorem 1. *Assume the following conditions hold:*

1) *The function $X(t, x)$ satisfies* (3).
2) *There exists $h_0 > 0$ such that the system* (2) *has a bounded on $\mathbb{R}$, exponentially stable* (*in the sense of Definition* 1) *solution $x_k^{h_0}$, which lies in the domain D together with its ρ – neighborhood for some $\rho > 0$.*
3) *Additionally,*

$$h_0 e^{C(\frac{\ln 4N}{\alpha}+1)}\Big[1 + (C + C^2)\Big(\frac{\ln 4N}{\alpha} + 1\Big)\Big] \leq \frac{\delta}{8}\,, \tag{8}$$

$$\frac{3N\delta}{2} < \rho, \tag{9}$$

$$h_0 \leq \frac{\rho}{4C}\,, \tag{10}$$

where N, δ and α are defined in (6) *and C is given by* (3).

Then for all h, $0 < h < h_0$, the system (2) *has a bounded solution on $\mathbb{R}$.*

We proceed with studying the conditions for the existence of a bounded solution of (1), given that (2) has such a solution for $t = kh_0$.

The following theorem holds.

Theorem 2. *Let the following conditions hold:*

1) *the function $X(t, x)$ satisfies the condition* 1) *of Theorem* 1;
2) $\exists\, h_0 > 0$ *such that the system* (7) *has a bounded on $\mathbb{Z}$, uniformly in k_0 exponentially stable solution which belongs to the domain D together with its ρ neighborhood.*

Then, if the inequalities (9)–(10) *hold, the system* (1) *has a bounded solution defined on* $\mathbb{R}$.

We now proceed with the study the conditions on the existence of a bounded on $\mathbb{R}$ solution of the system (2) under the assumption that the system (1) has such a solution.

Theorem 3. *Let the following conditions hold:*

1) *The function* $X(t,x)$ *satisfies the condition* 1) *of* 1*;*
2) *The system* (1) *has a bounded on* $\mathbb{R}$, *asymptotically stable uniformly in* $t_0 \in \mathbb{R}$ *solution* $x(t)$, *which lies in the domain* D *with some* ρ *– neighborhood.*

Then there exists h_0 *such that for all* $0 < h \le h_0$ *the system* (2) *has a bounded on* $\mathbb{R}$ *solution* $x^h(t)$, *and*

$$\sup_{t\in R} \big|x^h(t) - x(t)\big| \to 0, \quad h \to 0 \tag{11}$$

The next example shows that the asymptotic stability of a bounded solution $x(t)$ is essential, and without this assumption we can get a qualitatively different behavior of solutions of differential and difference equations.

Example 1. Consider the differential equation

$$\ddot{x} + x = 0 \tag{12}$$

with the general solution

$$x(t) = C_1 \cos t + C_2 \sin t.$$

Its solutions are bounded, stable, but not asymptotically stable. The corresponding differential equation has the form

$$x((k+2)h) - 2x((k+1)h) + 2x(kh) = 0, \tag{13}$$

with the general solution

$$x_k^h = C_1 2^{\frac{hk}{2}} \cos \frac{hk\pi}{4} + C_2 2^{\frac{hk}{2}} \sin \frac{hk\pi}{4}\,.$$

We see that for all steps $h > 0$, all solutions of this equation (except the trivial one) are unbounded.

Oscillatory Properties of Solutions to Two-Dimensional Emden–Fowler Type Systems

Monika Dosoudilová

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic
E-mail: dosoudilova@fme.vutbr.cz

Alexander Lomtatidze

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic;
Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: lomtatidze@fme.vutbr.cz

Jiří Šremr

Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: sremr@ipm.cz

On the half-line $[0,+\infty[$, we consider the system

$$\boxed{\begin{aligned} u' &= g(t)|v|^{1/\alpha}\operatorname{sgn} v,\\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned}} \tag{1}$$

where $\alpha > 0$ and $p, g\colon [0,+\infty[\to \mathbb{R}$ are locally Lebesgue integrable functions.

A pair (u,v) is said to be *a solution to system* (1) *on the interval* $I \subseteq [0,+\infty[$ if the functions $u,v\colon I \to \mathbb{R}$ are absolutely continuous on every compact interval contained in I and satisfy equalities (1) almost everywhere in I. In the paper [5], Mirzov proved that all non-extendable solutions to system (1) are defined on the whole interval $[0,+\infty[$. Therefore, speaking about a solution to system (1), we assume without loss of generality that it is defined on the whole interval $[0,+\infty[$. Mirzov also proved (see, e.g., [4, Theorem 9.3]) that all non-zero solutions (u,v) to system (1) are proper, i.e., the inequality $\sup\{|u(\tau)|+|v(\tau)|:\ t\le\tau<+\infty\}>0$ holds for every $t\ge 0$.

Definition 1. A solution (u,v) to system (1) is called *non-trivial* if $u \not\equiv 0$ on any neighbourhood of $+\infty$. We say that a non-trivial solution (u,v) to system (1) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

It is well known (see [5, Theorem 1.1]) that a certain analog of Sturm's theorem holds for system (1) under the additional assumption

$$g(t) \ge 0 \quad \text{for a.e. } t \ge 0. \tag{2}$$

In particular, if inequality (2) holds and system (1) has an oscillatory solution, then any other non-trivial solution is also oscillatory. Moreover, under assumption (2), if (u,v) is an oscillatory solution to system (1), then, together with u, the function v also oscillates. On the other hand, it is clear that if $g\equiv 0$ on some neighbourhood of $+\infty$, then all non-trivial solutions to system (1) are non-oscillatory.

Therefore, we assume throughout the paper that inequality (2) holds and

$$\operatorname{meas}\big\{\tau \ge t:\ g(\tau) > 0\big\} > 0 \quad \text{for every } t \ge 0. \tag{3}$$

Definition 2. We say that system (1) is *oscillatory* if all its non-trivial solutions are oscillatory.

We first assume that the coefficient g is non-integrable on $[0,+\infty[$, i.e.,

$$\int_0^\infty g(s)\,\mathrm{d}s = +\infty. \tag{4}$$

Let

$$f_1(t) := \int_0^t g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (2), (3), and (4), we have $\lim_{t\to+\infty} f_1(t) = +\infty$ and there exists a number $t_g \geq 0$ such that $f_1(t) > 0$ for $t > t_g$ and $f_1(t_g) = 0$. Since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, we can assume without loss of generality that $t_g = 0$, i.e., $f_1(t) > 0$ for $t > 0$.

For any $\kappa > \alpha$, $\beta > 0$, and $\lambda < \alpha$, we put

$$k_1(t;\kappa,\beta,\lambda) := \frac{1}{f_1^{\kappa\beta}(t)} \int_0^t \big[f_1^\beta(t) - f_1^\beta(s)\big]^\kappa f_1^\lambda(s) p(s)\,\mathrm{d}s \quad \text{for } t > 0, \tag{5}$$

$$c_1(t;\lambda) := \frac{\alpha-\lambda}{f_1^{\alpha-\lambda}(t)} \int_0^t \frac{g(s)}{f_1^{\lambda+1-\alpha}(s)} \bigg(\int_0^s f_1^\lambda(\xi) p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \quad \text{for } t > 0. \tag{6}$$

Theorem 1. *Let conditions* (2), (3), *and* (4) *hold,* $\kappa > \alpha$, $\beta > 0$, $\lambda < \alpha$, *and either*

$$\limsup_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) = +\infty \tag{7}$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) < +\infty, \\ \text{the function } c_1(\,\cdot\,;\lambda) \text{ does not possess a finite limit as } t \to +\infty. \end{cases} \tag{8}$$

Then system (1) *is oscillatory.*

Observe that condition (7) with $\beta = 1$, $\lambda = 0$ and $g \equiv 1$ reduces to the condition

$$\limsup_{t\to+\infty} \frac{1}{t^\kappa} \int_0^t (t-s)^\kappa p(s)\,\mathrm{d}s = +\infty \quad \text{for some } \kappa > \alpha \tag{9}$$

which is the half-linear extension of the classical Kamenev linear oscillation criterion (see [2]). Conditions (8) then give a possible counterpart of the oscillation criterion (9).

It is well known that system (1) is oscillatory provided that the function

$$M : t \longmapsto \frac{1}{f_1(t)} \int_0^t g(s) \bigg(\int_0^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \tag{10}$$

is bounded from below in some neighbourhood of $+\infty$ and does not have a finite limit as $t \to +\infty$ (see, e.g., [4, Theorem 12.3]). However, Theorem 1 can be applied also in the case, where the lower limit of the function M given by (10) is $-\infty$.

Now we formulate a Hartman–Wintner type result which follows from Theorem 1. For any $\lambda < \alpha$ and $\nu < 1$, we put

$$\widetilde{c}_1(t;\lambda,\nu) := \frac{1-\nu}{f_1^{1-\nu}(t)} \int\limits_0^t \frac{g(s)}{f_1^{\nu}(s)} \bigg(\int\limits_0^s f_1^{\lambda}(\xi)p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t > 0. \tag{11}$$

Corollary 1. *Let conditions* (2), (3), *and* (4) *hold,* $\lambda < \alpha$, $\nu < 1$, *and either*

$$\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu).$$

Then system (1) *is oscillatory.*

Observe that Corollary 1 with $\lambda = 0$ and $\nu = 0$ coincide with the above-mentioned Mirzov's result, namely Theorem 12.3 from [4]. On the other hand, it is worth mentioning that Corollary 1 with $g \equiv 1$, $\lambda = 0$ and $\nu = 1-\alpha$ is in compliance with Theorem 1.1 stated in [3].

Unlike the above part, we assume in what follows that g is integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,\mathrm{d}s < +\infty. \tag{12}$$

Let

$$f_2(t) := \int\limits_t^{+\infty} g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (2), (3), and (12), we have $\lim_{t\to+\infty} f_2(t) = 0$ and $f_2(t) > 0$ for $t \geq 0$.

For any $\kappa > \alpha$, $\beta > 0$, and $\lambda > \alpha$, we put

$$k_2(t;\kappa,\beta,\lambda) := f_2^{\kappa\beta}(t) \int\limits_0^t \big[f_2^{-\beta}(t) - f_2^{-\beta}(s)\big]^{\kappa} f_2^{\lambda}(s)p(s)\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{13}$$

$$c_2(t;\lambda) := (\lambda-\alpha) f_2^{\lambda-\alpha}(t) \int\limits_0^t \frac{g(s)}{f_2^{\lambda+1-\alpha}(s)} \bigg(\int\limits_0^s f_2^{\lambda}(\xi)p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{14}$$

Theorem 2. *Let conditions* (2), (3), *and* (12) *hold,* $\kappa > \alpha$, $\beta > 0$, $\lambda > \alpha$, *and either*

$$\limsup_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) = +\infty \tag{15}$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) < +\infty, \\ \text{the function } c_2(\,\cdot\,;\lambda) \text{ does not possess a finite limit as } t\to+\infty. \end{cases} \tag{16}$$

Then system (1) *is oscillatory.*

Analogously to the "non-integrable" case, the following Hartman–Wintner type result can be derived from Theorem 2. For any $\lambda > \alpha$ and $\nu > 1$, we put

$$\widetilde{c}_2(t;\lambda,\nu) := (\nu-1) f_2^{\nu-1}(t) \int\limits_0^t \frac{g(s)}{f_2^{\nu}(s)} \bigg(\int\limits_0^s f_2^{\lambda}(\xi)p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{17}$$

Corollary 2. *Let conditions* (2), (3), *and* (12) *hold,* $\lambda > \alpha$, $\nu > 1$, *and either*

$$\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu).$$

Then system (1) *is oscillatory.*

As far as we know, a Hartman–Wintner type result for the half-linear equation

$$\boxed{\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0} \tag{18}$$

in the case, where

$$\int\limits_0^{+\infty} r^{\frac{1}{1-q}}(s)\,\mathrm{d}s < +\infty \tag{19}$$

is satisfied, is known only under the additional assumption that $p(t) \geq 0$ for a. e. $t \geq 0$ (see survey given in [1, Section 2.2]). We can exclude this additional assumption and derive from Corollary 2 the following statement.

Corollary 3. *Let* $\lambda > q-1$ *and relation* (19) *hold. Then each of following two conditions is sufficient for oscillation of* (18)*:*

$$\lim_{t\to+\infty} R(t)\int\limits_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int\limits_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty,$$

$$-\infty < \liminf_{t\to+\infty} R(t)\int\limits_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int\limits_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s <$$
$$< \limsup_{t\to+\infty} R(t)\int\limits_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int\limits_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s,$$

where $R(t) := \int\limits_t^{+\infty} r^{\frac{1}{1-q}}(s)\,\mathrm{d}s$ *for* $t \geq 0$.

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Continuous Dependence of the Minimum of the Bolza Type Functional on the Initial Data in Nonlinear Optimal Control Problems with Distributed Delay

Pridon Dvalishvili

I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: pridon.dvalishvili@tsu.ge

Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition. Suppose that $O\subset\mathbb{R}^n_x$, $V\subset\mathbb{R}^r_u$ are open sets and $X_0\subset O$, $P\subset\mathbb{R}^l_p$ are compact sets. Let the $(1+n)$-dimensional function $F(t,x,u,p)=(f^0,f)^T$ be continuous on the set $I\times O\times V\times P$ and continuously differentiable with respect to $x\in O$, where $I=[t_0,t_1]$. By $\Delta(V)$ we denote collection of compact sets $U\subset V$. Let $0\le\sigma_1<\sigma_2$ be a given number and let Φ be the set of initial functions $\varphi(t)\in O$, $t\in[t_0-\sigma_2,t_0]$. Next, let $\Omega(U)$ be the set of measurable control functions $u(t)\in U$, $t\in I$, where $U\in\Delta(V)$ and let Q be the set of continuous scaler functions $q(t,x)$, $(t,x)\in I\times O$.

To each $\mu=(\sigma,x_0,\varphi,p)\in\Lambda=[\sigma_1,\sigma_2]\times X_0\times\Phi\times P$ we put into correspondence the controlled differential equation with distributed delay

$$\dot x(t)=\int\limits_{-\sigma}^{0} f\big(t,x(t+s),u(t),p\big)\,ds,\ \ t\in I,\ \ u\in\Omega(U) \tag{1}$$

with the initial condition

$$x(t)=\varphi_0(t),\ \ t\in[t_0-\sigma,t_0),\ \ x(t_0)=x_0, \qquad (2).$$

Definition 1. Let $w=(\mu,u)=(\sigma,x_0,\varphi,p,u)\in W=\Lambda\times\Omega(U)$ be a given element. A function $x(t)=x(t;w)$, $t\in[t_0-\sigma_2,t_1]$, is called a solution of equation (1) with initial condition (2), or a solution corresponding to element w defined on $[t_0-\sigma,t_1]$, if it satisfies condition (2), is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) everywhere on $[t_0,t_1]$.

Definition 2. Let $\mu\in\Lambda$ be a fixed element. A control $u\in\Omega(U)$ is said to be admissible if for the element $w=(\mu,u)$ there exists the corresponding solution $x(t)=x(t;w)$ defined on the interval $[t_0-\sigma_2,t_1]$.

The set of admissible control $u(t)$ is denoted by $\Omega(\mu;U)$.

Definition 3. Let an element $\mu=(\sigma,x_0,\varphi_0,p)\in\Lambda$ and a function $q\in Q$ be given. A control $u_0(t;\mu,U,q)\in\Omega(\mu;U)$ is called optimal if

$$J(u_0(\,\cdot\,;\mu,U,q),\sigma,p,q)=\inf\big\{J(u(\,\cdot\,),\sigma,p,q):\ u\in\Omega(\mu;U)\big\},$$

where

$$J(u(\,\cdot\,),\sigma,p,q)=q(t_1,x(t_1))+\int\limits_{t_0}^{t_1}\int\limits_{-\sigma}^{0} f^0(t,x(t+s),u(t),p)\,ds\,dt \tag{3}$$

and $x(t)=x(t;w)$, $w=(\mu,u)$.

The problem (1)–(3) is called an optimal control problem with the distributed delay. The control $u_0(t;\mu,U,q)$ is called a solution of the problem (1)–(3).

To formulate the main result we introduce the following notations: by M we denote the set of continuous functions $x(t)\in O$, $t\in I_1=[t_0-\sigma_2,t_0)\cup(t_0,t_1]$ with $cl(x(I_1))\subset O$,

$$F\big(t,x(\,\cdot\,),u,\sigma,p\big)=\int\limits_{-\sigma}^{0}F(t,x(t+s),u,p)\,ds,\quad G\big(t,x(\,\cdot\,),U,\sigma,p\big)=\big\{F(t,x(\,\cdot\,),u,\sigma,p):\ u\in U\big\},$$

$$t\in I,\ \ x(\,\cdot\,)\in M,\ \ U\in\Delta(V),\ \ \sigma\in[\sigma_1,\sigma_2],\ \ p\in P.$$

Theorem. *Let $\mu_0=(\sigma_0,x_{00},\varphi_0,p_0)\in\Lambda$ be a fixed element and let $U_0\in\Delta(V)$ be a given set. Let the following conditions hold:*

1. *$\Omega(\mu_0;U_0)$ is not empty;*
2. *there exists a compact set $K\subset O$ such that for every $u\in\Omega(\mu_0;U_0)$ we have*

$$x(t;w_0)\in K,\ \ t\in[t_{00}-\sigma_2,t_1],$$

where $w_0=(\mu_0,u)$;

Then for any $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)$ such that for every

$$\mu=(\sigma,x_0,\varphi,p)\in\Lambda,\ \ q\in Q,\ \ U\in\Delta(V)$$

the conditions

$$|\sigma-\sigma_0|+|x_0-x_{00}|+\|\varphi-\varphi_0\|+\|q-q_0\|+D(U,U_0)\le\delta$$

are satisfied and the set $G(t,x(\,\cdot\,),U,\sigma,p)$ is convex for each $(t,x(\,\cdot\,))\in I\times M$.

Then the optimal control problem (1)–(3) has a solution $u_0(t;\mu,U,q)\in\Omega(U)$ and the following inequality

$$\Big|J\big(u_0(\,\cdot\,;\mu_0,U_0,q_0),\sigma_0,p_0,q_0\big)-J\big(u_0(\,\cdot\,;\mu,U,q),\sigma,p,q\big)\Big|\le\varepsilon$$

is fulfilled. Here

$$\begin{aligned}\|\varphi-\varphi_0\|&=\sup\big\{|\varphi(t)-\varphi_0(t)|:\ t\in[t_0-\sigma_2,t_0]\big\},\\ \|q-q_0\|&=\sup\big\{|q(t,x)-q_0(t,x)|:\ (t,x)\in I\times K\big\},\\ D(U,U_0)&=\sup\big\{|u'-u''|:\ u'\in U,\ u''\in U_0\big\}.\end{aligned}$$

Theorem is proved by scheme given in [1–3].

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Asymptotic Representation of Solutions of Second-Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

V. M. Evtukhov

Odessa National University, Odessa, Ukraine
E-mail: evmod@i.ua

E. S. Vladova

Odessa State Academy of Civil Engineering and Architecture, Odessa, Ukraine
E-mail: lena@gavrilovka.com.ua

We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_2(y'), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, the $\varphi_i : \Delta(Y_i^0) \to]0,+\infty[$ $(i = 1,2)$ are some twice continuously differentiable functions, where $\Delta(Y_i^0)$ is some one-sided neighborhood of the point Y_i^0, Y_i^0 is equal to either 0 or $\pm\infty$.

This equation has been considered in the works of Evtukhov V. M. [6]–[9] in case, when $\varphi_i(z)$ were power-law functions, and in the works of Belozerova M. A. [1]–[4] in cases, when $\varphi_i(z)$ were slowly or regularly varying functions when $z \to Y_i^0$, $i = \overline{1,2}$ (see Seneta [10]).

In the equation (1), we suppose that the function $\varphi_1(z)$ satisfies the conditions:

$$\lim_{\substack{z\to Y_1^0\\ z\in\Delta(Y_1^0)}} \frac{z\varphi_1'(z)}{\varphi_1(z)} = \lambda \ \ (\lambda \in \mathbb{R}), \tag{2}$$

and the function $\varphi_2(z)$ is following:

$$\begin{gathered} \varphi_2'(z) \neq 0 \ \text{ when } \ z \in \Delta(Y_2^0), \qquad \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \varphi_2(z) = \Phi_2^0, \ \ \Phi_2^0 \in \{0,+\infty\}, \\ \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \frac{\varphi_2''(z)\varphi_2(z)}{[\varphi_2'(z)]^2} = 1. \end{gathered} \tag{3}$$

The fulfilment of requirements (2), (3) means that the function $\varphi_1(z)$ is regularly or slowly varying when $z \to Y_1^0$, and the function $\varphi_2(z)$ is rapidly varying when $z \to Y_2^0$. With such assumptions for the functions $\varphi_i(z)$, $i = \overline{1,2}$, we are unaware of any results about the asymptotic behavior of solutions neither for the equation (1), not for any of its specific cases.

A solution y of the equation (1) is called a $P_\omega(\Lambda_0)$-solution, where $-\infty \le \Lambda_0 \le +\infty$, if it is defined on some interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions:

$$\begin{gathered} \lim_{t\uparrow\omega} y(t) = Y_1^0, \quad \lim_{t\uparrow\omega} \varphi_2(y'(t)) = \Phi_2^0, \\ \lim_{t\uparrow\omega} \frac{\varphi_2'(y'(t))}{\varphi_2(y'(t))}\, \frac{y''(t)y(t)}{y'(t)} = \Lambda_0. \end{gathered} \tag{4}$$

The aim of the paper is to derive necessary and sufficient conditions for the existence of $P_\omega(\Lambda_0)$-solutions of the equation (1) when $\Lambda_0 \in \mathbb{R}\setminus\{0\}$, and also to establish asymptotic formulas for such solutions and their derivatives of the first order.

Let us introduce notation needed in forthcoming considerations.

First, we set:

$$\mu_i^0 = \begin{cases} 1, & \text{if } Y_i^0 = +\infty, \ \text{or} Y_i^0 = 0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is a right neighborhood of the point } 0, \\ -1, & \text{if } Y_i^0 = -\infty, \ \text{or} Y_i^0 = 0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is a left neighborhood of the point } 0. \end{cases}$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t-\omega, & \text{if } \omega < +\infty, \end{cases}$$

$$J(t) = \begin{cases} \displaystyle\int_A^t p(\tau)\,d\tau, & \text{if } \lambda \neq -\Lambda_0, \\ \displaystyle\int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, & \text{if } \lambda = -\Lambda_0, \end{cases} \qquad \beta = \begin{cases} 1+\lambda\Lambda_0^{-1}, & \text{if } \lambda \neq -\Lambda_0, \\ -\Lambda_0^{-1}, & \text{if } \lambda = -\Lambda_0, \end{cases}$$

where the integration limit $A \in \{\omega, a\}$ is chosen so as to ensure that the corresponding integral J tends either to zero or to infinity when $t \uparrow \omega$.

Next, we set numbers A_i^* $(i = 1, 2)$:

$$A_1^* = \begin{cases} 1, & \text{if } \omega = \infty, \\ -1, & \text{if } \omega < \infty, \end{cases} \qquad A_2^* = \begin{cases} 1, & \text{if } A = a, \\ -1, & \text{if } A = \omega. \end{cases}$$

Theorem 1. *Let $\Lambda_0 \in \mathbb{R}\setminus\{0\}$. Then, for the existence of $P_\omega(\Lambda_0)$-solutions of the equation* (1)*, it is necessary and sufficient that*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = -\Lambda_0\beta \tag{5}$$

and sign conditions be satisfied

$$\begin{gathered} A_1^* > 0 \ \textit{ if } \ Y_1^0 = \pm\infty, \quad A_1^* < 0 \ \textit{ if } \ Y_1^0 = 0, \\ A_2^*\beta > 0 \ \textit{ if } \ \Phi_2^0 = 0, \quad A_2^*\beta < 0 \ \textit{ if } \ \Phi_2^0 = \pm\infty, \end{gathered} \tag{6}$$

$$\mu_1^0\mu_2^0 A_1^* > 0 \ \textit{ and } \ \alpha_0\mu_2^0 A_2^*\beta > 0. \tag{7}$$

Moreover, each solution of this kind admits the following asymptotic representation when $t \uparrow \omega$

$$\frac{y(t)}{y'(t)} = \pi_\omega(t)[1+o(1)], \tag{8}$$

$$\frac{1}{\varphi_1(y(t))\varphi_2'(y'(t))} = \alpha_0\,\frac{\pi_\omega(t)p(t)}{\Lambda_0}\,[1+o(1)], \tag{9}$$

moreover, for $\omega = +\infty$, there exists a one-parameter family of such solutions, if $\Lambda_0 > 0$, and there exists a two-parameter family of such solutions, if $\Lambda_0 < 0$; for $\omega < +\infty$, there exists a one-parameter family of such solutions, if $\Lambda_0 > 0$.

We will introduce auxiliary notation and conditions that will enable us to rewrite the asymptotic formulas (8), (9) more conveniently.

Definition (see [4])**.** We say that a function $\theta : \Delta(U^0) \to]0, +\infty[$, $U^0 \in \{0, \pm\infty\}$ satisfies condition S if for any continuously differentiable function $l : \Delta(U^0) \to]0, +\infty[$ such that

$$\lim_{\substack{z\to U^0 \\ z\in\Delta(U^0)}} \frac{z\,l'(z)}{l(z)} = 0,$$

the following asymptotic formula is fulfilled:

$$\theta(zl(z)) = \theta(z)[1+o(1)] \ \text{ if } \ z \to U^0 \ \ \big(z \in \Delta(U^0) \ \big).$$

Since the $\varphi_1(z)$ is a regularly varying function of the λ-order when $z \to Y_1^0$, then for this function the following representation is executed:

$$\varphi_1(z) = |z|^\lambda \theta_1(z), \tag{10}$$

where the function $\theta_1(z)$ is slowly varying when $z \to Y_1^0$.

Let us introduce notation for the rapidly varying function $\varphi_2(z)$:

$$\psi(z) = \int\limits_B^z \frac{ds}{\varphi_2(s)}\,, \ \text{ where } \ B = \begin{cases} Y_2^0, & \text{if } \displaystyle\int\limits_b^{Y_2^0} \frac{ds}{\varphi_2(s)} \ \text{ converges,} \\ b, & \text{if } \displaystyle\int\limits_b^{Y_2^0} \frac{ds}{\varphi_2(s)} \ \text{ diverges,} \end{cases} \tag{11}$$

where b is any number from the interval $\Delta(Y_2^0)$ and $\Psi^0 = \lim\limits_{z \to Y_2^0} \psi(z)$.

It is evident that the function $\psi(z)$ is also rapidly varying when $z \to Y_2^0$, consequently, $|\psi^{-1}(z)|$ is a slowly varying function when $z \to \Psi^0$.

Theorem 2. *Let $\Lambda_0 \in \mathbb{R} \setminus \{0\}$ and functions $\theta_1(z)$, $|\psi^{-1}(z)|$ satisfy the condition S. Then each $P_\omega(\Lambda_0)$-solution (if any) of the equation* (1) *admits the asymptotic representations when $t \uparrow \omega$*

$$y(t) = \mu_1^0 \Big| \pi_\omega(t) \psi^{-1}\big(\mu_2^0 |J(t)|^{\frac{1}{\beta}}\big) \Big| [1+o(1)],$$

$$\frac{1}{\varphi_2'(y'(t))} = -\mu_2^0 \Big| \pi_\omega(t) \psi^{-1}(\mu_2^0 |J(t)|^{\frac{1}{\beta}}) \Big|^\lambda \Big| \Lambda_0^{-1} \pi_\omega(t) p(t) \theta_1\big(\mu_1^0 |\pi_\omega(t)|\big) \Big| [1+o(1)].$$

Theorems 1 and 2 were proved using the results for cyclic systems of differential equations with regularly varying nonlinearities established in [5].

As an example illustrating our results, consider the differential equation:

$$y'' = \alpha_0 p(t) |y|^\lambda \big| \ln |y| \big|^\gamma e^{-\sigma |y'|^\delta} |y'|^{1-\delta} \operatorname{sign} y', \tag{12}$$

where $\alpha_0 \in \{1,-1\}$, $\delta, \sigma \in \mathbb{R} \setminus \{0\}$, $\lambda, \gamma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function.

Equation (12) belongs to the type of equations (1), where $\varphi_1(z) = |z|^\lambda |\ln |z||^\gamma$, $\varphi_2(z) = e^{-\sigma |z|^\delta} |z|^{1-\delta} \operatorname{sign} z$. The function $\varphi_1(z)$ is properly varying of the λ-order, when $z \to Y_2^0$, and the function $\varphi_2(z)$, in case $\delta > 0$, is rapidly varying when $z \to \pm\infty$, and in case $\delta < 0$, it is rapidly varying when $z \to 0$.

For the $\varphi_2(z)$, the function $\psi(z)$ which was defined in (11) has the following form:

$$\psi(z) = \frac{1}{\sigma\delta}\, e^{\sigma |z|^\delta}.$$

In their turn, the function $\theta_1(z)$, which was defined in (10), and the function $\psi^{-1}(z)$ have the following form:

$$\theta_1(z) = \big| \ln |z| \big|^\gamma, \quad \psi^{-1}(z) = \ln^{\frac{1}{\delta}} |\sigma\delta z|^{\frac{1}{\sigma}}$$

and satisfy the condition S.

For the equation (12), the condition (4) from the definition of $P_\omega(\Lambda_0)$-solution has the following form:

$$\lim_{t\uparrow\omega} \sigma\delta|y'|^\delta \frac{yy''(t)}{[y'(t)]^2} = -\Lambda_0.$$

Therefore, for the equation (12), Theorems 1 and 2 readily imply the following assertion.

Corollary. *Let $\Lambda_0 \in \mathbb{R}\setminus\{0\}$. Then for the existence of $P_\omega(\Lambda_0)$-solution of the equation (12), it is necessary and sufficient that the conditions (5)–(7) be satisfied. Moreover, each solution of this kind admits the following asymptotic representation when $t\uparrow\omega$:*

$$y(t) = \mu_1^0 \Big|\pi_\omega(t)\ln^{\frac{1}{\delta}}\big|\mu_2^0\sigma\delta|J(t)|^{\frac{1}{\beta}}\big|^{\frac{1}{\delta}}\Big|[1+o(1)],$$

$$|y'(t)|^\delta = \frac{1}{\sigma}\bigg[\lambda\ln\Big|\pi_\omega(t)\ln^{\frac{1}{\delta}}\big|\mu_2^0\sigma\delta|J(t)|^{\frac{1}{\beta}}\big|^{\frac{1}{\delta}}\Big| + \ln\Big|\Lambda_0^{-1}\pi_\omega(t)p(t)\big|\ln|\mu_1^0\pi_\omega(t)|\big|^\gamma\Big|\bigg] + o(1),$$

moreover, for $\omega=+\infty$, there exists one-parameter family of solutions, if $\Lambda_0>0$, and there exists two-parameter family of solutions, if $\Lambda_0<0$; for $\omega<+\infty$, there exists one-parameter family, if $\Lambda_0>0$.

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Invariant Manifolds of a Certain Class of Differential Equations

P. V. Feketa, Yu. M. Perestyuk

Taras Shevchenko National University of Kiev, Kiev, Ukraine
E-mail: petro.feketa@gmail.com; perestyuk@gmail.com

We have established sufficient conditions for the existence of invariant toroidal manifolds of a certain class of linear extensions of dynamical system on torus. A similar problem for a sufficiently wide class of impulsive differential equations with non-fixed impulses also have been investigated.

We consider a system of differential equations, defined in the direct product of a torus $\mathcal{T}_m$ and an Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{1}$$

where $\varphi = \operatorname{coll}(\varphi_1, \dots, \varphi_m)$, $x = \operatorname{coll}(x_1, \dots, x_n)$, $A(\varphi)$ and $f(\varphi)$ are continuous 2π-periodic with respect to each of the variable φ_j, $j = 1, \dots, m$ matrix and vector functions, respectively, defined on the torus $\mathcal{T}_m$.

We assume that the vector function $a(\varphi)$ satisfies the Lipschitz's condition

$$\big\|a(\varphi) - a(\psi)\big\| \leq \ \mathcal{L}\|\varphi - \psi\|, \tag{2}$$

for each $\varphi, \psi \in \mathcal{T}_m$ and some constant $\alpha > 0$.

Let $\varphi_t(\varphi)$, $\varphi_0(\varphi) = \varphi$ be the solution of the first of equations (1).

Consider the system of equations

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)) \tag{3}$$

that depends on φ as a parameter.

By invariant toroidal manifold of system (1) we will understand the set $x = u(\varphi), u(\varphi) \in C(\mathcal{T}_m)$, where $u(\varphi)$ is such function that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution of system of equations (3) for each $\varphi \in \mathcal{T}_m$.

Deep research regarding the existence of invariant toroidal manifolds of differential equations were made by A. M. Samoilenko and the results of these studies are summarized in the classical monograph [1]. The main approach to the study of toroidal manifolds of system of equations (1) is based on the concept of Green–Samoilenko function of the invariant tori problem introduced in [1].

Let $\Omega^t_\tau(\varphi), \Omega^\tau_\tau(\varphi) = E$ be a fundamental matrix of system (3) and $C(\varphi)$ be a matrix function from the space $C(\mathcal{T}_m)$.

Let

$$G_0(\tau, \varphi) = \begin{cases} \Omega^0_\tau(\varphi)C(\varphi_\tau(\varphi)), & \tau \leq 0, \\ -\Omega^0_\tau(\varphi)E - C(\varphi_\tau(\varphi)), & \tau > 0. \end{cases} \tag{4}$$

Function $G_0(\tau, \varphi)$ is called Green–Samoilenko function of the invariant tori problem (1) if the following estimate holds

$$\int\limits_{-\infty}^{\infty} \|G_0(\tau, \varphi)\| \, d\tau \leq K < \infty, \ \ \varphi \in \mathcal{T}_m. \tag{5}$$

If system of equations (1) has a Green–Samoilenko function, it's invariant toroidal manifold may be represented as

$$x = u(\varphi) = \int\limits_{-\infty}^{\infty} G_0(\tau, \varphi) f(\varphi_\tau(\varphi)) \, d\tau, \ \ \varphi \in \mathcal{T}_m.$$

Consider two classes of differential equations for which a Green–Samoilenko functions exist, so the invariant toroidal sets exist as well.

Let the matrix $A(\varphi_\tau(\varphi))$ from the system (1) commutes with its integral (Lappo–Danilevsky case, see [2, Part 2, § 13] for details).

$$A(\varphi_t(\varphi))\int\limits_{\tau}^{t} A(\varphi_s(\varphi))\,ds = \int\limits_{\tau}^{t} A(\varphi_s(\varphi))\,ds \cdot A(\varphi_t(\varphi)) \tag{6}$$

for $t \geq \tau$.

Then

$$\Omega_\tau^t(\varphi) = e^{\int\limits_{\tau}^{t} A(\varphi_s(\varphi))\,ds}$$

is a fundamental matrix of homogeneous system (3) and the following theorem holds.

Theorem 1. *Suppose that for any $t \geq \tau \geq 0$ a matrix $A(\varphi_t(\varphi))$ commutes with its integral. If all the eigenvalues of matrix*

$$A_0 = \lim_{t\to\infty} \frac{1}{t}\int\limits_{\tau}^{t} A(\varphi_s(\varphi))\,ds \tag{7}$$

are on the left half of the complex plane λ, then system (1) has an asymptotically stable invariant toroidal manifold $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, for any $f(\varphi) \in C(\mathcal{T}_m)$.

Consider the case where the matrix $A(\varphi)$ in system (1) is such that the largest eigenvalue of the matrix

$$\widehat{A}(\varphi) = \frac{1}{2}\big(A(\varphi) + A^T(\varphi)\big),$$

where $A^T(\varphi)$ is a matrix transposed to $A(\varphi)$, $\Lambda(\varphi)$ is negative in ω-limit points of any solution $\varphi_t(\varphi)$ of the first equation from (1).

Using the Vazhevsky inequality [2], we see that in this case the function

$$G_0(\tau,\varphi) = \begin{cases} \Omega_\tau^0(\varphi), & \tau \leq 0, \\ 0, & \tau > 0 \end{cases} \tag{8}$$

satisfies the estimate

$$\|G_0(\tau,\varphi)\| \leq Ke^{-\gamma|\tau|}, \ \ \tau \in R,$$

and it is a Green–Samoilenko function of the invariant tori problem. Thus, the system of equations (1) has an asymptotically stable invariant toroidal manifold

$$x = u(\varphi) = \int\limits_{-\infty}^{0} \Omega_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau, \ \ \varphi \in \mathcal{T}.$$

Finally, we will develop the conditions for the existence of invariant toroidal sets of impulsive system of differential equations that undergo impulsive perturbation at the moments when the phase point meets a given set in the phase space

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \ \ \varphi \notin \Gamma, \qquad \Delta x\big|_{\varphi\in\Gamma} = B(\varphi)x + g(\varphi). \tag{9}$$

Suppose the set Γ is a smooth $(m-1)$-dimensional submanifold of the torus $\mathcal{T}_m$ dimension and is determined by the equation $\Phi(\varphi) = 0$, where $\phi(\varphi)$ is a continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ function.

Let $t_i(\varphi)$, $i \in Z$, be solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$, which are the moments of impulsive perturbations in system (9), and assume that uniformly with respect to $t \in R$ there exists a limit

$$\lim_{T\to\infty} \frac{i(t,t+T)}{T} = \rho, \tag{10}$$

where $i(a,b)$ is a number of points $t_i(\varphi)$ in the interval (a,b).

Theorem 2. *Let a matrix $A(\varphi)$ satisfy the Lappo–Danilevsky condition for any $t \ge \tau \ge 0$ and uniformly with respect to $t \in R$ the finite limit* (10) *exist.*

Then, if

$$\gamma + \rho \ln \alpha < 0, \tag{11}$$

$$\gamma = \max_j \operatorname{Re} \lambda_j(A_0), \quad \alpha^2 = \max_{\varphi\in\mathcal{T}_m} \max_j \lambda_j (E + B(\varphi))^T (E + B(\varphi)),$$

then system of equations (9) *has an asymptotically stable invariant toroidal set*

$$x = u(\varphi) = \int_{-\infty}^{0} G_0(\tau,\varphi) f(\varphi_\tau(\varphi))\,d\tau + \\ + \sum_{t_i(\varphi)<0} G_0(t_i(\varphi)+0,\varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \ \ \varphi \in \mathcal{T}_m.$$

Note that the conditions of Theorem 2 can be weakened by requiring the inequality (11) to be fulfilled not on the whole surface of a torus, but only in ω-limit sets of all solutions of the first equation from (9).

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Existence and Uniqueness of a Periodic Solution to the Second Order Differential Equation with an Attractive Singularity

Robert Hakl

Institute of Mathematics, AS CR, Brno, Czech Republic
E-mail: `hakl@ipm.cz`

Manuel Zamora

Departamento de Matemática, Universidad del Bío-Bío, Chile
E-mail: `mzamora@ubiobio.cl`

1 Statement of the Problem and the Main Result

Our results deal with the following family of equations

$$u'' + \frac{g(t)}{u^\lambda} = h(t)u^\delta, \tag{1}$$

where $\lambda > 0$, $\delta \in [0,1)$, $g, h \in L_\omega$, and g is a non-negative function not necessarily essentially bounded from below by some positive constant. We establish a relation between the order of the singularity λ, the order of the nonlinearity δ, and the regularity p of the function $[h]_+$ guaranteeing the existence of an ω-periodic solution to (1). Furthermore, one can easily check that the obtained results establish a link between the results obtained both in [2] and [3].

