E ISSN 1512-3391

A. RAZMADZE MATHEMATICAL INSTITUTE
of I. Javakhishvili Tbilisi State University

International Workshop on the Qualitative Theory of Differential Equations

QUALITDE – 2013

December 20 – 22, 2013

Tbilisi, Georgia

ABSTRACTS

Dedicated to the 100th birthday anniversary
of Professor Levan Magnaradze

22.06.1913 – 06.02.2002

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Feedback Stabilization of Complex System Behavior

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1 Introduction

The famous Hodgkin–Huxley neuron model [4] is the first mathematical model describing neural excitation transmission derived from the angle of physics and lays the basis of electrical neurophysiology. FitzHugh–Nagumo equation [3, 7], which is a simplification of Hodgkin-Huxley model, describes the generation and propagation of the nerve impulse along the giant axon of the squid. Based on the finite propagating speed in the signal transmission between the neurons, the following coupled FitzHugh–Nagumo neural system is proposed [5]:

$$\left|\begin{array}{l} \dot{u}_1 = -u_1(u_1-1)(u_1-a) - u_2 + cf(u_3), \\ \dot{u}_2 = b(u_1 - \gamma u_2), \\ \dot{u}_3 = -u_3(u_3-1)(u_3-a) - u_4 + cf(u_1), \\ \dot{u}_4 = b(u_3 - \gamma u_4), \end{array}\right. \tag{1}$$

where a, b, γ are positive constants, $u_{1,2}$ represent transmission variables, and $u_{3,4}$ are receiving variables; c measures the coupling strength, $f \in C^3$, $f(0) = 0$, $f'(0) = 1$. We shall take $f(x) = \tan h(x)$ in our investigation. System (1) is symmetric.

2 Edge of Chaos of Coupled FitzHugh–Nagumo CNN Model

We apply the following constructive algorithm [1, 2] for studying the dynamics of (1):

1. Map coupled FitzHugh–Nagumo system (1) into the associated discrete- space version, which we shall call coupled FitzHugh–Nagumo Cellular Neural Networks (CNN) model [6]:

$$\left|\begin{array}{l} \dfrac{du_j^1}{dt} = -u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + cf(u_j^3), \\ \dfrac{du_j^2}{dt} = b(u_j^1 - \gamma u_j^2), \\ \dfrac{du_j^3}{dt} = -u_j^3(u_j^3-1)(u_j^3-a) - u_j^4 + cf(u_j^1), \\ \dfrac{du_j^4}{dt} = b(u_j^3 - \gamma u_j^4), \quad j = 1, \ldots, n. \end{array}\right. \tag{2}$$

The system is transformed into a system of ordinary differential equations which is identified as the state equations of a CNN with appropriate templates. We map the variables u_1, u_2, u_3 and u_4 into CNN layers [6] such that the state voltage of a CNN cell at a grid point is $u_j^i, i = 1, 2, 3, 4$, $n = M.M$, M is number of the cells.

2. Find the equilibrium points of (2). According to the theory of dynamical systems the equilibrium points $\widehat{u}_j^i$ of (2) are these for which:

$$\left| \begin{array}{l} -u_j^1(u_j^1 - 1)(u_j^1 - a) - u_j^2 + c\tan h(u_j^3) = 0, \\ \ b(u_j^1 - \gamma u_j^2) = 0, \\ -u_j^3(u_j^3 - 1)(u_j^3 - a) - u_j^4 + c\tan h(u_j^1) = 0, \\ \ b(u_j^3 - \gamma u_j^4) = 0. \end{array} \right. \tag{3}$$

Equation (3) may have one, two,three or four real roots $\widehat{u}_j^1$, $\widehat{u}_j^2$, $\widehat{u}_j^3$, $\widehat{u}_j^4$ respectively. In general, these roots are functions of the cell parameters a, b, c, γ. In other words, we have $\widehat{u}_j^i = \widehat{u}_j^i(a, b, c, \gamma)$, $i = 1, 2, 3, 4$. We shall consider only the equilibrium point $E_0 = (0, 0, 0, 0)$.

3. Calculate now the Jacobian matrix of (3) about equilibrium point E_0. In our particular case the associate linear system in a sufficient small neighborhood of the equilibrium point E_0 can be given by

$$\frac{dz}{dt} = DF(E_0)z$$

$DF(E_0) = J$ is the Jacobian matrix of each of the equilibrium points and can be computed by:

$$J_{p,s} = \left. \frac{\partial F_p}{\partial u_s} \right|_{u=E_0}, \quad 1 \le p, s \le n. \tag{4}$$

4. Calculate the trace $\mathrm{Tr}(E_0) = \sum_{q=1}^{N} \lambda_q$. In the equilibrium point $E_0 = (0, 0, 0, 0)$ trace is $\mathrm{Tr}(0, 0, 0, 0) = -a - b\gamma - a - b\gamma = -2(a + b\gamma)$.

Definition 1. Stable and Locally Active Region SLAR(E) at the equilibrium point E_0 for coupled FitzHugh–Nagumo CNN model (2) is such that $\mathrm{Tr} < 0$.

In our particular case we have: $\mathrm{Tr}(0, 0, 0, 0) = -2(a + b\gamma) < 0$ for all a, b, γ positive. Therefore in the equilibrium point $E_0 = (0, 0, 0, 0)$ we have stable and locally active region.

5. Edge of chaos.

We shall identify the edge of chaos domain (EC) in the cell parameter space by using the following definition [1, 2]:

Definition 2. Coupled FitzHugh–Nagumo CNN model is said to be operating on the edge of chaos EC iff there is at least one equilibrium point E_0, which belongs to SLAR(E).

The following theorem then holds:

Theorem 1. *CNN model of coupled FitzHugh–Nagumo system* (1) *is operating in the edge of chaos regime for all a, b and γ positive. For this parameter values there is at least one equilibrium point which belongs to* SLAR(E).