In the paper [1], the authors studied the equation (1) with $\delta = 0$ and proved that if $\overline{h} > 0$, $g(t) \geq 0$ for a.e. $t \in \mathbb{R}$, and

$$\left(\frac{\omega}{4}\int_0^\omega [h]_+(s)\,ds\right)^\lambda \int_0^\omega h(s)\,ds \leq \int_0^\omega g(s)\,ds \tag{2}$$

holds, then (1) (with $\delta = 0$) has at least one ω-periodic solution. In spite of the fact that the condition (2) is unimprovable, in general, as shown in [1, Counter-example 4.1], for some particular cases it can be weakened (see Corollary 4 below). However, the main importance of the present paper lies in the answer to the cases when the function g is positive almost everywhere in $\mathbb{R}$ but it is not uniformly essentially bounded from below by some positive constant. In particular, to that cases when g can be estimated by some polynomial with isolated zeroes (see Corollary 5).

For convenience, we are going to introduce a list of notation which is used throughout:

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are the sets of all natural, integer, and real numbers, respectively;

C_ω is the Banach space of ω-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}$, endowed with the norm $\|u\|_C = \max\{|u(t)| : t \in [0,\omega]\}$;

L^p_ω, where $1 \leq p < +\infty$, is the Banach space of ω-periodic functions $h : \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable on $[0,\omega]$ in the p-th power, endowed with the norm $\|h\|_p = \big(\int_0^\omega |h(t)|^p\,dt\big)^{1/p}$;

$L_\omega = L^1_\omega$;

L^∞_ω is the Banach space of ω-periodic essentially bounded functions $h : \mathbb{R} \to \mathbb{R}$, endowed with the norm $\|h\|_\infty = \text{ess sup}\big\{|h(t)| : t \in [0,\omega]\big\}$;

if $h \in L_\omega$ then

$$\overline{h} = \frac{1}{\omega}\int_0^\omega h(s)\,ds, \quad [h]_+(t) = \frac{|h(t)| + h(t)}{2}, \quad [h]_-(t) = \frac{|h(t)| - h(t)}{2} \quad \text{for a.e. } t \in \mathbb{R}.$$

By an ω-periodic solution to the equation (1) we understand a positive function $u \in C_\omega$ which is absolutely continuous together with its first derivative on every compact interval of $\mathbb{R}$ and satisfies (1) almost everywhere on $\mathbb{R}$. To formulate our main result we need the following notation

Notation 1. Let $g, h \in L_\omega$, $g(t) \geq 0$ for a.e. $t \in \mathbb{R}$, $\sigma \geq 0$. Then, for every $t \in \mathbb{R}$, we define

$$G(t,\sigma) = \lim_{x \to t_+} \int\limits_x^{t+\frac{\omega}{2}} \frac{g(s)}{(s-t)^\sigma}\,ds + \lim_{x \to t_-} \int\limits_{t+\frac{\omega}{2}}^{x+\omega} \frac{g(s)}{(t+\omega-s)^\sigma}\,ds,$$

$$H_-(t,\sigma) = \lim_{x \to t_+} \int\limits_x^{t+\frac{\omega}{2}} \frac{[h]_-(s)}{(s-t)^\sigma}\,ds + \lim_{x \to t_-} \int\limits_{t+\frac{\omega}{2}}^{x+\omega} \frac{[h]_-(s)}{(t+\omega-s)^\sigma}\,ds.$$

Note that, for every fixed $t \in \mathbb{R}$, the limits in Notation 1 exist and each of them is either finite or equal to $+\infty$.

Theorem 1. *Let $[h]_+ \in L^p_\omega$, $p \in [1,+\infty)$,*

$$g(t) \geq 0 \quad \text{for a.e. } t \in \mathbb{R}, \ \ \overline{g} > 0, \tag{3}$$

and let there exist a function $\varphi \in L^q_\omega$, $q \in [1,+\infty)$, such that $[h]_+(t) \leq \varphi(t) g^{\frac{q-1}{q}}(t)$ for a.e. $t \in \mathbb{R}$[1]*. Let, moreover,*

$$G\Big(t, \frac{(2p-1)(\lambda+\delta)q}{(1-\delta)p}\Big) + H_+ H_-\Big(t, \frac{(2p-1)(\lambda+\delta)(q-1)}{(1-\delta)p}\Big) > H_+^q \|\varphi\|_q^q \quad \text{for } t \in \mathbb{R}, \tag{4}$$

where

$$H_+ = \Big[z(p)\Big(\frac{(1-\delta)p}{2p-1}\Big)^{\frac{p-1}{p}} (1-\delta)\|[h]_+\|_p\Big]^{\frac{\lambda+\delta}{1-\delta}}, \quad z(p) = \begin{cases} \Big(\dfrac{p-1}{(1+\delta)p-1}\Big)^{\frac{p-1}{p}} & \text{if } p > 1, \\ 1 & \text{if } p = 1. \end{cases} \tag{5}$$

Then there exists a unique ω-periodic solution to (1) *if and only if $\overline{h} > 0$.*

It can be easily verified that there is no ω-periodic solution to (1) provided (3) is fulfilled and $\overline{h} \leq 0$.

Further, we would like to emphasize that the equation (1) has at most one ω-periodic solution provided $g, h \in L_\omega$ are such that (3) is satisfied, $\delta \in [0,1)$, and $\lambda > 0$. As far as we know such a result is new for this type of equations.

In conclusion, our results can be represented as a genuine relation between the order of the singularity, the order of the nonlinear term, and the regularity of the input functions involved, existing in the class of differential equations with attractive singularity.

2 Corollaries and Examples

Below we discuss some particular cases illustrating the result obtained. The first assertion shows that Theorem 1, being applied when the function g is uniformly bounded from below by a positive constant, yields as a corollary the results proven in [2, 3].

Corollary 1. *Let $[h]_+ \in L^p_\omega$, $p \in [1,+\infty)$, $g(t) \geq g_0 > 0$ for a.e. $t \in \mathbb{R}$, and let $\frac{\lambda+\delta}{1-\delta} \geq \frac{1}{2p-1}$. Then there exists a unique ω-periodic solution to* (1) *if and only if $\overline{h} > 0$.*

[1]If $q = 1$ then we put $g^{\frac{q-1}{q}}(t) = 1$ for $t \in \mathbb{R}$.

In the case when $[h]_+ \in L^\infty_\omega$ then $[h]_+ \in L^p_\omega$ for every $p \in [1,+\infty)$. Therefore, from Corollary 1 we obtain the following assertion:

Corollary 2. *Let $[h]_+ \in L^\infty_\omega$ and let $g(t) \geq g_0 > 0$ for a.e. $t \in \mathbb{R}$. Then there exists a unique ω-periodic solution to* (1) *if and only if $\overline{h} > 0$.*

Example 1. In [3], there was proven that the equation

$$u'' + \frac{1}{u^\lambda} = h(t) \tag{6}$$

with $[h]_+ \in L^p_\omega$ and $\overline{h} > 0$, has a unique ω-periodic solution if $\lambda \geq 1/(2p-1)$. Moreover, there is also established an example showing that for every $\lambda \in (0, 1/(2p-1))$, there exists $h \in L^p_\omega$ with $\overline{h} > 0$ such that (6) has no ω-periodic solution.

Corollary 1 says that if a sub-linear term is added to (6), the condition $\lambda \geq 1/(2p-1)$ can be weakened. In particular,

$$u'' + \frac{1}{u^\lambda} = h(t)u^\delta \tag{7}$$

has a unique ω-periodic solution for any $\lambda > 0$ if $\delta \in [1/(2p), 1)$, provided $[h]_+ \in L^p_\omega$, $\overline{h} > 0$. On the other hand, a relation of this type was proven in [2] by using the method of lower and upper functions in the case when $p = 1$. Note here that Corollary 1 joins both results of [2] and [3] and establishes a relation between the orders of the singularity and nonlinearity, and the regularity of the input function, guaranteeing the existence of a unique ω-periodic solution to (7).

As it was mentioned in Example 1, in the case when $\lambda < 1/(2p-1)$, an additional condition on h is required in order to guarantee the existence of an ω-periodic solution to (6). One of such a condition can be obtained from Theorem 1 immediately by putting $\varphi \equiv [h]_+$ and $q = p$. Then we have the following assertion:

Corollary 3. *Let $[h]_+ \in L^p_\omega$, $p \in [1,+\infty)$, and let $\frac{\lambda+\delta}{1-\delta} < \frac{1}{2p-1}$. Let, moreover,*

$$\frac{2(1-\delta)}{1+\lambda-2p(\lambda+\delta)}\Big(\frac{\omega}{2}\Big)^{\frac{1+\lambda-2p(\lambda+\delta)}{1-\delta}} + H_+\Big(\frac{2}{\omega}\Big)^{\frac{(2p-1)(\lambda+\delta)(p-1)}{(1-\delta)p}} \int\limits_0^\omega [h]_-(s)\,ds > H_+^p \|[h]_+\|_p^p,$$

where H_+ is given by (5)*. Then there exists a unique ω-periodic solution to* (7) *if and only if $\overline{h} > 0$.*

If $p = 1$ and $\delta = 0$, then Corollary 3 results in the following consequence.

Corollary 4. *Let $[h]_+ \in L_\omega$ and let $\lambda < 1$. Let, moreover,*

$$\bigg(\frac{\omega}{4}\int\limits_0^\omega [h]_+(s)\,ds\bigg)^\lambda \int\limits_0^\omega h(s)\,ds < \frac{\omega}{2^\lambda(1-\lambda)}\,. \tag{8}$$

Then there exists a unique ω-periodic solution to (6) *if and only if $\overline{h} > 0$.*

It is worth mentioning here that the condition (8) improves the condition (2), which was obtained in [1], for the equation (6).

However, as it was mentioned in the introduction, the main importance of Theorem 1 can be distinguished for the cases when g has zeroes at isolated points. Thus, the essence of the condition (4) is different to the one established in [1, Corollary 4.2]. That gives also an answer to the open problem [1, Open problem 4.2].

Corollary 5. *Let $h \in L^p_\omega$, $p \in [1, +\infty)$, and let there exist $c > 0$, $\alpha_i, \beta_i \geq 0$, $t_i \in \mathbb{R}$ $(i = 1, \dots, n)$ such that $t_1 < t_2 < \cdots < t_n < t_1 + \omega$ and*

$$g(t) \geq c(t_{i+1} - t)^{\alpha_{i+1}}(t - t_i)^{\beta_i} \quad \text{for a.e. } t \in (t_i, t_{i+1}), \;\; i = 1, \dots, n-1,$$
$$g(t) \geq c(t_1 + \omega - t)^{\alpha_1}(t - t_n)^{\beta_n} \quad \text{for a.e. } t \in (t_n, t_1 + \omega).$$

Let, moreover,

$$\frac{\lambda + \delta}{1 - \delta} > \frac{(1 + \gamma_0)(1 + \gamma p)}{(1 + \gamma)(2p - 1)} \;\; \text{if } p > 1, \quad \frac{\lambda + \delta}{1 - \delta} \geq 1 + \gamma_0 \;\; \text{if } p = 1,$$

where $\gamma_0 = \max\big\{\min\{\alpha_i, \beta_i\} : \; i = 1, \dots, n\big\}$, $\gamma = \max\big\{\alpha_i, \beta_i : \; i = 1, \dots, n\big\}$. Then there exists a unique ω-periodic solution to (1) if and only if $\overline{h} > 0$.

Now, we are going to present an example of a differential equation to which our result can be efficiently applied. To illustrate the result, we have selected a particular case of a physical model studied in [4, Section 5]. The dynamic of a trapless 3D Bose–Einstein condensate with variable scattering length can be ruled by the equation (1), where g models the S-wave scattering length, which is assumed to vary ω-periodically in time. In our case, a non-negative g corresponds to attractive interactions between the elementary particles and h is an external force (usually $\delta = 0$). Then the existence of an ω-periodic solution to (1) is interpreted as a bound state of the condensate with external trap. To simplify the model, the function g can be considered as a polynomial function, which may have several zeroes. However, we also can investigate the problem in the case when g is trigonometric. Obviously, the latter case seems to be more useful in applications but also more complicated to study analytically. Nevertheless, according to Corollary 5, it is sufficient to check the condition (4) in a neighborhood of each zero of g. Thus, having approximated g by a Taylor polynomial, the problem is reduced to the much simpler one - the polynomial case. This makes our result efficiently applicable in several cases.

Example 2. Consider the equation

$$u'' + \frac{g(t)}{u^\lambda} = h(t), \tag{9}$$

where

$$g(t) = ct^\alpha(\omega - t)^\alpha \quad \text{for } t \in [0, \omega),$$
$$g(t) = g(t - k\omega) \quad \text{for } t \in [k\omega, (k+1)\omega), \;\; k \in \mathbb{Z} \setminus \{0\},$$

c, α, and λ are positive numbers, $\overline{h} > 0$. According to Corollary 5 one can conclude that (9) has a unique ω-periodic solution if one of the following conditions is fulfilled:

- $[h]_+ \in L^\infty_\omega$ and $\lambda > \alpha/2$;
- $[h]_+ \in L^p_\omega$, $p \in (1, +\infty)$, and $\lambda > \frac{1+\alpha p}{2p-1}$;
- $[h]_+ \in L_\omega$ and $\lambda \geq 1 + \alpha$.

Note that the result for the case when $[h]_+ \in L^\infty_\omega$ is obtained by applying Corollary 5 for $p \in (1, +\infty)$ sufficiently large.

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The Infinite Analogues of Perron's Effect of Value Change in Characteristic Exponents

N. A. Izobov

Department of Differential Equations, Institute of Mathematics,
National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: `izobov@im.bas-net.by`

A. V. Il'in

Moscow State University, Moscow, Russia
E-mail: `iline@cs.msu.su`

We consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in R^n, \quad t \geq t_0, \tag{1}$$

with bounded, for example, infinitely differentiable on the semi-axis $[t_0, +\infty)$ coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A) < 0$. The above system is a linear approximation for the nonlinear system

$$\dot{y} = A(t)y + f(t,y), \quad y \in R^n, \quad t \geq t_0, \tag{2}$$

with an infinitely differentiable in time t and variable $y_1, \dots, y_n$ perturbation $f : [t_0, +\infty) \times R^n \Longrightarrow R^n$ of order $m > 1$ smallness in the neighborhood of the origin $y = 0$ and admissible growth outside of the origin, satisfying the condition

$$\|f(t,y)\| \leq C_f \|y\|^m, \quad C_f = const > 0, \quad y \in R^n, \quad t \geq t_0. \tag{3}$$

The goal of the known (partial) Perron's effects of sign and value change [1], [2, pp. 50, 51] of characteristic exponents is to construct both a two-dimensional linear system (1) with concrete negative characteristic exponents and the perturbations $f(t,y)$ of second order $m = 2$ such that a part of nontrivial solutions of system (2) have already the positive as well as concrete exponents. Thus, this effect is partial one. In our previous works [3], [4] which were preceded by the works [5]–[9] written jointly with S. K. Korovin, we have obtained a general and a complete finite-dimensional Perron's effect which has been realized for arbitrarily given:

1) negative characteristic exponents $\lambda_1 \leq \cdots \leq \lambda_n < 0$ of the system of linear approximation (1);

2) positive characteristic exponents $0 < \beta_1 \leq \cdots \leq \beta_n$ of all nontrivial solutions of system (2);

3) order $m > 1$. There naturally arises the question on the existence of different infinite analogues of that effect.

The first such an infinite version is contained in the following

Theorem 1 ([10], [11]). *For any parameters $m > 1$ and $\lambda_1 \leq \lambda_2 < 0$ and for arbitrary nonempty finite or bounded countable sets $\beta_i \subset [\lambda_i, +\infty)$, $i = 1, 2$, satisfying the condition of separation $\sup \beta_1 \leq in\beta_2$, there exist:*

1) *the two-dimensional system of linear approximation (1) with bounded infinitely differentiable on the semi-axis $[1, +\infty)$ coefficients and characteristic exponents $\lambda_1(A) = \lambda_1 \leq \lambda_2(A) = \lambda_2$;*

2) *the infinitely differentiable in its arguments t, y_1, y_2 and satisfying the condition* (3) *perturbation $f:[1,+\infty)\times R^2\to R^2$ of order $m>1$, such that all nontrivial solutions $y(t,c)$, $y(1,c)=c$ of the nonlinear two-dimensional perturbed system* (2) *are infinitely extendable to the right, and their characteristic exponents $\lambda[y(\,\cdot\,,c)]$ form the sets*

$$\big\{\lambda[y(\,\cdot\,,c)]:\ c_2=0,\ c_1\neq 0\big\}=\beta_1,\quad \big\{\lambda[y(\,\cdot\,,c)]:\ c_2\neq 0\big\}=\beta_2,\ \ c=(c_1,c_2)\in R^2$$

Nevertheless, for systems (1) and (2) constructed in the proof of that theorem, the limiting set

$$\Lambda_0(A,f)\equiv \operatorname*{Lim}_{\rho\to+\infty}\big\{\lambda[y(\,\cdot\,,c)]:\ 0<\|c\|<\rho\big\}$$

consists of no more than two different positive numbers.

The second, more important version of Perron's infinite effect is connected with its realization for nontrivial solutions of perturbed systems (2) starting in the arbitrarily small neighborhood of the origin $y=0$, that is, connected with the construction of an infinite set $\Lambda_0(A,f)$ on the positive semi-axis $(0,+\infty)$. Indeed, investigations (see, for e.g., [12, pp. 232–242], [13], [14, pp. 277–326]) dealt with the linear approximation (1) of exponential and conditional stability and also instability of a zero solution $y\equiv 0$ of system (2) are reduced to the construction of accessible boundaries of characteristic exponents of infinitely extendable solutions of that system starting at the moment $t=t_0$ in the above-mentioned neighborhood of the origin. Such estimates allow one, in particular, to define signs of characteristic exponents of these solutions. Therefore, there arises the need to realize Perron's infinite effect in any arbitrarily small neighborhood of the origin.

It should be noted here that the effect of values replacement of negative characteristic exponents of system (1) by an infinite set $\Lambda_0(A,f)=\beta$ of characteristic exponents of nontrivial solutions from an arbitrarily small neighborhood of the origin of the exponentially stable (with an e.s. zero solution) nonlinear system (2) and perturbation (3) was realized in [15]–[17]. But it turned out that the set β belonged to the negative semi-axis $(-\infty,0)$ (instead of the necessary positive $(0,+\infty)$), whereas the set of characteristic exponents of solutions of system (2) starting outside of the origin remained unexplored. As for the Perron's effect of values change and its various analogues, it suggests, firstly, an infinite extension of all solutions of the nonlinear system (2) with perturbation (3) which for $m>1$ is not, as a rule, realized and, secondly, the positiveness of all characteristic exponents of these solutions.

Thus, the infinite (countable) version of Perron's effect of values change in any neighborhood of the origin realizes the following

Theorem 2. *For any parameters $m>1$, $\lambda_1\leq\lambda_2<0$ and an arbitrary countable closed from the above set $\beta\subset[\lambda_1,+\infty)$ with the properties $\lambda_2\leq b\equiv\sup\beta\in\beta$ there exist:*

1) *the two-dimensional linear system* (1) *with bounded infinitely differentiable on the semi-axis $[1,+\infty)$ coefficients and characteristic exponents $\lambda_1(A)=\lambda_1\leq\lambda_2(A)=\lambda_2$;*

2) *satisfying the condition* (3)*, the infinitely differentiable in time t and variables y_1, y_2 perturbation $f:[1,+\infty)\times R^2\to R^2$, such that all nontrivial solutions*

$$y(t,c),y(1,c)=c\in R^2\setminus\{0\},$$

of the nonlinear system (2) *with perturbation* (3) *are infinitely extendable to the right, and their characteristic exponents form the set $\Lambda(A,f)=\Lambda_0(A,f)=\beta$ and take the values*

$$\lambda[y(\cdot,c)]=b,\ \ \forall\,c\notin I\equiv\big\{x\in R^2:\ |x|\leq 1,\ x_2=0\big\}.$$

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Some Properties of Solutions and Approximate Algorithms for One System of Nonlinear Partial Differential Equations

Temur Jangveladze

I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Georgian Technical University, Tbilisi, Georgia
E-mail: `tjangv@yahoo.com`

In mathematical modeling of many natural processes nonlinear nonstationary differential models are received very often. One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. In the quasistationary case the corresponding system of Maxwell's equations has the form [1]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \quad \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the electro-conductivity of the substance. The first vector equation of system (1) describes the process of diffusion of the magnetic field and the second equation describes the change of the temperature at the expense of Joule's heating.

For a more thorough description of electromagnetic field propagation in the medium, it is desirable to take into consideration different physical effects, first of all heat conductivity of the medium has to be taken into consideration. In this case the same process is described by the following system:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \quad \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2 + \operatorname{div}(\kappa \operatorname{grad} \theta), \tag{2}$$

where κ is a coefficient of heat conductivity. As a rule this coefficient is a function of argument θ as well.

Many other processes are described by system of the type (1) and (2) and many works are dedicated on investigation and numerical resolution of the initial-boundary value problems for these type models (see, for example, [2]–[14] and references therein).

In the domain $(0,1) \times (0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{gathered} \frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2, \\ U(0,t) = 0, \quad U(1,t) = \psi > 0, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x) \geq v_0 > 0, \end{gathered} \tag{3}$$

where U_0 and V_0 are known functions defined on $[0,1]$ and ψ and v_0 are constants.

It is not difficult to verify that if $\alpha \neq 1$ and $V_0(x) = v_0$, then the following functions

$$U(x,t) = \psi x, \quad V(x,t) = \big[v_0^{1-\alpha} + (1-\alpha)\psi^2 t\big]^{\frac{1}{1-\alpha}} \tag{4}$$

are solutions of the problem (3). But if $\alpha > 1$ in the finite time $t_0 = \delta_0^{1-\alpha}/\psi^2(\alpha-1)$ the function becomes infinity. This example shows that solution of problem (3) with smooth initial and boundary conditions can be blown up in the finite time.

In the domain $\Omega \times (0,T)$, where $\Omega = (0,1)$, let us consider the following system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^{\alpha}\,\frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = V^{\alpha}\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}\,. \tag{5}$$

Many facts that obtained for (3) problem are valid for (5) too. In particular, functions $U(x,t)$ and $V(x,t)$ defined by (4) satisfy system (5). From this one can deduce that for system (5), analogical to (3) problem, adding the following boundary conditions, if $\alpha > 1$, the theorem of global solvability does not take place:

$$\frac{\partial V(x,t)}{\partial x}\bigg|_{x=0} = \frac{\partial V(x,t)}{\partial x}\bigg|_{x=1} = 0.$$

It is well-known that the general method for construction of economic algorithms for multi-dimensional problems of mathematical physics is a decomposition method (see, for example, [15] and references therein). Complex nonlinearity dictates also to split along the physical process and investigate basic model by them. In particular, it is logical to split system (2) in two models. In first Joule's rule, while in second process of thermal conductivity are considered. Investigation of splitting along the physical processes in one-dimensional case is the natural beginning of studding this issue. In this direction the first step was made in the work [3].

Let us consider initial-boundary value problem for system (5), where $-1/2 \le \alpha \le 1/2$, $\alpha \ne 0$, with usual initial and following boundary conditions:

$$U(x,t) = \frac{\partial V(x,t)}{\partial x} = 0, \quad (x,t) \in \partial\Omega \times [0,T].$$

If we denote $V^{\frac{1}{2}} = W$, $2\alpha = \gamma$, then problem (5) can be rewritten in the following equivalent form [3]:

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(W^{\gamma}\,\frac{\partial U}{\partial x}\Big),\\
\frac{\partial W}{\partial t} = \frac{1}{2}\,W^{\gamma-1}\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\,\Big(\frac{\partial W}{\partial x}\Big)^2,\\
U(x,t) = \frac{\partial W(x,t)}{\partial x} = 0, \ \ (x,t) \in \partial\Omega \times [0,T],\\
U(x,0) = U_0(x), \quad W(x,0) = W_0(x) = V_0^{1/2}(x).
\end{gathered} \tag{6}$$

Let us introduce the notations:

$$\begin{gathered}
\omega_\tau = \big\{t_j = j\tau,\ j = 0,1,\dots,N,\ \tau = T/N\big\},\\
y_t = \frac{y^{j+1} - y^j}{\tau}\,, \quad y_{1t} = \frac{y_1^{j+1} - y^j}{\tau}\,, \quad y_{2t} = \frac{y_2^{j+1} - y^j}{\tau}\,,\\
y = \eta_1 y_1 + \eta_2 y_2, \quad \eta_1 + \eta_2 = 1, \quad \eta_1 > 0, \quad \eta_2 > 0.
\end{gathered}$$

Correspond to the problem (6) following additive averaged semi-discrete schemes:

$$\begin{gathered}
u_{1t} = \frac{d}{dx}\Big(w_1^{\gamma}\,\frac{du_1}{dx}\Big), \quad \eta_1 w_{1t} = \frac{1}{2}\,w_1^{\gamma-1}\Big(\frac{du_1}{dx}\Big)^2,\\
u_{2t} = \frac{d}{dx}\Big(w_2^{\gamma}\frac{du_2}{dx}\Big), \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\,\Big(\frac{dw_2}{dx}\Big)^2
\end{gathered} \tag{7}$$

and

$$\begin{gathered}
u_t = \frac{d}{dx}\Big[(\eta_1 w_1^{\gamma} + \eta_2 w_2^{\gamma})\,\frac{du}{dx}\Big],\\
\eta_1 w_{1t} = \frac{1}{2}\,w_1^{\gamma-1}\Big(\frac{du}{dx}\Big)^2, \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\,\Big(\frac{dw_2}{dx}\Big)^2,
\end{gathered} \tag{8}$$

with suitable initial and boundary conditions.

The following statement takes place.

Theorem 1. *If problem* (6) *has a sufficiently smooth solution and* $-1 \le \gamma \le 1$, *then the solutions of the schemes* (7) *and* (8) *converge in the norm of the space* $L_2(0,1)$ *to the solution of problem* (6) *as* $\tau \to 0$ *and the following estimate is true*

$$\|u^j - U(t_j)\| + \|w^j - W(t_j)\| = O(\tau^{\frac{1}{2}}).$$

Let us consider first type initial-boundary value problem for the following model system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V\,\frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \Big(\frac{\partial U}{\partial x}\Big)^2.$$

The semi-discrete and finite difference second order accuracy schemes with respect of space step is constructed and studied in [6] for this case of nonlinearity. In [4] more general finite difference schemes including second order accuracy two-level scheme and three-level type scheme are also studied.

Let us introduce the grids:

$$\omega_{h\tau} = \overline{\omega}_h \times \omega_\tau, \quad \omega^*_{h\tau} = \omega^*_h \times \omega_\tau,$$

where

$$\overline{\omega}_h = \big\{x_i = ih,\ i = 0,1,\dots,M,\ h = 1/M\big\}, \quad \omega_h = \overline{\omega}_h \setminus \{x_0, x_M\},$$
$$\omega^*_h = \big\{x^*_i = (i-1/2)h,\ i = 1,2,\dots,M\big\}.$$

Let us introduce also scalar-products, norms and well-known notations:

$$(y,z) = \sum_{i=1}^{M-1} y_i z_i h, \quad (y,z] = \sum_{i=1}^{M} y_i z_i h, \quad \|y\| = (y,y)^{1/2}, \quad \|y]| = (y,y]^{1/2},$$
$$y_x = \frac{y_{i+1}-y_i}{h}, \quad y_{\overline{x}} = \frac{y_i - y_{i-1}}{h}, \quad y_t = \frac{y^{j+1}-y^j}{\tau}, \quad y_{\overline{t}t} = \frac{y^{j+1} - 2y^j + y^{j-1}}{\tau^2},$$
$$y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma) y^j$$

and consider the following finite-difference scheme:

$$\begin{gathered} u_t + \mu\tau u_{\overline{t}t} = \big(v^{(\sigma)} u^{(\sigma)}_{\overline{x}}\big)_x, \quad v_t + \mu\tau v_{\overline{t}t} = \big(u^{(\sigma)}_{\overline{x}}\big)^2, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = U_0(x), \quad v(x,0) = V_0(x), \\ u(x,\tau) = U_0(x) + \tau (V U_{\overline{x}})_x\Big|_{t=0}, \quad v(x,\tau) = V_0(x) + \tau (U_{\overline{x}})^2\Big|_{t=0}. \end{gathered} \tag{9}$$

In the (9) discrete function u is defined on $\omega_{h\tau}$ and v is defined on $\omega^*_{h\tau}$.

The following statement takes place.

Theorem 2. *If* $\sigma - 0.5 \ge \mu \ge 0$ *and problem has sufficiently smooth solution, then finite difference scheme* (9) *converges as* $\tau \to 0$, $h \to 0$, *and the following estimate is true*

$$\|U^j - u^j\| + \|V^j - v^j]| = O\big(\tau^2 + h^2 + (\sigma - 0.5 - \mu)\tau\big).$$

It is clear that from Theorem 2 we get the following result: If $\sigma = 0.5$, $\mu = 0$ or $\sigma = 1$, $\mu = 0.5$ then convergence is the second order $O(\tau^2 + h^2)$.

Various numerical experiments using above mentioned discrete models are carried out. These experiments agree with theoretical investigations.

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Existence of Singular Solutions for Second Order Singular Differential Equations

Jaroslav Jaroš

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius Unviversity, Bratislava, Slovakia
E-mail: Jaroslav.Jaros@fmph.uniba.sk

Takaŝi Kusano

Professor Emeritus at Hiroshima University, Department of Mathematics, Faculty of Science, 739-8526 Higashi-Hiroshima, Japan
E-mail: kusanot@zj8.so-net.ne.jp

We consider second order singly singular differential equations of the types

$$\big(p(t)|x'|^{\alpha}\big)' + q(t)x^{-\beta} = 0, \tag{E}$$

and

$$\big(p(t)|x'|^{-\alpha}\big)' + q(t)x^{\beta} = 0, \tag{F}$$

under the assumption that

(a) α and β are positive constants;

(b) $p, q : [0,\infty) \to (0,\infty)$ are continuous functions.

By a solution on an interval J of (E) or (F) we mean a function $x : J \to (0,\infty)$ which is continuously differentiable on J together with $p(t)|x'(t)|^{\alpha}$ or $p(t)|x'(t)|^{-\alpha}$ and satisfies (E) or (F) there. If J is an unbounded interval of the form $[t_0,\infty)$, then $x(t)$ is said to be a *proper solution.* If J is a bounded interval of the form $[t_0,T)$ and $x(t)$ cannot be extended to the right beyond T, then $x(t)$ is called a *singular solution at* T. In this paper our attention is focused on singular solutions of (E) and (F) which are decreasing in their intervals of existence.

As is easily seen any singular solution $x(t)$ at T of (E) or (F) on $[t_0,T)$ has one of the following asymptotic behaviors

$$\lim_{t\to T-0} x(t) = A, \quad \lim_{t\to T-0} x'(t) = 0, \quad \text{for some } A > 0; \tag{I}$$

$$\lim_{t\to T-0} x(t) = A, \quad \lim_{t\to T-0} x'(t) = -\infty, \quad \text{for some } A > 0; \tag{II}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = 0; \tag{III}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = -B, \quad \text{for some } B > 0; \tag{IV}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = -\infty. \tag{V}$$

A singular solution satisfying (I) or (II) is termed a *white hole solution* or a *black hole solution*, respectively, while a singular solution satisfying (III), (IV) or (V) is termed an *extinct solution at* T *of the first kind, of the second kind or of the third kind*, respectively. Notice that the notion of black hole and white hole solutions was introduced by the present authors in [2] and [3].

It can be shown that equation (E) has white hole solutions but not black hole ones, whereas equation (F) may have black hole solutions but not white hole ones.

Theorem 1. *Equation* (E) *always has white hole solutions. More precisely, for any given $T > 0$ and $A > 0$ there exists a decreasing solution $x(t)$ of* (E) *satisfying* (I)*.*

Theorem 2. *Equation* (F) *has black hole solutions if and only if $\alpha > 1$, in which case, for any given $T > 0$ and $A > 0$ there exists a decreasing solution $x(t)$ of* (F) *satisfying* (II)*.*

The situations in which (E) and (F) possess extinct solutions of the second kind can be completely characterized as follows.

Theorem 3. *Equation* (E) *has extinct solutions of the second kind if and only if $\beta < 1$, in which case, for any given $T > 0$ and $B > 0$ there exists a decreasing solution $x(t)$ of* (E) *satisfying* (IV)*.*

Theorem 4. *Equation* (F) *always has extinct solutions of the second kind. More precisely, for any given $T > 0$ and $B > 0$ there exists a decreasing solution $x(t)$ of* (F) *satisfying* (IV)*.*

All of the above four theorems are verified by solving the appropriate integral equations with the help of the Schauder fixed point therems in Banach spaces. For example, the integral equations to be solved in Theorem 1 and Theorem 4 are

$$x(t) = A + \int_t^T \bigg(\frac{1}{p(s)} \int_s^T q(r)x(r)^\beta \, dr \bigg)^{-\frac{1}{\alpha}} ds,$$

and

$$x(t) = \int_t^T \bigg[\frac{1}{p(s)} \bigg(p(T)B^{-\alpha} + \int_s^T q(r)x(r)^{-\beta} \, dr \bigg) \bigg]^{\frac{1}{\alpha}} ds,$$

respectively.

It remains to ask whether (E) and (F) possess extinct solutions of the first and/or the third kinds. One easily sees that (E) (or (F)) cannot admit extinct solutions of the third kind (or the first kind). Information about the existence of such extinct solutions is provided by the following theorems in which the concept of *regularly varying functions at finite points*, defined below, plays a crucial role

Definition.

(i) A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be *regularly varying at infinity of index ρ* (in the sense of Karamata) if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0$$

(ii) Let $T > 0$ be a finite constant. A measurable function $f : [0, T) \to (0, \infty)$ is said to be *regularly varying of index ρ at T* if $f(T - \tau^{-1})$, as a function of τ, is regularly varying of index $-\rho$ at infinity in the sense of Karamata.

The definition and some basic properties of regularly varying functions in the sense of Karamata can be found in [1, 6]. See also [5]. The concept of regularly varying functions at finite points has recently been introduced by the present authors [4].

The totality of regularly varying functions of index ρ at T is denoted by $\mathrm{RV}_T(\rho)$.The symbol SV_T is often used for $\mathrm{RV}_T(0)$, in which case its members are called *slowly varying functions at T*. By definition a function $f(t) \in \mathrm{RV}_T(\rho)$ is expressed in the form

$$f(t) = (T - t)^\rho L(t), \quad t \in [t_0, T), \quad \text{for some } L \in \mathrm{SV}_T.$$

Note that any positive continuous function on $[a, \infty)$ is slowly varying at any point $T \in (a, \infty)$, that is, a member of SV_T for any $T > 0$.

It is an easy task to show that most of the basic properties of regularly varying functions at infinity can be carried over to regularly varying functions at finite points. For instance, the Karamata integration theorem is translated into the following propositon, which is also referred to as the Karamata integration theorem for regularly varying functions at finite points.

Proposition. *Let* $L \in \mathrm{SV}_T$.

(i) *If* $\rho < -1$, *then*

$$\int\limits_a^t (T-s)^\rho L(s)\,ds \sim -\frac{1}{\rho+1}(T-t)^{\rho+1}L(t), \quad t \to T-0.$$

(ii) *If* $\rho > -1$, *then*

$$\int\limits_t^T (T-s)^\rho L(s)\,ds \sim \frac{1}{\rho+1}(T-t)^{\rho+1}L(t), \quad t \to T-0.$$

(iii) *If* $\rho = -1$, *then*

$$l(t) = \int\limits_a^t \frac{L(s)}{T-s}\,ds \in \mathrm{SV}_T \ \text{ and } \lim_{t\to T-0}\frac{L(t)}{l(t)} = 0,$$

and if $L(t)/(T-t)$ *is integrable in a left neighborhood of* T, *then*

$$m(t) = \int\limits_t^T \frac{L(s)}{T-s}\,ds \in \mathrm{SV}_T \ \text{ and } \lim_{t\to T-0}\frac{L(t)}{m(t)} = 0.$$

Applying the Schauder–Tychonoff fixed point theorem in combination with the above proposition to solve the integral equations

$$x(t) = \int\limits_t^T \bigg(\frac{1}{p(s)}\int\limits_s^T q(r)x(r)^\beta\,dr\bigg)^{-\frac{1}{\alpha}} ds, \tag{IE}$$

$$x(t) = \int\limits_t^T \bigg(\frac{1}{p(s)}\int\limits_s^T q(r)x(r)^{-\beta}\,dr\bigg)^{\frac{1}{\alpha}} ds, \tag{IF}$$

we are able to find criteria for the existence of extinct solutions of the first and the third kinds for (E) and (F) belonging to $\mathrm{RV}_T(\rho)$ with positive ρ.

Theorem 5. *Assume that* $\beta < \min\{\alpha, 1\}$. *Then, for any given* $T > 0$, *equation* (E) *has an extinct solution* $x(t)$ *at* T *of the first kind which belongs to the class* $\mathrm{RV}_T(\rho)$ *with*

$$\rho = \frac{\alpha+1}{\alpha+\beta}$$

and enjoys the exact asymptotic behavior

$$x(t) \sim \Big[\frac{(T-t)^{\alpha+1}p(t)^{-1}q(t)}{\alpha(\rho-1)\rho^\alpha}\Big]^{\frac{1}{\alpha+\beta}}, \quad t \to T-0.$$

Theorem 6. *Assume that $\alpha > \max\{\beta, 1\}$. Then, for any given $T > 0$, equation* (F) *has an extinct solution $x(t)$ at T of the third kind which belongs to the class* $\mathrm{RV}_T(\rho)$ *with*

$$\rho = \frac{\alpha - 1}{\alpha + \beta}$$

and enjoys the exact asymptotic behavior

$$x(t) \sim \Big[\frac{(T-t)^{\alpha-1}p(t)q(t)^{-1}}{(\alpha(1-\rho))^{-1}\rho^{\alpha}}\Big]^{\frac{1}{\alpha+\beta}}, \quad t \to T-0.$$

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On the Periodic Problem for the Nonlinear Telegraph Equation with a Boundary Condition of Poincare

Otar Jokhadze, Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: `jokha@rmi.ge; khar@rmi.ge`

In a plane of independent variables x and t in the strip $\Omega := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ t \in \mathbb{R}\}$ consider the problem of finding a solution $U(x,t)$ of the telegraph equation with power nonlinearity of the form

$$L_\lambda U := U_{tt} - U_{xx} + 2aU_t + cU + \lambda |U|^\alpha U = F(x,t), \ \ (x,t) \in \Omega, \tag{1}$$

satisfying the Poincare homogeneous boundary condition

$$\gamma_1 U_x(0,t) + \gamma_2 U_t(0,t) + \gamma_3 U(0,t) = 0, \ \ t \in \mathbb{R}, \tag{2}$$

and the Dirichlet condition

$$U(l,t) = 0, \ \ t \in \mathbb{R}, \tag{3}$$

respectively, for $x = 0$ and $x = l$, and also the periodicity condition with respect to variable t

$$U(x, t+T) = U(x,t), \ \ x \in [0,l], \ \ t \in \mathbb{R}, \tag{4}$$

with constant real coefficients a, c, γ_i, $i = 1,2,3$, and parameter $\lambda \neq 0$, where $\gamma_1\gamma_2 \neq 0$. Here $T := const > 0$, $\alpha := const > 0$; F is a given, while U is an unknown real T-periodic with respect to time functions.