3 Stabilizing Feedback Control for Coupled FitzHugh–Nagumo CNN Model

Let us extend the model (2) by adding to each cell the local linear feedback:

$$\left|\begin{aligned} \frac{du_j^1}{dt} &= -u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + cf(u_j^3) - ku_j^1, \\ \frac{du_j^2}{dt} &= b(u_j^1 - \gamma u_j^2), \\ \frac{du_j^3}{dt} &= -u_j^3(u_j^3-1)(u_j^3-a) - u_j^4 + cf(u_j^1) - ku_j^3, \\ \frac{du_j^4}{dt} &= b(u_j^3 - \gamma u_j^4), \quad j = 1, \dots, n, \end{aligned}\right. \tag{5}$$

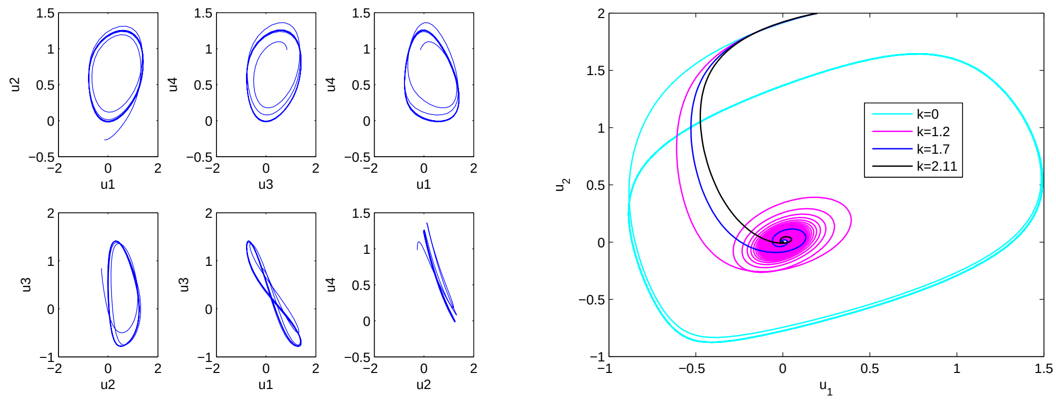
where k is the feedback controls coefficient, which is assumed to be equal for all cells. The problem is to prove that this simple and available for the implementation feedback can stabilize the coupled FitzHugh–Nagumo CNN model. In the following we present a proof of this statement and give sufficient condition on the feedback coefficient values which provide stability of the CNN nonlinear model (5).

As a first step, we examine the the stability conditions of the system (5), linearized in the neighborhood of the zero equilibrium point E_0. This system in a vector-matrix form is given by

$$\frac{dz}{dt} = J(k)z,$$

$J(k)$ is the Jacobian matrix of the controlled CNN in E_0:

$$J(k) = \begin{bmatrix} -(a+k) & -1 & c & 0 \\ b & -b\gamma & 0 & 0 \\ c & 0 & -(a+k) & -1 \\ 0 & 0 & b & -b\gamma \end{bmatrix}. \tag{6}$$



(a) (b)

Figure 1. (a) EC phenomena for CNN model (2); (b) Phase trajectory $u_1 - u_2$ for different values on the feedback coefficient CNN model (5).

Theorem 2. *Let the parameters a, b and γ of coupled FitzHugh–Nagumo CNN system and feedback coefficient k (5) have positive values. Then its linearized in E_0 model (6) is asymptotically stable for all*

$$k > \sqrt{\left(\frac{(b-1)^2}{8b\gamma}\right)^2 + c^2} + \frac{(b-1)^2}{8b\gamma} - a. \tag{7}$$

Acknowledgements

This paper is partially supported by the bilateral joint research project between Bulgarian Academy of Sciences and Israel Academy of Sciences.

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On a Two-Point Boundary Value Problem for Systems of Nonlinear Generalized Ordinary Differential Equations with Singularities

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For the nonlinear generalized system with singularities

$$dx_i = f_i(t, x_1, \dots, x_n)da_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n), \tag{1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \ \ (i = 1, \dots, n_0), \ \ x_i(b-) = 0 \ \ (i = n_0+1, \dots, n), \tag{2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of the desired solution x; $a_i : [a,b] \to \mathbb{R}$ is a nondecreasing function, and $f_i :]a,b[\times \mathbb{R}^n \to \mathbb{R}$ is a function belonging to the local Carathéodory class $\mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n, \mathbb{R}; a_i)$ for every $i \in \{1, \dots, n\}$.

We investigate the question of solvability of the problem (1), (2), when the system (1) has singularities. Singularity is understood in a sense that the components of the vector-function f may have non-integrable components on the boundary points a and b, in general.

The interest to the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1, 2] and references therein).

Basic notation and definitions. $\mathbb{N}_{1n_0} = \{1, \dots, n_0\}$, $\mathbb{N}_{2n_0} = \{1+n_0, \dots, n\}$. $\mathbb{R} =]-\infty, +\infty[$. $\mathbb{R}^{n\times m}$ is the set of all real $n \times m$-matrices. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n \times m$-matrices. $\mathbb{R}^n = R^{n\times 1}$. $r(X)$ the spectral radius of $X \in \mathbb{R}^{n\times n}$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function X at the point t. $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ with bounded variation components. $\mathrm{BV}_{loc}(]a,b[\,, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X :]a,b[\to \mathbb{R}^{n\times m}$ with bounded variation components on every close interval from $[a,b]$.

If $\alpha \in \mathrm{BV}([a,b], \mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1,2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ ($t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}}$) is the set of all points from $[a,b]$ for which $d_m\alpha(t) \neq 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : \ t \in D_{\alpha m}\}$ ($m = 1,2$). If $\beta \in \mathrm{BV}([a,b], \mathbb{R})$, then

$$\nu_{\alpha m \beta j} = \max\bigg\{ d_j\beta(t_{\alpha m l}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m} \bigg\}$$

$(j, m = 1,2)$; here $t_{\alpha 20} = a-1$, $t_{\alpha 1 n_{\alpha 1}+1} = b+1$.

g_c is the continuous part of a function $g : [a,b] \to \mathbb{R}$, and $D_g = \{t \in [a,b] : d_1g(t) + d_2g(t) \neq 0\}$.

Integrals are understood in the Kurzweil–Stieltjes sense (see, [2]).

$L^p_{loc}([a,b], \mathbb{R}; g)$ $(1 \le p < +\infty)$ and $L^{+\infty}_{loc}([a,b], \mathbb{R}; g)$ are the standard spaces of functions.

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}$, then $\mathrm{Car}([a,b] \times D_1, D_2; g)$ is the Carathéodory class, i.e., the set of all mappings $f : [a,b] \times D_1 \to D_2$ such that:(i) the function $f(\,\cdot\,, x) : [a,b] \to D_2$ is $\mu(g)$-measurable for every $x \in D_1$; (ii) the function $f(t, \cdot\,) : D_1 \to D_2$ is continuous for $\mu(g)$-almost every $t \in [a,b]$, and $\sup\big\{|f(\,\cdot\,, x)| : \ x \in D_0\big\} \in L([a,b], R; g)$ for every compact $D_0 \subset D_1$; $\mathrm{Car}_{loc}(]a,b[\,\times D_1, D_2; g)$ is the set of all mappings $f :]a,b[\,\times D_1 \to D_2$ the restriction of which on every closed interval $[c,d]$ of $]a,b[$ belongs to $\mathrm{Car}([c,d] \times D_1, D_2; g)$.

We assume that $a_i : I_i \to \mathbb{R}$ $(i = 1,\dots,n)$ are nondecreasing functions, and $f_i \in \mathrm{Car}_{loc}(I_i \times \mathbb{R}^n, \mathbb{R}^n; a_i)$ $(i = 1,\dots,n)$, where $I_i =]a,b]$ if $i \in \mathbb{N}_{1n_0}$ and $I_i = [a,b[$ if $i \in \mathbb{N}_{2n_0}$.

A vector-function $(x_i)_{i=1}^n$, $x_i \in \mathrm{BV}_{loc}(I_i,\mathbb{R})$ $(i = 1,\dots,n)$, is said to be a solution of the system (1) if $x_i(t) = x_i(s) + \int\limits_s^t f_l(\tau, x_1(\tau),\dots,x_n(\tau))\, da_i(\tau)$ for $s < t$; $s,t \in I_i$ $(i = 1,\dots,n)$.

A solution $(x_i)_{i=1}^n$ of the system (1) is said to be a solution of the problem (1), (2) if one-sided limits $x_i(a+)$ $(i \in \mathbb{N}_{1n_0})$ and $x_i(b-)$ $(i \in \mathbb{N}_{2n_0})$ exist and the equalities (2) hold.

Let $b_{il} : I_i \to \mathbb{R}$ $(i,l = 1,\dots,n)$ be nondecreasing functions. A vector-function $(x_i)_{i=1}^n$, $x_i \in \mathrm{BV}(I_i,\mathbb{R})$ $(i = 1,\dots,n)$, is said to be a solution of the system of differential inequalities $dx_i(t) \le \sum\limits_{l=1}^n x_l(t)\, db_{il}(t)$ for $t \in I_i$ $(i = 1,\dots,n)$, if it satisfies the corresponding integral inequalities.

We assume that $\det(1 + (-1)^j d_j a_i(t)) \ne 0$ for $t \in I_i$ $(j = 1,2;\ i = 1,\dots,n)$. This inequalities guarantee the unique solvability of the Cauchy problem for the corresponding equation. By $\gamma_\beta(\,\cdot\,,s)$ we denote the unique solution of the Cauchy problem $d\gamma(t) = \gamma(t)d\beta(t)$, $\gamma(s) = 1$.

Definition. A matrix-function $C = (c_{il})_{i,l=1}^n$, $c_{il} \in \mathrm{BV}_{loc}(I_i,\mathbb{R}_+)$ $(i,l = 1,\dots,n)$ belongs to the set $\mathcal{U}(a+,b-;a_1,\dots,a_n;n_0)$ if the system

$$\operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) dx_i(t) \le \sum_{l=1}^n c_{il}(t) x_l(t)\, da_i(t) \ \text{ for } \ t \in I_i \ (i = 1,\dots,n)$$

has no nontrivial nonnegative solution satisfying the condition (2).

Theorem. *Let*

$$f_i(t,x_1,\dots,x_n)\operatorname{sgn}\left(\left(n_0 + \frac{1}{2} - i\right)x_i\right) \le -b_i(t)|x_i| +$$

$$+ \sum_{l=1}^n \eta_{il}(t)|x_l| \ \text{ for } \ \mu(a_{ic})\text{-a.e. } t \in I_i \ \text{ and for all } \ t \in D_{a_i},\ \ x_k \in \mathbb{R} \ \ (i,k = 1,\dots,n), \tag{3}$$

$$(-1)^j f_i(t,x_1,\dots,x_n) d_2 a_i(t) \operatorname{sgn}\left(x_i + (-1)^j f_i(t,x_1,\dots,x_n) d_j a_i(t)\right) \le$$

$$\le -b_i(t)|x_i| + \sum_{l=1}^n \eta_{il}(t)|x_l| \ \text{ for } \ t \in I_i \ \ (j = 1,2;\ \ i \in \mathbb{N}_{3-j\,n_0}), \tag{4}$$

where $\eta_{il} \in L_{loc}(I_i,\mathbb{R};a_i)$, $b_i \in L_{loc}(I_i,\mathbb{R}_+;a_i)$ $(i,l = 1,\dots,n)$. *Let, moreover,* $C = (c_{il})_{i,l=1}^n \in \mathcal{U}(a+,b-;a_1,\dots,a_n;n_0)$,

$$\lim_{t\to a+} b_i(t) d_2 a_i(t) < 1, \ \ \lim_{t\to a+}\ \limsup_{k\to\infty} \gamma_{\alpha_i}(t, a + 1/k) = 0 \ \ (i \in \mathbb{N}_{1n_0}), \tag{5}$$

$$\lim_{t\to b-} b_i(t) d_1 a_i(t) < 1, \ \ \lim_{t\to b-}\ \limsup_{k\to\infty} \gamma_{\alpha_i}(t, b - 1/k) = 0 \ (i \in \mathbb{N}_{2n_0}), \tag{6}$$

where

$$\alpha_i(t) \equiv \int_{c_0}^t b_i(\tau)\, da_i(\tau), \quad c_{il}(t) \equiv \int_a^t \eta_{il}(\tau)\, da_i(\tau) \ \ (i,l = 1,\dots,n), \ \ c_0 = (a+b)/2.$$