Remark 1. Let $\Omega_T := \Omega \cap \{0 < t < T\}$, $f := F\big|_{\overline{\Omega}_T}$. Easy to see that if $U \in C^2(\overline{\Omega})$ is a classical solution of the problem (1)–(4), then function $u := U\big|_{\overline{\Omega}_T}$ represents a classical solution of the following nonlocal problem

$$L_\lambda u = f(x,t), \ \ (x,t) \in \Omega_T, \tag{5}$$

$$\gamma_1 u_x(0,t) + \gamma_2 u_t(0,t) + \gamma_3 u(0,t) = 0, \ \ u(l,t) = 0, \ \ 0 \leq t \leq T, \tag{6}$$

$$(B_0 u)(x) = 0, \ \ (B_0 u_t)(x) = 0, \ \ x \in [0,l], \tag{7}$$

where $(B_\beta w)(x) := w(x,0) - \exp(-\beta T) w(x,T)$, $\beta \in \mathbb{R}$, $x \in [0,l]$, and, vice versa, if $f \in C(\overline{\Omega}_T)$ and $u \in C^2(\overline{\Omega}_T)$ is a classical solution of the problem (5)–(7), then function $U \in C^2(\overline{\Omega})$, being T-periodic with respect to time continuation of function u from the domain Ω_T into the strip Ω, will be a classical solution of the problem (1)–(4), if $f(x,0) = f(x,T)$, $x \in [0,l]$.

Definition 1. Let $f \in C(\overline{\Omega}_T)$ be a given function. Let $\Gamma_1 : x = 0$, $0 \leq t \leq T$, $\Gamma_2 : x = l$, $0 \leq t \leq T$. Function u is called a strong generalized solution of the problem (5)–(7) of the class C, if $u \in C(\overline{\Omega}_T)$ and there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2) := \big\{w \in C^2(\overline{\Omega}_T) : (\gamma_1 w_x + \gamma_2 w_t + \gamma_3 w)\big|_{\Gamma_1} = 0, \ w\big|_{\Gamma_2} = 0\big\}$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{\Omega}_T)$, while $B_0 u_n \to 0$ and $B_0 u_{nt} \to 0$ as $n \to \infty$, respectively, in the spaces $C^1([0,l])$ and $C([0,l])$.

Remark 2. It is obvious that classical solution of the problem (5)–(7) from the space $C^2(\overline{\Omega}_T)$ is a strong generalized solution of this problem of the class C.

To the periodic problem for nonlinear hyperbolic equations with boundary conditions of Dirichlet or Roben there is devoted comprehensive literature (see, e.g., [1–11] and the bibliography therein). In the present work it is investigated the periodic with respect to time problem (5)–(7), when the direction of derivative in the boundary condition does not coincide with the direction of the normal. The periodic problem is reduced to the one nonlocal with respect to time problem for solution of which it is proved a priori estimate. For the theorem of existence it is used representations of solutions of problems of Cauchy, Goursat and Darboux in different parts of the domain under consideration. The question of uniqueness is also considered.

When the following conditions are fulfilled

$$\lambda > 0, \;\; a > 0, \;\; c > 0; \;\; \gamma_1\gamma_2 < 0, \;\; \gamma_3\gamma_2 > 0, \tag{8}$$

then for the strong generalized solution u of the problem (5)–(7) of the class C it is proved the following a priori estimate

$$\|u\|_{C(\overline{\Omega}_T)} \le c\|f\|_{C(\overline{\Omega}_T)} \tag{9}$$

with positive constant $c = c(a, c, \gamma_i, l, T)$, not depending on functions u and f.

Remark 3. Note that the question of solvability of the problem (5)–(7) is reduced to the question of obtaining uniform with respect to parameter $\tau \in [0, 1]$ a priori estimate of the strong generalized solution of the following equation

$$v_{tt} - v_{xx} + \tau(c - a^2)v + \tau\lambda \exp(-\alpha at)|v|^\alpha v = \tau \exp(at)f(x, t), \;\; (x, t) \in \overline{\Omega}_T, \tag{10}$$

satisfying the boundary

$$\gamma_1 v_x(0, t) + \gamma_2 v_t(0, t) + (\gamma_3 - a\gamma_2)v(0, t) = 0, \;\; v(l, t) = 0, \;\; 0 \le t \le T, \tag{11}$$

and nonlocal conditions

$$(B_a v)(x) = 0, \;\; (B_a v_t)(x) = 0, \;\; x \in [0, l]. \tag{12}$$

For obtaining uniform with respect to τ priori estimate for the solution of the problem (10)–(12) it is sufficient instead of (8) to require fulfilment of the following more restrictive conditions

$$\lambda > 0, \;\; a > 0, \;\; c \ge a^2, \;\; \gamma_1\gamma_2 < 0, \;\; \gamma_3\gamma_2^{-1} \ge a. \tag{13}$$

The following theorem is valid.

Theorem. *Let conditions* (13) *be fulfilled and* $f \in C(\overline{\Omega}_l)$. *Then the problem* (5)–(7) *has at least one strong generalized solution* u *of the class* C *in the sense of Definition* 1 *which belongs to the space* $C^1(\overline{\Omega}_l)$, *and when* $f \in C^1(\overline{\Omega}_l)$, *this solution is classical.*

Note that the problem (5)–(7) can not have more than one strong generalized solution of the class C in the domain Ω_l, if there hold conditions (13) and

$$|a^2 - c| < \frac{1}{c_0}, \;\; 0 < \lambda < \lambda_0,$$

where $\lambda_0 := (1 - c_0|a^2 - c|)(c_0 M_0)^{-1}$, $M_0 := (1 + \alpha)(2c_1\|f\|_{C(\overline{\Omega}_l)})^\alpha$, and c_0, c_1 are definite positive constants, depending on a, γ_i, l.

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Stability of Linear Impulsive Itô Equations with Delay

Ramazan I. Kadiev

Dagestan Research Center of the Russian Academy of Sciences,
Department of Mathematics, Dagestan State University, Makhachkala, Russia
E-mail: kadiev_r@mail.ru

Arcady Ponosov

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences,
P. O. Box 5003, N-1432 Ås, Norway
E-mail: arkadi@nmbu.no

Stability analysis of Itô equations with aftereffect attracts attention of many researchers (see e.g. [2, 3, 8, 10, 11]). On the other hand, stability of impulsive functional differential equations is popular in the literature as well (see e.g. [1, 4]). In [7] we considered these two classes of equations together using the framework based on Azbelev's W-method (see e.g. [2, 3, 5, 6]).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By $\mathbb{E}$ we denote the expectation on this probability space. The scalar stochastic processes $\mathcal{B}_i$, $i = 1, 2, \dots, m$ form the standard m-dimensional Brownian motion on $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. [9]).

Below we use the following spaces:

- L_∞ consists of all scalar, essentially bounded functions on $[0, \infty)$ with the usual norm;
- $\overline{L}_q^n$ consists of all n-dimensional progressively measurable (with respect to the above stochastic basis) stochastic processes on $[0, \infty)$, whose trajectories are almost surely (a.s.) locally q-integrable if $1 \leq q < \infty$ and a.s. locally bounded if $q = \infty$;
- k^n contains all $\mathcal{F}_0$-measurable n-dimensional random variables, and we will put $k^1 = k$ in the sequel;
- D^n contains all n-dimensional stochastic processes on $[0, \infty)$, which could be represented as $x(t) = x(0) + \int\limits_0^t f_0(s)\,ds + \sum\limits_{i=1}^m \int\limits_0^t f_i(s)\,d\mathcal{B}_i(s)$, where $x(0) \in k^n$, $f_0 \in \overline{L}_1^n$, $f_i \in \overline{L}_2^n$ $(i = 1, 2, \dots, m)$.

Consider the following scalar Itô equation with impulses

$$
\begin{gathered}
dx(t) = \Big[-a(t)x(t) + \sum_{k=1}^{m_0} b_k(t)x(h_k(t)) \Big] dt + \sum_{i=1}^{m} \Big[\sum_{k=1}^{m_i} c_{ik}(t)x(g_{ik}(t)) \Big] d\mathcal{B}_i(t) \ \ (t \geq 0), \\
x(s) = \varphi(s) \ \ (s < 0),
\end{gathered} \tag{1}
$$

$$
x(\mu_j) = A_j x(\mu_j - 0), \ \ j = 1, 2, 3, \dots, \ \ \text{a.s.}, \tag{2}
$$

where $x(t, \omega) \in R^1$; μ_j, A_j, $j = 1, 2, 3, \dots$, are real numbers such that $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, $\lim\limits_{j\to\infty} \mu_j = \infty$; $a, b_k \in \overline{L}_1^1$ for $k = 1, 2, \dots, m_0$, $c_{ij} \in \overline{L}_2^1$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, m_i$; h_k, g_{ij} are Borel measurable functions on $[0, \infty)$ such that $h_k(t) \leq t$, $g_{ij}(t) \leq t$ $(t \in [0, \infty))$ a.s. for $k = 1, 2, \dots, m_0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, m_i$; φ is a scalar $\mathcal{F}_0$-measurable stochastic process with a.s. essentially bounded trajectories.

We remark that under these assumptions Eq. (1)–(2) is a special case of the general stochastic functional differential equation considered in [5, 6]. In particular, any $x(0) \in k$ gives rise to a unique (up to the $\mathbb{P}$-equivalence) solution of this equation. We denote this solution by $x_\varphi(t, x_0)$.

Definition 1. The trivial solution of Eq. (1)–(2) is called exponentially p-stable with respect to the initial conditions if $\mathbb{E}|x_\varphi(t,x_0)|^p \le \overline{c}(\mathbb{E}|x_0|^p + \operatorname*{vrai\,sup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^p)\exp\{-\beta t\}$ $(t \ge 0)$ for some positive constants $\overline{c}$, β and all x_0 and φ described above.

Theorem 1. *Suppose that there exist positive numbers A, ρ, σ, δ, α such that $|A_j| \le A$, $\rho \le \mu_{j+1} - \mu_j \le \sigma$ for $j = 1,2,\dots$, $a(t) \ge \alpha$ $(t \in [0,\infty))$ a.s., $|b_k(t)| \le \overline{b}_k(t)$ $(t \in [0,\infty))$ a.s., where $\overline{b}_k \in L_\infty$ a.s. $k = 1,2,\dots,m_0$, $|c_{ij}(t)| \le \overline{c}_{ij}(t)$ $(t \in [0,\infty))$ a.s., where $\overline{c}_{ij} \in L_\infty$ for $i = 1,2,\dots,m$, $j = 1,2,\dots,m_i$, $t - h_k(t) < \delta$, $t - g_{ij}(t) < \delta$ $(t \in [0,\infty))$ almost everywhere for $k = 1,2,\dots,m_0$, $i = 1,2,\dots,m$, $j = 1,2,\dots,m_i$ and*

$$\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)} \operatorname*{vrai\,sup}_{t\ge0}\Big\{\sum_{k=1}^{m_0}\overline{b}_k(t)\Big\} + \\ + \Big[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\Big]^{1/2} c_p \operatorname*{vrai\,sup}_{t\ge0}\Big\{\sum_{i=1}^{m}\sum_{k=1}^{m_i}\overline{c}_{ik}(t)\Big\} < 1, \quad (3)$$

where c_p is the positive constant from the Burkholder–Davis–Gandy inequalities (see e.g. [9]). Then the trivial solution of Eq. (1)–(2) is exponentially $2p$-stable with respect to the initial conditions.

Sketch of the Proof

The proof of the theorem is based on Azbelev's W-transform of Eq. (1)-(2) (see e. g. [2], [3], [5]), which uses the so-called 'reference equation'. Normally, it is an equation which already possesses the desired asymptotic properties, but which is simpler than the given equation. The W-method works if an integral operator, which results from the substitution of the solutions of the reference equations into the given equation, is invertible.

Following this idea, we observe first of all that in our case it is sufficient to show that under the assumptions of the theorem the following inequality holds

$$\big(\mathbb{E}|x_\varphi(t,x_0)|^{2p}\big)^{1/2p} \le N\exp\{-\beta t\}\Big[\|x_0\|_{k_{2p}} + \Big(\operatorname*{vrai\,sup}_{s<0}\mathbb{E}|\varphi(s)|^{2p}\Big)^{1/2p}\Big], \quad (4)$$

where N and β are positive numbers.

Eq. (1) can be rewritten as follows:

$$dx(t) = \Big[-a(t)x(t) + \sum_{k=1}^{m_0} b_k(t)(S_{h_k}x)(t) + f_0(t)\Big]\,dt + \sum_{i=1}^{m}\Big[\sum_{k=1}^{m_i} c_{ik}(t)(S_{g_{ik}}x)(t) + f_i(t)\Big]\,d\mathcal{B}_i(t) \ \ (t\ge0), \quad (5)$$

where

$$f_0(t) = \sum_{k=1}^{m_0} b_k(t)\varphi_{h_k}(t), \quad f_i(t) = \sum_{k=1}^{m_i} c_{ik}(t)\varphi_{g_{ik}}(t) \ \text{ for } \ i = 1,2,\dots,m,$$

$$(S_hx)(t) = \begin{cases} x(h(t)), & \text{if } h(t) \ge 0,\\ 0, & \text{if } h(t) < 0,\end{cases} \qquad \varphi_h(t) = \begin{cases} 0, & \text{if } h(t) \ge 0,\\ \varphi(h(t)), & \text{if } h(t) < 0.\end{cases}$$

The operator S_h is a stochastic analogue of *the inner superposition operator* [2]. In the stochastic case, if $h(t)$ is progressively measurable with respect to the given stochastic basis and $h(t) \le t$ $(t \in [0,+\infty))$ a.s., then the operator S_h maps the space D^n into the space $\overline{L}^n_\infty$, and it is also Volterra [6].

Note that the solution $x_\varphi(t,x_0)$ of Eq. (2), (5) satisfies (4) if the solution $y(t)$ of the equation

$$dy(t) = \Big[\big(-a(t)+\beta\big)y(t) + \sum_{k=1}^{m_0} b_k(t)\exp\big\{\beta(t-h_k(t))\big\}(S_{h_k}y)(t) + f_0(t)\exp\{\beta t\}\Big]dt +$$

$$+\sum_{i=1}^{m}\Big[\sum_{k=1}^{m_i}c_{ik}(t)\exp\big\{\beta(t-g_{ik}(t))\big\}(S_{g_{ik}}y)(t)+f_i(t)\exp\{\beta t\}\Big]d\mathcal{B}_i(t)\ \ (t\geq 0),\tag{6}$$

$$y(\mu_j)=A_jy(\mu_j-0),\ \ j=1,2,3,\dots,\ \ \text{a.s.},\tag{7}$$

where β is some positive number, satisfies the inequality

$$\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}\leq N\Big[\|y(0)\|_{k_{2p}}+\Big(\operatorname{vrai}\sup_{s<0}\mathbb{E}|\varphi(s)|^{2p}\Big)^{1/2p}\Big],$$

for some positive N.

To see this, we observe that there is the one-to-one correspondence $x_\varphi(t,x_0)=\exp\{-\beta t\}y(t)$ with $x_0=y(0)$ between the solutions of Eq. (2), (5) and Eq. (6)–(7).

Consider the following reference equation

$$\begin{aligned}dy(t)&=\big[-a(t)+\overline{f}_0(t)\big]dt+\sum_{i=1}^{m}\overline{f}_i(t)d\mathcal{B}_i(t)\ \ (t\geq 0),\\ y(\mu_j)&=A_jy(\mu_j-0),\ \ j=1,2,3,\dots,\ \ \text{a.s.},\end{aligned}\tag{8}$$

coupled with Eq. (7), under the assumptions that $\overline{f}_0\in\overline{L}_1$, $\overline{f}_i\in\overline{L}_2$ for $i=1,2,\dots,m$.

It can be shown (see e.g. [6]) that any $y(0)\in k$ gives rise to a unique solution (up to the P-equivalence) of Eq. (8). A direct calculation yields the explicit representation of this solution:

$$\begin{aligned}y(t)=\exp\Big\{-\int_0^t a(s)\,ds\Big\}y(0)+\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\overline{f}_0(s)\,ds+\\ +\sum_{i=1}^{m}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\overline{f}_i(s)\,d\mathcal{B}_i(s)\ \ (t\geq 0).\end{aligned}$$

Using this representation we can rewrite Eq. (6), (7) as follows:

$$\begin{aligned}y(t)=\exp\Big\{-\int_0^t a(s)\,ds\Big\}y(0)+\beta\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jy(s)\,ds+\\ +\sum_{k=1}^{m_0}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jb_k(s)\exp\big\{\beta(s-h_k(s))\big\}(S_{h_k}y)(s)\,ds+\\ +\sum_{i=1}^{m}\sum_{k=1}^{m_i}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\exp\big\{s-g_{ik}(s)\big\}\,d\mathcal{B}_i(s)+\\ +\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jf_0(s)\,ds+\sum_{i=1}^{m}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jf_i(s)\,d\mathcal{B}_i(s)\ \ (t\geq 0).\end{aligned}$$

A direct estimation procedure (see [7] for the details) ends up with the following inequality:

$$\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}\leq\|y(0)\|_{k_{2p}}+K\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}+d\Big(\operatorname{vrai}\sup_{s<0}\mathbb{E}|\varphi(s)|^{2p}\Big)^{1/2p},$$

where

$$K=\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)}\Big[\beta+\exp\{\beta\delta\}\operatorname{vrai}\sup_{t\geq 0}\Big\{\sum_{k=1}^{m_0}|b_k(t)|\Big\}\Big]+$$

$$+\Big[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\Big]^{1/2} c_p \exp\{\beta\delta\} \operatorname*{vrai\,sup}_{t\geq 0}\Big\{\sum_{i=1}^{m}\sum_{k=1}^{m_i}|c_{ik}(t)|\Big\},$$

$$d=\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)}\,d_0+c_p\Big[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\Big]^{1/2}\sum_{i=1}^{m}d_i.$$

By the assumptions of the theorem, there exists $\beta>0$ such that $K<1$. This completes the proof.

Consider now a vector version of Eq. (1)–(2), where we assume that $x(\omega,t)\in R^n$, $a(t)$ is an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_1^1$; $b_k(t)$ ia an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_1^1$ for $k=1,2,\dots,m_0$; $c_{ij}(t)$ is an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_2^2$ for $i=1,2,\dots,m$, $j=1,\dots,m_i$; A_j is an $n\times n$-matrix with real entries. In addition, we assume that for some functions $\widehat{b}_k$, $k=1,2,\dots,m_0$, $\widehat{c}_{ij}$, $i=1,2,\dots,m$, $j=1,2,\dots,m_i$, belonging to L_∞, the following estimates hold true: $\|b_k(t)\|\leq\widehat{b}_k(t)$ $(t\in[0,+\infty))$ a.s. for $k=1,2,\dots,m_0$, $\|c_{ij}(t)\|\leq\widehat{c}_{ij}$ $(t\in[0,+\infty))$ a.s. for $i=1,2,\dots,m$, $j=1,2,\dots,m_i$.

Theorem 2. *Suppose that there exist positive numbers A, ρ, σ, δ, α such that $\|A_j\|\leq A$, $\rho\leq\mu_{j+1}-\mu_j\leq\sigma$ for $j=1,2,\dots,$ $\big\|\exp\big\{-\int_s^t a(\tau)\,d\tau\big\}\big\|\leq\exp\{-\alpha(t-s)\}$ $(0\leq s\leq t<\infty)$ a.s., $t-h_k(t)<\delta$, $t-g_{ij}(t)<\delta$ $(t\in[0,\infty))$ almost everywhere for $k=1,2,\dots,m_0$, $i=1,2,\dots,m$, $j=1,2,\dots,m_i$, and in addition, the estimate* (3) *is valid where $\overline{b}_k(t)$ and $\overline{c}_{ik}(t)$ are replaced by $\widehat{b}_k(t)$ and $\widehat{c}_{ik}(t)$, respectively. Then the trivial solution of the vector version of Eq.* (1)–(2) *is exponentially $2p$-stable w.r.t. the initial function.*

The proof of this result essentially coincides with the proof of Theorem 1 if one replaces absolute values with vector norms and $n\times n$-matrix norms, respectively.

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On the Largest Lyapunov Exponent of the Linear Differential System with Parameter-Multiplier

M. V. Karpuk

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: m.vasilitch@gmail.com

Consider the n-dimensional ($n \geq 2$) linear system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with piecewise continuous on the half-line $t \geq 0$ coefficient matrix $A(\,\cdot\,) : [0, +\infty) \to \mathbb{R}^n$. Denote the class of all such systems by $\mathcal{M}_n^*$. We identify the system (1) and it's coefficient matrix and therefore write $A \in \mathcal{M}_n^*$. Along with (1) we consider the one-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{2}$$

of linear differential systems with a scalar parameter-multiplier $\mu \in \mathbb{R}$. Denote by $\mathcal{K}_n^*$ class of families (2) generated by systems $A \in \mathcal{M}_n^*$. Fixing in the family (2) the value of parameter μ we obtain the linear differential system which we denote by $\langle \mu \rangle_A$. Denote by $\lambda_1(\mu A) \leq \cdots \leq \lambda_n(\mu A)$ the Lyapunov exponents [1, p. 34], [2, p. 63] of the system $\langle \mu \rangle_A$.

V. I. Zubov in [3, p. 408, Problem 1] set the following problem: find out how the Lyapunov exponents of the systems (1) and (2) are related. Emphasize that in [3] in the formulation of the problem it is not necessary that the coefficient matrix of (1) to be bounded. Therefore exponent $\lambda_i(\mu A)$, $i = 1, \dots, n$, can take improper values $-\infty$ and $+\infty$. Hence the function $\lambda_i(\mu A)$ of a variable $\mu \in \mathbb{R}$ is a mapping $\mathbb{R} \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, +\infty\}$. We call $\lambda_i(\mu A)$ the i-th Lyapunov exponent of the family (2).

In other words the problem of Zubov can be formulated as: for every $i = 1, \dots, n$ give a complete description of the set $\mathcal{L}_i^n \overset{\text{def}}{=} \{\lambda_i(\mu A) : \mathbb{R} \to \overline{\mathbb{R}} \mid A \in \mathcal{M}_n^*\}$ of i-th Lyapunov exponents of the families from $\mathcal{K}_n^*$.

In this article the problem of Zubov is solved for the largest Lyapunov exponent $\lambda_n(\mu A)$ on the assumption that $\lambda_n(\mu A)$ is not identically equal to $+\infty$ on any of the half-lines.

Note that for families of linear differential systems

$$\mathrm{d}x/\mathrm{d}t = A(t, \mu)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{3}$$

with continuous in the variables t, μ and bounded on the half-line $t \geq 0$ for every fixed $\mu \in \mathbb{R}$ coefficient matrix $A(t, \mu) : [0, +\infty) \times \mathbb{R} \to \operatorname{End} \mathbb{R}^n$, a similar problem is solved in [4]. It is proved that for every $i = 1, \dots, n$ function $\lambda(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ is the i-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$ and have an upper semicontinuous minorant. In the paper [4] it is proved that this result holds in a more general situation – for the Lyapunov exponents of families of morphisms of Millionshchikov bundles.

Despite the fact that the dependence on the parameter in the families (2) is linear, the description of the largest Lyapunov exponents of families from $\mathcal{K}_n^*$ is similar to the description of the largest Lyapunov exponents in the general case of families (3).

We consider $\overline{\mathbb{R}}$ with a natural (order) topology, so that $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1,1]$. Choose such a homeomorphism $\ell : \overline{\mathbb{R}} \to [-1,1]$ in a standard way:

$$\ell(x) = \begin{cases} \dfrac{x}{|x|+1}, & \text{if } x \in \mathbb{R}, \\ \operatorname{sgn}(x), & x = \pm\infty. \end{cases}$$

Since the mapping ℓ performs an order-preserving homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$, we say that function $f : \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the Baire class $\mathcal{K}$ if composition $\ell \circ f$ belongs to the class $\mathcal{K}$.

Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [5, p. 223–224] if, for each $r \in \mathbb{R}$, the preimage of the interval $[r, +\infty)$ under the mapping f is a G_δ-set.

The following theorem describes the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$ from the viewpoint of the Baire clasification.

Theorem 1. *Function $\lambda_n(\mu A) : \mathbb{R} \to \overline{\mathbb{R}}$ of a variable μ belongs to the class $({}^*, G_\delta)$ for any system $A \in \mathcal{M}_n^*$.*

Using an inequality similar to the Lyapunov inequality we get

Lemma 1. *Suppose that for some $\mu_0 \neq 0$ the largest Lyapunov exponent of the system $\langle \mu_0 \rangle_A$ of a family* (2) *is non-positive* (*can be $-\infty$*)*. Then the largest Lyapunov exponent of the system $\langle \mu \rangle_A$ is non-negative for any $\mu \in \mathbb{R}$ such that $\mu\mu_0 \leq 0$.*

The following theorem shows that assertions of Theorem 1 and Lemma 1 give us a sharp description of the restriction on some half-line of the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$.

Theorem 2. *For any non-negative on some half-line function $f(\,\cdot\,) : \mathbb{R} \to \overline{\mathbb{R}}$ of the class $({}^*, G_\delta)$, there exist a system $A \in \mathcal{M}_n^*$ such that the largest Lyapunov exponent* (*as a function of $\mu \in \mathbb{R}$*) *of the system $\langle \mu \rangle_A$, coincides with $f(\,\cdot\,)$ on this half-line and is identically zero on the other half-line.*

Using the Lemma 1 we get further description of the properties of the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$.

Lemma 2. *Suppose that for some $\mu_0 \neq 0$ the largest Lyapunov exponent of the system $\langle \mu_0 \rangle_A$ of a family* (2) *is finite and equals $\lambda \in \mathbb{R}$. Then the largest Lyapunov exponent of the system $\langle \mu \rangle_A$ satisfies the inequality $\lambda_n(\mu A) \geq \lambda\mu/\mu_0$ for any $\mu \in \mathbb{R}$ such that $\mu\mu_0 \leq 0$.*

Using Lemma 1, Theorem 1 and Lemma 2 with some additional considerations we obtain

Theorem 3. *Function $\lambda_n(\mu A) : \mathbb{R} \to \overline{\mathbb{R}}$ of the variable μ is non-negative on some half-line, vanishes at zero and belongs to a class $({}^*, G_\delta)$ for any system $A \in \mathcal{M}_n^*$. Moreover, suppose that $\lambda_n(\mu A)$ takes at least one finite value on that half-line. Then there exist such a real number $b \in \mathbb{R}$ that the inequality $\lambda_n(\mu A) \geq b\mu$ holds for all $\mu \in \mathbb{R}$.*

Theorem 3 shows that the largest Lyapunov exponent of each family from $\mathcal{K}_n^*$ is non-negative on some half-line, vanishes at zero, belongs to a class $({}^*, G_\delta)$ and satisfies alternative: 1) it exceeds some linear function $b\mu$, or 2) it identically equals $+\infty$ on some half-line. In the first case these conditions are sufficient as shows the following theorem.

Theorem 4. *For each non-negative on some half-line function $f(\,\cdot\,) : \mathbb{R} \to \overline{\mathbb{R}}$ which vanishes at zero, belongs to the class $({}^*, G_\delta)$ and satisfies the inequality $f(\mu) \geq b\mu$ for any $\mu \in \mathbb{R}$ and some fixed $b \in \mathbb{R}$, there exist such a system $A \in \mathcal{M}_n^*$ that the largest Lyapunov exponent* (*as a function of μ*) *of the system $\langle \mu \rangle_A$ coincides with $f(\,\cdot\,)$.*

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On the Solvability of One Nonlocal in Time Problem for Multidimensional Wave Equations with Power Nonlinearity

Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: khar@rmi.ge

In the space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t, in the cylindrical domain $D_T=\Omega\times(0,T)$, where Ω is a Lipshitz domain in $\mathbb{R}^n$, consider a nonlocal problem on finding solution $u(x,t)$ of the following equation

$$Lu:=u_{tt}-\sum_{i=1}^{n}u_{x_ix_i}+2au_t+cu+\lambda|u|^{\alpha}u=F(x,t),\ \ (x,t)\in D_T, \tag{1}$$

satisfying the homogeneous boundary condition for the side boundary $\Gamma:=\partial\Omega\times(0,T)$ of the cylinder D_T

$$\Big(\frac{\partial u}{\partial\nu}+\sigma u\Big)\Big|_{\Gamma}=0, \tag{2}$$

and the homogeneous nonlocal conditions

$$K_\mu u:=u(x,0)-\mu u(x,T)=0,\ \ x\in\Omega, \tag{3}$$

$$K_\mu u_t:=u_t(x,0)-\mu u_t(x,T)=0,\ \ x\in\Omega, \tag{4}$$

where F is a given function; α, λ, μ, a, c, σ are given constants and $\alpha>0$, $\lambda\mu\neq0$; $\frac{\partial}{\partial\nu}$ is the derivative with respect to the outer normal to ∂D_T, $n\geq2$.

Let

$$\overset{\circ}{C}{}^2_\mu(\overline{D}_T):=\Big\{v\in C^2(D_T):\ \Big(\frac{\partial v}{\partial\nu}+\sigma v\Big)\Big|_{\Gamma}=0,\ K_\mu v=0,\ K_\mu v_t=0\Big\},$$

$$\overset{\circ}{W}{}^1_{2,\mu}(D_T):=\big\{v\in W^1_2(D_T):\ K_\mu v=0\big\},$$

where W^1_2 is the well-known Sobolev space and the equality $K_\mu v=0$ is understood in the sense of the trace theory.

Definition. Let $F\in L_2(D_T)$. We call function u a strong generalized solution of the problem (1)–(4) of the class W^1_2 if $u\in\overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and there exists a sequence of functions $u_m\in\overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ such that $u_m\to u$ in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and $Lu_m\to F$ in the space $L_2(D_T)$.

It is obvious that a classical solution of the problem (1)-(4) of the space $C^2(\overline{D}_T)$ represents a strong generalized solution of this problem of the class W^1_2.

Theorem. *Let $\lambda>0$, $|\mu|<1$ and $a\geq0$, $c\geq a^2$, $\sigma>0$. Then for any $F\in L_2(D_T)$, if the exponent of nonlinearity $\alpha<\frac{2}{n-1}$, then the problem* (1)–(4) *has at least one strong generalized solution of the class W^1_2.*

Note that under the conditions of the Theorem there exists a positive number $\lambda_0=\lambda_0(F,a,c,\sigma,\mu,T)$ such that for $0<\lambda<\lambda_0$ the problem (1)–(4) can not have more than one strong generalized solution of the class W^1_2.

We also note that even in the linear case, i.e. for $\lambda=0$, the problem (1)–(4) is not always well-posed. For example, when $\lambda=0$ and $|\mu|=1$, the corresponding to (1)–(4) homogeneous problem may have infinite number of linearly independent solutions.

Positive Periodic Solutions of Singular in Phase Variables Differential Systems

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: `kig@rmi.ge`

Let $\omega > 0$,

$$\mathbb{R}_{0+} =]0, +\infty[\,, \quad \mathbb{R}^n_{0+} = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 > 0, \dots, x_n > 0\Big\},$$

and $f_i : \mathbb{R} \times \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i = 1, \dots, n)$ be ω-periodic in the first argument continuous functions. Consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \ \ (i = 1, \dots, n). \tag{1}$$

A solution $(u_i)_{i=1}^n$ of the system (1) with ω-periodic components $u_i : \mathbb{R} \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ is called a positive ω-periodic solution of the system (1).

The problem on the existence of a positive ω-periodic solution has been investigated earlier mainly only for regular differential systems, i.e., for the systems whose right sides are continuous, or satisfy the local Carathéodory conditions on the set $\mathbb{R} \times \mathbb{R}^n_+$, where

$$\mathbb{R}_+ = [0, +\infty[\,, \quad \mathbb{R}^n_+ = \big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 \geq 0, \dots, x_n \geq 0\big\}$$

(see [1, 2] and the references therein).

Theorems below on the existence of a positive ω-periodic solution of the system (1) cover the cases in which the system under consideration has singularities in phase variables, in particular, the case where for arbitrary i and $k \in \{1, \dots, n\}$ the equality

$$\lim_{x_k \to 0} \big|f_i(t, x_1, \dots, x_n)\big| = +\infty \ \text{ for } \ x_j > 0 \ \ (j = 1, \dots, n; \ \ j \neq k)$$

is fulfilled.

In Theorems 1 and 2 it is assumed, respectively, that the functions f_i $(i = 1, \dots, n)$ on the set $\mathbb{R} \times \mathbb{R}^n_{0+}$ satisfy the inequalities

$$\sigma_i\big(f_i(t, x_1, \dots, x_n) - p_i(t)x_i\big) \geq q_i(t, x_i) \ \ (i = 1, \dots, n) \tag{2}$$

and

$$q_i(t, x_i) \leq \sigma_i\big(f_i(t, x_1, \dots, x_n) - p_i(t)x_i\big) \leq \\ \leq \sum_{k=1}^n p_{ik}(t, x_1 + \dots + x_n)x_k + q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n). \tag{3}$$

Here,

$$\sigma_i \in \{-1, 1\} \ \ (i = 1, \dots, n),$$

$p_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n)$ are ω-periodic continuous functions, $p_{ik} : \mathbb{R} \times \mathbb{R}_{0+} \to \mathbb{R}_+$ and $q_i : \mathbb{R} \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are ω-periodic in the first and nonincreasing in the second argument continuous functions, and $q_0 : \mathbb{R} \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is an ω-periodic in the first argument and nonincreasing in the last n arguments continuous function. Moreover, p_i and q_i $(i=1,\dots,n)$ satisfy the conditions

$$\sigma_i\int\limits_0^\omega p_i(s)\,ds<0 \ \ (i=1,\dots,n), \tag{4}$$

$$\max\big\{q_i(t,x):0\le t\le\omega\big\}>0 \ \text{ for } \ x>0. \tag{5}$$

Along with (1) we consider the auxiliary differential system

$$\frac{du_i}{dt}=(1-\lambda)\big(p_i(t)u_i+\sigma_i q_i(t,u_i)\big)+\lambda f_i(t,u_1,\dots,u_n)+\sigma_i\varepsilon \ \ (i=1,\dots,n), \tag{6}$$

depending on the parameters $\lambda>0$ and $\varepsilon>0$.

Theorem 1 (**Principle of a priori boundedness**). *Let the inequalities* (2) *be fulfilled and let there exist positive constants ε_0 and ρ such that for arbitrary $\lambda\in[0,1]$ and $\varepsilon\in]0,\varepsilon_0]$ every positive ω-periodic solution $(u_i)_{i=1}^n$ of the system* (6) *admits the estimates*

$$u_i(t)<\rho \ \ (i=1,\dots,n).$$

Then the system (1) *has at least one positive ω-periodic solution.*

By $X=(x_{ik})_{i,k=1}^n$ we denote the $n\times n$ matrix with components $x_{ik}\in\mathbb{R}$ $(i,k=1,\dots,n)$, and by $r(X)$ we denote the spectral radius of the matrix X. For any continuous ω-periodic function $p:\mathbb{R}\to\mathbb{R}$, satisfying the condition

$$\int\limits_0^\omega p(s)\,ds\neq 0,$$

we put

$$g_\omega(p)(t,s)=\bigg(\exp\bigg(-\int\limits_0^\omega p(\tau)\,d\tau\bigg)-1\bigg)^{-1}\exp\bigg(\int\limits_s^t p(\tau)\,d\tau\bigg) \ \text{ for } t \text{ and } s\in\mathbb{R}.$$

Theorem 2. *Let the inequalities* (3) *and*

$$\lim_{x\to+\infty} r(H(x))<1 \tag{7}$$

be fulfilled, where $H(x)=(h_{ik}(x))_{i,k=1}^n$ and

$$h_{ik}(x)=\max\bigg\{\int\limits_t^{t+\omega}\big|g_\omega(p_i)(t,s)\big|p_{ik}(s,x)\,ds:\ 0\le t\le\omega\bigg\} \ \ (i,k=1,\dots,n).$$

Then the system (1) *has at least one positive ω-periodic solution.*

This theorem can be proved on the basis of Theorem 1 and Theorem 3.1 of [3].

Now we pass to the case, where

$$\sigma_i p_i(t)\le 0 \ \text{ for } \ t\in\mathbb{R}, \ \ p_i(t)\not\equiv 0 \ \ (i=1,\dots,n) \tag{8}$$

and the inequalities (3) have the form

$$q_i(t,x_i)\le\sigma_i\big(f_i(t,x_1,\dots,x_n)-p_i(t)x_i\big)\le \\ \le|p_i(t)|\sum_{k=1}^n h_{ik}(x)x_k+q_0(t,x_1,\dots,x_n) \ \ (i=1,\dots,n), \tag{9}$$

where $h_{ik}:\mathbb{R}_{0+}\to\mathbb{R}_{0+}$ $(i,k=1,\dots,n)$ are continuous nonincreasing functions, and σ_i, q_i $(i=1,\dots,n)$ and q_0 are the numbers and functions satisfying the above conditions.

From Theorem 2 it follows the following corollary.

Corollary 1. *If along with* (8) *and* (9) *the inequality* (7) *is fulfilled, where* $H(x) = (h_{ik}(x))_{i,k=1}^n$, *then the system* (1) *has at least one positive* ω*-periodic solution.*

As an example, we consider the differential systems

$$\frac{du_i}{dt} = \sigma_i\Big(\sum_{k=1}^n p_{ik}u_k + f_{0i}(t,u_1,\dots,u_n)\Big) \ \ (i=1,\dots,n) \tag{10}$$

and

$$\frac{du_i}{dt} = \sigma_i \sum_{k=1}^n \big(p_{ik}u_k + q_{ik}(t)u_k^{-\alpha_k}\big), \tag{11}$$

where $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$, p_{ik} $(i,k=1,\dots,n)$ are the constants satisfying the inequalities

$$p_{ii}<0, \ \ p_{ik} \geq 0 \ \ (i\neq k; \ \ i,k=1,\dots,n), \tag{12}$$

$f_{0i}:\mathbb{R}\times\mathbb{R}_{0+}^n \to \mathbb{R}_+$ $(i=1,\dots,n)$ are ω-periodic in the first argument continuous functions, and $q_{ik}:\mathbb{R}\to\mathbb{R}_+$ $(i,k=1,\dots,n)$ are ω-periodic continuous functions.

Corollary 2. *Let on the set* $\mathbb{R}\times\mathbb{R}_{0+}^n$ *the inequalities*

$$q_i(t,x_i) \leq f_{0i}(t,x_1,\dots,x_n) \leq q_0(t,x_1,\dots,x_n) \ \ (i=1,\dots,n)$$

be fulfilled, where $q_0:\mathbb{R}\times\mathbb{R}_{0+}^n\to\mathbb{R}_+$ *is an* ω*-periodic in the first and nonincreasing in the last* n *arguments continuous function, and* $q_i:\mathbb{R}\times\mathbb{R}_{0+}\to\mathbb{R}_+$ $(i=1,\dots,n)$ *are* ω*-periodic in the first and nonincreasing in the second argument continuous functions satisfying the conditions* (5)*. Then for the existence of at leat one positive* ω*-periodic solution of the system* (10) *it is necessary and sufficient that the real parts of eigenvalues of the matrix*

$$(p_{ik})_{i,k=1}^n \tag{13}$$

be negative.