Then the problem (1), (2) *is solvable.*

Corollary. *Let the conditions* (3)–(6) *hold, where the functions* a_i $(i = 1,\dots,n)$ *have no more than a finite number of points of discontinuity,* $b_i \in L_{loc}(I_i,\mathbb{R}_+;a_i)$, $\alpha_i(t) \equiv \int\limits_{c_0}^t b_i(\tau)\, da_i(\tau)$,

$$\int_a^t \eta_{il}(\tau)\, da_i(\tau) \equiv \int_c^t h_{il}(\tau)\, d\beta_l(\tau) \ \ (i,l = 1,\dots,n),$$

$c_0 = (a+b)/2$; β_l $(l = 1,\dots,n)$ are nondecreasing functions, $h_{ii} \in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i \neq l;\ i,l = 1,\dots,n)$; $1 \le \mu \le +\infty$. Let, moreover, $r(\mathcal{H}) < 1$, where $\mathcal{H} = (\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ is the $3n \times 3n$-matrix defined by $\mathcal{H}_{j+1\,m+1} = (\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)})_{i,k=1}^n$ $(j,m = 0,1,2)$, $\xi_{ij} = (s_j(\beta_i)(b) - s_j(\beta_i)(a))^{\frac{1}{\nu}}$ $(j = 0,1,2;\ i = 1,\dots,n)$; $\lambda_{k0i0} = (\frac{4}{\pi^2})^{\frac{1}{\nu}}\xi_{k0}^2$ if $s_0(\beta_i)(t) \equiv s_0(\beta_k)(t)$, $\lambda_{k0i0} = \xi_{k0}\xi_{i0}$ if $s_0(\beta_i)(t) \not\equiv s_0(\beta_k)(t)$, $\lambda_{kmij} = \xi_{km}\xi_{ij}$ if $m^2 + j^2 > 0$ and $mj = 0$ $(j,m = 0,1,2)$, $\lambda_{kmij} = \big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\big)^{\frac{1}{\nu}}$ $(j,m = 1,2)$ $(i,k = 1,\dots,n)$; $1/\mu + 2/\nu = 1$. Then the problem (1), (2) *is solvable.*

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Project No. GNSF/ST09_175_3-101).

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On the Nonlinear Boundary Value Problems for Systems of Impulsive Equations with Finite Points of Impulses Actions

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We consider the general nonlinear boundary value problem for the system of impulsive equations with finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l = 1,\dots,m_0); \tag{2}$$

$$h(x) = 0, \tag{3}$$

where $-\infty < a < \tau_1 < \cdot < \tau_{m_0} \leq b < \infty$, m_0 is a natural number, $f \in \operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, $I_l : R^n \to \mathbb{R}^n$ $(l = 1,\dots,m_0)$ are continuous operators, and $h : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

We give the Conti–Opial type theorems (among them effective sufficient conditions) for the solvability of the problem which are analogous to ones given in [1] (see also the references therein) for ordinary differential equations.

Basic notation and definitions. $\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$; $|X| = (|x_{ij}|)_{i,j}^{n,m}$. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n\times m$-matrices. $I_{n\times n}$ is the identity $n\times n$-matrix. $r(X)$ is the spectral radius of $X \in \mathbb{R}^{n\times n}$. $\mathbb{R}^n = \mathbb{R}^{n\times 1}$.

$X(t-)$ and $X(t+)$ are one-sided limits of the matrix-function X at the point t.

$C([a,b],\mathbb{R}^n;m_0)$ is the set of all vector-functions $x : [a,b] \to \mathbb{R}^n$, having the one sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,\dots,m_0)$, whose restrictions to every closed interval from $[a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}$ is continuous. $C_s([a,b],\mathbb{R}^n;m_0)$ is the Banach space with the norm $\|x\|_s = \sup\{\|x(t)\| : t \in [a,b]\}$.

$\widetilde{C}([a,b],\mathbb{R}^n;m_0)$ is the set of all matrix-functions $x : [a,b] \to \mathbb{R}^n$, having the one sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,\dots,m_0)$, whose restrictions to an arbitrary closed interval from $[a,b] \setminus \{\tau_k\}_{k=1}^m$ is absolutely continuous.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$. An operator $\varphi : C([a,b],\mathbb{R}^{n\times m};m_0) \to R^n$ is called nondecreasing if for every x, y such that $x(t) \leq y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a,b]$.

$\operatorname{Car}([a,b]\times D_1, D_2)$, where $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, is the standard Carathéodory class of all mappings $F : [a,b]\times D_1 \to D_2$; $\operatorname{Car}^0([a,b]\times D_1,D_2)$ is the set of all mappings F such that the matrix-function $F(.,x(.))$ is measurable for every vector-function $x \in C([a,b],D_1;m_0)$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}([a,b],\mathbb{R}^n;m_0)$ satisfying both the system (1) a. e. on $[a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}$ and the relation (2) for every $k \in \{1,\dots,m_0\}$.