Corollary 3. *If*

$$\max\big\{q_{ii}(t): \ 0\leq t\leq\omega\big\} > 0 \ \ (i=1,\dots,n),$$

then for the existence of at least one positive ω*-periodic solution of the system* (11) *it is necessary and sufficient that the real parts of eigenvalues of the matrix* (13) *be negative.*

The uniqueness of a positive ω-periodic solution of the system (1) can be proved only in the case where each function f_i has the singularity in the i-th phase variable only. More precisely, we consider the case when the system (1) has one of the following two forms:

$$\frac{du_i}{dt} = p_i(t)x_i + \sigma_i\big(f_{0i}(t,u_1,\dots,u_n) + q_i(t,u_i)\big) \ \ (i=1,\dots,n) \tag{14}$$

and

$$\frac{du_i}{dt} = \sigma_i\Big(\sum_{k=1}^n p_{ik}x_k + q_i(t,u_i)\Big) \ \ (i=1,\dots,n). \tag{15}$$

Here $\sigma_i\in\{-1,1\}$ $(i=1,\dots,n)$, p_{ik} $(i,k=1,\dots,n)$ are the constants satisfying the inequalities (12), $p_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ are ω-periodic continuous functions, $q_i:\mathbb{R}\times\mathbb{R}_{0+}\to\mathbb{R}_+$ $(i=1,\dots,n)$ are ω-periodic in the first and nonincreasing in the second argument functions, and $f_{0i}:\mathbb{R}\times\mathbb{R}_+^n\to\mathbb{R}_+$ are ω-periodic in the first argument continuous functions. Moreover, p_i and q_i $(i=1,\dots,n)$ satisfy the conditions (4) and (5).

Theorem 3. *Let on the set* $\mathbb{R} \times \mathbb{R}_+^n$ *the conditions*

$$\sigma_i\Big(f_{0i}(t,x_1,\dots,x_n) - f_{0i}(t,y_1,\dots,y_n)\Big)\operatorname{sgn}(x_i-y_i) \leq \sum_{k=1}^n p_{ik}(t)|x_k-y_k| \ \ (i=1,\dots,n)$$

be fulfilled, where $p_{ik} : \mathbb{R} \to \mathbb{R}_+$ $(i,k=1,\dots,n)$ *are* ω*-periodic continuous functions. If, moreover,*

$$r(H) < 1, \tag{16}$$

where $H = (h_{ik})_{i,k=1}^n$ *and*

$$h_{ik} = \max\bigg\{\int\limits_t^{t+\omega} \big|g_\omega(p_i)(t,s)\big|p_{ik}(s)\,ds: \ 0 \leq t \leq \omega\bigg\} \ \ (i,k=1,\dots,n),$$

then the system (14) *has a unique positive* ω*-periodic solution.*

Corollary 4. *Let the functions* p_i $(i=1,\dots,n)$ *satisfy the inequalities* (8) *and on the set* $\mathbb{R} \times \mathbb{R}_+^n$ *the conditions*

$$\sigma_i\Big(f_{0i}(t,x_1,\dots,x_n) - f_{0i}(t,y_1,\dots,y_n)\Big)\operatorname{sgn}(x_i-y_i) \leq |p_i(t)|\sum_{k=1}^n h_{ik}|x_k-y_k| \ \ (i=1,\dots,n)$$

be fulfilled, where h_{ik} $(i,k=1,\dots,n)$ *are nonnegative constants. If, moreover, the matrix* $H = (h_{ik})_{i,k=1}^n$ *satisfies the condition* (16)*, then the system* (14) *has a unique positive* ω*-periodic solution.*

Corollary 5. *For the existence of a unique positive* ω*-periodic solution of the system* (15) *it is necessary and sufficient that the real parts of eigenvalues of the matrix* (13) *be negative.*

Note that in the conditions of Theorem 3 and its corollaries, the functions q_i $(i=1,\dots,n)$ may have singularities of arbitrary order in the second argument. For example, in (14) and (15) we may assume that

$$q_i(t,x) = q_{i1}(t)x^{-\mu_{i1}} + q_{i2}(t)\exp(x^{-\mu_{i2}}) \ \ (i=1,\dots,n),$$

where $\mu_{i1} > 0$, $\mu_{i2} > 0$ $(i=1,\dots,n)$, and $q_{ik} : \mathbb{R} \to \mathbb{R}_+$ $(i=1,\dots,n;\ k=1,2)$ are ω-periodic continuous functions such that

$$\max\big\{q_{i1}(t) + q_{i2}(t): \ 0 \leq t \leq \omega\big\} > 0 \ \ (i=1,\dots,n).$$

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Positive Solutions of Periodic Type Boundary Value Problems for Nonlinear Hyperbolic Equations with Singularities in the Phase Variable

Tariel Kiguradze

Florida Institute of Technology, Melbourne, USA

e-mail: `tkigurad@fit.edu`

Let $a > 0$, $b > 0$,

$$D_{ab} = [0, a] \times [0, b], \quad D_b = \mathbb{R} \times [0, b].$$

In the rectangle D_{ab} and the strip D_b, respectively, consider the boundary value problems

$$u_{xy} = f(x, y, u), \tag{1}$$

$$u(0, y) = \lambda_1 u(a, y), \quad u(x, 0) = \lambda_2 u(x, b) \tag{2}$$

and

$$u_{xy} = p(x) u_y + q(x, y, u), \tag{3}$$

$$u(x + a, y) = u(x, y), \quad u(x, 0) = \lambda u(x, b). \tag{4}$$

Here

$$0 < \lambda_i < 1 \ \ (i = 1, 2), \quad 0 < \lambda < 1,$$

and $f : D_{ab} \times (0, +\infty) \to [0, +\infty)$, $p : \mathbb{R} \to \mathbb{R}$ and $q : D_b \to [0, +\infty)$ are continuous functions. Furthermore,

$$p(x + a) = p(x), \quad q(x + a, y, z) = q(x, y, z) \ \text{ for } \ (x, y) \in D_b, \ \ z > 0.$$

A function $u : D_{ab} \to (0, +\infty)$ ($u : D_b \to (0, +\infty)$ is called a *positive solution* of equation (1) (equation (3)) if it has continuous partial derivatives u_x, u_y, u_{xy} and satisfies equation (1) (equation (3)) in the rectangle D_{ab} (in the strip D_b).

A positive solution of equation (1) (equation (3)) satisfying the boundary conditions (2) (boundary conditions (4)) is called a *positive solution* of problem (1), (2) (problem (3), (4)).

The existence theorems formulated below cover the case where

$$\lim_{z \to 0} f(x, y, z) = +\infty, \quad \lim_{z \to 0} q(x, y, z) = +\infty,$$

i.e. the case, where equations (1) and (3) are singular with respect to the phase variable.

Similar results for ordinary differential equations are established in [1].

Introduce the functions

$$g_1(x, s) = \begin{cases} \dfrac{1}{1 - \lambda_1} & \text{for } \ 0 \le s \le x \le a \\ \dfrac{\lambda_1}{1 - \lambda_1} & \text{for } \ 0 \le x < s \le a \end{cases},$$

$$g_2(y, t) = \begin{cases} \dfrac{1}{1 - \lambda_2} & \text{for } \ 0 \le t \le y \le b \\ \dfrac{\lambda_2}{1 - \lambda_2} & \text{for } \ 0 \le y < t \le ba \end{cases}.$$

Theorem 1. *Let the inequality*

$$h_0(x,y,z) \le f(x,y,z) \le h_1(x,y,z)\Big(1+\frac{z}{v(x,y)}\Big)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where* $h_i : D_{ab}\times(0,+\infty)\to[0,+\infty)$ $(i=0,1)$ *are continuous functions nonincreasing with respect to the third variable, and* $v : D_{ab}\to(0,+\infty)$ *is a continuous function. Moreover, let*

$$\max\big\{h_0(x,y,z):\ (x,y)\in D_{ab}\big\}>0 \ \ for\ \ z>0, \tag{5}$$

$$\lim_{z\to+\infty} h^*(z)<1, \tag{6}$$

where

$$h^*(z)=\max\bigg\{\int_0^a\int_0^b \frac{g_1(x,s)g_2(y,t)}{v(x,y)}\,h_1(s,t,z)\,ds\,dt:\ (x,t)\in D_{ab}\bigg\}.$$

Then problem (1), (2) *has at least one positive solution.*

Corollary 1. *Let the inequality*

$$h_0(x,y,z)\le f(x,y,z)\le h_1(x,y,z)(1+z)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where* $h_i : D_{ab}\times(0,+\infty)\to[0,+\infty)$ $(i=0,1)$ *are continuous functions nonincreasing with respect to the third variable. Moreover, if* h_0 *satisfies condition* (5) *and* h_1 *satisfies the condition*

$$\lim_{z\to+\infty}\int_0^a\int_0^b h_1(x,y,z)\,dx\,dy<(1-\lambda_1)(1-\lambda_2),$$

then problem (1), (2) *has at least one positive solution.*

Corollary 2. *Let the inequality*

$$h_0(x,y,z)\le f(x,y,z)\le h_1(z)\Big(1+\frac{z}{v_0(x,y)}\Big)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where*

$$v_0(x,y)=\big((1-\lambda_1)x+\lambda_1 a\big)\big((1-\lambda_2)y+\lambda_2 b\big), \tag{7}$$

$h_0 : D_{ab}\times(0,+\infty)\to[0,+\infty)$ *is a continuous function nonincreasing with respect to the third variable and satisfying condition* (5), *and* $h_1 : (0,+\infty)\to(0,+\infty)$ *is a nonincreasing continuous function such that*

$$\lim_{z\to+\infty} h_1(z)<(1-\lambda_1)(1-\lambda_2). \tag{8}$$

Then problem (1), (2) *has at least one positive solution.*

Corollary 3. *Let the inequality*

$$h_0(x,y,z)\le f(x,y,z)-\frac{l_0 z}{v_0(x,y)}\le h_1(x,y,z)(1+z)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where* l_0 *is a nonnegative constant,* v_0 *is a function given by equality* (7), *and* $h_i : D_{ab}\times(0,+\infty)\to[0,+\infty)$ $(i=0,1)$ *are continuous functions nonincreasing with respect to the third variable. Furthermore, let* h_0 *satisfy condition* (5) *and let* h_1 *satisfy the condition*

$$\lim_{z\to+\infty}\int_0^a\int_0^b h_1(x,y,z)\,dx\,dy=0. \tag{9}$$

Then problem (1), (2) *has at least one positive solution if and only if*

$$l_0 < (1-\lambda_1)(1-\lambda_2). \tag{10}$$

Example 1. Consider the equation

$$u_{xy} = \frac{l_0}{v_0(x,y)}\,u + \sum_{k=1}^{m} l_k(x,y)u^{-\mu_k}, \tag{11}$$

where l_0 is a nonnegative constant, $\mu_k > 0$ $(k=1,\dots,m)$, and $l_k : D_{ab} \to (0,+\infty)$ $(k=1,\dots,m)$ are continuous functions. According to Corollary 3, problem (11), (2) has at least one positive solution if and only if inequality (10) holds.

This example demonstrates that condition (6) (condition (8)) in Theorem 1 (in Corollary 2) is unimprovable and it cannot be replaced by a the nonstrict inequality

$$\lim_{z\to+\infty} h^*(z) \le 1 \quad \Big(\lim_{z\to+\infty} h_1(z) \le (1-\lambda_1)(1-\lambda_2)\Big).$$

Set

$$P_0(x) = \exp\bigg(\int\limits_0^x p(s)\,ds\bigg), \ \ \lambda_0 = P_0(a). \tag{12}$$

On the basis of Corollaries 1–3 one can prove the following assertions on existence of a positive solution of problem (3), (4).

Corollary 4. *Let the inequality*

$$h_0(x,y,z) \le q(x,y,z) \le h_1(x,y,z)(P_0(x)+z),$$

hold on the set $D_{ab}\times(0,+\infty)$, *where* $h_i : D_{ab}\times(0,+\infty) \to [0,+\infty)$ $(i=0,1)$ *are continuous functions nonincreasing with respect to the third variable. Furthermore, if*

$$\lambda_0 < 1,$$

h_0 *satisfies condition* (5) *and* h_1 *satisfies the condition*

$$\lim_{z\to+\infty} \int\limits_0^a \int\limits_0^b h_1(x,y,z)\,dx\,dy < (1-\lambda_0)(1-\lambda),$$

then problem (3), (4) *has at least one positive solution.*

Corollary 5. *Let* $\lambda_0 < 1$ *and let the inequality*

$$h_0(x,y,z) \le q(x,y,z) \le h_1(z)\Big(P_0(x) + \frac{z}{w_0(x,y)}\Big)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where*

$$w_0(x,y) = \big((1-\lambda_0)x + \lambda_1 a\big)\big((1-\lambda)y + \lambda b\big), \tag{13}$$

$h_0 : D_{ab}\times(0,+\infty) \to [0,+\infty)$ *is a continuous function nonincreasing with respect to the third variable and satisfying condition* (5), *and* $h_1 : (0,+\infty) \to (0,+\infty)$ *is a nonincreasing continuous function such that*

$$\lim_{z\to+\infty} h_1(z) < (1-\lambda_0)(1-\lambda).$$

Then problem (3), (4) *has at least one positive solution.*

Corollary 6. *Let* $\lambda_0 < 1$ *and let the inequality*

$$h_0(x,y,z) \leq q(x,y,z) - \frac{l_0 z}{w_0(x,y)} \leq h_1(x,y,z)(1+z)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where* l_0 *is a nonnegative constant,* w_0 *is a function given by equality* (13)*, and* $h_i : D_{ab} \times (0,+\infty) \to [0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable and satisfying conditions* (5) *and* (9)*. Then problem* (3), (4) *has at least one positive solution if and only if*

$$l_0 < (1-\lambda_0)(1-\lambda). \tag{14}$$

Example 2. Let $p : \mathbb{R} \to \mathbb{R}$ be an a-periodic continuous function, and let λ_0, P_0 and w_0 be the number and functions given by (12) and (13). Furthermore, $\lambda_0 < 1$. Consider the equation

$$u_{xy} = p(x)u_y + \frac{l_0}{w_0(x,y)}\,u + \sum_{k=1}^{m} l_k(x,y)u^{-\mu_k}, \tag{15}$$

where l_0 is a nonnegative constant, $\mu_k > 0$ $(k = 1,\dots,m)$, and $l_k : D_b \to (0,+\infty)$ $(k = 1,\dots,m)$ are continuous functions a-periodic with respect to the first variable. According to Corollary 6, problem (15), (4) has at least one positive solution if and only if inequality (14) holds.

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Investigation and Numerical Solution of Some Systems of Partial Integro-Differential Equations

Zurab Kiguradze

I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

E-mail: zkigur@yahoo.com

Mathematical simulation, analysis, and numerical solution of diffusion problems describing various processes and phenomena are very important. To such problems belongs, for instance, mathematical modeling of diffusion of a magnetic field into a substance whose electric conductivity depends essentially on temperature. In a quasistationary case the corresponding system of Maxwell equations has the following form [1]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \quad \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the thermal heat capacity of the substance.

Numerous publications in the 20th century deal with the study of integro-differential equations of various kinds (see, for example, [2]–[13] and references therein). The system (1) can be reduced to the following integro-differential form [2]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\Big[a\Big(\int_0^t |\operatorname{rot} H|^2 \, d\tau\Big) \operatorname{rot} H\Big], \tag{2}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the system of the integro-differential equations (2) is complex. Equations and systems of type (2) still yield to the investigation for special cases (see, for example, [2]–[13] and references therein).

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, then we get the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial V}{\partial x}\Big], \tag{3}$$

where

$$S(x,t) = \int_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] \, d\tau. \tag{4}$$

In [10] some generalization of the system of type (2) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by, so-called, averaged integro-differential model, the (3), (4) type analog of which have the following form:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad \frac{\partial V}{\partial t} = a(S)\,\frac{\partial^2 V}{\partial x^2}\,, \tag{5}$$

where

$$S(t)=\int\limits_0^t\int\limits_0^1\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau. \tag{6}$$

The existence and uniqueness of the solutions of the initial-boundary value problems for the models of type (3), (4) and (5), (6) are studied in many works (see, for example, [2]–[5], [10]–[13] and reference therein).

Our aim is to study the asymptotic behavior of solutions as $t\to\infty$ and semi-discrete schemes for the initial-boundary value problem for systems (3), (4) and (5), (6).

In the domain $[0,1]\times[0,\infty)$ for the systems (3), (4) and (5), (6) we consider the following two kind of boundary conditions:

$$U(0,t)=V(0,t)=U(1,t)=V(1,t)=0,\ \ t\ge 0, \tag{7}$$

$$U(0,t)=V(0,t)=0,\quad U(1,t)=\psi_1,\quad V(1,t)=\psi_2,\ \ t\ge 0, \tag{8}$$

and usual initial conditions:

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1], \tag{9}$$

where $\psi_1=const\ge 0$, $\psi_2=const\ge 0$, $\psi_1^2+\psi_2^2\neq 0$ and U_0 and V_0 are given functions.

The following statement takes place.

Theorem 1. *If $a(S)=(1+S)^p$, $0<p\le 1$, U_0, $V_0\in H^3(0,1)\cap H_0^1(0,1)$ and they satisfy the coincident conditions, then for the solution of problems (3), (4), (7), (9) and (5)–(7), (9) the following asymptotic relation holds as $t\to\infty$*

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big|+\Big|\frac{\partial V(x,t)}{\partial x}\Big|+\Big|\frac{\partial U(x,t)}{\partial t}\Big|+\Big|\frac{\partial V(x,t)}{\partial t}\Big|\le C\exp\Big(-\frac{t}{2}\Big).$$

Here and below C denotes positive constants.

Theorem 2. *If $a(S)=(1+S)^p$, $0<p\le 1$, U_0, $V_0\in H^3(0,1)\cap H_0^1(0,1)$ and they satisfy the coincident conditions, then for the solution of problems (3), (4), (8), (9) and (5), (6), (8), (9) the following asymptotic relations hold as $t\to\infty$:*

$$\Big|\frac{\partial U(x,t)}{\partial x}-\psi_1\Big|+\Big|\frac{\partial V(x,t)}{\partial x}-\psi_2\Big|\le Ct^{-1-p},\quad \Big|\frac{\partial U(x,t)}{\partial t}\Big|+\Big|\frac{\partial V(x,t)}{\partial t}\Big|\le Ct^{-1}.$$

Now let us consider the semi-discrete scheme for problems (3), (4), (7), (9) and (5)–(7), (9). On $[0,1]$ let us introduce a net with mesh points denoted by $x_i=ih$, $i=0,1,\dots,M$, with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. The semi-discrete approximation at (x_i,t) are designed by $u_i=u_i(t)$ and $v_i=v_i(t)$. The exact solution to the problem at (x_i,t) is denoted by $U_i=U_i(t)$ and $V_i=V_i(t)$. At points $i=1,2,\dots,M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences.

Using usual notations let us correspond to those problems the following semi-discrete schemes:

$$\begin{gathered}\frac{du_i}{dt}=\big[a(s)u_{\overline{x},i}\big]_x,\quad \frac{dv_i}{dt}=\big[a(s)v_{\overline{x},i}\big]_x,\ \ i=1,2,\dots,M-1,\\ u_0(t)=u_M(t)=v_0(t)=v_M(t)=0,\\ u_i(0)=U_{0,i},\quad v_i(0)=V_{0,i},\ \ i=0,1,\dots,M,\end{gathered} \tag{10}$$

where

$$s_i(t)=\int\limits_0^t\big[(u_{\overline{x},i})^2+(v_{\overline{x},i})^2\big]\,d\tau,$$

or

$$s(t) = h \int\limits_0^t \sum_{k=1}^{M} \left[(u_{\overline{x},k})^2 + (v_{\overline{x},k})^2 \right] d\tau.$$

So, we obtained Cauchy problem (10) for nonlinear system of ordinary integro-differential equations.

It is not difficult to obtain the following estimates:

$$\|u(t)\|^2 + \int\limits_0^t \|u_{\overline{x}}]|^2 \, d\tau \le C, \quad \|v(t)\|^2 + \int\limits_0^t \|v_{\overline{x}}]|^2 d\tau \le C, \tag{11}$$

where

$$\|w(t)\|^2 = \sum_{i=1}^{M-1} w_i^2(t)h, \quad \|w_{\overline{x}}]|^2 = \sum_{i=1}^{M} w_{\overline{x},\,i}^2(t)h.$$

The a priori estimates (11) guarantee the global solvability of the problem (10).

The following statement is true.

Theorem 3. *If $a(S) \ge a_0 = const > 0$, $a'(S) \ge 0$, $a''(S) \le 0$ and problems (3), (4), (7), (9) and (5)–(7), (9) have a sufficiently smooth solution $U(x,t)$, $V(x,t)$, then the solution of problems (10), $u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$, $v = v(t) = (v_1(t), v_2(t), \dots, v_{M-1}(t))$ tends to $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$, $V = V(t) = (V_1(t), V_2(t), \dots, V_{M-1}(t))$ as $h \to 0$ and the following estimates are true:*

$$\|u(t) - U(t)\| \le Ch, \quad \|v(t) - V(t)\| \le Ch.$$

Now consider the two-dimensional case. Assume that the magnetic field has the following form $H = (U, V, 0)$ and $U = U(x,y,t)$, $V = V(x,y,t)$. System (2) takes the following form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial y} \left[a(S) \left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} \right) \right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) \right], \tag{12}$$

where

$$S(x,y,t) = \int\limits_0^t \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right)^2 d\tau.$$

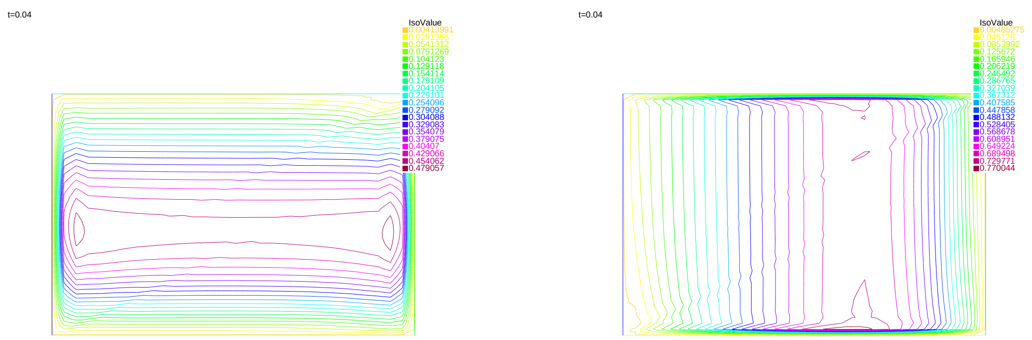

Figure 1. The numerical solution u (left) and v (right) at $t = 0$.

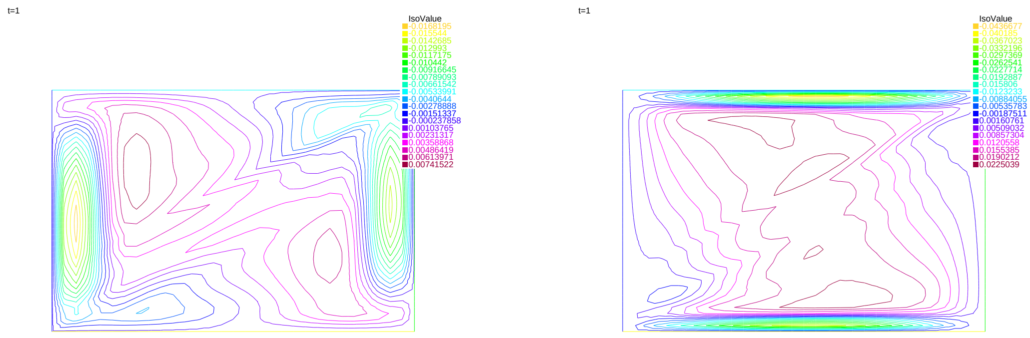

Figure 2. The numerical solution u (left) and v (right) at $t = 1$.

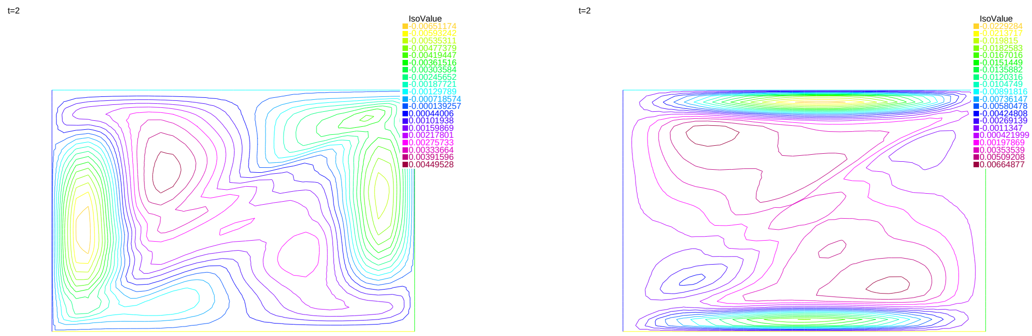

Figure 3. The numerical solution u (left) and v (right) at $t = 2$.

We have carried out numerous numerical experiments for systems (3), (4); (5), (6) and (12) with different kind of initial-boundary value problems. In pictures (Figures 1–3) below there are numerical solutions for two-dimensional system (12) with homogeneous Dirichlet boundary conditions.

From these figures can be deduced that when time is increasing solution is dying in two-dimensional case too like one-dimensional case, that we proved theoretically (Theorem 1).

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On One Formula of Computation of Uniform Means of Piecewise Continuous Functions on the Semiaxis

A. V. Konyukh

Belarusian State Economic University, Minsk, Belarus
E-mail: al3128@gmail.com

We denote by CP a class of piecewise continuous functions $a(\,\cdot\,):[0,+\infty)\to\mathbb{R}$, and by $m(a;s,t)$ we denote an integral mean of a function $a(\,\cdot\,)\in CP$ on a segment $[s,t]$, i.e. a quantity $m(a;s,t)\stackrel{\rm def}{=}(t-s)^{-1}\int_s^t a(\xi)d\xi$. We also denote by CPB a subclass of CP, consisting of bounded on the semiaxis functions.

In paper [1], in particular, some formulae for computation of lower $\underline{\underline{a}}$ and upper $\overline{\overline{a}}$ integral means of a function $a(\,\cdot\,)\in CP$, i.e. of quantities

$$\underline{\underline{a}}\stackrel{\rm def}{=}\varliminf_{t-s\to+\infty} m(a;s,t)\ \text{ and }\ \overline{\overline{a}}\stackrel{\rm def}{=}\varlimsup_{t-s\to+\infty} m(a;s,t) \tag{1}$$

are obtained. As well as a function $a(\,\cdot\,)$ belongs to CP, the values of $\underline{\underline{a}}$ and $\overline{\overline{a}}$ may be infinite ($-\infty$ or $+\infty$). All these values are obviously finite for functions $a(\,\cdot\,)\in CPB$. The general result of the paper [1] concerning computation of the quantities (1) for a function $a(\,\cdot\,)\in CP$ is that in case of their finiteness the same formulae, known for functions $a(\,\cdot\,)\in CPB$ ([2, p. 117] and [3, p. 66]), are valid. The assumption of a finite value of the quantities (1) is significant [1].

In this paper, in addition to the formulae of [1], one more formula for computation of the quantities (1) for functions of the class CP, the validity of which for functions of the class CPB was established earlier in [4] and [5], is obtained.

The properties of the quantities (1) are important in connection with the study of the lower $\underline{\beta}[x]$ and upper $\overline{\beta}[x]$ Bohl exponents [6, . 171–172; 7] of nonzero solutions $x(\,\cdot\,)$ of the linear differential system

$$\dot{x}=A(t)x,\ \ x\in\mathbb{R}^n,\ \ t\geqslant 0, \tag{2}$$

which are defined by formulae:

$$\underline{\beta}[x]=\varliminf_{t-s\to+\infty}\frac{1}{t-s}\ln\frac{\|x(t)\|}{\|x(s)\|}\ \text{ and }\ \overline{\beta}[x]=\varlimsup_{t-s\to+\infty}\frac{1}{t-s}\ln\frac{\|x(t)\|}{\|x(s)\|}\,, \tag{3}$$

and used in the Lyapunov exponent theory. In particular, choosing the function in (1) as $a(\tau)\equiv(\ln\|x(\tau)\|)'$, we obtain the quantities (3).

Following [1], for the fuction $a(\,\cdot\,)\in CP$ we denote by $T(a)$ a set of all two-dimensional sequences $((s_k,t_k))_{k\in\mathbb{N}}$ such that $t_k-s_k\to+\infty$ when $k\to+\infty$ and there exists $\lim\limits_{k\to+\infty} m(a;s_k,t_k)$, and we denote by $S(a)$ a subset of all sequences $((s_k,t_k))_{k\in\mathbb{N}}$ of $T(a)$, for which the additional condition $s_k\to+\infty$ holds. By the definitions of the lower and the upper limits, the definitions (1) of the uniform integral means $a(\,\cdot\,)$ may be written as follows

$$\underline{\underline{a}}=\inf_{((s_k,t_k))\in T(a)}\lim_{k\to+\infty} m(a;s_k,t_k)\ \text{ and }\ \overline{\overline{a}}=\sup_{((s_k,t_k))\in T(a)}\lim_{k\to+\infty} m(a;s_k,t_k). \tag{4}$$

It is, in particular, shown in [1], that the following equalities are valid

$$\underline{\underline{a}}=\inf_{((s_k,t_k))\in S(a)}\lim_{k\to+\infty} m(a;s_k,t_k)\ \text{ and }\ \overline{\overline{a}}=\sup_{((s_k,t_k))\in S(a)}\lim_{k\to+\infty} m(a;s_k,t_k). \tag{5}$$

Definitions (5), in comparison with definitions (4), constrict the class of two-dimensional sequences, which could be taken for evaluation of the limit of integral averages. The class of such sequences may be even more essentially constricted [4, 5], as it is shown below.

We fix a sequence $\delta = (\delta_k)_{k\in\mathbb{N}}$ such that $\delta_{k+1} - \delta_k \to +\infty$ when $k \to +\infty$ (every such sequence δ we will hereinafter call the rapidly increasing). We denote $\Delta_i = [\delta_i, \delta_{i+1}]$, $i \in \mathbb{N}$, and will write $s \approx t \pmod{\delta}$, or shorter $s \approx t$, if s and t belong for some i to the same segment Δ_i. It is shown in [4, 5], that for the function $a(\,\cdot\,) \in CPB$ (and this condition is essentially used in the proof) its lower and upper uniform integral means may be evaluated under the conditions $t - s \to +\infty$ and $s \approx t \pmod{\delta}$. This statement in the theorem stated below, is transferred to the class of functions CP. We denote by $S(a;\delta)$ for fixed rapidly increasing sequence δ and a function $a(\,\cdot\,) \in CP$ a subset of those sequences $((s_k, t_k))_{k\in\mathbb{N}}$ of $S(a)$, for which the condition $s_k \approx t_k$ (mod *delta*) holds.

Theorem. *For every function $a(\,\cdot\,) \in CP$ and every fixed rapidly increasing sequence δ the following equalities hold: if $\underline{\underline{a}} > -\infty$, then*

$$\underline{\underline{a}} = \varliminf_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \inf_{((s_k,t_k))\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k), \tag{6}$$

and if $\overline{\overline{a}} < +\infty$, then

$$\overline{\overline{a}} = \varlimsup_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \sup_{((s_k,t_k))\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k). \tag{7}$$

Let us emphasize the importance of restrictions $\underline{\underline{a}} > -\infty$ and $\overline{\overline{a}} < +\infty$ for the validity of the formulae (6) and (7), respectively. Indeed, for example, the equality (7) does not hold for the sequence $\delta = (\delta_k)_{k\in\mathbb{N}}$, where $\delta_k = k^2$, $k \in \mathbb{N}$, and the function $a(\,\cdot\,)$, given by the equalities: $a(t) = -k^2$ when $t \in [(2k-1)^2, (2k)^2 - 1)$, $a(t) = k^2$ when $t \in [(2k)^2 - 1, (2k)^2)$ and $a(t) = 1$ when $t \in [(2k)^2, (2k+1)^2)$, $k \in \mathbb{N}$.

In fact, for the so-defined function $a(\,\cdot\,)$ we have: $\overline{\overline{a}} = +\infty$, since, as is easily seen,

$$m(a;(2k)^2-1,(2k+1)^2) = (k^2+4k+1)/(4k+2) \to +\infty \ \text{ for } \ k \to +\infty.$$

On the other hand, the integral mean $m(a;s_i,t_i) = 1$, if $s_i, t_i \in [\delta_{2k}, \delta_{2k+1}]$, $k \in \mathbb{N}$, and $m(a;s_i,t_i) \leqslant k^2 - k^2(t_i - s_i - 1) = -k^2(t_i - s_i - 2)$, if $s_i, t_i \in [\delta_{2k-1}, \delta_{2k}]$, $k \in \mathbb{N}$, and, therefore, in this case $m(a;s_i,t_i) \leqslant 0$ when $t_i - s_i \geqslant 2$. That is why for the sequence δ and the function $a(\,\cdot\,)$ holds the equality

$$\varlimsup_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \sup_{(s_k,t_k)\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k) = 1,$$

i.e. the first equality in (7) does not holds.

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Asymptotic Representations of One Class of Singular Solutions of Second-Order Differential Equations

L. I. Kusik

Odessa National Marine University, Odessa, Ukraine
E-mail: `ludakusik@mail.ru`

Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[\times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is a continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} $(i \in \{0, 1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0, 1\})$ is either 0 or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} 1, & \text{if eigher } Y_i = +\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is right neighborhood of the point } 0, \\ -1, & \text{if eigher } Y_i = -\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0 \mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \ \text{ and } \ \mu_0 \mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in a left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1). \tag{3}$$

We study Eq. (1) on class $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, that are defined as follows.

Definition 1. A solution y of Eq. (1) on interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition

$$\lim_{t \uparrow \omega} \frac{[y'(t)]^2}{y(t) y''(t)} = \lambda_0.$$

We put

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty \end{cases}$$

and impose a restriction on the function f.

Definition 2. We say that a function f satisfies condition $(RN)_{\lambda_0}$ as $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ if there exist a number $\alpha_0 \in \{-1, 1\}$, a continuous function $p : [a, \omega[\to]0, +\infty[$ and continuous functions $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, 1)$ of orders σ_i $(i = 0, 1)$ regular varying as $z \to Y_i$ $(i = 0, 1)$ such that for arbitrary continuously differentiable functions $z_i : [a, \omega[\to \Delta_{Y_i}$ $(i = 0, 1)$ satisfying the conditions

$$\lim_{t \uparrow \omega} z_i(t) = Y_i \ \ (i = 0, 1),$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_0'(t)}{z_0(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_1'(t)}{z_1(t)} = \frac{1}{\lambda_0 - 1},$$

one has representation

$$f\big(t, z_0(t), z_1(t)\big) = \alpha_0 p(t) \varphi_0(z_0(t)) \varphi_1(z_1(t)) [1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{4}$$

Note that the choice of α_0 and the functions p and $\varphi_i (i = 0, 1)$ in Definition 2 depends on the choice of $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

Assuming $\lambda_0 \in \mathbb{R}\setminus\{0, 1\}$ and the function satisfies the condition $(RN)_{\lambda_0}$, we introduce auxiliary functions I_0, I_1, Q and a nonzero real number γ by the formulas

$$I_0(t) = \int_{A_0}^{t} p(\tau)\, d\tau, \quad I_1(t) = \int_{A_1}^{t} \pi_\omega(\tau)p(\tau)\, d\tau,$$

$$Q(t) = \begin{cases} I_0(t) & \text{for } 1 - \sigma_0\lambda_0 - \sigma_1 \neq 0, \\ \pi_\omega^{-1}(t)I_1(t) & \text{for } 1 - \sigma_0\lambda_0 - \sigma_1 = 0, \end{cases} \quad \gamma = \begin{cases} 1 - \sigma_0\lambda_0 - \sigma_1 & \text{if } 1 - \sigma_0\lambda_0 - \sigma_1 \neq 0, \\ \lambda_0 - 1 & \text{if } 1 - \sigma_0\lambda_0 - \sigma_1 = 0, \end{cases}$$

where the integration limits $A_i \in \{a; \omega\}$ $(i = 0, 1)$ are chosen so as to ensure that the integrals I_i $(i = 0, 1)$ tend either to zero or to $\pm\infty$ as $t \uparrow \omega$.

In paper [1] the following theorem was formulated and proved.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ and let the function f satisfy condition $(RN)_{\lambda_0}$, moreover, let the orders σ_i $(i = 0, 1)$ of the functions φ_i $(i = 0, 1)$ regularly varying as $y^{(i)} \to Y_i$ $(i = 0, 1)$ satisfy the condition $\sigma_0 + \sigma_1 \neq 1$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the differential equation* (1)*, it is necessary and, if one of the conditions*

$$\text{eigher } \lambda_0 \neq \sigma_1 - 1, \ \text{ or } \lambda_0 = \sigma_1 - 1 \ \text{ and } (1 - \sigma_1)(1 - \sigma_0 - \sigma_1) > 0,$$

is satisfied, sufficient that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)p(t)}{Q(t)} = \frac{\gamma}{\lambda_0 - 1}, \tag{5}$$

$$\mu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}} = Y_0, \quad \mu_1 \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0-1}} = Y_1$$

and the sign conditions

$$\alpha_0\mu_1\gamma Q(t) > 0, \quad \mu_0\mu_1\lambda_0(\lambda_0 - 1)\pi_\omega(t) > 0 \ \text{ for } t \in [a, \omega[$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)}{\varphi_0(y(t))\varphi_1(y'(t))} = \alpha_0\gamma Q(t)[1 + o(1)], \quad \frac{y'(t)}{y(t)} = \frac{\lambda_0(1 + o(1))}{(\lambda_0 - 1)\pi_\omega(t)} \ \text{ as } t \uparrow \omega,$$

and such solutions form a one-parameter family if $\lambda_0(1 - \sigma_0 - \sigma_1) < 0$ and two-parameter family if

$$\lambda_0(1 - \sigma_0 - \sigma_1) > 0 \ \text{ and } \mu_0\mu_1(\lambda_0 + 1 - \sigma_1)\lambda_0 > 0.$$

Since $\omega \le +\infty$ this theorem describes the asymptotic behavior as regular and singular $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Eq. (1) (about definitions of regular and singular solutions see [2]). We specify the conditions of existence and asymptotic behavior of some singular $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Eq. (1) in the neighborhood of a singular point. Assuming that $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ and the function f satisfies condition $(RN)_{\lambda_0}$, in which ω exchange to $t_* \in [a, \omega[$, $\lim_{t\uparrow t_*} p(t) = const \neq 0$, set a question about the occurence of $P_{t_*}(Y_0, Y_1, \lambda_0)$-solutions of Eq. (1). In this case

$$\pi_\omega(t) = t - t_*, \quad \pi_\omega(t) < 0 \ \text{ as } t \in [a, t_*[\,,$$

$$I_0(t) \sim p(t_*)(t - t_*), \quad I_1(t) \sim \frac{p(t_*)(t - t_*)^2}{2} \ \text{ as } t \uparrow t_*,$$

$$Q(t) \sim \begin{cases} p(t_*)(t - t_*), & \text{if } 1 - \sigma_0\lambda_0 - \sigma_1 \neq 0 \\ \dfrac{p(t_*)(t - t_*)}{2}, & \text{if } 1 - \sigma_0\lambda_0 - \sigma_1 = 0, \end{cases} \quad \text{as } t \uparrow t_*.$$

Moreover, from limiting relation (5) it follows that Eq. (1) has no $P_{t_*}(Y_0, Y_1, \lambda_0)$-solution if $1 - \sigma_0\lambda_0 - \sigma_1 = 0$.