Definition. Let $\ell : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ such that $|P(t,x)| \le \Phi(t)$ for a.e. $[a,b]$ and $x \in \mathbb{R}^n$;

(b) $|J_l(x)| \le \Psi_l$ for $x \in \mathbb{R}^n$ $(l=1,\dots,m_0)$;

(c) $\det(I_{n\times n}+G_l) \ne 0$ $(l=1,\dots,m_0)$ and the problem

$$\frac{dx}{dt} = A(t)x,\ \ x(\tau_l+)-x(\tau_l-) = G_l x(\tau_l)\ \ (l=1,\dots,m_0);\ \ |\ell(x)| \le \ell_0(x) \tag{4}$$

has only the trivial solution for every matrix-function $A\in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices $G_1,\dots,G_{m_0}$ for which there exists a sequence $y_k\in\widetilde{C}([a,b],\mathbb{R}^n;m_0)$ $(k=1,2,\dots)$ such that $\lim\limits_{k\to+\infty}\int_a^t (P(\tau,y_k(\tau))-A(\tau))\,d\tau=0$ uniformly on $[a,b]$ and $\lim\limits_{k\to+\infty} J_l(y_k(\tau_l))=G_l$ $(l=1,\dots,m_0)$.

Theorem 1. *Let the conditions*

$$\|f(t,x)-P(t,x)x\| \le q\big(t,\|x\|\big)\ \ \text{a.e. on}\ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\},\ \ x\in\mathbb{R}^n, \tag{5}$$

$$\|I_l(x)-J_l(x)x\| \le \beta_l(\|x\|)\ \ \text{for}\ \ x\in\mathbb{R}^n\ \ (l=1,\dots,m_0), \tag{6}$$

$$|h(x)-\ell(x)| \le \ell_0(x)+\ell_1(\|x\|_s)\ \ \text{for}\ \ x\in\mathrm{BV}([a,b],\mathbb{R}^n) \tag{7}$$

hold, where $\ell : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators, the pair $(P,\{J_l\}_{l=1}^{m_0})$ satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l \in C(\mathbb{R}_+,\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that

$$\limsup_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\| + \int_a^b q(t,\rho)\,dt + \sum_{l=1}^{m_0}\beta_l(\rho)\bigg) < 1. \tag{8}$$

Then the problem (1), (2); (3) *is solvable.*

Theorem 2. *Let the conditions* (5)–(8), *$P_1(t) \le P(t,x) \le P_2(t)$ for a.a. $t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}$ and $x\in\mathbb{R}^n$, and $J_{1l} \le I_k(x) \le J_{2l}$ for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$ hold, where $P\in\mathrm{Car}^0([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([a,b],\mathbb{R}^{n\times n})$ $(i=1,2)$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$, $\ell : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; $q\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l\in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1\in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function. Let, moreover, the inequalities in* (c) *of definition hold and the problem* (4) *have only the trivial solution for every matrix-function $A\in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that $P_1(t)\le A(t)\le P_2(t)$ for a.a. $t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}$ and $x\in\mathbb{R}^n$, and $J_{1l}\le G_l\le J_{2l}$ for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$. Then the problem* (1), (2); (3) *is solvable.*

Remark. Theorem 2 is interesting only in the case when $P\notin\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because otherwise it follows from Theorem 1.

Corollary 1. *Let the conditions* (5)–(8) *hold, where $P(t,x)\equiv P(t)$, $P\in L([a,b],\mathbb{R}^{n\times n})$, $J_l(x)\equiv J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ are constant matrices, $\ell_0(x)\equiv 0$, $\ell : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n$ is a linear continuous operator, $q\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l\in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1\in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function. Let, moreover, $\det(I_{n\times n}+J_l)\ne 0$ $(l=1,\dots,m_0)$ and the homogeneous impulsive problem $dx/dt=P(t)x$, $x(\tau_l+)-x(\tau_l-)=J_l x(\tau_l)$ $(l=1,\dots,m_0)$; $\ell(x)=0$ have only the trivial solution. Then the problem* (1), (2); (3) *is solvable.*

We give the effective conditions for the solvability of the problem (1), (2); (3).

For every matrix-function $X \in L([a,b], \mathbb{R}^{n\times n})$ and constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we introduce the operators $\big[(X, Y_1, \dots, Y_{m_0})(t)\big]_0 \equiv I_{n\times n}$, $\big[(X, Y_1, \dots, Y_{m_0})(t)\big]_{i+1} \equiv \int\limits_a^t X(\tau)\big[(X, Y_1, \dots, Y_{m_0})(\tau)\big]_i\, d\tau + \sum\limits_{a\le\tau_l<t} Y_l\big[(X, Y_1, \dots, Y_{m_0})(\tau_l)\big]_i$ $(i = 0, 1, \dots)$.

Corollary 2. *Let the matrix-function $P \in L([a,b], \mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+, \mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x) \equiv 0$, and $\ell(x) \equiv \int\limits_a^b d\mathcal{L}(t)\cdot x(t)$, $\mathcal{L} \in L([a,b], \mathbb{R}^{n\times n})$. Let, moreover, there exist natural numbers k and m such that the matrix $M_k = -\sum\limits_{i=0}^{k-1}\int\limits_a^b d\mathcal{L}(t)\cdot\big[(P, J_l, \dots, J_{m_0})(t)\big]_i$ is nonsingular and $r(M_{k,m}) < 1$, where $M_{k,m} = \big[(|P|, |J_1|, \dots, |J_{m_0}|)(b)\big]_m + \sum\limits_{i=0}^{m-1}\big[(|P|, |J_1|, \dots, |J_{m_0}|)(b)\big]_i \int\limits_a^b dV(M_k^{-1}\mathcal{L})(t)\cdot\big[(|P|, |J_1|, \dots, |J_{m_0}|)(t)\big]_k$. Then the problem (1), (2); (3) is solvable.*

Corollary 3. *Let the matrix-function $P \in L([a,b], \mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+, \mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x) \equiv 0$, and $\ell(x) \equiv \sum\limits_{j=1}^{n_0}\mathcal{L}_j x(t_j)$, $t_j \in [a,b]$, $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$. Let, moreover, there exist natural numbers k and m such that the matrix $M_k = \sum\limits_{j=1}^{n_0}\sum\limits_{i=0}^{k-1}\mathcal{L}_j\big[(P_0, J_l, \dots, J_{m_0})(t_j)\big]_i$ is nonsingular and $r(M_{k,m}) < 1$, where $M_{k,m} = \big[(|P|, |J_l|, \dots, |J_{m_0}|)(b)\big]_m + \Big(\sum\limits_{i=0}^{m-1}\big[(|P|, |J_l|, \dots, |J_{m_0}|)(b)\big]_i\Big)\cdot\sum\limits_{j=1}^{n_0}|M_k^{-1}\mathcal{L}_j|\Big[(|P|, |J_l|, \dots, |J_{m_0}|)(t_j)\Big]_k$. Then the problem (1), (2); (3) is solvable.*

Corollary 4. *Let the matrix-function $P \in L([a,b], \mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+, \mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x) \equiv 0$, and $\ell(x) \equiv \sum\limits_{j=1}^{n_0}\mathcal{L}_j x(t_j)$, $t_j \in [a,b]$, $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$. Let, moreover, $\det\Big(\sum\limits_{j=1}^{n_0}\mathcal{L}_j\Big) \neq 0$ and $r\big(\mathcal{L}_0\cdot V(A)(b)\big) < 1$, where $\mathcal{L}_0 = I_{n\times n} + \Big|\Big(\sum\limits_{j=1}^{n_0}\mathcal{L}_j\Big)^{-1}\Big|\cdot\sum\limits_{j=1}^{n_0}|\mathcal{L}_j|$ and $A = \int\limits_a^b|P(t)|\,dt + \sum\limits_{l=1}^{m_0}|J_l|$. Then the problem (1), (2); (3) is solvable.*

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

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On the General Nonlinear Boundary Value Problems for Systems of Discrete Equations

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There are considered the problem on the solvability of the system of nonlinear discrete equations

$$\Delta y(l-1)=g(l,y(l),y(l-1)) \ \text{ for } \ l\in\mathbb{N}_{m_0} \tag{1}$$

under the boundary value condition

$$\zeta(y)=0, \tag{2}$$

where $m_0\geq 2$ is a fixed natural number, $g\in\mathrm{Car}(\mathbb{N}_{m_0}\times\mathbb{R}^n,\mathbb{R}^n)$, and $\zeta:E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

We give the Conti–Opial type theorems (among them effective sufficient conditions) for the solvability of the problem which are analogous to ones given in [1] (see also the references therein) for ordinary differential equations.

Basic Notation and Definitions

$\mathbb{N}=\{1,2,\dots\}$, $\mathbb{N}_0=\{0,1,\dots\}$, $\mathbb{Z}$ is the set of all integers. If $m\in\mathbb{N}$, then $\mathbb{N}_m=\{1,\dots,m\}$, $\widetilde{\mathbb{N}}_m=\{0,1,\dots,m\}$. $\mathbb{R}=\,]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$. $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$; $|X|=(|x_{ij}|)_{i,j}^{n,m}$. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n\times m$-matrices. $I_{n\times n}$ is the identity $n\times n$-matrix. $r(X)$ is the spectral radius of $X\in\mathbb{R}^{n\times n}$. $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

$E(J,\mathbb{R}^{n\times m})$, where $J\subset\mathbb{Z}$, is the space of all matrix-functions $Y=(y_{ij})_{i,j=1}^{n,m}:J\to\mathbb{R}^{n\times m}$ with the norm $\|Y\|_J=\max\big\{\|Y(l)\|:\ l\in J\big\}$, $|Y|_J=(|y_{ij}|_J)_{i,j=1}^{n,m}$.

Δ is the difference operator of the first order, i.e. $\Delta Y(k-1)=Y(k)-Y(k-1)$ for $Y\in E(\widetilde{\mathbb{N}}_l,\mathbb{R}^{n\times m})$, $k\in\mathbb{N}_l$; If Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0)=O_{n\times m}$, or $Y(l)=O_{n\times m}$, respectively, if it is necessary.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is said to be positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$; if, in addition, the spaces are partially ordered, then the operator g is called nondecreasing if $g(x)\leq g(y)$ for every $x,y\in B_1$ such that $x\leq y$.

If $J\subset\mathbb{Z}$, $D_1\subset\mathbb{R}^n$ and $D_2\subset\mathbb{R}^{n\times m}$, then $\mathrm{Car}(J\times D_1,D_2)$ is the discrete Carathéodory class, i.e., the set of all mappings $F:J\times D_1\to D_2$ such that the function $F(j,\cdot\,):D_1\to D_2$ is continuous for every $j\in J$.

By a solution of the difference problem (1), (2) we understand a vector-function $y\in\mathrm{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ satisfying both the system (1) for $i\in\{1,\dots,m_0\}$ and the boundary value condition (2).

Definition 1. Let $\mathcal{L}:E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}^n$ be a linear continuous operator, and let $\mathcal{L}:E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}_+^n$ be a positive homogeneous operator. We say that a pair (G_1,G_2), consisting of matrix-functions $G_j\in\mathrm{Car}(\mathbb{N}_{m_0}\times\mathbb{R}^{2n},\mathbb{R}^{n\times n})$ $(j=1,2)$, satisfy the Opial condition with respect to the pair $(\mathcal{L},\mathcal{L}_0)$ if:

(a) there exist a matrix-function $\Phi \in E(\mathbb{N}_{m_0}, \mathbb{R}^n_+)$ such that $|G_j(l,x,y)| \le \Phi(l)$ for $x, y \in \mathbb{R}^n$ ($j = 1, 2$; $l = 1, \dots, m_0$);

(b) $\det\big(I_{n\times n} + (-1)^j B_j(l)\big) \neq 0$ ($j = 1, 2$; $l = 1, \dots, m_0$) and the problem

$$\Delta y(l-1) = B_1(l)y(l) + B_2(l)y(l-1) \ \ (l \in \mathbb{N}_{m_0}), \ \ |\mathcal{L}(y)| \le \mathcal{L}_0(y) \tag{3}$$

has only the trivial solution for every matrix-functions $B_j \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ ($j = 1, 2$) for which there exists a sequences $x_k, y_k \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ ($k = 1, 2, \dots$) such that

$$\lim_{k\to+\infty} G_j(l, x_k(l), y_k(l)) = B_j(l) \ \ (j = 1, 2; \ \ l = 1, \dots, m_0).$$