Theorem 2. *Let $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ and let the function f satisfy condition $(RN)_{\lambda_0}$, moreover, let the orders σ_i $(i=0,1)$ of the functions φ_i $(i=0,1)$ regularly varying as $y^{(i)} \to Y_i$ $(i=0,1)$ satisfy the condition $\sigma_0+\sigma_1 \neq 1$ and in representation* (4) $\lim\limits_{t\uparrow t_*} p(t) = const \neq 0$. *Then, for the existence of $P_{t_*}(Y_0,Y_1,\lambda_0)$-solutions of the differential equation* (1) *it is necessary together with* (2) *that conditions*

$$\mu_0 \lim_{t\uparrow t_*}(t_*-t)^{\frac{2-\sigma_1}{1-\sigma_0-\sigma_1}} = Y_0, \quad \mu_1 \lim_{t\uparrow t_*}(t_*-t)^{\frac{1+\sigma_0}{2-\sigma_1}} = Y_1, \tag{7}$$

$$\alpha_0\mu_1(1-\sigma_0-\sigma_1)(1+\sigma_0) < 0, \quad \mu_0\mu_1(1-\sigma_0-\sigma_1)(2-\sigma_1) < 0 \tag{8}$$

hold. Each solution of this kind admits the asymptotic representations

$$\begin{aligned} \frac{y'(t)}{\varphi_0(y(t))\varphi_1(y'(t))} &= \frac{\alpha_0 p(t_*)(1-\sigma_0-\sigma_1)}{1+\sigma_0}\,(t-t_*)[1+o(1)], \\ \frac{y'(t)}{y(t)} &= \frac{(2-\sigma_1)(1+o(1))}{(1-\sigma_0-\sigma_1)(t-t_*)} \ \ \text{as } \ t\uparrow t_*. \end{aligned} \tag{9}$$

If together with (2), (6)–(8) *one of the conditions*

$$\text{eigher } \ \frac{2-\sigma_1}{1+\sigma_0} \neq \sigma_1 - 1, \ \ \text{or } \ \frac{2-\sigma_1}{1+\sigma_0} = \sigma_1 - 1 \ \ \text{and } \ (1-\sigma_1)(1-\sigma_0-\sigma_1) > 0$$

hold, then differential equation (1) *has $P_{t_*}(Y_0,Y_1,\frac{2-\sigma_1}{1+\sigma_0})$-solutions, that admits the asymptotic representations* (9) *as $t\uparrow t_*$, and such solutions form a one-parameter family if $(2-\sigma_1)(1+\sigma_0)(1-\sigma_0-\sigma_1)<0$ and two-parameter family if*

$$(2-\sigma_1)(1+\sigma_0)(1-\sigma_0-\sigma_1) > 0 \ \ \text{and } \ \mu_0\mu_1(2-\sigma_1)(3-2\sigma_1+\sigma_0-\sigma_0\sigma_1) > 0.$$

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Existence and Uniqueness of Solutions of Stochastic Differential Equations Driven by Standard and Fractional Brownian Motions

Anatoly Levakov, Maksim Vaskouski

Belarusian State University, Minsk, Belarus
E-mail: `levakov@tut.by; vaskovskii@bsu.by`

Let us fix a probability space $(\Omega,\mathcal{F},P)$ with a filtration $(\mathcal{F}_t)$, let a random variable $\xi:\Omega\to R^d$, r-dimensional standard Brownian motion $W(t,\omega)$, m-dimensional fractional Brownian motion $B^H(t,\omega)$ with the Hurst index $H\in(\frac{1}{2},1)$, random functions $f:R_+\times R^d\times\Omega\to R^d$, $g:R_+\times R^d\times\Omega\to R^{d\times r}$, $\sigma:R_+\times R^d\times\Omega\to R^{d\times m}$ satisfy the following conditions:

1) the standard Brownian motion $W(t,\omega)$ is $\mathcal{F}_t$-Brownian motion, the fractional Brownian motion $B^H(t,\omega)$ and random variable $\xi(\omega)$ are $\mathcal{F}_0$-measurable;

2) the processes $W(t,\omega)$, $B^H(t,\omega)$ and random variable $\xi(\omega)$ are independent;

3) the processes $f(t,X,\omega)$, $g(t,X,\omega)$, $\sigma(t,X,\omega)$ are measurable and $\mathcal{F}_t$-adapted for any fixed $X\in R^d$.

We consider a stochastic differential equation

$$dX(t,\omega)=f\big(t,X(t,\omega),\omega\big)dt+g\big(t,X(t,\omega),\omega\big)dW(t,\omega)+\sigma\big(t,X(t,\omega),\omega\big)dB^H(t,\omega), \tag{1}$$

with an initial condition

$$X(0,\omega)=\xi(\omega). \tag{2}$$

A mapping $h:R_+\times R^d\times\Omega\to R^{d\times l}$ has *linear growth* if for any $T\in R_+$ there exists an $\mathcal{F}_0$-measurable random variable $M_T(\omega)$ such that for almost all $\omega\in\Omega$ for all $(t,X)\in[0,T]\times R^d$ there holds the inequality

$$\big|h(t,X,\omega)\big|\le M_T(\omega)(1+|X|).$$

A mapping $h:R_+\times R^d\times\Omega\to R^{d\times l}$ is called *bounded* if for any $T\in R_+$ there exists an essentially bounded $\mathcal{F}_0$-measurable random variable $Q_T(\omega)$ such that for almost all $\omega\in\Omega$ for all $(t,X)\in[0,T]\times R^d$ there holds the inequality

$$\big|h(t,X,\omega)\big|\le Q_T(\omega).$$

Let $\alpha,\beta\in(0,1]$. We say that a mapping $h:R_+\times R^d\times\Omega\to R^{d\times l}$ satisfies (α,β)-*Holder condition* if for any $T\in R_+$ there exists an $\mathcal{F}_0$-measurable random variable $K_T(\omega)$ such that for almost all $\omega\in\Omega$ for any t, $s\in[0,T]$, $X,Y\in R^d$ there holds the inequality $|h(t,X,\omega)-h(s,Y,\omega)|\le K_T(\omega)(|t-s|^\alpha+|X-Y|^\beta)$.

A mapping $h:R_+\times R^d\times\Omega\to R^{d\times l}$ satisfies *local Lipschitz condition* if for any a, $T\in R_+$ there exists an $\mathcal{F}_0$-measurable random variable $L_{a,T}(\omega)$ such that for almost all $\omega\in\Omega$ for any $t\in[0,T]$, $X,Y\in R^d$, $|X|\le a$, $|Y|\le a$ there holds the inequality

$$\big|h(t,X,\omega)-h(t,Y,\omega)\big|\le L_{a,T}(\omega)|X-Y|.$$

A random process $\eta(t,\omega)$, $t\in R_+$, $\omega\in\Omega$, has Holder-continuous trajectories of order $\alpha\in(0,1]$ if for almost all $\omega\in\Omega$ and for all $T\in R_+$ there exists a constant $C(T,\omega)$ such that for any $t,s\in[0,T]$ there holds the inequality

$$\big|\eta(t,\omega)-\eta(s,\omega)\big|\le C(T,\omega)|t-s|^{\alpha}.$$

Condition A. We say that Condition A holds if the functions f, g satisfy local Lipschitz condition and have linear growth, the mapping σ satisfies $(\delta,1)$-Holder condition with $\delta>1-H$.

For any $\alpha\in(0,1/2)$, $t\in R_+$ and mapping $h:R_+\to R^d$ denote

$$|h(t)|_{\alpha}=|h(t)|+\int\limits_0^t\frac{|h(t)-h(s)|}{(t-s)^{\alpha+1}}\,ds.$$

Denote by $E_0(\zeta)$ the conditional expectation $E(\zeta|\mathcal{F}_0)$ of a random variable ζ with respect to the σ-algebra $\mathcal{F}_0$.

Definition 1. Solution of Eq. (1) with initial condition (2) is an $\mathcal{F}_t$-adapted process $X(t,\omega)$, $t\in R_+$, $\omega\in\Omega$, which has Holder continuous trajectories of any order $\alpha\in(1-H,\min\{\delta,\frac{1}{2}\})$ almost surely, and such that for any $T>0$, $\alpha\in(1-H,\min\{\delta,\frac{1}{2}\})$, $p\ge2$ with probability 1 there holds the inequality

$$\int\limits_0^T E_0\big(|X(t,\omega)|_{\alpha}^p\big)\,dt<\infty,$$

and for any $t\in R_+$ there holds almost surely the equality

$$X(t,\omega)=\xi(\omega)+\int\limits_0^t f\big(s,X(s,\omega),\omega\big)\,ds+\\ +\int\limits_0^t g\big(s,X(s,\omega),\omega\big)\,dW(s,\omega)+\int\limits_0^t\sigma\big(s,X(s,\omega),\omega\big)\,dB^H(s,\omega),$$

where integral with respect to standard Brownian motion is the Ito integral, integral with respect fractional Brownian motion is the pathwise Riemann–Stieltjes integral [1].

Definition 2. We say that a solution $X(t,\omega)$ of Eq. (1) with initial condition (2) is *unique* if for any solution $Y(t,\omega)$ of Eq. (1) with initial condition (2) there holds the equality

$$P\big(X(t,\omega)=Y(t,\omega)\ \ \forall\,t\in R_+\big)=1.$$

Theorem. *If Condition A holds, then Eq.* (1) *with initial condition* (2) *has a unique solution.*

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Periodic Solutions to Second-Order Duffing Type Equations

Alexander Lomtatidze

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic;
Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: lomtatidze@fme.vutbr.cz

Jiří Šremr

Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: sremr@ipm.cz

We are interested in the question on the existence and uniqueness of a **positive** solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u + (-1)^i q(t,u)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega).} \tag{1_i}$$

Here, $p \in L([0,\omega])$, $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $i \in \{1,2\}$. Under a solution to problem (1), as usually, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere and verifies periodic conditions.

Definition 1. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \leq 0 \quad \text{for } t \in [0,\omega]\big)$$

is fulfilled.

Definition 2. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a nontrivial sign-constant solution.

Let us introduce the following hypothesis:

$$\left.\begin{array}{l} q(t,x) \geq q_0(t,x) \quad \text{for a.e. } t \in [0,\omega] \text{ and all } x \geq 0, \\ q_0\colon [0,\omega] \times [0,+\infty[\, \to \mathbb{R} \text{ is a Carathéodory function,} \\ q_0(t,\cdot)\colon [0,+\infty[\, \to \mathbb{R} \text{ is non-decreasing for a.e. } t \in [0,\omega]. \end{array}\right\} \tag{H_1}$$

Theorem 1_1. *Let $p \in \mathcal{V}^-(\omega)$, $q(\cdot,0) \equiv 0$, and hypothesis (H_1) be fulfilled. Let, moreover, there exist a function $\alpha \in AC^1([0,\omega])$ satisfying*

$$\alpha(t) > 0 \quad \textit{for } t \in [0,\omega],$$
$$\alpha''(t) \geq p(t)\alpha(t) - q(t,\alpha(t))\alpha(t) \quad \textit{for a.e. } t \in [0,\omega],$$
$$\alpha(0) = \alpha(\omega), \quad \alpha'(0) \geq \alpha'(\omega).$$

Then problem (1_1) has at least one positive solution u such that

$$u(t_u) \leq \alpha(t_u) \quad \textit{for some } t_u \in [0,\omega].$$

Theorem 1_2. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(\,\cdot\,,0) \equiv 0$, and there exists a function $\beta \in AC^1([0,\omega])$ satisfying*

$$\begin{gathered}\beta(t) > 0 \quad \text{for } t \in [0,\omega],\\ \beta''(t) \le p(t)\beta(t) + q(t,\beta(t))\beta(t) \quad \text{for a.e. } t \in [0,\omega],\\ \beta(0) = \beta(\omega), \quad \beta'(0) \le \beta'(\omega).\end{gathered}$$

Then problem (1_2) has at least one positive solution u such that

$$u(t) \le \beta(t) \quad \text{for } t \in [0,\omega].$$

Corollary 1_1. *Let $p \in \mathcal{V}^-(\omega)$, $q(\,\cdot\,,0) \equiv 0$, hypothesis (H_1) be satisfied, and*

$$\lim_{x\to+\infty} \int_0^\omega q_0(s,x)\,\mathrm{d}s = +\infty. \tag{2}$$

Then problem (1_1) has at least one positive solution.

Corollary 1_2. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(\,\cdot\,,0) \equiv 0$, hypothesis (H_1) be satisfied, and*

$$\lim_{x\to+\infty} \int_E q_0(s,x)\,\mathrm{d}s = +\infty \quad \text{for every } E \subseteq [0,\omega], \ \operatorname{meas} E > 0. \tag{3}$$

Then problem (1_2) has at least one positive solution.

Assumption (3) in Corollary 1_2 is optimal and cannot be weakened to assumption (2). However, assuming (2) instead of (3), problem (1_2) may still have a positive solution under a more restrictive assumption on p than $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. More precisely, the following statement holds.

Corollary 2. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, $q(\,\cdot\,,0) \equiv 0$, and hypothesis (H_1) be satisfied. Let, moreover, condition (2) hold and there exist $x_0 \ge 0$ such that*

$$q_0(t,x_0) \ge 0 \quad \text{for a.e. } t \in [0,\omega].$$

Then problem (1_2) has at least one positive solution.

The next statements show that, under a stronger assumption on q than (H_1), the assumptions $p \in \mathcal{V}^-(\omega)$ and $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ in the above results are necessary. Introduce the hypothesis:

$$\left.\begin{array}{l}\text{For every } b > a > 0 \text{ there exists } h_{cd} \in L([0,\omega]) \text{ such that}\\ h_{ab}(t) \ge 0 \quad \text{for a.e. } t \in [0,\omega], \quad h_{ab} \not\equiv 0,\\ q(t,x) \ge h_{ab}(t) \quad \text{for a.e. } t \in [0,\omega] \text{ and all } x \in [a,b].\end{array}\right\} \tag{H_2}$$

Proposition 1_1. *Let hypothesis (H_2) hold. If problem (1_1) has a positive solution then the inclusion $p \in \mathcal{V}^-(\omega)$ is satisfied.*

Proposition 1_2. *Let hypothesis (H_2) hold. If problem (1_2) has a positive solution then the condition $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ is satisfied.*

Now we give two uniqueness type results for problem (1). Introduce the following hypothesis:

$$\left.\begin{array}{l}\text{For every } b > a > 0 \text{ and } c > 0, \text{ there exists } h_{abc} \in L([0,\omega]) \text{ such that}\\ h_{abc}(t) > 0 \quad \text{for a.e. } t \in [0,\omega],\\ q(t,x+c) - q(t,x) \ge h_{abc}(t) \quad \text{for a.e. } t \in [0,\omega] \text{ and all } x \in [a,b].\end{array}\right\} \tag{H_3}$$

Theorem 2. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(\,\cdot\,,0) \equiv 0$, and hypothesis (H_3) hold. Then problem* (1) *has at most one positive solution.*

A quite stronger assertion can be proved under the assumption that $p \in \operatorname{Int} \mathcal{V}^+(\omega)$. On the other hand, hypothesis (H_3) can be weakened to the following one:

$$\left.\begin{array}{l}\text{For every } \ b > a > 0 \ \text{ and } \ c > 0, \ \text{ there exists } \ h_{abc} \in L([0,\omega]) \ \text{ such that}\\ h_{abc}(t) \geq 0 \quad \text{for a.e. } \ t \in [0,\omega], \ \ h_{abc} \not\equiv 0,\\ q(t,x+c) - q(t,x) \geq h_{abc}(t) \quad \text{for a.e. } \ t \in [0,\omega] \ \text{ and all } \ x \in [a,b].\end{array}\right\} \tag{H_4}$$

Theorem 3. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, hypothesis (H_4) hold, and*

$$q(t,0) \geq 0 \quad \text{for a.e. } \ t \in [0,\omega].$$

Then problem (1_2) *has at most one positive solution. Moreover, any non-trivial solution to this problem is either positive or negative.*

If q in (1) is a function with separated variables, we arrive at the problem

$$\boxed{u'' = p(t)u + (-1)^i h(t)\varphi(u)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),} \tag{4_i}$$

where $p, h \in L([0,\omega])$, $\varphi \in C(\mathbb{R})$, and $i \in \{1,2\}$. This problem covers a rather wide class of equations and serves us as a model problem to illustrate the results.

Theorem 4_1. *Let $p \in \mathcal{V}^-(\omega)$, $\varphi(0) = 0$, and*

$$h(t) \geq 0 \quad \text{for a.e. } \ t \in [0,\omega], \quad h \not\equiv 0. \tag{5}$$

Let, moreover, at least one of the following conditions be fulfilled:

(a) *The inequality*

$$\liminf_{x \to +\infty} \varphi(x) > -\infty$$

holds and there exists $c > 0$ such that $p(t) \leq h(t)\varphi(c)$ for a.e. $t \in [0,\omega]$.

(b) *The equality*

$$\lim_{x \to +\infty} \varphi(x) = +\infty \tag{6}$$

holds.

Then problem (4_1) *has at least one positive solution.*

Theorem 4_2. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $\varphi(0) = 0$, and at least one of the following conditions be fulfilled:*

(a) *There exists $c > 0$ such that $p(t) + h(t)\varphi(c) \geq 0$ for a.e. $t \in [0,\omega]$.*

(b) *Condition* (6) *holds and*

$$h(t) > 0 \quad \text{for a.e. } \ t \in [0,\omega]. \tag{7}$$

Then problem (4_2) *has at least one positive solution.*

Theorem 5. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $\varphi(0) = 0$, φ is increasing on $[0,+\infty[$, and relations* (6) *and* (7) *be satisfied. Then the problem* (4) *has a unique positive solution.*

If we assume that the function φ in (4_2) is even and u is a solution to problem (4_2) the the function $-u$ is its solution, as well. Therefore, we get the following multiplicity type result.

Theorem 6. *Let* $p \in \operatorname{Int} \mathcal{V}^+(\omega)$*,* $\varphi(0) = 0$*,* φ *is increasing on* $[0, +\infty[$ *, and relations* (5) *and* (6) *be satisfied. Then the problem*

$$u'' = p(t)u + h(t)\varphi(|u|)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has exactly three solutions (positive, negative, and trivial).

Finally, we consider the problem with two "super-linear" terms

$$\boxed{u'' = p(t)u + (-1)^i h(t)|u|^\lambda \operatorname{sgn} u + f(t)|u|^\mu \operatorname{sgn} u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),} \tag{8_i}$$

where $p, h, f \in L([0, \omega])$, $\lambda, \mu > 1$, and $i \in \{1, 2\}$. The next statements follow from Theorems 1_1 and 1_2.

Theorem 7_1. *Let* $p \in \mathcal{V}^-(\omega)$*,* $\lambda > \mu > 1$*, relation* (5) *hold, and there exists a number* $c > 0$ *such that*

$$[f(t)]_+ \leq ch(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{9}$$

Then problem (8_1) *has at least three solutions* (*positive, negative, and trivial*)*.*

Theorem 7_2. *Let* $p \in \operatorname{Int} \mathcal{V}^+(\omega)$*,* $\lambda > \mu > 1$*, relation* (5) *hold, and there exists a number* $c > 0$ *such that*

$$[f(t)]_- \leq ch(t) \quad \text{for a.e. } t \in [0, \omega]. \tag{10}$$

Then problem (8_2) *has at least three solutions* (*positive, negative, and trivial*)*.*

Remark 1. If

$$f(t) \geq 0 \quad \text{for a.e. } t \in [0, \omega]$$

then inequality (10) is trivially satisfied and we can claim in Theorem 7_2 that problem (8) has exactly three solutions.

Theorem 8_1. *Let* $p \in \mathcal{V}^-(\omega)$*,* $\lambda > \mu > 1$*, condition* (7) *hold, and*

$$[f]_+^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}} \in L([0, \omega]). \tag{11}$$

Then problem (8_1) *has at least three solutions* (*positive, negative, and trivial*)*.*

Theorem 8_2. *Let* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$*,* $\lambda > \mu > 1$*, condition* (7) *hold, and*

$$[f]_-^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}} \in L([0, \omega]). \tag{12}$$

Then problem (8_2) *has at least three solutions* (*positive, negative, and trivial*)*.*

Remark 2. Observe that if there exists $c > 0$ such that inequality (9) (respectively, (10)) holds then inclusion (11) (respectively, (12)) is trivially satisfied.

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On Γ-Ultimate Classes of Perturbations

E. K. Makarov

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: `jcm@im.bas-net.by`

Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with a piecewise continuous bounded coefficient matrix A and with the Cauchy matrix X_A. Together with system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with a piecewise continuous bounded perturbation matrix Q. We use the notation $\lambda_n(A+Q)$ for the higher exponent of system (2). Let $\mathfrak{M}$ be an arbitrary class of perturbations. The number $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}\}$ is the attainable upper bound of exponents of system (2) with perturbations in the class $\mathfrak{M}$. The problem of finding $\Lambda(\mathfrak{M})$ for various $\mathfrak{M}$ specified by a given smallness condition is an important problem in the theory of Lyapunov characteristic exponents [1, p. 157], [2], [3, p. 46]. It was studied, e.g., in [4]–[15]. In numerous cases, an algorithm similar to the algorithm for the computation of the sigma-exponent [4] can be constructed for $\Lambda(\mathfrak{M})$. In some other cases [5], [6], [13]–[15], formulas similar to those for the computation of the central [1, p. 99], [12], [13] and exponential [15] exponents hold. For the set $\mathfrak{M}_0[\theta]$ of all perturbations satisfying the estimate $\|Q(t)\| \leq N_Q e^{-\sigma\theta(t)}$, where $N_Q \geq 0$, $\sigma > 0$, are numbers depending on Q and $\theta : [0,+\infty[\, \to]0,+\infty[$ is a fixed piecewise continuous function monotone increasing to $+\infty$ such that $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\theta(t) < +\infty$, it was proved in [5], [6] that

$$\Lambda(\mathfrak{M}_0[\theta]) = \lim_{\delta\to+0} \overline{\lim_{k\to\infty}} \frac{1}{t_k(\delta)} \sum_{j=0}^{k} \ln \big\|X_A(t_{j+1}(\delta), t_j(\delta))\big\|, \tag{3}$$

where, for each $\delta > 0$, the sequence $t_j(\delta)$, $j \in \mathbb{N}$, referred to as the δ-characteristic sequence for the perturbation class $\mathfrak{M}_0[\theta]]$, is defined by the recursion formula $t_{j+1}(\delta) = t_j(\delta) + \delta\theta(t_j(\delta))$, and any nonnegative number can be taken for $t_0(\delta) \geq 0$.

The perturbation classes $\mathfrak{M}$ for which $\Lambda(\mathfrak{M})$ admits a representation of the form (3) were called limit classes in [5], [6]. In the report we present sufficient conditions for the considered class of piecewise continuous bounded perturbations to have similar properties.

For an arbitrary set S and for any $n \in \mathbb{N}$, by $S^{n\times n}$ we denote the set of all $n \times n$-matrices with entries in S. In $\mathbb{R}^{n\times n}$, we fix the spectral norm $\|\cdot\|$, and by $\mathrm{KC}_n(\mathbb{R}^+)$ we denote the linear space of all bounded piecewise continuous matrix functions defined everywhere on the positive half-line $\mathbb{R}^+ := [0,+\infty[$ and ranging in $\mathbb{R}^{n\times n}$.

A function $\gamma \in \mathrm{KC}_1(\mathbb{R}^+)$ is said to be strictly positive iff the condition $\inf\limits_{t\in J} \gamma(t) > 0$ holds for every finite interval $J \subset \mathbb{R}^+$.

Following the approach suggested in [16], we interpret a one-dimensional smallness class as an arbitrary linear subspace $\mathfrak{s} \subset \mathrm{KC}_1(\mathbb{R}^+)$ that contains at least one strictly positive function and satisfies the following fullness condition: together with any element β, the set $\mathfrak{s}$ contains all functions $\varphi \in \mathrm{KC}_1(\mathbb{R}^+)$ such that $|\varphi(t)| \leq |\beta(t)|$ for all $t \geq 0$. For each $n \in \mathbb{N}$, the smallness class of dimension $n \times n$ corresponding to the one-dimensional class $\mathfrak{s}$ is defined as the set of matrices

$\mathfrak{s}^{n\times n}$. By [16], the smallness class $\mathfrak{s}^{n\times n}$ can be equivalently defined as the set of $Q \in \mathrm{KC}_n(\mathbb{R}^+)$ such that $\|Q\| \le \beta$ for some $\beta \in \mathfrak{s}$.

A system of generators for a one-dimensional class $\mathfrak{s}$ is defined as a subset $\mathcal{K} \subset \mathfrak{s}$ that consists of strictly positive functions and has the property that, for each $\beta \in \mathfrak{s}$, there exists $\varphi \in \mathcal{K}$ such that $|\beta| \le C\varphi$ for some $C > 0$ depending on φ and β.

Example 1. For the one-dimensional class $\mathfrak{M}_0[\theta]$, a system of generators is given by the one-parameter family of functions $\{\exp(-\sigma\theta(t)) : \sigma > 0\}$, which can be restricted to the countable family $\exp(-\theta(t)/k)$, $k \in \mathbb{N}$.

Let $\mathbb{T}$ be the set of all sequences of times $t_k \ge 1$, $k \in \mathbb{N}\cup\{0\}$, monotone increasing to $+\infty$. By $\mathbb{T}_0$ we denote the subset of $\mathbb{T}$ that consists of subsequences satisfying the condition $\lim\limits_{k\to+\infty} t_k^{-1}t_{k+1} = 1$ of slow growth [17] and the condition $\lim\limits_{k\to+\infty}(t_{k+1}-t_k) = +\infty$. For arbitrary $\beta \in \mathrm{KC}_1(\mathbb{R}^+)$, $N \ge 0$, and $\tau \in \mathbb{T}$, let

$$\Omega(A,\tau) = \overline{\lim_{k\to\infty}}\, \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln \|X_A(t_{i+1},t_i)\|,$$

$$\Delta_N(\beta,\tau) = \overline{\lim_{k\to\infty}}\, \frac{1}{t_k} \int_{t_0}^{t_k} K_N^\tau(s)\beta(s)\,ds,$$

where the $t_k \ge 1$, $k \in \mathbb{N}\cup\{0\}$, are elements of the sequence τ, $K_N^\tau(s) = e^{N(s-t_k)}$ for $s \in]t_k, t_{k+1}]$, $k \in \mathbb{N}$, and $K_N^\tau(s) = 0$ for $s \le t_0$. If $\beta \in \mathrm{KC}_1(\mathbb{R}^+)$ is a strictly positive function, then we introduce the additional notation

$$\gamma(\beta,\tau) = \overline{\lim_{k\to\infty}}\, \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln \frac{2}{\sin\varphi_i}, \quad \varphi_i = \min\bigg\{\frac{\pi}{2}, e^{-2N_A} \int_{t_i-1}^{t_i} \beta(s)\,ds\bigg\};$$

Theorem 1. *Let $\mathfrak{M} = \mathfrak{s}^{n\times n}$ be some smallness class of perturbations, and let $\mathcal{K}$ be a system of generators of the corresponding one-dimensional class $\mathfrak{s}$. If there exists a set $\Gamma \subset \mathbb{T}_0$ such that the inequality $\inf\limits_{\beta\in\mathcal{K}} \gamma(\beta,\tau) = 0$ holds for each sequence $\tau \in \Gamma$, and for any $\beta \in \mathcal{K}$ and $M > 0$, there exists a sequence $\tau \in \Gamma$ satisfying the condition $\Delta_M(\beta,\tau) = 0$, then*

$$\Lambda(\mathfrak{M}) = \sup_{\beta\in\mathcal{K}} \sup_{M>0} \inf_{\tau\in\mathcal{R}_M^\beta} \Omega(A,\tau) = \sup_{\tau\in\Gamma} \Omega(A,\tau),$$

where $\mathcal{R}_M^\beta = \{\tau \in \Gamma :\ \Delta_M(\beta,\tau) = 0\}$.

Example 2. For the limit class $\mathfrak{M}_0[\theta]$ with a system of generators $\mathcal{K}$ consisting of the functions $\beta_\sigma(t) = \exp(-\sigma\theta(t))$, $\sigma > 0$, $t \ge 0$, for the set Γ, one can take the set of all δ-characteristic sequences for $\delta \in]0,\delta_0]$ with an arbitrary $\delta_0 > 0$.

Definition 1. An arbitrary smallness class $\mathfrak{M}$ is called a Γ-ultimate (or Γ-limit) class if there exists a set $\Gamma \subset \mathbb{T}$ such that the relation

$$\Lambda(\mathfrak{M}) = \sup_{\tau\in\Gamma} \Omega(A,\tau)$$

holds for every system (1).

Definition 2. A one-dimensional smallness class $\mathfrak{s}$ is said to be radical if, together with each element β, the set $\mathfrak{s}$ contains all of its powers β^ε, $\varepsilon \in]0,1]$.

Example 3. The condition to be radical holds for classes of exponential and infinitesimal perturbations as well as for the classes $\mathfrak{M}_0[\theta]$. The classes of sigma-perturbations and perturbations integrable on the half-line do not satisfy that condition.

Remark. In Definition 2, instead of the requirement of all positive powers of elements with exponents less than unity to belong to the class $\mathfrak{s}$, it suffices to require that only the roots (radicals) of them with any positive integer power belong to $\mathfrak{s}$; i.e., it suffices to consider the values $\varepsilon = 1/k$ for $k \in \mathbb{N}$.

Theorem 2. *If a one-dimensional smallness class $\mathfrak{s}$ is radical, consists of functions tending to zero at infinity, and has a system of generators $\mathcal{K}$ where each element β is a continuous function with exact zero Lyapunov exponent and satisfies the condition*

$$\int\limits_{t-1}^{t} \ln\beta(s)\,ds \geq C_\beta \ln\beta(t), \tag{4}$$

for some $C_\beta > 0$ and for all sufficiently large t; then for each $n \in \mathbb{N}$ the smallness class $\mathfrak{M} = \mathfrak{s}^{n\times n}$ is Γ-limit.

Example 4. One can readily see that the assumptions of Theorem 2 are satisfied for all limit classes $\mathfrak{M}$ considered in [5], [6]. In that case, condition (4) can be reduced to the inequality

$$\int\limits_{t-1}^{t} \theta(s)\,ds \leq C_\beta\theta(t), \tag{5}$$

whose validity for $C_\beta = 1$ is provided by the monotone growth of the function θ. Condition (5) with $C_\beta > 0$ is valid for some nonmonotone functions as well satisfying the condition $\theta(t)/t \to 0$ as $t \to +\infty$, for example, for $\theta_1(t) = (1 + \sin^2 t)\ln t$. This permits one to use Theorem 2 for the proof of the Γ-limit property of the classes $\mathfrak{M}_0[\theta]$ with such functions.

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On the Green Operator of the General Linear Boundary Value Problem for a Class of Functional Differential Equations with Continuous and Discrete Times

V. P. Maksimov

Perm State University, Perm, Russia
E-mail: maksimov@econ.psu.ru

1 Introduction

We consider here a system of functional differential equations (FDE, FDS) that is a typical one met with in mathematical modeling various dynamic processes and covers many kinds of dynamic models with aftereffect (integrodifferential, delayed differential, differential difference, difference) and impulsive disturbances [2, 4, 5, 8]. The equations of the system contain simultaneously terms depending on continuous time, $t \in [0,T]$, and discrete, $t \in \{0, t_1, \dots, t_N, T\}$. The interest of researchers in such "continuous-discrete systems" (CDS) is growing constantly (see, for instance, [1, 9, 6] and references therein).

First we descript in detail a class of continuous-discrete functional differential equations (CDFDE) with linear Volterra operators and appropriate spaces where those are considered. We concerned with the representation of general solution to the system using the Cauchy operator and the fundamental matrix. Next the setting of the general linear boundary value problem (BVP) for CDFDE is given, and conditions for the unique solvability of BVP are formulated. Finally we propose a representation to solutions of the uniquely solvable BVP and discuss some properties of the corresponding Green operator.

2 A class of Continuous-Discrete Functional Differential Systems

Fix a segment $[0,T] \subset R$. By $L^n = L^n[0,T]$ we denote the space of summable functions $v : [0,T] \to R^n$ under the norm $\|v\|_{L^n} = \int\limits_0^T |v(s)|_n \, ds$, where $|\cdot|_n$ stands for the norm of R^n.

Given set $\{\tau_1, \dots, \tau_m\}, 0 < \tau_1 < \cdots < \tau_m < T$, the space $DS^n(m) = DS^n[0, \tau_1, \dots, \tau_m, T]$ is defined (see [3, 5]) as the space of piecewise absolutely continuous functions $y : [0,T] \to R^n$ representable in the form

$$y(t) = \int\limits_0^t v(s) \, ds + y(0) + \sum_{k=1}^m \chi_{[\tau_k, T]}(t) \Delta y(\tau_k),$$

where $v \in L^n$, $\Delta y(\tau_k) = y(\tau_k) - y(\tau_k - 0)$, $\chi_{[\tau_k,T]}(t)$ is the characteristic function of the segment $[\tau_k, T]$: $\chi_{[\tau_k,T]}(t) = 1$ if $t \in [\tau_k, T]$ and $\chi_{[\tau_k,T]}(t) = 0, t \notin [\tau_k, T]$. Thus the elements of $DS^n(m)$ are the functions being absolutely continuous on each $[0, \tau_1), [\tau_1, \tau_2), \dots, [\tau_m, T]$ and continuous from the right at the points $\tau_1, \dots, \tau_m$. Under the norm

$$\|y\|_{DS^n(m)} = \|\dot{y}\|_{L^n} + |y(0)|_n + \sum_{k=1}^m |\Delta y(\tau_k)|_n$$

the space $DS^n(m)$ is Banach.

Let us fix a set $J = \{t_0, t_1, \dots, t_\mu\}, 0 = t_0 < t_1 < \dots < t_\mu = T$.

$FD^\nu(\mu) = FD^\nu\{t_0, t_1, \dots, t_\mu\}$ denotes the space of functions $z : J \to R^\nu$ under the norm

$$\|z\|_{FD^\nu(\mu)} = \sum_{i=0}^{\mu} |z(t_i)|_\nu.$$

We consider the system

$$\begin{aligned} \dot{y} &= \mathcal{T}_{11}y + \mathcal{T}_{12}z + f, \\ z &= \mathcal{T}_{21}y + \mathcal{T}_{22}z + g, \end{aligned} \tag{1}$$

where the linear operators $\mathcal{T}_{ij}$, $i, j = 1, 2$, are defined as follows:

$$\mathcal{T}_{11} : DS^n(m) \to L^n; \tag{$\mathcal{T}_{11}$}$$

$$(\mathcal{T}_{11}y)(t) = \int_0^t K^1(t,s)\dot{y}(s)\,ds + A_0^1(t)y(0) + \sum_{\{k:\ \tau_k<t\}} A_k^1(t)\Delta y(\tau_k),\ \ t \in [0,T].$$

Here the elements $k_{ij}^1(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}^1(t,s)| \le \kappa(t)$, $i, j = 1, \dots, n$, $\kappa(\,\cdot\,)$ is summable on $[0,T]$, $(n \times n)$-matrices $A_0^1, \dots, A_m^1$ have elements summable on $[0,T]$,

$$\mathcal{T}_{12} : DS^n(m) \to L^n;\ \ (\mathcal{T}_{12}z)(t) = \sum_{\{j:\ t_j \le t-\Delta_1\}} B_j^1(t)z(t_j),\ \ t \in [0,T], \tag{$\mathcal{T}_{12}$}$$

where elements of matrices B_j^1, $j = 0, 1, \dots, \mu,$, are summable on $[0,T]$, $\Delta_1 \ge 0$.

$$\mathcal{T}_{21} : DS^n(m) \to L^n; \tag{$\mathcal{T}_{21}$}$$

$$(\mathcal{T}_{21}y)(t_i) = \int_0^{\max\{t_i-\Delta_2\}} K_i^2(s)\dot{y}(s)ds + A_{i0}^2 y(0) + \sum_{\{k:\ \tau_k<t_i\}} A_{ik}^2 \Delta y(\tau_k),\ \ i = 0, 1, \dots, \mu,$$

with measurable and essentially bounded on $[0,T]$ elements of matrices K_i^2 and constant $(\nu \times n)$-matrices A_{ik}^2, $i = 0, 1, \dots, \mu$, $k = 0, 1, \dots, m$; $\Delta_2 \ge 0$,

$$\mathcal{T}_{22} : DS^n(m) \to L^n;\quad (\mathcal{T}_{22}z)(t) = \sum_{j=0}^{i-1} B_{ij}^2 z(t_j)\ i = 1, \dots, \mu, \tag{$\mathcal{T}_{22}$}$$

with constant $(\nu \times \nu)$-matrices B_{ij}^2.

In what follows we will use some results from [4, 5] concerning the equation

$$\dot{y} = \mathcal{T}_{11}y + f \tag{2}$$

and the results of [2] concerning the equation

$$z = \mathcal{T}_{22}z + g. \tag{3}$$

The general solution of (2) has the form

$$y(t) = Y(t)\alpha + \int_0^t C_1(t,s)f(s)\,ds, \tag{4}$$

with arbitrary $\alpha \in R^{n+mn}$, where $Y(\,\cdot\,)$ is the fundamental matrix, $C_1(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

As for equation (3), it has the immediate analogs of the above terms. Thus, the general solution of (3) has the representation

$$z(t_i) = Z(t_i)\beta + (C_2 g)(t_i), \ \ i = 0, 1, \dots, \mu, \tag{5}$$

with arbitrary $\beta \in R^{\nu}$, where $Z(\,\cdot\,)$ is the fundamental matrix, $C_2(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

As is shown in [7], under the assumption that $\Delta_1 + \Delta_2 \neq 0$, the general solution $x = \begin{pmatrix} y \\ z \end{pmatrix} \in DS^n(m) \times FD^{\nu}(\mu)$ of (1) has the form

$$x = \mathcal{X} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \mathcal{C} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{6}$$

where the fundamental matrix $\mathcal{X}$ is expressed in terms of the fundamental matrices Y and Z by the equality

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix} = \begin{pmatrix} H_{11}Y & H_{12}Z \\ H_{21}Y & H_{22}Z \end{pmatrix}, \tag{7}$$

the Cauchy operator $\mathcal{C}$ is expressed in terms of the Cauchy operators C_1 and C_2:

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} = \begin{pmatrix} H_{11}C_1 & H_{12}C_2 \\ H_{21}C_1 & H_{22}C_2 \end{pmatrix}, \tag{8}$$

$$\begin{gathered} H_{11} = (I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}, \quad H_{12} = -(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}C_1\mathcal{T}_{12}, \\ H_{21} = C_2\mathcal{T}_{21}(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}, \quad H_{22} = I + C_2\mathcal{T}_{21}(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}C_1\mathcal{T}_{12}. \end{gathered} \tag{9}$$

3 General Boundary Value Problem

The general linear BVP is the system (1) supplemented by linear boundary conditions

$$\ell x = \ell \begin{pmatrix} y \\ z \end{pmatrix} = \gamma, \ \ \gamma \in R^N, \tag{10}$$

where $\ell : DS^n(m) \times FD^{\nu}(\mu) \to R^N$ is the linear bounded vector functional. Let us give the representation of ℓ:

$$\ell \begin{pmatrix} y \\ z \end{pmatrix} = \int\limits_0^T \Phi(s)\dot{y}(s)\,ds + \Psi_0 y(0) + \sum_{k=1}^{m} \Psi_k \Delta y(\tau_k) + \sum_{j=0}^{\mu} \Gamma_j z(t_j). \tag{11}$$

Here Ψ_k, $k = 0, 1, \dots, m$, are constant $(N \times n)$-matrices, Γ_j, $j = 0, 1, \dots, \mu$ are constant $(N \times \nu)$-matrices, Φ is $(N \times n)$-matrix with measurable and essentially bounded on $[0, T]$ elements. We assume that the components $\ell_i : DS^n(m) \times FD^{\nu}(\mu) \to R$, $i = 1, \dots, N$ are linearly independent.