Theorem 1. *Let the conditions*

$$\big\|g(l,x,y) - G_1(l,x,y)x - G_2(l,x,y)y\big\| \le \alpha\big(l, \|x\| + \|y\|\big) \ \ \text{for } l \in \mathbb{N}_{m_0}, \ \ x, y \in \mathbb{R}^n, \tag{4}$$

$$\big|\zeta(y) - \mathcal{L}(y)\big| \le \mathcal{L}_0(y) + \ell_1\big(\|y\|_{m_0}\big) \ \ \text{for } \ y \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \tag{5}$$

hold, where $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{L}_0 : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators, the pair (G_1, G_2) satisfies the Opial condition with respect to the pair $(\mathcal{L}, \mathcal{L}_0)$; $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing vector-function such that

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\Big(\|\ell_1(\rho)\| + \sum_{l=1}^{m_0} \alpha(l,\rho)\Big) < 1. \tag{6}$$

Then the problem (1), (2) *is solvable.*

Theorem 2. *Let the conditions* (4)–(6) *and $P_{j1}(l) \le G_j(l,x,y) \le P_{j2}(l)$ for $l \in \mathbb{N}_{m_0}$, $x, y \in \mathbb{R}^n$ $(j=1,2)$ hold, where $P_{j1}, P_{j2} \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$ $(j=1,2)$, $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{L}_0 : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; and $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing vector-function. Let, moreover, the inequalities in* (c) *of Definition* 1 *and the problem* (3) *have only the trivial solution for every matrix-functions B_1 and B_2 from $E(\mathbb{N}_{m_0}, \mathbb{R}^n)$ such that $P_{j1}(l) \le B_j(l) \le P_{j2}(l)$ for $l \in \mathbb{N}_{m_0}$ $(j = 1, 2)$. Then the problem* (1), (2) *is solvable.*

Remark. Theorem 2 is interesting only in the case when $G_j(l, \cdot, \cdot) \notin C(\mathbb{R}^{2n}, \mathbb{R}^{n\times n})$ for some $j \in \{1, 2\}$ and $l \in \{1, \dots, m_0\}$, because it immediately follows from Theorem 1 in the case when $G_j \in \mathrm{Car}(\mathbb{N}_{m_0} \times \mathbb{R}^{2n}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$.

Corollary 1. *Let the conditions* (4)–(6) *hold, where $G_j(l,x,y) \equiv G_j(l)$ $(j = 1, 2)$; $G_1, G_2, \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$; $\mathcal{L}_0(y) \equiv 0$, $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear continuous operator; $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing function. Let, moreover, $\det\big(I_{n\times n} + (-1)^j G_j(l)\big) \neq 0$ for $l \in \mathbb{N}_{m_0}$ $(j = 1, 2)$ and the problem $\Delta y(l-1) = G_1(l)y(l) + G_2(l)y(l-1)$ $(l \in \mathbb{N}_{m_0})$, $\mathcal{L}(y) = 0$ have only the trivial solution. Then the problem* (1), (2) *is solvable.*

We give the effective conditions for the solvability of the problem (1), (2).

On the set $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n}) \times E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times})$ we introduce the operators by the following way. If $G_1, G_2 \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ and, in addition, $\det(I_{n\times n} + G_2(l)) \neq 0$ $(l = 1, \dots, m_0)$, then we assume

$$\big[(G_1, G_2)(l)\big]_0 \equiv I_n, \ \ \big[(G_1, G_2)(l)\big]_k \equiv -\sum_{i=l+1}^{m_0} \big(G_1(i) + G_2(i+1)\big)\times$$
$$\times\big(I_n + G_2(i)\big)^{-1}\big[(G_1, G_2)(i)\big]_{k-1} \ \ (k = 1, 2, \dots);$$

$$V_1(G_1,G_2)(l) \equiv \sum_{i=l+1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big|,$$

$$V_{k+1}(G_1,G_2)(l) \equiv \sum_{i=l+1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big| \cdot V_k(G_1,G_2)(i) \ \ (k=1,2,\dots).$$

Theorem 3. *Let the conditions* (4)–(6) *hold, where* $G_j(l,x,y) \equiv G_j(l)$ $(j=1,2)$*;* $G_1,G_2,\in E(\mathbb{N}_{m_0},\mathbb{R}^n)$*;* $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+,\mathbb{R}_+)$ *is a function nondecreasing in the second variable,* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *is a nondecreasing function;* $\mathcal{L}_0(y) \equiv 0$*;* $\mathcal{L}(y) \equiv \sum\limits_{j=1}^{n_0} L_j y(k_j)$*,* n_0 *is a natural number,* $k_j \in \widetilde{\mathbb{N}}_{m_0}$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$*. Let, moreover,* $\det\big(I_{n\times n}+(-1)^j G_j(l)\big) \neq 0$ *for* $l \in \mathbb{N}_{m_0}$ $(j=1,2)$ *and there exist natural numbers* k *and* m *such that* $\det(M_k) \neq 0$ *and* $r(M_{k,m})<1$*, where*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1} L_j\big(I_n+G_2(k_j+1)\big)^{-1}\big[(G_1,G_2)(k_j)\big]_i, \quad M_{k,m} = V_m(G_1,G_2)(0)+$$
$$+\sum_{i=0}^{m-1} \big|\,\big[(G_1,G_2)(\,\cdot\,)\big]_i\big|_{\widetilde{\mathbb{N}}_{m_0}} \cdot \sum_{j=1}^{n_0} |M_k^{-1}L_j|\big(I_n+G_2(k_j+1)\big)^{-1} V_k(G_1,G_2)(k_j).$$