BVP (1), (10) is well-defined if $N = n + mn + \nu$. In such a situation, BVP (1), (10) is uniquely solvable for any f, g if and only if the matrix

$$\ell\mathcal{X} = (\ell\mathcal{X}^1, \dots, \ell\mathcal{X}^{n+mn+\nu}), \tag{12}$$

where $\mathcal{X}^j$ is the j-th column of $\mathcal{X}$, is nonsingular, i.e.

$$\det \ell\mathcal{X} \neq 0. \tag{13}$$

Theorem. *Suppose that $N = n + mn + \nu$. Then BVP* (1), (10) *is uniquely solvable for any f, g if and only if* (13) *holds where $N \times N$-matrix $\ell\mathcal{X}$ is defined by* (12), (11), (7), (9)*. In the case that* (13) *takes place, a solution to* (1), (10) *has the representation*

$$x = \begin{pmatrix} y \\ z \end{pmatrix} = \mathcal{X}(\ell\mathcal{X})^{-1}\gamma + \mathcal{G} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{14}$$

where the Green operator $G : L^n \times FD^\nu(\mu) \to DS^n(m) \times FD^\nu(\mu)$ *is defined by the equality*

$$\mathcal{G} = \mathcal{C} - \mathcal{X}(\ell\mathcal{X})^{-1}\ell\mathcal{C}. \tag{15}$$

The representation (15) allows one to study properties of the components $\mathcal{G}_{11}, \dots, \mathcal{G}_{22}$ to

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{pmatrix}.$$

Denote by Λ_{ij}, $i, j = 1, 2$, the components of the operator $\Lambda = (\ell\mathcal{X})^{-1}\ell : DS^n(m) \times FD^\nu(\mu) \to R^{n+mn} \times R^\nu$ and introduce operators $\mathcal{X}^\Lambda_{ij}$, $i, j = 1, 2$, by the equalities

$$\mathcal{X}^\Lambda_{ij} = \sum_{k=1}^{2} \mathcal{X}_{ik}\Lambda_{kj}, \quad i, j = 1, 2.$$

Thus, for $\mathcal{G}_{ij}$, we have

$$\mathcal{G}_{ij} = \mathcal{C}_{ij} - \sum_{k=1}^{2} \mathcal{X}^\Lambda_{ik}\mathcal{C}_{kj}, \quad i, j = 1, 2.$$

With this equality it can be established, in particular, that $\mathcal{G}_{11}$ is an integral operator, and some useful relationships for its kernel can be derived.

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Functions of Bounded Semivariation and the Kurzweil Integral

Giselle Antunes Monteiro

Institute of Mathematics, Academy of Sciences of the Czech Republic, Praha, Czech Republic
E-mail: gam@math.cas.cz

Different notions of variation appear when we deal with problems in infinite dimension. Among them, the semivariation is commonly found in the study of convolution, integral equations and measure differential equations. In the setting of Stieltjes-type integral, the functions of bounded semivariation play an important role and are usually connected with results on the existence of the corresponding integral. In this notes we are particularly interested in its connection with the integral due to Kurzweil (see [4]).

To introduce the definition of a semivariation, we need to recall that, given an interval $[a,b]$, a division of $[a,b]$ is a finite set of the form

$$D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\},\quad a=\alpha_0<\alpha_1<\cdots<\alpha_{\nu(D)}=b,$$

where $\nu(D)\in\mathbb{N}$ corresponds to the number of subintervals in which $[a,b]$ is divided. The set of all finite divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

In what follows, X and Y are Banach spaces and $L(X,Y)$ stands for the Banach space of bounded linear operators from X to Y. By $\|\cdot\|_X$ and $\|\cdot\|_{L(X,Y)}$ we denote the norm in X and the usual operator norm in $L(X,Y)$, respectively.

The *semivariation* of a function $F:[a,b]\to L(X,Y)$ on $[a,b]$ is defined by

$$\mathrm{SV}_a^b(F)=\sup\big\{V(F,D,[a,b]):\ D\in\mathcal{D}[a,b]\big\},$$

where,

$$V(F,D,[a,b])=\sup\bigg\{\Big\|\sum_{j=1}^{\nu(D)}\big[F(\alpha_j)-F(\alpha_{j-1})\big]x_j\Big\|_Y:\ x_j\in X,\ \|x_j\|_X\le 1\bigg\},$$

for $D\in\mathcal{D}[a,b]$. If $\mathrm{SV}_a^b(F)<\infty$, we say that the function F is of bounded semivariation on $[a,b]$. The set of all functions $F:[a,b]\to L(X,Y)$ of bounded semivariation on $[a,b]$ is denoted by $SV([a,b],L(X,Y))$.

It is not hard to see that

$$\mathrm{SV}_a^b(F)\le \mathrm{var}_a^b(F)$$

holds for $F:[a,b]\to L(X,Y)$, where $\mathrm{var}_a^b(F)$ stands for the total variation of F on $[a,b]$, i.e.

$$\mathrm{var}_a^b(F)=\sup\bigg\{\sum_{j=1}^{\nu(D)}\big\|F(\alpha_j)-F(\alpha_{j-1})\big\|_{L(X,Y)}:\ D\in\mathcal{D}[a,b]\bigg\}.$$

Denoting by $BV([a,b],L(X,Y))$ the set of all functions $F:[a,b]\to L(X,Y)$ such that $\mathrm{var}_a^b(F)<\infty$, we have

$$BV([a,b],L(X,Y))\subseteq SV([a,b],L(X,Y)).$$

One can show that semivariation and total variation are equivalent provided the space Y has a finite dimension (see [7]).

The following proposition summarizes some basic properties of functions of bounded semivariation.

Proposition 1. *Let* F, $G \in SV([a,b], L(X,Y))$ *and* $\lambda \in \mathbb{R}$. *Then:*

- F *is bounded on* $[a,b]$*;*
- $\mathrm{SV}_a^b(F+G) \le \mathrm{SV}_a^b(F) + \mathrm{SV}_a^b(G)$ *and* $\mathrm{SV}_a^b(\lambda F) = |\lambda|\,\mathrm{SV}_a^b(F)$*;*
- $\mathrm{SV}_a^b(F) \le \mathrm{SV}_a^c(F) + \mathrm{SV}_c^b(F)$ *for* $c \in [a,b]$.
- *The function* $t \in [a,b] \longmapsto \mathrm{SV}_a^t(F)$ *is nondecreasing.*

As a consequence, it follows that, $SV([a,b], L(X,Y))$ is a linear space. Moreover, it is a Banach space with respect to the norm given by

$$\|F\|_{SV} = \|F(a)\|_{L(X,Y)} + \mathrm{SV}_a^b(F) \ \text{ for } \ F \in SV([a,b], L(X,Y)).$$

In literature we can usually find the semivariation characterized as follows (see, for instance, [1] and [3, 3.6, Chapter I]).

Proposition 2. *The semivariation of a function* $F : [a,b] \to L(X,Y)$ *is given by*

$$\mathrm{SV}_a^b(F) = \sup\Big\{ \mathrm{var}_a^b\,(y^* \circ F) : \ y^* \in Y^*, \ \|y^*\|_{Y^*} \le 1 \Big\}.$$

where, for $y^* \in Y^*$, *the function* $(y^* \circ F) : [a,b] \to X^*$ *is given by*

$$(y^* \circ F)(t)(x) = y^*(F(t)x) \ \text{ for } \ t \in [a,b], \ \ x \in X.$$

Our aim is to present a new characterization of the semivariation by the means of the abstract Kurzweil-Stieltjes integral introduced by Š. Schwabik in [6]. For the reader's convenience, we will recall its definition.

As usual, a partition of $[a,b]$ is a tagged division $P = (\xi, D)$ where $D \in \mathcal{D}[a,b]$ with $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ and $\xi_j \in [\alpha_{j-1}, \alpha_j]$ for $j = 1, \dots, \nu(D)$. Furthermore, given a positive function $\delta : [a,b] \to \mathbb{R}^+$ (called a gauge on $[a,b]$), a partition $P = (\xi, D)$ is said to be δ-fine if

$$[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \ \text{ for } \ j = 1, \dots, \nu(D).$$

Let $F : [a,b] \to L(X)$, $g : [a,b] \to X$ be given. The abstract Kurzweil-Steltjes integral $\int_a^b F\,\mathrm{d}[g]$ exists if there is $I \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\big\|\Sigma(F, \Delta g, P) - I\big\|_X < \varepsilon \ \text{ for all } \delta\text{-fine partitions } P \text{ of } [a,b],$$

where $\Sigma(F, \Delta g, P) = \sum\limits_{j=1}^{\nu(D)} F(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})]$. In such case we write $I = \int_a^b F\,\mathrm{d}[g]$.

Basic properties of the Kurzweil–Stieltjes integral in abstract spaces can be found, for example, in [5] and [6].

We are now ready to state our main result.

Main Theorem. *If* $F \in SV([a,b], L(X))$, *then*

$$\mathrm{SV}_a^b(F) = \sup\bigg\{ \bigg\| F(b)g(b) - \int_a^b F\,\mathrm{d}[g] \bigg\|_X ; \ g \in S_L([a,b], X), \ \|g\|_\infty \le 1 \bigg\},$$

where $S_L([a,b], X)$ *denotes the set of all finite step functions* $g : [a,b] \to X$ *which are left-continuous on* $(a,b]$ *and such that* $g(a) = 0$.

The proof follows closely the ideas presented in [2] where, using the Young integral in Hilbert spaces, an analogous characterization of variation is presented.

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New Convergence Theorem for the Abstract Kurzweil–Stieltjes Integral

Giselle Antunes Monteiro, Umi Mahnuna Hanung, Milan Tvrdý

Institute of Mathematics, Academy of Sciences of the Czech Republic, Praha, Czech Republic
E-mail: gam@math.cas.cz; hanungum@ugm.ac.id; tvrdy@math.cas.cz

In theory of Riemann integral, the impact of the Bounded Convergence Theorem, also called *Arzelà* or *Arzelà–Osgood* or *Osgood Theorem*, is comparable to the importance of the Lebesgue Dominated Convergence Theorem in the theory of the Lebesgue integration. In this work we are concerned with the abstract Kurzweil–Stieltjes integral, that is, the Stieltjes type integral for functions with values in a Banach space introduced by Š. Schwabik in [6]. Our aim is to present the Bounded Convergence Theorem in this abstract setting.

To make our statement more precise we need to fix some notations.

In what follows, X is the Banach space and $L(X)$ is the Banach space of all bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X, while $\|\cdot\|_\infty$ stands for the supremum norm. Furthermore, $BV([a,b],X)$ denotes the set of functions valued in X of bounded variation on $[a,b]$ and $G([a,b],X)$ denotes the set of regulated functions.

Throughout the paper by $\int\limits_a^b \mathrm{d}[F]\,g$ we understand the abstract Kurzweil–Stieltjes integral of $g:[a,b]\to X$ with respect to $F:[a,b]\to L(X)$ in the sense of [6].

Main Theorem (Bounded Convergence Theorem). *Let $g\in G([a,b],X)$, a sequence $\{g_n\}\subset G([a,b],X)$ and $K\in[0,\infty)$ be such that*

$$\lim_{n\to\infty} g_n(t)=g(t) \ \ \textit{for} \ \ t\in[a,b]$$

and

$$\|g_n\|_\infty\le K<\infty \ \ \textit{for} \ \ n\in\mathbb{N}.$$

Then for any $F\in BV([a,b],L(X))$ and $n\in\mathbb{N}$ the integrals $\int\limits_a^b \mathrm{d}[F]\,g$, $\int\limits_a^b \mathrm{d}[F]\,g_n$ exist and

$$\lim_{n\to\infty}\int\limits_a^b \mathrm{d}[F]\,g_n=\int\limits_a^b \mathrm{d}[F]\,g.$$

In the case of real valued functions, the proof of such convergence result is based either on Arzelà's Lemma or on other sophisticated tools (cf. e.g. [2, Theorem II.19.3.14]) that cannot be extended to the case of Banach space-valued functions. Nevertheless, a paper by J. W. Lewin [3], in which an elementary proof of Bounded Convergence Theorem is given for the Riemann integral, offered some enlightenment to this topic.

Our approach is inspired by some of the ideas presented in [3] encompassing some new concepts that we will present below.

Let J be a bounded interval in $\mathbb{R}$. We say that a finite set $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}\subset J$ is a *generalized division of* J if $\alpha_0<\alpha_1<\dots<\alpha_{\nu(D)}$. The set of all generalized divisions of the interval J is denoted by $\mathcal{D}^*(J)$.

Let $f:[a,b]\to X$ and let J be an arbitrary subinterval of $[a,b]$. Then we define the variation of f on J by

$$\operatorname{var}_J f = \sup_{D\in\mathcal{D}^*(J)} \bigg\{ \sum_{j=1}^{\nu(D)} \big\|f(\alpha_j)-f(\alpha_{j-1})\big\|_X \bigg\}.$$

If $\operatorname{var}_J f < \infty$, we say that f is of bounded variation on J and we write $f\in BV(J,X)$. Let us note that for intervals this definition coincides with that used by Gordon in [1]. Furthermore, it is easy to see that it coincides also with the usual (Jordan's) notion of the variation if J is a compact interval.

Making use of the variation on arbitrary intervals we introduce the variation over elementary sets. Recall that a bounded set $E\subset\mathbb{R}$ is an elementary set if it is a finite union of intervals. Moreover, given an elementary set E we can determine a collection of intervals $\{J_k: k=1,\dots,m\}$ such that $E=\bigcup_{k=1}^m J_k$ and the union $J_k\cup J_\ell$ is not an interval whenever $k\neq\ell$. Such collection, called *minimal decomposition*, is uniquely determined and the intervals forming this collection are pairwise disjoint.

Definition. Given a function $f\colon[a,b]\to X$ and an elementary subset E of $[a,b]$, the variation of f over E is

$$\operatorname{var}(f,E)=\sum_{k=1}^m \operatorname{var}_{J_k} f,$$

where $\{J_k\colon k=1,\dots,m\}$ is the minimal decomposition of E.

Now, we present an analogue to Lewin's lemma from [3]. In comparison with Lewin's original version, we replace the Lebesgue measure by the variation of a given function over elementary sets. For the proof, we needed to extend the Jordan's decomposition of functions of bounded variation (for the classical setting, see [2, Theorem I.7.1]) to the abstract setting.

Lemma. *Let $\{A_n\}$ be a sequence of subsets of $[a,b]$ such that $A_{n+1}\subseteq A_n$, $n\in\mathbb{N}$, and $\bigcap_n A_n=\emptyset$. Given $f\in BV([a,b],X)$, for $n\in\mathbb{N}$ put*

$$v_n=\sup\big\{\operatorname{var}(f,E)\colon E \text{ is an elementary subset of } A_n\big\}.$$

Then $\lim\limits_{n\to\infty} v_n = 0$.

In order to apply the previous lemma in the proof of our main result we need to introduce the notion of the Kurzweil–Stieltjes integral over elementary sets.

Definition. Let $F:[a,b]\to L(X)$, $g:[a,b]\to X$ and an elementary subset E of $[a,b]$ be given. The Kurzweil–Stieltjes integral of g with respect to F over E is given by

$$\int_E \mathrm{d}[F]g=\int_a^b \mathrm{d}[F](g\,\chi_E)$$

provided the integral on the right-hand side exists in the sense of [6].

Many basic properties of the integral defined above are immediate consequences of what is known for the abstract Kurzweil–Stieltjes integral, see [4] and [6]. Moreover, the integral over elementary sets in terms of its minimal decomposition can be calculated as follows.

Proposition. *Let $F\in BV([a,b],L(X))$, $g:[a,b]\to X$ and an elementary subset E of $[a,b]$ be such that the integral $\int_E \mathrm{d}[F]\,g$ exists. Then*

$$\int_E \mathrm{d}[F]\,g=\sum_{k=1}^m \int_{J_k} \mathrm{d}[F]\,g,$$

where $\{J_k\colon k=1\dots,m\}$ *is the minimal decomposition of* E.

If we assume in addition that F *is continuous on* $[a,b]$, *then*

$$\bigg\|\int\limits_E \mathrm{d}[F]\,g\bigg\|_X \le \operatorname{var}(F,E)\Big(\sup_{t\in E}\|g(t)\|_X\Big).$$

This proposition together with the analogue of Lewin's lemma mentioned above were the main tools that enabled us to prove the main result of this communication.

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The Focal Boundary Value Problem for Strongly Singular Higher-Order Nonlinear Functional-Differential Equations

S. Mukhigulashvili

Mathematical Institute of the Academy of Sciences of the Czech Republic, Branch in Brno, Brno, Czech Republic;
Ilia State University, Faculty of Physics and Mathematics, Tbilisi, Georgia
E-mail: mukhig@ipm.cz

Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, m), \quad u^{(j-1)}(b) = 0 \ \ (j = m+1, \dots, n). \tag{2}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a,b]$ functions, to the set $L_{loc}(]a,b])$. By $u^{(i-1)}(a)$ we denote the right limit of the function $u^{(i-1)}$ at the point a.

In the paper [2] the Agarwal–Kiguradze type theorems (see [1]) are proved for the focal two-point boundary problem for the linear differential equation with deviating arguments, which guarantee Fredholm's property for such problems. Here, on the basis of previous papers we prove a priori boundedness principle for the problem (1), (2) from which follow several sufficient conditions of solvability of this problem.

We use the following notations.

$R^+ = [0, +\infty[$;

$[x]_+$ is the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a,b])$ is the space of functions $y :]a,b] \to R$, which are integrable on $[a+\varepsilon, b]$ for arbitrary small $\varepsilon > 0$;

$L_\alpha(]a,b])$ ($L^2_\alpha(]a,b])$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha$ functions $y :]a,b] \to R$, with the norm

$$\|y\|_{L_\alpha} = \int_a^b (s-a)^\alpha |y(s)|\,ds \quad \bigg(\|y\|_{L^2_\alpha} = \bigg(\int_a^b (s-a)^\alpha y^2(s)\,ds \bigg)^{1/2} \bigg);$$

$M(]a,b])$ is the set of the measurable functions $\tau :]a,b] \to]a,b]$;

$\widetilde{L}^2_\alpha(]a,b])$ is the Banach space of $y \in L_{loc}(]a,b])$ functions, with the norm

$$\|y\|_{\widetilde{L}^2_\alpha} \equiv \max\bigg\{ \bigg[\int_a^t (s-a)^\alpha \bigg(\int_s^t y(\xi)\,d\xi \bigg)^2 ds \bigg]^{1/2} : \ a \leq t \leq b \bigg\};$$

$L_n(]a,b])$ is the Banach space of $y \in L_{loc}(]a,b])$ functions, with the norm

$$\|y\|_{L_n} = \sup\bigg\{ (s-a)^{m-1/2} \int_s^t (\xi-a)^{n-2m} |y(\xi)|\,d\xi : \ a < s \leq t \leq b \bigg\} < +\infty;$$

$\widetilde{C}^{n-1}_{loc}(]a,b])$ is the space of the functions $y:]a,b]\to R$, which are continuous (absolutely continuous) together with $y',y'',\dots,y^{(n-1)}$ on $[a+\varepsilon,b]$ for arbitrarily small $\varepsilon>0$;

$\widetilde{C}^{n-1,m}(]a,b])$ is the space of the functions $y\in\widetilde{C}^{n-1}_{loc}(]a,b])$ such that

$$\int\limits_a^b |x^{(m)}(s)|^2\,ds<+\infty;$$

$C^{m-1}_1(]a,b])$ is the Banach space of the functions $y\in C^{m-1}_{loc}(]a,b])$ such that

$$\limsup_{t\to a}\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}<+\infty\ \ (i=1,\dots,m),$$

with the norm:

$$\|x\|_{C^{m-1}_1}=\sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}}:\ a<t\le b\Big\};$$

$\widetilde{C}^{m-1}_1(]a,b])$ is the Banach space of the functions $y\in\widetilde{C}^{m-1}_{loc}(]a,b])$ such that:

$$\|x\|_{\widetilde{C}^{m-1}_1}=\|x\|_{C^{m-1}_1}+\bigg(\int\limits_a^b |x^{(m)}(s)|^2\,ds\bigg)^{1/2};$$

$D_n(]a,b]\times R^+)$ is the set of such functions $\delta:]a,b]\times R^+\to L_n(]a,b])$ that $\delta(t,\cdot\,):R^+\to R^+$ is nondecreasing for every $t\in]a,b]$, and $\delta(\,\cdot\,,\rho)\in L_n(]a,b])$ for any $\rho\in R^+$. A solution of problem (1), (2) is sought in the space $\widetilde{C}^{n-1,m}(]a,b])$.

Define the operator $P:C^{m-1}_1(]a,b])\times C^{m-1}_1(]a,b])\to L_{loc}(]a,b])$, by the equality

$$P(x,y)(t)=\sum_{j=1}^m p_j(x)(t)y^{(j-1)}(\tau_j(t))\ \text{ for }\ a<t\le b,$$

where $p_j:C^{m-1}_1(]a,b])\to L_{loc}(]a,b])$, and $\tau_j\in M(]a,b])$. Also for any $\gamma>0$ define the set A_γ by the relation $A_\gamma=\big\{x\in\widetilde{C}^{m-1}_1(]a,b]):\ \|x\|_{\widetilde{C}^{m-1}_1}\le\gamma\big\}$.

Following the article [2] of Kiguradze and Půža, we introduce the following definitions.

Definition 1. Let γ_0 and γ be the positive numbers. We say that the continuous operator $P:C^{m-1}_1(]a,b])\times C^{m-1}_1(]a,b])\to L_n(]a,b])$ is γ_0, γ consistent with boundary condition (2) if:

(i) for any $x\in A_{\gamma_0}$ and almost all $t\in]a,b]$ the inequality

$$\sum_{j=1}^m \big|p_j(x)(t)x^{(j-1)}(\tau_j(t))\big|\le\delta\big(t,\|x\|_{\widetilde{C}^{m-1}_1}\big)\|x\|_{\widetilde{C}^{m-1}_1}\tag{3}$$

holds, where $\delta\in D_n(]a,b]\times R^+)$.

(ii) for any $x\in A_{\gamma_0}$ and $q\in\widetilde{L}^2_{2n-2m-2}(]a,b])$ the equation

$$y^{(n)}(t)=\sum_{j=1}^m p_j(x)(t)y^{(j-1)}(\tau_j(t))+q(t)$$

under boundary conditions (2), has the unique solution y in the space $\widetilde{C}^{n-1,m}(]a,b])$ and $\|y\|_{\widetilde{C}^{m-1}_1}\le\gamma\|q\|_{\widetilde{L}^2_{2n-2m-2}}$.

Definition 2. We say that the operator P is γ consistent with boundary condition (2), if the operator P is γ_0, γ consistent with boundary condition (2) for any $\gamma_0 > 0$.

In the sequel, it is assumed that the operator F_p is defined by the equality

$$F_p(x)(t) = \Big| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t) x^{(j-1)}(\tau_j(t))(t) \Big|,$$

continuously acting from $C_1^{m-1}(]a,b])$ to $L_{\widetilde{L}^2_{2n-2m-2}}(]a,b])$, and $\widetilde{F}_p(t,\rho) \equiv \sup\{F_p(x)(t) : \|x\|_{C_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ for each $\rho \in [0,+\infty[$. Then the following theorem is valid.

Theorem 1. *Let the operator P be γ_0, γ consistent with boundary condition* (2), *and there exist a positive number $\rho_0 \leq \gamma_0$ such that*

$$\big\| \widetilde{F}_p\big(\,\cdot\,, \min\{2\rho_0, \gamma_0\}\big) \big\|_{\widetilde{L}^2_{2n-2m-2}} \leq \gamma_0/\gamma.$$

Let moreover, for any $\lambda \in]0,1[$, an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t)$$

under the conditions (2), *admit the estimate $\|x\|_{\widetilde{C}_1^{m-1}} \leq \rho_0$. Then problem* (1), (2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,b])$.*

From this theorem it follow different theorems with efficient sufficient conditions for the solvability of problem (1), (2). Here we give one of them. Define the operators $h_j : C_1^{m-1}(]a,b]) \times]a,b] \times]a,b] \to L_{loc}(]a,b] \times]a,b])$, $f_j : C_1^{m-1}(]a,b]) \times [a,b] \times M(]a,b]) \to C_{loc}(]a,b] \times]a,b])$ $(j=1,\dots,m)$ by the equalities

$$h_1(x,t,s) = \Big| \int_s^t (\xi-a)^{n-2m} \Big[(-1)^{n-m} p_1(x)(\xi)\Big]_+ \, d\xi \Big|,$$

$$h_j(x,t,s) = \Big| \int_s^t (\xi-a)^{n-2m} p_j(x)(\xi)\, d\xi \Big|, \quad j=\overline{2,m},$$

$$f_j(x,c,\tau_j)(t,s) = \Big| \int_s^t (\xi-a)^{n-2m} |p_j(x)(\xi)| \Big| \int_{\xi}^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}\, d\xi_1 \Big|^{1/2} d\xi \Big|.$$

Then the following theorem is true.

Theorem 2. *Let the continuous operator $P : C_1^{m-1}(]a,b]) \times C_1^{m-1}(]a,b]) \to L_n(]a,b])$ admit the condition* (3) *where $\delta \in D_n(]a,b] \times R^+)$, $\tau_j \in M(]a,b])$ and the numbers $\gamma_0 \in]a,b]$, $l_j > 0$, $\bar{l}_j > 0$, $\gamma_j > 0$ $(j=1,\dots,m)$ be such that the inequalities*

$$(t-a)^{2m-j} h_j(x,t,s) \leq l_j, \quad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_j} f_j(x,a,\tau_j)(t,s) \leq \bar{l}_j$$

for $a < t \leq s \leq b$, $\|x\|_{\widetilde{C}_1^{m-1}} \leq \gamma_0$

hold. Let, moreover, the operator F and function $\eta \in D_{2n-2m-2}(]a,b] \times R^+)$ be such that condition

$$\Big| F(x)(t) - \sum_{j=1}^{m} p_j(x)(t) x^{(j-1)}(\tau_j(t))(t) \Big| \leq \eta\big(t, \|x\|_{\widetilde{C}_1^{m-1}}\big)$$

and inequality

$$\|\eta(\,\cdot\,,\gamma_0)\|_{\widetilde{L}^2_{2n-2m-2}} < \frac{\gamma_0}{r_n}$$

are fulfilled, where

$$r_n = \Big(1 + \sum_{j=1}^{m} \frac{(2m-2j+1)^{-1/2}}{(m-j)!}\Big)\,\frac{2^{m-1}(2n-2m-1)}{(\nu_n - B)(2m-1)!!}\,.$$

Then problem (1), (2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

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Positive Solutions of Nonlinear Boundary Value Problems for Singular in Phase Variables Two-Dimensional Differential Systems

Nino Partsvania

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: `ninopa@rmi.ge`

Bedřich Půža

Institute of Informatics, Faculty of Business and Management, Brno University of Technology, Brno, Czech Republic
E-mail: `puza@fbm.vutbr.cz`

Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_{0+} =]0, +\infty[$, and $f_i : [0, a] \times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ be measurable in the first and continuous in the last two arguments functions.

Consider the two-dimensional differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \;\; (i = 1, 2) \tag{1}$$

with the nonlinear boundary conditions

$$\int\limits_0^a \varphi(s, u_1(s))\, d\sigma(s) = c, \quad u_2(a) = \psi(u_1(a)), \tag{2}$$

where $c \geq 0$, $\varphi : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing in the second argument function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, and $\sigma : [0, a] \to \mathbb{R}$ is a nondecreasing function such that

$$\sigma(a) - \sigma(0) = 1.$$

An absolutely continuous vector function $(u_1, u_2) : [0, a] \to \mathbb{R}^2_+$ is said to be **a positive solution of the differential system** (1) if it satisfies the inequalities

$$u_i(t) > 0 \;\text{ for }\; 0 < t < a \;\; (i = 1, 2),$$

and almost everywhere on $]0, a[$ the equalities (1) are fulfileed.

A positive solution of the system (1) satisfying the conditions (2) is said to be **a positive solution of the problem** (1), (2).

We investigate the problem (1), (2) in the case where the functions f_i $(i = 1, 2)$ on the set $]0, a[\times \mathbb{R}^2_{0+}$ admit the estimates

$$\begin{gathered} g_{10}(t) \leq -x^{\lambda_1} y^{-\mu_1} f_1(t, x, y) \leq g_1(t), \\ g_{20}(t) \leq -x^{\lambda_2} y^{\mu_2} f_2(t, x, y) \leq g_2(t), \end{gathered} \tag{3}$$

where λ_i and μ_i $(i = 1, 2)$ are non-negative constants, and $g_{i0} :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$, $g_i :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$ are integrable functions.

If $\lambda_i > 0$ for some $i \in \{1, 2\}$, then in view of (3) we have

$$\lim_{x \to 0} f_i(t, x, y) = +\infty \;\text{ for }\; 0 < t < a, \;\; y > 0.$$

And if $\mu_2 > 0$, then

$$\lim_{y\to 0} f_2(t,x,y) = +\infty \ \text{ for } \ 0 < t < a, \ \ x > 0.$$

Consequently, in both cases the system (1) has the singularity in at least one phase variable.

Boundary value problems for singular in phase variables second order nonlinear differential equations arise in different fields of natural science and are the subject of numerous studies (see e.g. [1], [3]–[7] and the references therein). In the recent paper by I. Kiguradze [2], optimal conditions are obtained for the solvability of the Cauchy–Nicoletti type nonlinear problems for singular in phase variables differential systems. As for the problems of the type (1), (2), they still remain unstudied in the above-mentioned singular cases.

Let

$$\nu_0 = \frac{\mu_1}{1+\mu_2}\,, \quad \nu = 1 + \lambda_1 + \lambda_2\nu_0.$$

On the set $\big\{(t,x,y):\ 0 \le t \le a,\ x > 0,\ y \ge 0\big\}$ we introduce the functions

$$w_0(t,x,y) = \bigg[x^\nu + \nu\int_t^a g_{10}(s)\bigg(x^{\lambda_2}y^{1+\mu_2} + (1+\mu_2)\int_s^a g_{20}(\tau)\,d\tau\bigg)^{\nu_0} ds\bigg]^{\frac{1}{\nu}},$$

$$w(t,x,y) = \bigg[y^{1+\mu_2} + (1+\mu_2)\int_t^a w_0^{-\lambda_2}(s,x,y)g_2(s)\,ds\bigg]^{\frac{1}{1+\mu_2}},$$

$$w_1(t,x,y) = \bigg[x^{1+\lambda_1} + (1+\lambda_1)\int_t^a w^{\mu_1}(s,x,y)g_1(s)\,ds\bigg]^{\frac{1}{1+\lambda_1}},$$

Theorem 1. *Let*

$$\lim_{x\to+\infty} \varphi(t,x) = +\infty \ \ \textit{uniformly with respect to } t \in [0,a],$$

and let for some $\delta > 0$ the inequality

$$c \ge \int_0^a \varphi\big(s, w_1(s,\delta,\psi(\delta)\big)\,d\sigma(s)$$

hold. Then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If*

$$c < \int_0^a \varphi\big(s, w_0(s,0,0)\big)\,d\sigma(s),$$

then the problem (1), (2) *has no positive solution.*

The particular cases of (2) are the nonlocal boundary conditions

$$\sum_{k=1}^m \ell_k u^{\mu_k}(a_k) = c, \quad u_2(a) = \psi(u_1(a)), \tag{4}$$

where $\ell_k > 0$, $\mu_k > 0$, $0 \le a_k \le a$ $(k = 1,\dots,m)$.

Theorems 1 and 2 imply the following corollary.

Corollary 1. *If for some $\delta > 0$ the inequality*

$$c \geq \sum_{k=1}^{m} \ell_k w_1\big(a_k, \delta, \psi(\delta)\big)$$

holds, then the problem (1), (4) *has at least one positive solution. And if*

$$c < \sum_{k=1}^{m} \ell_k w_0(a_k, 0, 0),$$

then the problem (1), (4) *has no positive solution.*

Corollary 2. *For an arbitrary $c > 0$, the differential system* (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = c, \quad u_2(a) = 0.$$

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Approximate Optimal Control in the Feedback Form for Impulse Parabolic Problem with Quickly-Oscillating Coefficients

Mykola Perestyuk, Oleksiy Kapustyan, Alina Rusina

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

E-mail: pmo@univ.kiev.ua; kapustyanav@gmail.com; rusina.alina@gmail.com

One of the most important problems of the optimal control theory of distributed parameter systems is to obtain synthesis, i.e. optimal control in the feedback form. Wide class of both distributed and lumped control systems without constraints was studied in the earliest works. In this paper, optimal control function which has been expressed in the feedback form is found for the parabolic equations with quickly-oscillating coefficients and controlled impulsive perturbation at a fixed time. The exact formula for the synthesis was found and its approximate form that lies in substitution of quickly-oscillating parameters with homogenized and infinite sum with finite was justified.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $\varepsilon\in(0,1)$ be a small parameter, $Q=(0,T)\times\Omega$, $\theta\in(0,T)$, be a fixed time of impulsive perturbation, $a\neq-1$, $b\in\mathbb{R}$, $c>0$, $d>0$, be fixed.

We consider the parabolic equations with quickly-oscillating coefficients and controlled impulse at a fixed time

$$\begin{cases}\dfrac{\partial y}{\partial t}=A^\varepsilon y+u(t,x), \quad (t,x)\in Q,\\ y\big|_{\partial\Omega}=0,\\ y\big|_{t=0}=y_0^\varepsilon,\end{cases}\tag{1}$$

$$y(\theta+0,x)-y(\theta,x)=ay(\theta,x)+bw(x) \ \text{ for almost all } \ x\in\Omega,\tag{2}$$

$$J(y,u,w)=\int\limits_\Omega y^2(T,x)\,dx+c\int\limits_Q u^2(t,x)\,dt\,dx+d\int\limits_\Omega w^2(x)\,dx\to\inf,\tag{3}$$

where $A^\varepsilon=\operatorname{div}(a^\varepsilon\nabla)$, $a^\varepsilon(x)=a(\frac{x}{\varepsilon})$, a is measurable, periodic matrix which satisfies the conditions of uniform ellipticity and boundedness: $\exists\, v_1>0$, $v_2>0$, $\forall\,\eta\in\mathbb{R}^n$,

$$v_1\sum_{i=1}^n\eta_i^2\le\sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j\le v_2\sum_{i=1}^n\eta_i^2.\tag{4}$$

Let $\{X_i^\varepsilon\}$, $\{\lambda_i^\varepsilon\}$ be solutions of the spectral problem

$$\begin{cases}A^\varepsilon X_i^\varepsilon=-\lambda_i^\varepsilon X_i^\varepsilon,\\ X_i^\varepsilon\big|_{\partial\Omega}=0,\end{cases}\tag{5}$$

$\{X_i^\varepsilon\}\subset H_0^1(\Omega)$ is a orthonormalized basis in $L^2(\Omega)$, $0<\lambda_1^\varepsilon\le\lambda_2^\varepsilon\le\cdots$, $\lambda_i^\varepsilon\to\infty$, $i\to\infty$.

A norm and a scalar product in $L^2(\Omega)$ are denoted by $\|\cdot\|$ i $(\cdot\,,\,\cdot)$ respectively.

We are looking for a solution of (1)–(3) in the following form

$$y^\varepsilon(t,x)=\sum_{i=1}^\infty y_i^\varepsilon(t)X_i^\varepsilon(x), \quad u^\varepsilon(t,x)=\sum_{i=1}^\infty u_i^\varepsilon(t)X_i^\varepsilon(x), \quad w^\varepsilon(x)=\sum_{i=1}^\infty w_i^\varepsilon X_i^\varepsilon(x).$$

Then we obtain a countable system of one-dimensional impulse optimal control problems:

$$\begin{cases} \dfrac{d}{dt}\,y_i^\varepsilon(t) = -\lambda_i^\varepsilon y_i^\varepsilon + u_i^\varepsilon(t), \\ y_i^\varepsilon(0) = y_i^{0\varepsilon}, \\ y_i^\varepsilon(\theta+0) - y_i^\varepsilon(\theta) = a y_i^\varepsilon(\theta) + b w_i^\varepsilon, \\ (y_i^\varepsilon(T))^2 + c\displaystyle\int_0^T (u_i^\varepsilon(t))^2\,dt + d(w_i^\varepsilon)^2 \to \inf, \end{cases} \tag{6}$$

where $y_i^{0\varepsilon} = (y_0^\varepsilon, X_i^\varepsilon)$. By using method of Pontryagin's Maximum Principle [2] we obtain that problem (6) has the unique solution

$$\forall\, t \in [0,T], \quad u_i^\varepsilon(t) = -\frac{(a+1)dy_i^{0\varepsilon}e^{-\lambda_i^\varepsilon T}K_i^\varepsilon(t)}{cd + cb^2e^{-2\lambda_i^\varepsilon(T-\theta)} + d\int_0^T (K_i^\varepsilon(t))^2\,dt}\,, \tag{7}$$

$$\forall\, t \in [0,T], \quad w_i^\varepsilon = \frac{bce^{-\lambda_i^\varepsilon(T-\theta)}}{dK_i^\varepsilon(t)}\,u_i^\varepsilon(t), \tag{8}$$

where

$$K_i^\varepsilon(t) = \begin{cases} (a+1)e^{-\lambda_i^\varepsilon T}e^{\lambda_i^\varepsilon t}, & t \in [0,\theta], \\ e^{-\lambda_i^\varepsilon T}e^{\lambda_i^\varepsilon t}, & t \in (\theta, T]. \end{cases}$$

Denote

$$\alpha_i^\varepsilon(t) = -\frac{(a+1)de^{-\lambda_i^\varepsilon T}K_i^\varepsilon(t)}{cd + cb^2e^{-2\lambda_i^\varepsilon(T-\theta)} + d\int_0^T (K_i^\varepsilon(t))^2\,dt}\,.$$

Thus we obtain a control in the feedback form

$$u_i^\varepsilon[t, y_i^\varepsilon(t)] = \beta_i^\varepsilon(t)y_i^\varepsilon(t), \tag{9}$$

$$w_i^\varepsilon[y_i^\varepsilon(\theta)] = \frac{bc}{d(a+1)}\,\beta_i^\varepsilon(\theta)y_i^\varepsilon(\theta), \tag{10}$$

where

$$\beta_i^\varepsilon(t) = e^{\lambda_i^\varepsilon t}\alpha_i^\varepsilon(t)\bigg(1 + \int_0^t e^{\lambda_i^\varepsilon s}\alpha_i^\varepsilon(s)\,ds\bigg)^{-1}, \ \text{ when } \ t \in [0,\theta],$$

$$\beta_i^\varepsilon(t) = e^{\lambda_i^\varepsilon t}\alpha_i^\varepsilon(t)\bigg(a+1+(a+1)\int_0^\theta e^{\lambda_i^\varepsilon s}\alpha_i^\varepsilon(s)\,ds + \frac{b^2c\alpha_i^\varepsilon(\theta)}{(a+1)d}\,e^{\lambda_i^\varepsilon\theta} + \int_\theta^t e^{\lambda_i^\varepsilon s}\alpha_i^\varepsilon(s)\,ds\bigg)^{-1}, \quad \text{when } \ t \in (\theta, T]. \tag{11}$$

Note that β_i^ε is uniformly bounded on $[0,T]$

$$\exists\,\beta > 0 \ \ \forall\, i \geq 1, \ \ \forall\,\varepsilon \in (0,1), \ \ \sup_{t\in[0,T]} |\beta_i^\varepsilon(t)| \leq \beta. \tag{12}$$

Thus, the synthesis of a problem (1)–(3) is determinate by

$$u^\varepsilon\big[t,x,y^\varepsilon(t,x)\big]=\sum_{i=1}^\infty \beta_i^\varepsilon(t)\big(y^\varepsilon(t),X_i^\varepsilon\big)X_i^\varepsilon(x), \tag{13}$$

$$w^\varepsilon\big[x,y^\varepsilon(\theta,x)\big]=\sum_{i=1}^\infty \gamma_i^\varepsilon\big(y^\varepsilon(\theta),X_i^\varepsilon\big)X_i^\varepsilon(x), \tag{14}$$

where β_i^ε is defined by (11), and

$$\gamma_i^\varepsilon=\frac{bc}{d(a+1)}\,\beta_i^\varepsilon(\theta).$$

Let us construct approximate homogenized synthesis. Let constant matrix a^0 be homogenized for $a(\frac{x}{\varepsilon})$, $A^0=\operatorname{div}(a^0\nabla)$, $\{\lambda_i^0\}$, $\{X_i^0\}$ be solutions of the appropriate spectral problem

$$\begin{cases} A^0X_i^0=-\lambda_i^0X_i^0,\\ X_i^0\big|_{\partial\Omega}=0,\end{cases}$$

where spectrum of A^0 is simple

$$0<\lambda_1^0<\lambda_2^0<\cdots<\lambda_k^0<\cdots,\ \ \lambda_i^0\to\infty,\ \ i\to\infty. \tag{15}$$

Then for all $i\geq 1$ limits hold [3]

$$\lambda_i^\varepsilon\to\lambda_i^0,\ \ X_i^\varepsilon\to X_i^0\ \text{ in }\ L^2(\Omega)\ \text{ when }\ \varepsilon\to 0. \tag{16}$$

We assume that

$$y_0^\varepsilon\to y_0\ \text{ weekly in }\ L^2(\Omega)\ \text{ when }\ \varepsilon\to 0. \tag{17}$$

From (11) we obtain that for all $t\in[0,T]$ and all $i\geq 1$,

$$\beta_i^\varepsilon(t)\to\beta_i^0(t),\ \ \gamma_i^\varepsilon\to\gamma_i^0=\frac{bc}{d(a+1)}\,\beta_i^0(\theta)\ \text{ for }\ \varepsilon\to 0, \tag{18}$$

where $\beta_i^0(t)$ is defined by (11) and substitute λ_i^ε with λ_i^0.

Let us consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t}=A^\varepsilon y+u_N^0[t,x,y], \quad (t,x)\in Q,\\ y\big|_{\partial\Omega}=0,\\ y\big|_{t=0}=y_0^\varepsilon,\end{cases} \tag{19}$$

$$y(\theta+0,x)-y(\theta,x)=ay(\theta,x)+bw_N^0[x,y(\theta)]\ \text{ for almost all }\ x\in\Omega, \tag{20}$$

where for $y\in L^2(Q)$, $z\in L^2(\Omega)$,

$$u_N^0[t,x,y]=\sum_{i=1}^N \beta_i^0(t)(y(t),X_i^0)X_i^0(x),$$

$$w_N^0[x,z]=\sum_{i=1}^N \gamma_i^0(z,X_i^0)X_i^0(x).$$

From (12)

$$\forall\, t\in[0,T],\ \ \|u_N^0[t,x,y]\|\leq\beta\|y\|, \tag{21}$$

the problem (19) has a unique solution in $W(0,\theta)$ [4]. Whereas $W(0,\theta)\subset C([0,\theta];L^2(\Omega))$ and the inequality holds

$$\big\|w_N^0[x,z]\big\| \le \frac{bc}{d(a+1)}\,\beta\|z\|, \tag{22}$$

then there exist a unique solution $y=y_N^\varepsilon(t,x)\in W_\theta(0,T)$ of the impulse problem (19), (20).

The main result of this paper is the following theorem.

Theorem. *Let the above assumptions be satisfied. Then for all $\eta>0$, $\exists\,\overline{N}\ge 1$, $\overline{\varepsilon}\in(0,1)$ such that $\forall\,N\ge\overline{N}$, $\forall\,\varepsilon\in(0,\overline{\varepsilon})$,*

$$\Big|J(y^\varepsilon,u^\varepsilon,w^\varepsilon)-J\Big(y_N^\varepsilon,u_N^0\big[t,x,y_N^\varepsilon(t,x)\big],w_N^0\big[x,y_N^\varepsilon(\theta,x)\big]\Big)\Big|<\eta, \tag{23}$$

where $\{y^\varepsilon,u^\varepsilon,w^\varepsilon\}$ is an optimal process for the problem (1)–(3), y_N^ε is a solution of the problem (19), (20).

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Fixed Point Problem Associated with State-Dependent Impulsive Boundary Value Problems

Irena Rachůnková, Jan Tomeček

Faculty of Science, Palacký University, Olomouc, Czech Republic
E-mail: irena.rachunkova@upol.cz; jan.tomecek@upol.cz

1 Formulation of Problem

In the literature most of impulsive boundary value problems deals with impulses at fixed times. This is the case that moments, where impulses act in state variables, are known. The theory of these impulsive problems is widely developed and presents direct analogies with methods and results for problems without impulses. A different situation arises, when impulse moments satisfy a predetermined relation between state and time variables. This case, which is represented by state-dependent impulses, is studied here, where we are interested in a system of n ($n \in \mathbb{N}$) nonlinear ordinary differential equations of the first order with state-dependent impulses and general linear boundary conditions on the interval $[a,b] \subset \mathbb{R}$. The main reason that boundary value problems with state-dependent impulses are developed significantly less than those with impulses at fixed moments is that new difficulties with an operator representation of the problem appear when examining state-dependent impulses. Therefore almost all existence results for boundary value problems with state-dependent impulses have been reached for periodic problems which can be transformed to fixed point problems of corresponding Poincaré maps in $\mathbb{R}^n$. Hence, the difficulties with a construction of a functional space and an operator have been cleared in the periodic case. Other types of boundary value problems with state-dependent impulses have been studied very rarely.

We construct and investigate a fixed point problem in some subset Ω of the Sobolev space $(\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n))^{p+1}$ and we provide conditions for its solvability. The existence of such fixed point allows us to construct a solution of the system of differential equations

$$z'(t) = f(t, z(t)), \quad \text{a.e. } t \in [a,b] \subset \mathbb{R}, \tag{1}$$

subject to the state-dependent impulse conditions

$$z(t+) - z(t-) = J_i(t, z(t-)), \ \text{ where } \ t = \gamma_i(z(t-)), \ \ i = 1, \dots, p, \tag{2}$$

and the general linear boundary condition

$$\ell(z) = c_0. \tag{3}$$

Problem (1)–(3) is studied under the assumptions

$$\left.\begin{aligned}
& n \geq 2, \ \ f \in \mathrm{Car}([a,b] \times \mathbb{R}^n; \mathbb{R}^n), \\
& c_0 \in \mathbb{R}^n, \ \ J_i \in \mathbb{C}([a,b] \times \mathbb{R}^n; \mathbb{R}^n), \ \ \gamma_i \in \mathbb{C}(\mathbb{R}^n; \mathbb{R}), \ \ i = 1, \dots, p, \\
& \ell : \mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^n) \to \mathbb{R}^n \ \text{ is a linear bounded operator, i.e.} \\
& \ell(z) = Kz(a) + \int_a^b V(t)\, \mathrm{d}[z(t)], \ \ z \in \mathbb{G}_{\mathrm{L}}([a,b]; \mathbb{R}^n), \\
& \text{where } \ K \in \mathbb{R}^{n \times n}, \ \ V \in \mathbb{BV}([a,b]; \mathbb{R}^{n \times n}), \ \ k = 1, \dots, n, \ \ n, p \in \mathbb{N}.
\end{aligned}\right\} \tag{4}$$

$\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ is a Banach space (equipped with the sup-norm) of left-continuous regulated (i.e. having finite one-sided limits at each point) on $[a,b]$ vector valued functions.

A mapping $z:[a,b]\to\mathbb{R}^n$ is a *solution* of problem (1)–(3) if for each $i\in\{1,\dots,p\}$ there exists a unique $\tau_i\in(a,b)$ such that

$$\tau_i=\gamma_i(z(\tau_i)),$$

$a<\tau_1<\tau_2<\dots<\tau_p<b$, the restrictions $z\big|_{[a,\tau_1]}, z\big|_{(\tau_1,\tau_2]},\dots,z\big|_{(\tau_p,b]}$ are absolutely continuous, z satisfies (1) for a.e. $t\in[a,b]$ and fulfils conditions (2) and (3).

2 Transversality Conditions

First, let us formulate conditions which guarantee that each possible solution of problem (1)–(3) in some region crosses each barrier γ_i at the unique impulse point τ_i, $i=1,\dots,p$. To this end consider positive real numbers μ_j, $j=1,\dots,n$, and denote

$$A=\Big\{(x_1,\dots,x_n)^T\in\mathbb{R}^n:\ |x_j|\le\mu_j,\ j=1,\dots,n\Big\}. \tag{5}$$

We assume that

$$\left.\begin{array}{l}\text{there exist disjoint subintervals } [a_i,b_i] \text{ of the interval } (a,b) \text{ such that}\\ a_1<\dots<a_p,\ \ a_i\le\gamma_i(x)\le b_i,\ \ i=1,\dots,p,\ \ x\in A,\end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l}\text{for each } i\in\{1,\dots,p\},\ \ j\in\{1,\dots,n\} \text{ there exists } \lambda_{ij}\in[0,\infty) \text{ such that}\\ \text{for each } x=(x_1,\dots,x_n)^T,\ \ y=(y_1,\dots,y_n)^T\in A\\ \big|\gamma_i(x)-\gamma_i(y)\big|\le\displaystyle\sum_{j=1}^n\lambda_{ij}|x_j-y_j|.\end{array}\right\} \tag{7}$$

Further we choose positive real numbers ρ_j, $j=1,\dots,n$, such that

$$\sum_{j=1}^n\lambda_{ij}\rho_j<1,\ \ i=1,\dots,p. \tag{8}$$

Under conditions (5)–(8), which we call *transversality conditions*, we can define the set

$$B=\Big\{u=(u_1,\dots,u_n)^T\in\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n):\ \|u_j\|_\infty\le\mu_j,\ \|u_j'\|_\infty\le\rho_j,\ j=1,\dots,n\Big\} \tag{9}$$

and prove that for each $i\in\{1,\dots,p\}$, the functional

$$\mathcal{P}_i:B\to(a,b),\ \ \mathcal{P}_iu=\tau_i, \tag{10}$$

is continuous. Here, for given $u\in B$, $\tau_i\in(0,T)$ is a unique root of the function $\gamma_i(u(t))-t$.

3 Fixed Point Problem and Existence Results

One of the basic results in our approach is a connection between a (discontinuous) solution z of problem (1)–(3) and a fixed point $(u_1,\dots,u_{p+1})$ of an operator $\mathcal{G}$ which operates on the set

$$\Omega=B^{p+1}\subset X=\big(\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n)\big)^{p+1}.$$

The space X, equipped with the norm $\|(u_1,\dots,u_{p+1})\|_X=\sum\limits_{k=1}^{p+1}\|u_k\|_{1,\infty}$ for $(u_1,\dots,u_{p+1})\in X$, is a Banach space. Under the assumptions

$$\det K\neq0, \tag{11}$$

$$\exists\,\widetilde f\in\mathbb{R}:\ |f(t,x)|\le\widetilde f\quad\text{for a.e. } t\in[a,b],\ \text{ all } x\in\mathbb{R}^n, \tag{12}$$

we define the operator $\mathcal{F}^* : \Omega \to X$,

$$(\mathcal{F}^*u)_k(t) = \int\limits_a^b G(t,s) \sum_{i=1}^{p+1} \chi_{(\tau_{i-1},\tau_i)}(s) f(s,u_i(s))\,\mathrm{d}s + \sum_{i=k}^{p} G_1(t,\tau_i) J_i(\tau_i,u_i(\tau_i)) + \\ + \sum_{i=1}^{k-1} G_2(t,\tau_i) J_i(\tau_i,u_i(\tau_i)) + K^{-1}c_0, \ \ k = 1,\dots,p+1, \tag{13}$$

where

$$\tau_i = \mathcal{P}_i u_i, \ \ i = 1,\dots,p, \ \ \tau_0 = a, \ \ \tau_{p+1} = b, \tag{14}$$
$$G_1(t,\tau) = -K^{-1}V(\tau), \quad G_2(t,\tau) = -K^{-1}V(\tau) + I, \ \ t,\tau \in [a,b],$$
$$G(t,\tau) = \begin{cases} G_1(t,\tau), & a \le t \le \tau \le b, \\ G_2(t,\tau), & a \le \tau < t \le b, \end{cases}$$

and K and V are from (4), I is the identity matrix. Unfortunatelly, $\mathcal{F}^*$ is not compact on Ω. We can overcome this obstacle by redefining the operator $\mathcal{F}^*$ by means of the operator $\mathcal{G} : \Omega \to X$,

$$(\mathcal{G}u)_k(t) = \begin{cases} (\mathcal{F}^*u)_k(\tau_{k-1}) + \int\limits_{\tau_{k-1}}^{t} f(s,u_k(s))\,\mathrm{d}s, & t < \tau_{k-1}, \\ (\mathcal{F}^*u)_k(t), & \tau_{k-1} \le t \le \tau_k, \\ (\mathcal{F}^*u)_k(\tau_k) + \int\limits_{\tau_k}^{t} f(s,u_k(s))\,\mathrm{d}s, & t > \tau_k, \ \ k = 1,\dots,p+1. \end{cases} \tag{15}$$

Now we are ready to prove the following theorem.

Theorem 1. *Let $\mu_j \in \mathbb{R}$, A be given by* (5)*, and let γ_i, λ_{ij} and ρ_j, $i = 1,\dots,p$, $j = 1,\dots,n$, satisfy* (6)*,* (7) *and* (8)*. Further, assume that conditions* (11)*,* (12) *and*

$$\gamma_i(x + J_i(t,x)) \le \gamma_i(x), \ \ (t,x) \in [a,b] \times A, \ \ i = 1,\dots,p, \tag{16}$$

hold. If $u = (u_1,\dots,u_{p+1})$ is a fixed point of the operator $\mathcal{G}$, then a function z defined by

$$z(t) = \begin{cases} u_1(t), & t \in [a,\mathcal{P}_1u_1], \\ u_2(t), & t \in (\mathcal{P}_1u_1,\mathcal{P}_2u_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{p+1}(t), & t \in (\mathcal{P}_pu_p,b]. \end{cases} \tag{17}$$

is a solution of problem (1)–(3)*.*

In order to get the existence result we assume in addition that

$$\exists\, \widetilde{J}_i \in (0,\infty) : \ |J_i(t,x)| \le \widetilde{J}_i, \ \ (t,x) \in [a,b] \times \mathbb{R}^n, \ \ i = 1,\dots,p, \tag{18}$$
$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \ \forall\, x,y \in A : \ |x-y| < \delta \Longrightarrow \big\|f(\,\cdot\,,x) - f(\,\cdot\,,y)\big\|_\infty < \varepsilon, \tag{19}$$
$$V \in \mathbb{C}([a_i,b_i];\mathbb{R}^{n\times n}), \ \ i = 1,\dots,p. \tag{20}$$

Here A is from (5) and $[a_i,b_i]$, $i = 1,\dots,p$, are from (6).

The main result is contained in the following theorem.

Theorem 2. *Assume that* (11), (12) *and* (18) *hold and that numbers* μ_j, ρ_j, $j = 1, \dots, n$, *satisfy*

$$\left.\begin{aligned} \mu_j \geq {} & |K^{-1}| \sup_{s\in[a,b]} |V(s)| \widetilde{f}(b-a) + 2\widetilde{f}(b-a) + \\ & + |K^{-1}| \sup_{s\in[a,b]} |V(s)| \sum_{k=1}^{p} \widetilde{J_k} + \sum_{k=1}^{p} \widetilde{J_k} + |K^{-1}c_0|, \\ \rho_j \geq {} & \widetilde{f}, \ \ j = 1, \dots, n. \end{aligned}\right\} \tag{21}$$

Further assume that conditions (6), (7), (8), (16), (19) *and* (20) *hold. Then the operator* $\mathcal{G}$ *has a fixed point in* Ω *and problem* (1)–(3) *has at least one solution* u *such that*

$$\|u\|_\infty \leq \max\{\mu_1, \dots, \mu_n\}.$$

These results are based on the papers [1]–[6]. Proofs of Theorems 1 and 2 can be found in [6].

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Survey of Buffer Phenomenon

Nikolay Rozov

Lomonosov Moscow State University, Moscow, Russia
E-mail: rozov@rozov.mccme.ru

1. *Buffer phenomenon* is an important example of mutually enriching interaction of theoretical research of a mathematical model for a real event and deep penetration into the essence of this event. Detailed investigation of such a phenomenon allowed introduction of new elements into the interpretation of "nonlinear world" notion.

   Oscillatory objects with distributed parameters are found in different fields of science, new hardware and modern technologies. Dynamics of such objects is simulated by systems of partial differential equations with boundary conditions. Stable cycle corresponding to a self-oscillatory regime is a periodic in time solution.

   Such a *boundary-value problem* contains also the parameters and it is essential to determine the number of coexisting self-oscillatory process for different values of parameters. Hence, it is a purely mathematical problem: studying *the dependence of a number of stable cycles on parameters* in a boundary-value problem.

2. *Buffer phenomenon* in a mathematical model of a distributed oscillatory system is observed when the considered boundary-value problem under proper choice of the values of parameters can contain any finite preliminarily fixed number of different stable cycles. In general case, buffer phenomenon of a parameter-dependent dynamic system has the following property: any *a priori* chosen finite number of single-type attractors exist in the system's phase space when the parameters are chosen properly.

   Obviously, the problem on investigation of time-periodic regimes in oscillatory objects with distributed parameters first was stated by A. A. Vitt [1].

3. Detailed statement of strict mathematical theory of buffer phenomenon can be found in papers and monographs [2–6]. The considered mathematical models are nonlinear boundary-value problems for the systems of partial differential equations of hyperbolic or parabolic type. It is essential that buffer phenomenon itself is specific to bifurcation process, in the course of which unlimited increase of the number of coexisting stable attractors takes place.

   The conducted research showed that buffer phenomenon is "typical" of rather broad class of mathematical models that adequately describe many nonlinear oscillatory processes in natural science (radiophysics [7, 8], mechanics [9], optics [10], combustion theory [11], ecology [12], neurodynamics [13]). Besides, relation of buffer phenomenon to such nontrivial phenomena as turbulence and dynamic chaos has been traced [14–16].

   The study of typical scenarios of accumulation of attractors in different dynamic systems is quite topical. Four scenarios of this kind have been discovered so far: Vitt, Turing, Hamilton, and homoclinic mechanisms of accumulation of attractors.

4. The situation in which *Vitt mechanism* is implemented is typical of a large class of physical processes described by hyperbolic equations. It consists in the following.

   Assume that in the problem of stability of equilibrium zero-state of some hyperbolic system there is a critical case of denumerable number of eigenvalues, and when parameters of the system change, a part of spectrum points is successively displaced to the right complex half-plane. Then in case of no certain resonant correlations between the system's eigenfrequencies, is observed unlimited accumulation of quasiharmonic stable cycles, and each cycle originates from zero-state of equilibrium as an unstable one, and then it acquires stability, rising its amplitude [3, 5, 7, 8].

5. *Turing mechanism*: when parameters change, each individual cycle first gains stability and then loses it once again. Thus, though the total number of attractors grows, their set is constantly renovated. As it is shown in [6], such situation is implemented in reaction-diffusion-type systems under proportional decrease of diffusion coefficients, but it can also show up in the systems with delay under unlimited increase of delay time.

6. As to finite-dimensional systems, the elementary mechanism of buffer onset is *Hamilton scenario* illustrated in [17, 18] by 2D-mappings from mechanics and systems of ordinary differential equations that are close to 2D-Hamiltonian ones. It should be noted that Hamilton mechanism has been the less studied one, though it is illustrated by many examples like pendulum-type equations with time-periodic small additional components [19].

7. In the case of systems of ordinary differential equations there are other, much more complex, mechanisms of accumulation of stable cycles that result from so-called homoclinic contacts existing in such systems; such mechanisms can also be conventionally called *homoclinic*. Among many results obtained for the systems with homoclinic structures, let us comment on three of them [20–22].

8. Note that buffer phenomenon in self-excited oscillators with a section of long two-wire line in a feedback circuit has been experimentally shown to be feasible [2, 8].

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On a Reduction of a Linear Homogeneous Differential System with Oscillating Coefficients of Some Special Kind

S. A. Shchogolev

I. I. Mechnikov Odessa National University, Odessa, Ukraine
E-mail: `sergas1959@gmail.com`

Let $G(\varepsilon_0) = \big\{t, \varepsilon:\ 0 < \varepsilon < \varepsilon_0,\ -L\varepsilon^{-1} \le t \le L\varepsilon^{-1},\ 0 < L < +\infty\big\}$.

Definition 1. We say that a function $f(t, \varepsilon)$ belongs to the class $S(m, \varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$ if

1) $f : G(\varepsilon_0) \to \mathbf{C}$;

2) $f(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect to t;

3) $d^k f(t, \varepsilon)/dt^k = \varepsilon^k f_k^*(t, \varepsilon)$ $(0 \le k \le m)$,

$$\|f\|_{S(m,\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t, \varepsilon)| < +\infty.$$

By a slowly varying function we mean a function from $S(m, \varepsilon_0)$.

Definition 2. We say that a function $f(t, \varepsilon, \theta(t, \varepsilon))$ belongs to the class $F(m, l, \varepsilon_0, \theta)$ $(m, l \in \mathbf{N} \cup \{0\})$ if this function can be represented as

$$f\big(t, \varepsilon, \theta(t, \varepsilon)\big) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp\big(in\theta(t, \varepsilon)\big),$$

and

1) $f_n(t, \varepsilon) \in S(m, \varepsilon_0)$;

2) $\|f\|_{F(m,l,\varepsilon_0,\theta)} \stackrel{def}{=} \|f_0\|_{S(m,\varepsilon_0)} + \sum\limits_{n=-\infty}^{\infty} |n|^l \|f_n\|_{S(m,\varepsilon_0)} < +\infty$, particular

$$\|f\|_{F(m,0,\varepsilon_0,\theta)} = \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m,\varepsilon_0)};$$

3) $\theta(t, \varepsilon) = \int\limits_0^t \varphi(\tau, \varepsilon)\, d\tau$, $\varphi(t, \varepsilon) \in \mathbf{R}^+$, $\varphi(t, \varepsilon) \in S(m, \varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t, \varepsilon) = \varphi_0 > 0$.

We denote by $(A)_{jk}$ the element a_{jk} of the matrix $A = (a_{jk})_{j,k=\overline{1,n}}$.

We say that $(n \times n)$-matrix $A(t, \varepsilon, \theta)$ belongs to the class $F(m, l, \varepsilon, \theta)$ if all elements of this matrix are the functions of the class $F(m, l, \varepsilon, \theta)$. Then we define

$$\|A\|^*_{F(m,l,\varepsilon_0,\theta)} \stackrel{def}{=} \max_{1 \le j \le n} \sum_{k=1}^{n} \|(A)_{jk}\|_{F(m,l,\varepsilon_0,\theta)}.$$

Let $f(t,\varepsilon,\theta)\in F(m,l,\varepsilon_0,\theta)$. We denote $\forall\, n\in\mathbf{Z}$:

$$\Gamma_n[f]=\frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\exp(-in\theta)\,d\theta,$$

particular

$$\Gamma_0[f]=\frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\,d\theta.$$

Consider the following differential system

$$\frac{dx}{dt}=\big(\Lambda(t,\varepsilon)+\varepsilon A(t,\varepsilon)+\mu P(t,\varepsilon,\theta)\big)x, \tag{1}$$

where $x=\operatorname{colon}(x_1,\dots,x_n)$, $\Lambda(t,\varepsilon)=\operatorname{diag}(\lambda_1(t,\varepsilon),\dots,\lambda_n(t,\varepsilon))$, $\lambda_j-\lambda_k=i\omega_{jk}(t,\varepsilon)$, $\omega_{jk}\in\mathbf{R}$, $\omega_{jk}\in S(m,\varepsilon_0)$, $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,n}}$, $a_{jk}\in S(m-1,\varepsilon_0)$, $P(t,\varepsilon,\theta)=(p_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,n}}$, $p_{jk}\in F(m,l,\varepsilon_0,\theta)$, $\mu\in(0,\mu_0)\subset\mathbf{R}^+$.

We study the problem of the existence of the transformation of the kind

$$x=\big(E+\Phi(t,\varepsilon,\theta,\mu)\big)z, \tag{2}$$

where $\Phi\in F(m^*,l,\varepsilon^*,\theta)$ $(m^*\le m,\ \varepsilon^*\le\varepsilon_0)$, reducing the system (1) to

$$\frac{dz}{dt}=\big(\widetilde{\Lambda}(t,\varepsilon,\mu)+\varepsilon^2H(t,\varepsilon)+\mu\varepsilon B(t,\varepsilon,\mu)\big)z, \tag{3}$$

where $\widetilde{\Lambda}=\operatorname{diag}(\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_n)$, $H=(h_{jk})_{j,k=\overline{1,n}}$, $B=(b_{jk})_{j,k=\overline{1,n}}$, $\widetilde{\lambda}_j,h_{jk},b_{jk}\in S(m^*,\varepsilon^*)$. Means coefficients of the system (3) are slowly-varying, while the coefficients of the system (1) are oscillating.

Lemma 1. *Suppose that the system* (1) *satisfies the following condition*

$$\forall\,\nu\in\mathbf{Z},\ \ j,k=\overline{1,n}\ \ (j\ne k):\ \ \inf_{G(\varepsilon_0)}\big|\omega_{jk}(t,\varepsilon)-\nu\varphi(t,\varepsilon)\big|\ge\gamma>0.$$

Then $\exists\,\mu_1\in(0,\mu_0)$, $\exists\,\varepsilon_1\in(0,\varepsilon_0)$ such that $\forall\,\mu\in(0,\mu_1)$, $\forall\,\varepsilon\in(0,\varepsilon_1)$ exists the transformation of kind

$$x=\big(E+\widetilde{\Psi}(t,\varepsilon,\theta,\mu)\big)y,$$

where $\widetilde{\Psi}=(\widetilde{\psi}_{jk}(t,\varepsilon,\theta,\mu))_{j,k=\overline{1,n}}$, $\widetilde{\psi}_{jk}\in F(m-1,l,\varepsilon_1,\theta)$, reducing the system (1) *to*

$$\frac{dy}{dt}=\Big(\Lambda(t,\varepsilon)+\varepsilon\Lambda_1(t,\varepsilon)+\mu U(t,\varepsilon,\mu)+\varepsilon^2H(t,\varepsilon)+\mu\varepsilon V(t,\varepsilon,\theta,\mu)\Big)y,$$

where $\Lambda_1=\operatorname{diag}(a_{11},\dots,a_{nn})$, $H=(h_{jk})_{j,k=\overline{1,n}}$, $h_{jk}\in S(m-2,\varepsilon_1)$, $U=\operatorname{diag}(u_1,\dots,u_n)$, $u_j\in S(m,\varepsilon_1)$, $V=(v_{jk})_{j,k=\overline{1,n}}$, $v_{jk}\in F(m-1,l,\varepsilon_1,\theta)$.

Lemma 2. *Let we have the scalar linear non-homogeneous first-order differential equation*

$$\frac{dx}{dt}=\big(i\omega(t,\varepsilon)+\varepsilon\alpha(t,\varepsilon)+\mu u(t,\varepsilon)\big)x+\varepsilon v(t,\varepsilon,\theta),$$

where $\omega(t,\varepsilon)\in S(m,\varepsilon_1)$, $\omega(t,\varepsilon)\in\mathbf{R}^+$, $u(t,\varepsilon)\in S(m,\varepsilon_1)$, $\alpha(t,\varepsilon)\in S(m-1,\varepsilon_1)$, $v(t,\varepsilon,\theta)\in F(m-1,l,\varepsilon_1,\theta)$ and the following conditions

1) $\inf\limits_{G(\varepsilon_1)} |\omega(t,\varepsilon) - \nu\varphi(t,\varepsilon)| \geq \gamma > 0 \ \forall\,\nu \in \mathbf{Z}$;

2) *alternative holds: or* $\operatorname{Re} u(t,\varepsilon) \equiv 0$, *or* $\inf\limits_{G(\varepsilon_1)} |\operatorname{Re} u(t,\varepsilon)| = \gamma_1 > 0$.

Then $\exists\,\varepsilon_2 \in (0,\varepsilon_1)$, $\mu_2 \in (0,\mu_1)$ *such that* $\forall\,\mu \in (0,\mu_2)$, $\varepsilon \in (0,\varepsilon_2)$ *the equation* (7) *has a particular solution* $x(t,\varepsilon,\theta,\mu) \in F(m-1,l,\varepsilon_2,\theta)$, *and* $\exists\,K_2 \in (0,+\infty)$ *such that*

$$\big\|x(t,\varepsilon,\theta,\mu)\big\|_{F(m-1,l,\varepsilon_2,\theta)} \leq K_2 \big\|v(t,\varepsilon,\theta,\mu)\big\|_{F(m-1,l,\varepsilon_2,\theta)}.$$

Lemma 3. *Let the function*

$$f\big(t,\varepsilon,\theta(t,\varepsilon)\big) = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} f_n(t,\varepsilon) \exp\big(in\theta(t,\varepsilon)\big)$$

belong to the class $F(m-1,l,\varepsilon_1,\theta)$. *Then the function*

$$x\big(t,\varepsilon,\theta(t,\varepsilon)\big) = \varepsilon \int\limits_0^t f\big(\tau,\varepsilon,\theta(\tau,\varepsilon)\big)\,d\tau$$

belongs to the class $F(m-1,l,\varepsilon_1,\theta)$ *also, and* $\exists\,K_3 \in (0,+\infty)$ *such that*

$$\|x\|_{F(m-1,l,\varepsilon_1,\theta)} \leq K_3 \|f\|_{F(m-1,l,\varepsilon_1,\theta)}.$$

Theorem. *Suppose the system* (1) *is such that*

1) $\forall\,\nu \in \mathbf{Z}$, $j,k = \overline{1,n}$ $(j \neq k)$: $\inf\limits_{G(\varepsilon_0)} \big|\omega_{jk}(t,\varepsilon) - \nu\varphi(t,\varepsilon)\big| \geq \gamma > 0$;

2) *the elements* $u_j(t,\varepsilon,\mu)$ $(j = \overline{1,n})$ *of the diagonal matrix* $U(t,\varepsilon,\mu)$, *which are defined in Lemma* 1, *have the alternative:*

   or $\operatorname{Re}\,(u_j(t,\varepsilon,\mu) - u_k(t,\varepsilon,\mu)) \equiv 0$ $(j,k = \overline{1,n},\ j \neq k)$;
   or $\inf\limits_{G(\varepsilon_1)} |\operatorname{Re}\,(u_j(t,\varepsilon,\mu) - u_k(t,\varepsilon,\mu))| \geq \gamma_0 > 0$, *where* ε_1 *are defined in Lemma* 1.

Then $\exists_3 \in (0,\varepsilon_0)$, $\mu_3 \in (0,\mu_0)$ *such that* $\forall\,\varepsilon \in (0,\varepsilon_3)$, $\forall\,\mu \in (0,\mu_3)$ *exists the transformation of kind* (2), *where* $\Phi(t,\varepsilon,\theta,\mu) \in F(m-1,l,\varepsilon_3,\theta)$, *which reduces the system* (1) *to the form* (3), *where* $H \in S(m-2,\varepsilon_3)$, $B \in S(m-1,\varepsilon_3)$.

Asymptotic Representations of Solutions for One Class of Non-Linear Differential Equations of the Second Order

O. R. Shlyepakov

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: oleg@gavrilovka.com.ua

We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_2(y'), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $\varphi_i : \Delta(Y_i^0) \to]0, +\infty[$ $(i = 1, 2)$ are twice continuously differentiable functions which satisfy the conditions:

$$\varphi_1'(z) \neq 0 \ \text{ when } \ z \in \Delta(Y_1^0), \qquad \lim_{\substack{z\to Y_1^0\\ z\in\Delta(Y_1^0)}} \varphi_1(z) = \Phi_1^0, \quad \Phi_1^0 \in \{0, +\infty\},$$
$$\lim_{\substack{z\to Y_1^0\\ z\in\Delta(Y_1^0)}} \frac{\varphi_1''(z)\varphi_1(z)}{[\varphi_1'(z)]^2} = 1, \tag{2}$$

$$\lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \frac{z\varphi_2'(z)}{\varphi_2(z)} = \lambda \ \ (\lambda \in \mathbb{R}), \tag{3}$$

where $\Delta(Y_i^0)$ is some one-sided neighborhood of the point Y_i^0, Y_i^0 is equal to either 0 or $\pm\infty$.

From (2), (3) it follows that the function $\varphi_1(z)$ is rapidly varying when $z \to Y_1^0$, and the function $\varphi_2(z)$ is regularly or slowly varying when $z \to Y_2^0$ (see Seneta [14]).

This equation was considered in the works of Evtukhov V. M. and Belozerova M. A. [1, 2, 3, 4, 10] for the cases when the functions $\varphi_i(z)$ were power-law or regularly or slowly varying when $z \to Y_i^0$, $i = \overline{1, 2}$.

In the work of Kharkov V. M. [7], the following equation was considered

$$y'' = \alpha_0 p(t)\varphi_1(y), \tag{4}$$

where the function $\varphi_1(z)$ was rapidly or regularly varying when $z \to Y_1^0$. Equation (4) is a particular case of the equation (1), when $\varphi_2(z) \equiv 1$. In [7], the class of solutions was established. For that class, the necessary and sufficient conditions as well as the asymptotic formulas for solutions were derived.

In the works of Evtukhov V. M. and Drik N. G. [5, 6, 10], the particular case for the equation (1) was considered:

$$y'' = \alpha_0 p(t)e^{\sigma y}|y'|^{\lambda}. \tag{5}$$

Equation (1), if the function $\varphi_1(z)$ is rapidly varying when $z \to Y_1^0$, and the function $\varphi_2(z)$ is slowly or regularly varying when $z \to Y_2^0$, in particular, the equation (5), has wide application for describing different processes in physics. For example, differential equations appearing in the Linan's problem from combustion theory could be reduced to the equation (1), as well as Poisson nonlinear differential equations for cylindric symmetrical plasma of combustion products could be reduced to the equation (1) by means of several notations (see [15, 11, 12]).

If the function $\varphi_1(z)$ is rapidly varying when $z \to Y_1^0$, and the function $\varphi_2(z)$ is slowly or regularly varying when $z \to Y_2^0$, equation (1) is a generalization for both equations (4) and (5).

A solution y of the equation (1) is called a $P_\omega(\Lambda_0)$-solution, where $-\infty \le \Lambda_0 \le +\infty$, if it is defined on some interval $[t_0,\omega[\subset[a,\omega[$ and satisfies the following conditions:

$$\lim_{t\uparrow\omega}\varphi_1(y(t))=\Phi_1^0,\quad \lim_{t\uparrow\omega}y'(t)=Y_2^0,$$
$$\lim_{t\uparrow\omega}\frac{\varphi_1(y(t))}{\varphi_1'(y(t))}\,\frac{y''(t)}{[y'(t)]^2}=\Lambda_0, \tag{6}$$

Note that the class of $P_\omega(\Lambda_0)$-solutions corresponds to the class of $\widetilde{P}_\omega(\widetilde{\Lambda}_0)$-solutions that was introduced by Kharkov V. M. in the work [8] for the equation (4) in case when $\Lambda_0=\widetilde{\Lambda}_0-1$.

In the paper, for the equation (1), in case $\Lambda_0\in\mathbb{R}\setminus\{0\}$, the asymptotic formulas for $P_\omega(\Lambda_0)$-solutions were established and the necessary and sufficient conditions for their existence were derived.

Let us introduce functions and notation:

$$\psi(z)=\int_B^z\frac{ds}{\varphi_2(s)}\,,\quad \text{where } B=\begin{cases} Y_2^0, & \text{if } \displaystyle\int_b^{Y_2^0}\frac{ds}{\varphi_2(s)} \text{ converges},\\[2mm] b, & \text{if } \displaystyle\int_b^{Y_2^0}\frac{ds}{\varphi_2(s)} \text{ diverges},\end{cases}$$

and b is any number from the interval $\Delta(Y_2^0)$.

Since $\psi'(z)>0$ when $z\in\Delta(Y_2^0)$, then $\psi:\Delta(Y_2^0)\to\Delta(\Phi_2^0)$ is an increasing function where $\Phi_2^0=\lim\limits_{z\to Y_2^0}\psi(z)$, consequently, Φ_2^0 equals either to zero or to $\pm\infty$, $\Delta(\Phi_2^0)$ is one-sided neighborhood of Φ_2^0.

Next we set:

$$\mu=\begin{cases}1, & \text{if } Y_2^0=+\infty, \text{ or } Y_2^0=0 \text{ and } \Delta(Y_2^0) \text{ is a right neighborhood of } 0,\\ -1, & \text{if } Y_2^0=-\infty, \text{ or } Y_2^0=0 \text{ and } \Delta(Y_2^0) \text{ is a left neighborhood of } 0.\end{cases}$$

From the definition of $\varphi_1(z)$ it follows that $\varphi_1'(z)$ preserves the sign. Consequently, it is possible to introduce notation:

$$\rho=\operatorname{sign}\varphi_1'(z).$$

Also we set:

$$\pi_\omega(t)=\begin{cases}t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty,\end{cases}$$

$$J(t)=\begin{cases}\displaystyle\int_A^t p(\tau)\,d\tau, & \text{if } (1-\lambda)\Lambda_0\neq1,\\[2mm] \displaystyle\int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, & \text{if } (1-\lambda)\Lambda_0=1,\end{cases}$$

$$\beta=\begin{cases}1-\lambda-\Lambda_0^{-1}, & \text{if } (1-\lambda)\Lambda_0\neq1,\\ -1, & \text{if } (1-\lambda)\Lambda_0=1,\end{cases}$$

where the integration limit $A\in\{\omega,a\}$ is chosen so as to ensure that the corresponding integral J tends either to zero or to infinity when $t\uparrow\omega$.

Moreover, we set

$$A_1^* = \begin{cases} 1, & \text{if } \omega = \infty, \\ -1, & \text{if } \omega < \infty, \end{cases} \quad A_2^* = \begin{cases} 1, & \text{if } A = a, \\ -1, & \text{if } A = \omega. \end{cases}$$

By means of notations described above, we establish the necessary and sufficient conditions for the existence of $\Lambda_0 \in \mathbb{R} \setminus \{0\}$-solutions of the equation (1).

Theorem 1. *Let $\Lambda_0 \in \mathbb{R} \setminus \{0\}$. Then for the existence of $P_\omega(\Lambda_0)$-solutions of the equation* (1)*, it is necessary and, if*

$$\lambda \neq 1, \ \text{ or } \ \lambda = 1 \ \text{ and } \ \Lambda_0 > 0$$

it is also sufficient that

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'(t)}{J(t)} = -\beta$$

and the following sign conditions be satisfied

$$-A_1^* \Lambda_0 > 0 \ \text{ when } \ \Phi_1^0 = +\infty, \quad -A_1^* \Lambda_0 < 0 \ \text{ when } \ \Phi_1^0 = 0,$$
$$A_2^* \beta > 0 \ \text{ when } \ \Phi_2^0 = +\infty, \quad A_2^* \beta < 0 \ \text{ when } \ \Phi_2^0 = 0,$$
$$\operatorname{sign}\big[\mu A_1^* \Lambda_0\big] = -\rho \ \text{ and } \ \operatorname{sign}\big[\alpha_0 A_2^* \beta\big] = 1.$$

Moreover, each solution of this kind admits the asymptotic representation when $t \uparrow \omega$

$$\frac{\varphi_1(y(t))}{\varphi_1'(y(t)) y'(t)} = -\Lambda_0 \pi_\omega(t)[1 + o(1)], \tag{7}$$

$$\frac{y'(t)}{\varphi_1(y(t)) \varphi_2(y'(t))} = -\alpha \pi_\omega(t) p(t)[1 + o(1)], \tag{8}$$

moreover, when $\Lambda_0 > 0$, there exists one-parameter family of such solutions, and when $\Lambda_0 < 0$, there exists two-parameter family of such solutions, if $\omega = +\infty$ and $\lambda > 1$, or if $\omega < +\infty$ and $\lambda < 1$.

We will introduce auxiliary conditions that will enable us to simplify the asymptotic formulas (7), (8).

Definition (see [4])**.** We say that a function $\theta : \Delta(U^0) \to]0, +\infty[$, $U^0 \in \{0, \pm\infty\}$ satisfies condition S, if for any continuously differentiable function $l : \Delta(U^0) \to]0, +\infty[$ such that

$$\lim_{\substack{z \to U^0 \\ z \in \Delta(U^0)}} \frac{z\, l'(z)}{l(z)} = 0,$$

the following asymptotic formula is fulfilled

$$\theta(z l(z)) = \theta(z)[1 + o(1)] \ \text{ when } \ z \to U^0 \ \ (z \in \Delta(U^0)).$$

From the properties of regularly varying functions, the following representations are obtained:

$$\varphi_1'(\varphi_1^{-1}(z)) = |z| \theta_1(z),$$
$$\varphi_2(z) = |z|^\lambda \theta_2(z),$$

where the functions $\theta_i(z)$ $(i = 1, 2)$ are slowly varying.

Theorem 2. *Let $\Lambda_0 \in \mathbb{R} \setminus \{0\}$ and functions $\theta_i(z)$ $(i \in \{1, 2\})$ satisfy the condition S. Then each $P_\omega(\Lambda_0)$-solution* (*if any*) *of the differential equation* (1) *admits the following asymptotic formulas when $t \uparrow \omega$*

$$\varphi_1(y(t)) = \Big|\Lambda_0 \pi_\omega(t) \theta_1\big(|\pi_\omega(t)|^{\frac{-1}{\Lambda_0}}\big)\Big|^{\lambda - 1} \Big|\pi_\omega(t) p(t) \theta_2\big(|J(t)|^{\frac{1}{\beta}}\big)\Big|^{-1} [1 + o(1)],$$

$$y'(t) = \mu \Big|\Lambda_0 \pi_\omega(t) \theta_1\big(|\pi_\omega(t)|^{\frac{-1}{\Lambda_0}}\big)\Big|^{-1} [1 + o(1)].$$

To obtain the above results, the results for cyclic systems from the work [9] were used.

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On One Nonlinear Boundary Value Problem for Second Order Singular Functional Differential Equations

Zaza Sokhadze

A. Tsereteli State University, Kutaisi, Georgia
E-mail: z.soxadze@gmail.com

We consider the functional differential equation

$$u''(t) = f\big(t, u(\tau(t))\big) \tag{1}$$

with the nonlinear boundary conditions

$$u(a) = \varphi_1(u), \quad u'(b) = \varphi_2(u), \tag{2}$$

where $f :]a,b]\times]0,+\infty[\to \mathbb{R}_+$ and $\tau :]a,b] \to [a,b]$ are continuous functions, and $\varphi_i : C([a,b];\mathbb{R}_+) \to \mathbb{R}_+$ $(i = 0,1)$ are continuous functionals.

We are, especially, interested in the case, where equation (1) is singular in a phase variable, i.e. the case, where

$$\lim_{x\to 0} f(t,x) = +\infty \ \text{ for } \ t \in]a,b].$$

Theorem. *Let on the set* $]a,b]\times]0,+\infty[$ *the inequality*

$$p_0(t,x) \le -f(t,x) \le p_1(t,x)(1+x)$$

hold, and let on the set $C([a,b];\mathbb{R}_+)$ *the inequality*

$$\varphi_1(u) + (b-a)\varphi_2(u) \le \ell\|u\| + r$$

be fulfilled, where $p_i :]a,b]\times]0,+\infty[\to]0,+\infty[$ $(i = 0,1)$ *are continuous and nonincreasing in the second argument functions. If, moreover,* $\tau(t) \ge t$ *for* $a < t \le b$, $\ell < 1$ *and*

$$\lim_{x\to+\infty} \int_a^b (s-a)p_1(s,x)\,ds < 1-\ell,$$

then problem (1), (2) *has at least one positive on* $]a,b[$ *solution.*

As an example, we consider the problem

$$u''(t) = -\frac{p(t)}{u^{\lambda}(\tau(t))}\,, \tag{3}$$

$$u(a) = \int_a^b \psi_1(u(s))\,d\sigma_1(s), \quad u(b) = \int_a^b \psi_2(u(s))\,d\sigma_2(s), \tag{4}$$

where λ is a positive constant, $p :]a,b] \to]0,+\infty[$, $\tau :]a,b] \to]a,b]$ and $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1,2)$ are continuous functions, and $\sigma_i : [a,b] \to \mathbb{R}$ $(i = 1,2)$ are nondecreasing functions such that

$$\sigma_i(b) - \sigma_i(a) = 1 \ \ (i = 1,2).$$

From the above-formulated theorem we have the following result.

Corollary. *If*

$$\limsup_{x\to+\infty} \frac{\psi_i(x)}{x} < 1 \ \ (i = 1,2),$$

then problem (3), (4) *has at least one positive on* $]a,b[$ *solution.*

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Strong Time-Singular Boundary Value Problems for Second-Order Differential Equations

Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science, Palacký University, Olomouc, Czech Republic

E-mail: svatoslav.stanek@upol.cz

Let $T > 0$, $J = [0, T]$ and $\|x\| = \max\{|x(t)| : t \in J\}$ for $x \in C(J)$.

We are interested in the singular boundary value problem

$$u''(t) + p(t)u'(t) = a(t)p(t)f\big(t, u(t), u'(t)\big), \quad t \in (0, T], \tag{1}$$

$$u(0) = u(T), \quad u'(0) = 0, \tag{2}$$

where p, a and f satisfy the conditions

(H_1) $p \in C(0, T]$, $p > 0$ on $(0, T]$ and $\int\limits_0^T p(t)\,\mathrm{d}t = \infty$,

(H_2) $a \in C(J)$, $a(0) = 0$ and $a > 0$ on $(0, T]$,

(H_3) there exists $A > 0$ such that $f \in C(J \times [-A, A] \times \mathbb{R})$ and

$$f(t, -A, 0) \le 0, \quad f(t, A, 0) \ge 0 \ \text{ for } \ t \in J,$$

(H_4) there exists $W > 0$ such that for $t \in J$, $|x| \le A$ and $|y| \le W$, the estimate $|f(t, x, y)| < \frac{W}{\|a\|}$ holds.

We say that $u : J \to \mathbb{R}$ is *a solution of problem* (1), (2) if $u \in C^1(J) \cap C^2(0, T]$, u satisfies the boundary conditions (2) and (1) holds for $t \in (0, T]$.

The special case of (1) is the differential equation $u'' + \frac{u'}{t^\gamma} = t^\mu f(t, u, u')$, where $\gamma \ge 1$ and $\mu > -\gamma$.

We note that due to condition (H_1) equation (1) has a strong time singularity at $t = 0$, and since any constant function u on J satisfies (2) and $u'' + p(t)u' = 0$ on $(0, T]$, problem (1), (2) is at resonance.

Remark 1. If f satisfies the condition

(H_5) for $(t, x, y) \in J \times [-A, A] \times \mathbb{R}$ the estimate $|f(t, x, y)| \le \varphi(|y|)$ holds, where $\varphi \in C[0, \infty)$, φ is nondecreasing and $\lim\limits_{v \to \infty} \frac{\varphi(v)}{v} = 0$,

then f also satisfies (H_4) for some $W > 0$.

Remark 2. Let condition (H_3) be replaced by

(H_3^*) there exist $B, C \in \mathbb{R}$ such that $B < C$, $f \in C(J \times [B, C] \times \mathbb{R})$ and

$$f(t, B, 0) \le 0, \quad f(t, C, 0) \ge 0 \ \text{ for } \ t \in J.$$

Put $A = \frac{C-B}{2}$ and introduce $g \in C(J \times [-A, A] \times \mathbb{R})$ as

$$g(t, x, y) = f\Big(t, x + \frac{B+C}{2}, y\Big). \tag{3}$$

Then

$$g(t, -A, 0) \le 0, \ \ g(t, A, 0) \ge 0 \ \text{ for } \ t \in J.$$

Hence g satisfies condition (H_3). Moreover, v is a solution of problem

$$\left.\begin{aligned} v'' + p(t)v' = a(t)p(t)g(t, v, v'),\\ v(0) = v(T), \ \ v'(0) = 0, \end{aligned}\right\} \tag{4}$$

if and only if $u = v - \frac{B+C}{2}$ is a solution of (1), (2). Consequently, without loss of generality we can work with condition (H_3) which due to its symmetry is more convenient for our consideration.

Our proof of the solvability to problem (1), (2) is based on a combination of the sequential method and the Leray–Schauder degree method [3] with the diaginalization method [1, 2]. To this end we introduce a function $f_* \in C(J \times \mathbb{R}^2)$ as

$$f_*(t, x, y) = \begin{cases} f(t, A, y) + \dfrac{(x-A)\kappa}{x - A + 1} & \text{if } \ x > A,\\ f(t, x, y) & \text{if } \ -A \le x \le A,\\ f(t, -A, y) - \dfrac{(A+x)\kappa}{A + x - 1} & \text{if } \ x < -A, \end{cases}$$

where $\kappa = \frac{1}{2}\big(\frac{W}{\|a\|} - M\big)$, $M = \max\big\{|f(t, x, y)| : \ t \in J, \ |x| \le A, \ |y| \le W\big\}$. By (H_4), $\kappa > 0$.

Choose $t_0 \in (0, T)$ and let $J_0 = [t_0, T]$. Consider the auxiliary regular boundary value problem

$$u''(t) + p(t)u'(t) = a(t)p(t)f_*\big(t, u(t), u'(t)\big), \ \ t \in J_0, \tag{5}$$

$$u(t_0) = u(T), \ \ u'(t_0) = 0. \tag{6}$$

We say that u *is a solution of problem* (5), (6) if $u \in C^2(J_0)$, u satisfies (6) and (5) holds for $t \in J_0$.

In order to proof the solvability of problem (5), (6) we introduce an operator $\mathcal{L}_0 : C^1(J_0) \times \mathbb{R} \to C^2(J_0) \times \mathbb{R}$ by the formula

$$\mathcal{L}_0(x, c) = \bigg(c + \int_{t_0}^{t} (\mathcal{F}_0 x)(s)\,\mathrm{d}s, c + \int_{t_0}^{T} (\mathcal{F}_0 x)(s)\,\mathrm{d}s\bigg),$$

where

$$(\mathcal{F}_0 x)(t) = e^{v(t)} \int_{t_0}^{t} e^{-v(s)} a(s)p(s)f_*\big(s, x(s), x'(s)\big)\,\mathrm{d}s, \quad v(t) = \int_{t}^{T} p(s)\,\mathrm{d}s.$$

Lemma 1. *Let* (H_1)*–*(H_4) *hold. If* (x, c) *is a fixed point of* $\mathcal{L}_0$*, then* x *is a solution of problem* (5), (6) *and* $c = x(t_0)$.

Now, in order to prove the existence of a fixed point of $\mathcal{L}_0$, we introduce completely continuous operators $\mathcal{K}_0 : C^1(J_0) \times \mathbb{R} \times [0, 1] \to C^1(J_0) \times \mathbb{R}$ and $\mathcal{H}_0 : C^1(J_0) \times \mathbb{R} \times [0, 1] \to C^1(J_0) \times \mathbb{R}$,

$$\mathcal{K}_0(x, c, \lambda) = \bigg(c, c + (1-\lambda)x(t_0) + \lambda \int_{t_0}^{T} (\mathcal{F}_0 x)(s)\,\mathrm{d}s\bigg),$$

$$\mathcal{H}_0(x, c, \lambda) = \bigg(c + \lambda \int_{t_0}^{t} (\mathcal{F}_0 x)(s)\,\mathrm{d}s, c + \int_{t_0}^{T} (\mathcal{F}_0 x)(s)\,\mathrm{d}s\bigg),$$

and the set

$$\Omega = \Big\{(x,c) \in C^1(J_0) \times \mathbb{R}: \ |x(t)| < A+1, \ |x'(t)| < W \text{ for } t \in J_0, \ |c| < A+1\Big\}.$$

Lemma 2. *Let* (H_1)*–*(H_4) *hold. Then*

$$\deg\big(\mathcal{I}_0 - \mathcal{K}_0(\,\cdot\,,\,\cdot\,,1),\Omega,0\big) \neq 0,$$
$$\deg\big(\mathcal{I}_0 - \mathcal{H}_0(\,\cdot\,,\,\cdot\,,0),\Omega,0\big) = \deg\big(\mathcal{I}_0 - \mathcal{H}_0(\,\cdot\,,\,\cdot\,,1),\Omega,0\big),$$

*where "*deg*" stands for the Leray–Schauder degree and* $\mathcal{I}_0$ *is the identical operator on* $C^1(J_0)\times\mathbb{R}$.

Theorem 1. *Let* (H_1)*–*(H_4) *hold and let* $t_0 \in (0,T)$*. Then the equation*

$$u''(t) + p(t)u'(t) = a(t)p(t)f\big(t,u(t),u'(t)\big), \ \ t \in J_0 = [t_0,T], \tag{7}$$

has a solution u *satisfying* (6) *and* $|u| \le A$, $|u'| < W$ *on* J_0.

Sketch of the proof. Since $\mathcal{K}_0(\,\cdot\,,\,\cdot\,,1) = \mathcal{H}_0(\,\cdot\,,\,\cdot\,,0)$ and $\mathcal{H}_0(\,\cdot\,,\,\cdot\,,1) = \mathcal{L}_0(\,\cdot\,,\,\cdot\,)$, it follows from Lemma 2 that $\deg(\mathcal{I}_0 - \mathcal{L}_0(\,\cdot\,,\,\cdot\,),\Omega,0) \neq 0$. Hence there exists a fixed point (u,c) of $\mathcal{L}_0$, and therefore u is a solution of problem (5), (6) and $c = u(t_0)$ by Lemma 1. We use (H_3) and have $|u| \le A$, $|u'| < W$ on J_0. Hence $f_*(t,u(t),u'(t)) = f(t,u(t),u'(t))$ on J_0. Consequently, u is a solution of (7), (6).

Theorem 2. *Let* (H_1)*–*(H_4) *hold. Then problem* (1), (2) *has at least one solution* u *and* $|u(t)| \le A$, $|u'(t)| \le W$ *for* $t \in J$.

Sketch of the proof. Let $\{t_n\} \subset (0,T)$ be a decreasing sequence such that $\lim\limits_{n\to\infty} t_n = 0$ and let $J_n = [t_n,T]$. By Theorem 1 (for $t_0 = t_n$), for each $n \in \mathbb{N}$ the problem

$$\left.\begin{array}{c} u''(t) + p(t)u'(t) = a(t)p(t)f\big(t,u(t),u'(t)\big), \ \ t \in J_n \\ u(t_n) = u(T), \ \ u'(t_n) = 0, \end{array}\right\}$$

has a solution u_n and $|u_n| \le A$, $|u_n'| < W$ on J_n. Let

$$y_n(t) = \begin{cases} u_n(t) & \text{if } \ t \in J_n, \\ u_n(t_n) & \text{if } \ t \in [0,t_n). \end{cases}$$

Then $y_n \in C^1(J) \cap C^2(J_n)$, $|y_n| \le A$, $|y_n'| < W$ on J and $|y_n''| \le 2W \max\{p(t) : t \in J_n\}$ on J_n. Since $\{y_n(T)\}$ is a bounded sequence, we may assume without loss of generality that it is convergent and let $\lim\limits_{n\to\infty} y_n(T) = \gamma$. We now apply the diagonalization method to the sequence $\{y_n\}$ and obtain a function $u \in C^2(0,T]$ such that $|u| \le A$, $|u'| \le W$ on $(0,T]$, $u(T) = \gamma$ and u satisfies (1) for $t \in (0,T]$.

Next we prove that $\lim\limits_{t\to 0} u(t) = \gamma$ and setting $u(0) = \gamma$, $u \in C(J)$ and $u(0) = u(T)$. Finally, it can be proved that $\lim\limits_{t\to 0} u'(t) = 0$. Setting $u'(0) = 0$, $u \in C^1(J)$. Consequently, u is a solution of (1), (2) and $|u| \le A$, $|u'| \le W$ on J.

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Oscillations of Solutions of Second Order Linear Differential Equations and the Corresponding Difference Equations

O. Stanzhytskyi

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail: ostanzh@gmail.com

V. Mogilova

National Technical University of Ukraine "Kyiv Polytechnic Institute", Kyiv, Ukraine
E-mail: mogilova@gmail.com

A. Tkachuk

National University of Food Technologies, Kyiv, Ukraine
E-mail: tkachukam@ukr.net

The object of our investigation is establishing the conditions for oscillations of solutions of linear second order differential equations, provided the solutions of the corresponding difference equations oscillate. We also establish the converse result, namely, when the oscillation of the solutions of difference equations implies the oscillation of the solutions of the corresponding differential equations.

Consider the linear second order differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0. \tag{1}$$

The following equations are called the *functional difference equation* and the *difference equation*, corresponding to (1), respectively:

$$\triangle^2 x(t) + hp(t)\triangle x(t) + h^2 q(t)x(t) = 0, \tag{2}$$

$$\triangle_k^2 x(t_0) + hp(t_0+kh)\triangle_k x(t_0) + h^2 q(t_0+kh)x(t_0+kh) = 0. \tag{3}$$

Here

$$\begin{aligned}
\triangle x(t) &= x(t+h) - x(t),\\
\triangle^2 x(t) &= \triangle(\triangle x(t)) = x(t+2h) - 2x(t+h) + x(t),\\
\triangle_k x(t_0) &= x(t_0+(k+1)h) - x(t_0+kh),\\
\triangle_k^2 x(t_0) &= \triangle_k(\triangle_k x(t_0)).
\end{aligned}$$

Denote $x_k^h = x(t_0+kh)$ to be the solution of the equation (3), with $t_k = t_0 + kh$.

Definition 1. We say that the solution x_k^h of the equation (3) *changes sign* at t_k, if either of the following conditions hold:

1) $x_k^h x_{k+1}^h < 0$;

2) $x_k^h = 0$, $x_{k-1}^h x_{k+1}^h < 0$.

Definition 2. A solution x_k^h of (3) is called *oscillatory* on some interval if it has at least two changes of signs on this interval.

We study the equation (2) under the conditions that ensure the continuity of its solutions. Thus, we have the usual concept of a zero for the solutions of (2), and the notion of oscillations of its solutions is essentially the same as for the solutions of (1).

Now we present the main results about the relation between the oscillation of solutions of the equations (1), (2), (3). These equations are equivalent to the following systems

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -p(t)y - q(t)x, \end{cases} \tag{4}$$

$$\begin{cases} x(t+h) = x(t) + hy(t), \\ y(t+h) = y(t) - h(p(t)y(t) + q(t)x(t)), \end{cases} \tag{5}$$

$$\begin{cases} x_{k+1}^h = x_k^h + hy_k^h, \\ y_{k+1}^h = y_k^h - h(p(t_0+kh)y_k^h + q(t_0+kh)y_k^h). \end{cases} \tag{6}$$

Therefore, the solutions of the system (5) are uniquely determined by the initial functions $x = \varphi(t), y = \psi(t)$, $t \in [0,h]$ which satisfy the coherence condition

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0), \\ \psi(h) = \psi(0) - h(p(0)\psi(0) + q(0)\varphi(0)). \end{cases} \tag{7}$$

In what follows we assume that $\varphi, \psi \in C([0,h])$. The solutions of the system (6) are uniquely determined by the initial data

$$x_0^h(t_0) = x_0, \quad y_0^h(t_0) = y_0.$$

Theorem 1. *Let p and q in (1) be Lipschitz on $[0,a]$. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ the assertion holds:*

If $x(t)$ is a solution of (1), which starts at $t_0 \in [0,h]$ and has at least three zeros on the interval $[t_0,a)$, then the corresponding solution of the difference equation (3) oscillates on $[t_0,a]$.

Consider now the equation (2), or the equivalent system (5). The following result follows from Theorem 1 and Lemma 1.

Theorem 2. *Assume that p and q in (2) are Lipschitz on $[0,a]$. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ the following statement holds:*

Every solution of the system (5) with the initial functions $\varphi, \psi \in C([0,h])$, which satisfy the condition (7), has oscillatory first component on the $(0,a)$, provided that there exists $t_0 \in [0,h]$ such that the solution of the equation (1) with the initial data

$$x(t_0) = \varphi(t_0), \dot{x}(t_0) = \psi(t_0)$$

has at least three zeros on (t_0,a).

Consider the equation

$$\ddot{x} + p(t)x = 0 \tag{8}$$

and the corresponding functional difference equation

$$\triangle^2 x(t) + h^2 p(t)x(t) = 0, \tag{9}$$

and the difference equation

$$\triangle_k^2 x(t_0) + h^2 p(t_0 + kh)x(t_0 + kh) = 0 \tag{10}$$

with p satisfying the Lipschitz condition on $[0,a]$. Let

$$m = \min_{t\in[0,a]} p(t), \quad M = \max_{t\in[0,a]} p(t).$$

Assume

$$m > 0 \ \text{ and } \ a > \frac{3\pi}{\sqrt{m}}\,. \tag{11}$$

Then if

$$a - \overline{h} > \frac{3\pi}{\sqrt{m}}\,, \tag{12}$$

all solutions of (8) with the initial data $t_0 \in [0,\bar{h}]$ have at least three zeros on the interval $[t_0, a)$.

Corollary 1. *Let p be Lipschitz on $[0,a]$, and the conditions* (11) *and* (12) *hold. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ all solutions of equation* (10) *with the initial data given at $t_0 \in [0,h]$, oscillate on the $[t_0,a)$.*

Corollary 2. *Let p be Lipschitz on $[0,a]$, and the conditions* (11) *and* (12) *hold. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ every solution of the system*

$$\begin{cases} x(t+h) = x(t) + hy(t),\\ y(t+h) = y(t) - hp(t)x(t) \end{cases}$$

with the initial functions $\varphi,\psi \in C([0,h])$ satisfying the coherence condition

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0),\\ \psi(h) = \psi(0) - hp(0)\varphi(0)), \end{cases}$$

has an oscillatory first component on the interval $(0,a)$.

Assume the following conditions hold:

$$p(t) \ge 0, \ \ t \in [0,a], \tag{13}$$

$$p(t) \ \text{ is Lipschitz on } [0,a]. \tag{14}$$

The difference equation, corresponding to (8), is

$$\triangle_k^2 x + h^2 p(kh)x(kh) = 0. \tag{15}$$

The following theorem describes the relation between the oscillations of solutions of (8) and (15).

Theorem 3. *Let $p(t)$ satisfy the conditions* (13) *and* (14). *Then there exists h_0 such that for all $0 < h \le h_0$ the assertion holds:*

> *If x_k^h is a solution of* (15) *which has at least three changes of sign on the interval $[0,a]$, then the corresponding solution of differential equation* (8) *oscillates on $[0,a]$.*

Sensitivity Analysis of One Class of Controlled Functional Differential Equation Considering Variable Delay Perturbation and the Continuous Initial Condition

Tamaz Tadumadze

I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: tamaz.tadumadze@tsu.ge

Sensitivity analysis of the differential equation consists in finding an analytic relation between solutions of the original and perturbed equations. It is an important tool for assessing properties of the mathematical models. For example, in an immune model [1], it allows one to determine dependence of viruses concentrations on the model parameters. In the present work linear representation of the first order sensitivity coefficient is obtained with respect to perturbations of the initial data.

Let $I=[a,b]$ be a finite interval and let $\mathbb{R}_x^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition. Suppose that $O\subset\mathbb{R}_x^n$ and $U_0\subset\mathbb{R}_u^r$ are open sets. Let the n-dimensional function $f(t,x,y,u,v)$ satisfies the following conditions: for almost all $t\in I$, the function $f(t,\cdot\,):O^2\times U_0^2\to\mathbb{R}_x^n$ is continuously differentiable; for any $(x,y,u,v)\in O^2\times U_0^2$, the functions $f(t,x,y,u,v)$, $f_x(\cdot\,)$, $f_y(\cdot\,)$, $f_u(\cdot\,)$, $f_v(\cdot\,)$ are measurable on I; for arbitrary compacts $K\subset O$, $U\subset U_0$ there exists a function $m_{K,U}(\cdot\,)\in L(I,[0,\infty))$, such that for any $(x,y,u,v)\in K^2\times U^2$ and for almost all $t\in I$ the following inequality is fulfilled

$$|f(t,x,y,u)|+|f_x(\cdot\,)|+|f_y(\cdot\,)|+|f_u(\cdot\,)|+|f_v(\cdot\,)|\le m_{K,U}(t).$$

Further, let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t)$, $t\in I$, satisfying the conditions:

$$\tau(t)<t,\quad \dot\tau(t)>0,\quad \inf\big\{\tau(a):\ \tau\in D\big\}:=\widehat\tau>-\infty,\quad \sup\{\tau^{-1}(b):\tau\in D\}<\infty,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let Φ be the set of continuous initial functions $\varphi(t)\in O$, $t\in I_1=[\widehat\tau,b]$ and let Ω be the set of measurable bounded control functions $u(t)\in U_0$, $t\in I_1$, with $u(I_1)\subset U_0$.

To each element (initial date) $\mu=(t_0,\tau,\varphi,u)\in\Lambda=[a,b)\times D\times\Phi\times\Omega$ we assign the controlled delay functional differential equation

$$\dot x(t)=f\big(t,x(t),x(\tau(t)),u(t),u(\theta(t))\big) \tag{1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0], \tag{2}$$

where $\theta\in D$ is a fixed delay function. The condition (2) is said to be continuous initial condition since always $x(t_0)=\varphi(t_0)$.

Definition 1. Let $\mu=(t_0,\tau,\varphi,u)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat\tau,t_1]$, $t_1\in(t_0,b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\widehat\tau,t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a given element and $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$. Let us introduce the set of variations

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta u) : \ |\delta t_0| \le \alpha, \ \|\delta\tau\| \le \alpha, \ \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ |\lambda_i| \le \alpha, \ i = \overline{1,k}, \ \|\delta u\| \le \alpha \Big\}.$$

Here $\delta t_0 \in I - t_{00}$, $\delta\tau \in D - \tau_0$, $\|\delta\tau\| = \sup\{|\delta\tau(t)| : \ t \in I\}$, $\delta u \in \Omega - u_0$, $\delta\varphi_i \in \Phi - \varphi_0$, $i = \overline{1,k}$, are fixed functions, $\alpha > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1] \times V.$$

Theorem 1. *Let the following conditions hold:*

1) *the function $\varphi_0(t)$, $t \in I_1$, is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;*
2) *the function $f_0(z)$, $z = (t, x, y) \in I \times O^2$, is bounded, where $f_0(t, x, y) = f(t, x, y, u_0(t), u_0(\theta(t)))$;*
3) *there exist the finite limits*

$$\dot{\varphi}_0^- = \dot{\varphi}_0(t_{00}-), \quad \lim_{z \to z_0} f_0(z) = f_0^-, \ \ z \in (a, t_{00}] \times O^2,$$

where $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that

$$\delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^-$, where $V^- = \{\delta\mu \in V : \ \delta t_0 \le 0\}$, and

$$\delta x(t; \delta\mu) = Y(t_{00}; t)[\dot{\varphi}_0^- - f_0^-]\delta t_0 + \beta(t; \delta\mu), \tag{4}$$

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s); t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\, ds +$$

$$+ \int_{t_{00}}^{t} Y(s; t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\, ds + \int_{t_{00}}^{t} Y(s; t)\Big[f_{0u}[s]\delta u(s) + f_{0v}[s]\delta u(\theta(s))\Big]\, ds, \tag{5}$$

$$\lim_{\varepsilon \to 0} \frac{o(t; \varepsilon\delta\mu)}{\varepsilon} = 0 \ \ \textit{uniformly for} \ \ (t, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times V^-, \tag{6}$$

$Y(s; t)$ is $n \times n$-matrix function satisfying the system

$$Y_s(s; t) = -Y(s; t) f_{0x}[s] - Y(\gamma_0(s); t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s), \ \ s \in [t_{00}, t],$$

and the condition

$$Y(s; t) = \begin{cases} H, & s = t, \\ \Theta, & s > t, \end{cases}$$

$f_{0x}[s] = f_{0x}(s, x_0(s), x_0(\tau_0(s)))$, $\gamma_0(s)$ is the inverse function of $\tau_0(s)$; H is the identity matrix and Θ is the zero matrix.

Some comments

The function $\delta x(t;\delta\mu)$ is called the first order sensitivity coefficient and the expression (4) is linear representation of sensitivity coefficient. On the other hand, the function $\delta x(t;\delta\mu)$ is called the first variation of solution $x_0(t)$, $t\in[t_{00},t_{10}+\delta_2]$, and the expression (4) is called the variation formula. The variation formulas play an important role in proving the necessary optimality conditions [2–4]. The questions connected with the variation formulas and the sensitivity analysis for the various classes of differential equations are considered in [2–5].

The addend

$$\int\limits_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\,ds$$

in formula (5) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression

$$Y(t_{00};t)[\dot{\varphi}_0^- - f_0^-]\delta t_0$$

is the effect of continuous initial condition (2) and perturbation of the initial moment t_{00}.

The expression

$$Y(t_{00};t)\delta\varphi(t_{00})+\int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\,ds$$

in formula (5) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$\int\limits_{t_{00}}^{t} Y(s;t)\Big[f_{0u}[s]\delta u(s)+f_{0v}[s]\delta u(\theta(s))\Big]\,ds$$

in formula (5) is the effect of perturbation of the control function $u_0(t)$.

Theorem 2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot{\varphi}_0(t)$ and $f_0(z)$, $z\in I\times O^2$, be bounded. Moreover, there exist the finite limits*

$$\dot{\varphi}_0^+ = \dot{\varphi}_0(t_{00}+),\quad \lim_{z\to z_0} f_0(z)=f_0^+,\ \ z\in[t_{00},b)\times O^2.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\ \delta t_0\geq 0\}$, formula (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot{\varphi}_0^+ - f_0^+)\delta t_0+\beta(t;\delta\mu).$$

Theorem 3. *Let the assumptions of Theorems* 1 *and* 2 *be fulfilled. Moreover,*

$$\dot{\varphi}_0^- = \dot{\varphi}_0^+ := \widehat{\varphi}_0,\ \ f_0^- = f_0^+ := \widehat{f}_0.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2]\times V$ formula (3) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\widehat{\varphi}_0-\widehat{f}_0)\delta t_0+\beta(t;\delta\mu).$$

All assumptions of Theorem 3 are satisfied if: the functions $\dot{\varphi}_0(t)$, $u_0(t)$, $u_0(\theta(t))$ are continuous at the point t_{00} and the function $f(t,x,y,u,v)$ is continuous and bounded. Clearly, in this case $\widehat{\varphi}_0=\dot{\varphi}_0(t_{00})$ and $\widehat{f}_0=f(t_{00},\varphi_0(t_{00}),\varphi_0(\tau_0(t_{00})),u_0(t_{00}),u_0(\theta(t_{00})))$.

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Well-Posedness of the Cauchy Problem for Semilinear Stochastic Differential Functional Equations in Hilbert Spaces

Maksim Vaskouski

Belarusian State University, Minsk, Belarus
E-mail: vaskovskii@bsu.by

In the paper we investigate existence, uniqueness and continuous dependence of solutions on the initial data and right-hand sides of a stochastic differential functional equation

$$dX(t)=AX(t)dt+f(t,X_t)dt+g(t,X_t)dW(t),\ \ t\in R_+, \tag{1}$$

with an initial condition

$$X(t)=\xi(t),\ \ t\in[-h,0], \tag{2}$$

on a certain filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, $t\geq 0$. Here A is a generator of a C_0-semigroup $S(t)$, $t\geq 0$, on a separable Hilbert space H, W is a Wiener process with values in a separable Hilbert space U and with a covariance nuclear nonnegative operator Q, $f:R_+\times\Omega\times C_h\to H$, $g:R_+\times\Omega\times C_h\to L_2(U,H)$ are measurable functions, $X_t(\omega)=\{X(t+\tau,\omega)\mid -h\leq\tau\leq 0\}\in C_h$, $h\geq 0$, is a time of delay, $\xi:[-h,0]\times\Omega\to H$ is a continuous $\mathcal{F}_0$-measurable process, where $R_+=[0,\infty)$, $C_h=C([-h,0],H)$ for $h>0$, $C_h=H$ for $h=0$, $C([-h,0],H)$ is the Banach space of continuous functions $z:[-h,0]\to H$ equipped with the topology of uniform convergence, $L_2(U,H)$ is the space of Hilbert-Schmidt operators from U into H.

Let p and q be fixed real numbers, $p>q>2$.

Define the set $\Phi_{p,q}(E)$ of all mappings $(t,\omega,\varphi)\to z(t,\omega,\varphi)$, $t\in R_+$, $\omega\in\Omega$, $\varphi\in C_h$, with values in a certain separable Hilbert space E, such that for any fixed $\varphi\in C_h$ the process $z(t,\omega,\varphi)$ is measurable and $\mathcal{F}_t$-adapted, for any fixed $(t,\omega)\in R_+\times\Omega$ the function $\varphi\to z(t,\omega,\varphi)$ is continuous at $\varphi\in C_h$, and there hold:

local Lipschitz condition at $\varphi\in C_h$, i.e., for any $a\in R_+$ there exists a constant $q_z(a)$ such that for all $t\in[0,a]$, $r\in\{p,q\}$ and $(\mathcal{F},\beta(C_h))$-measurable random variables $\zeta,\psi:\Omega\to C_h$ with $\operatorname{ess\,sup}\|\zeta\|\leq a$, $\operatorname{ess\,sup}\|\psi\|\leq a$, the inequality

$$E\big(\|z(t,\zeta)-z(t,\psi)\|^r\big)\leq q_z(a)E\big(\|\zeta-\psi\|^r\big)$$

is valid, where $\beta(C_h)$ is the Borel σ-algebra on the space C_h;

linear growth condition at $\varphi\in C_h$, i.e., there exists a continuous function $k_z:R_+\to R_+$ such that for all $t\in R_+$, $r\in\{p,q\}$ and $(\mathcal{F},\beta(C_h))$-measurable random variable $\eta:\Omega\to C_h$ with $E(\|\eta(\omega)\|^p)<\infty$, the inequality

$$E\big(\|z(t,\eta)\|^r\big)\leq k_z(t)\big(1+E(\|\eta\|^r)\big)$$

holds.

Denote by $\mathcal{A}$ the set of all linear operators $A:D(A)\to H$ with dense domain $D(A)$ in H such that A generates a C_0-semigroup of linear operators $S(t)$, $t\geq 0$, on H.

Let $\mathcal{J}_0$ and $\mathcal{J}$ be the set of all continuous $\mathcal{F}_0$-measurable processes $\zeta:[-h,0]\times\Omega\to H$, with

$$E\Big(\sup_{t\in[-h,0]}\|\zeta(t)\|^p\Big)<\infty,$$

and the set of all continuous $\mathcal{F}_t$-adapted processes $Y : [-h,\infty) \times \Omega \to H$, with

$$E\Big(\sup_{t\in[-h,T]} \|Y(t)\|^q\Big) < \infty \ \ \forall\, T \geq 0,$$

respectively.

In further we shall assume that the right-hand sides f and g of Eq. (1) belong to $\Phi_{p,q}(H)$, $\Phi_{p,q}(L_2(U,H))$ respectively and the initial data ξ belongs to $\mathcal{J}_0$.

Definition 1. A mild solution of the Cauchy problem (1), (2) is a continuous $\mathcal{F}_t$-adapted process $X(t)$, $t \geq -h$, such that for any $t \geq 0$

$$P\bigg(\int\limits_0^t \big(\|f(s,X_s)\| + \|g(s,X_s)\|^2\big)\,ds < \infty\bigg) = 1$$

and almost surely for all $t \geq -h$ there holds: $X(t) = \xi(t)$ for $t \in [-h,0]$ and

$$X(t) = S(t)\xi(0) + \int\limits_0^t S(t-s)f(s,X_s)\,ds + \int\limits_0^t S(t-s)g(s,X_s)\,dW(s)$$

for $t \geq 0$.

Definition 2. A mild solution $X(t)$ of the Cauchy problem (1), (2) is called unique, if for any mild solution $Y(t)$ of (1), (2) there holds

$$P(X(t) = Y(t) \ \ \forall\, t \geq -h) = 1.$$

Consider a perturbed Cauchy problem

$$dX(t) = \widetilde{A}X(t)dt + \widetilde{f}(t,X_t)dt + \widetilde{g}(t,X_t)dW(t), \ \ t \in R_+, \tag{3}$$

$$X(t) = \widetilde{\xi}(t), \ \ t \in [-h,0], \tag{4}$$

where $(\widetilde{A},\widetilde{f},\widetilde{g},\widetilde{\xi}) \in \mathcal{A} \times \Phi_{p,q}(H) \times \Phi_{p,q}(L_2(U,H)) \times \mathcal{J}_0$.

Definition 3. A mild solution $X(t)$ of the Cauchy problem (1), (2) depends continuously on the initial data and right-hand sides, if for any $\varepsilon > 0$, $T \in R_+$ exists $\delta = \delta(\varepsilon,T)$ such that for any $(\widetilde{A},\widetilde{f},\widetilde{g},\widetilde{\xi}) \in \mathcal{A} \times \Phi_{p,q}(H) \times \Phi_{p,q}(L_2(U,H)) \times \mathcal{J}_0$, with

$$E\Big(\|\widetilde{f}(t,\varphi) - f(t,\varphi)\|^p + \|\widetilde{g}(t,\varphi) - g(t,\varphi)\|^p\Big) + \\ + \|\widetilde{S}(t) - S(t)\|^p + E\Big(\sup_{s\in[-h,0]} \|\widetilde{\xi}(s) - \xi(s)\|^p\Big) \leq \delta \ \ \forall\,(t,\varphi) \in [0,T] \times C_h,$$

there holds the inequality

$$E\Big(\sup_{t\in[-h,T]} \|\widetilde{X}(t) - X(t)\|^q\Big) \leq \varepsilon,$$

where $\widetilde{S}(t)$ is the semigroup, generated by $\widetilde{A}$, and $\widetilde{X}(t)$ is a mild solution of the Cauchy problem (3), (4).

Definition 4. We say that the Cauchy problem (1), (2) is well-posed if there exists a unique mild solution $X(t)$ and this mild solution depends continuously on the initial data and right-hand sides.

The main result of this paper is the following theorem on the well-posedness of the Cauchy problem (1), (2).

Theorem. *Let $(A,f,g,\xi) \in \mathcal{A} \times \Phi_{p,q}(H) \times \Phi_{p,q}(L_2(U,H)) \times \mathcal{J}_0$, then the Cauchy problem* (1), (2) *is well-posed and the mild solution $X(t)$ of* (1), (2) *belongs to $\mathcal{J}$.*

C o n t e n t s