Then the problem (1), (2) *is solvable.*

Corollary 2. *Let the conditions* (4)–(6) *hold, where* $G_j(l,x,y) \equiv G_j(l)$ $(j=1,2)$*;* $G_1,G_2,\in E(\mathbb{N}_{m_0},\mathbb{R}^n)$*,* $\det\big(I_{n\times n}+(-1)^j G_j(l)\big) \neq 0$ *for* $l \in \mathbb{N}_{m_0}$ $(j=1,2)$*;* $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+,\mathbb{R}_+)$ *is a function nondecreasing in the second variable,* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *is a nondecreasing function;* $\mathcal{L}_0(y) \equiv 0$*;* $\mathcal{L}(y) \equiv \sum\limits_{j=1}^{n_0} L_j y(k_j)$*,* n_0 *is a natural number,* $k_j \in \widetilde{\mathbb{N}}_{m_0}$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$*. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0} L_j\big(I_n+G_2(k_j+1)\big)^{-1}\Big) \neq 0$$

and $r(L_0M_0)<1$*, where*

$$L_0 = I_n + \Big|\Big(\sum_{j=1}^{n_0} L_{1j}\big(I_n+G_2(k_j+1)\big)^{-1}\Big)^{-1}\Big| \cdot \sum_{j=1}^{n_0} \big|L_j\big(I_n+G_1(k_j)\big)^{-1}\big|;$$

$$M_0 = \sum_{i=1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big|.$$

Then the problem (1), (2) *is solvable.*

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

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On Existence of Quasi-Periodic Solutions to a Nonlinear Higher-Order Differential Equation

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1 Introduction

The paper is devoted to the existence of oscillatory and non-oscillatory quasi-periodic, in some sense, solutions to the higher-order Emden–Fowler type differential equation

$$y^{(n)} + p_0\,|y|^k \operatorname{sgn} y = 0, \quad n > 2, \ \ k \in \mathbb{R}, \ \ k > 1, \ \ p_0 \neq 0. \tag{1}$$

A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1]. Results about the existence of solutions with special asymptotic behavior are contained in [2]–[8].

2 On Existence of Quasi-Periodic Oscillatory Solutions

Put

$$\alpha = \frac{n}{k-1}\,. \tag{2}$$

Theorem 1. *For any integer $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}}(x^* - x)^{-\alpha} h(\log(x^* - x)), \ \ -\infty < x < x^* \tag{3}$$

is a solution to equation (1)*.*

Corollary 1. *For any integer even $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}}(x - x^*)^{-\alpha} h(\log(x - x^*)), \ \ x^* < x < \infty \tag{4}$$

is a solution to equation (1)*.*

Corollary 2. *For any integer odd $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 < 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{\frac{1}{k-1}}(x - x^*)^{-\alpha} h(\log(x - x^*)), \ \ x^* < x < \infty \tag{5}$$

is a solution to equation (1)*.*

3 On Existence of Positive Solutions with Non-power Asymptotic Behavior

The existence of such non-oscillatory solutions was also proved.

For equation (1) with $p_0 = -1$ it was proved [4] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y = (x^* - x)^{-\alpha} h(\log(x^* - x)), \tag{6}$$

where α is defined by (2) and h is a positive periodic non-constant function on $\mathbf{R}$.

A similar result was also proved [4] about Kneser solutions, i.e. those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j y^{(j)}(x) > 0$ for $0 \le j < n$. Namely, if $p_0 = (-1)^{n-1}$, then for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y(x) = (x - x_*)^{-\alpha}\ h(\log(x - x_*)),$$

where h is a positive periodic non-constant function on $\mathbf{R}$.

Still it was not clear how large n should be for the existence of that type of positive solutions.

Theorem 2 ([8])**.** *If $12 \le n \le 14$, then there exists $k > 1$ such that equation* (1) *with $p_0 = -1$ has a solution $y(x)$ such that*

$$y^{(j)}(x) = (x^* - x)^{-\alpha - j} h_j(\log(x^* - x)), \quad j = 0, 1, \dots, n-1,$$

where α is defined by (2) *and h_j are periodic positive non-constant functions on $\mathbf{R}$.*

Remark. Computer calculations give approximate values of α. They are, with the corresponding values of k, as follows:

if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;

if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;

if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.

Corollary 3 ([8])**.** *If $12 \le n \le 14$, then there exists $k > 1$ such that equation* (1) *with $p_0 = (-1)^{n-1}$ has a Kneser solution $y(x)$ satisfying*

$$y^{(j)}(x) = (x - x_0)^{-\alpha - j} h_j(\log(x - x_0)), \quad j = 0, 1, \dots, n-1,$$

with periodic positive non-constant functions h_j on $\mathbf{R}$.

Acknowledgement

The work was partially supported by the Russian Foundation for Basic Researches (Grant 11-01-00989).

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Sufficiency Conditions for the Asymptotic Stability of Solutions of Linear Homogeneous Nonautonomous Differential Equation of Second-Order

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In this paper we consider the problem on the stability of the real linear homogeneous differential equation (LHDE) of second order

$$y'' + p(t)y' + q(t)y = 0 \ \ (t \in I), \tag{1}$$

provided that the roots $\lambda_i(t)$ $(i = \overline{1,2})$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \ \ (t \in I), \quad \int\limits_{t_0}^{+\infty} \lambda_i(t)\,dt = -\infty \ \ (i = \overline{1,2}), \tag{2}$$

and there are finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = \overline{1,2})$. Stability of the equation (1) is investigated by the way of its reduction to the equivalent system, which is led to an almost triangular form with the help of the linear transformation. Below we give the obtained by us results.

Theorem 1. *In the case of* $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$ *the trivial solution of the equation* (1) *is asymptotically stable. It is sufficient to assume that* $p(t), q(t) \in C_I$.

This theorem follows from the results of Lyapunov A. M.

Theorem 2. *Let the condition* (2) *hold for* $i = 2$, *and let:*

(1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_2(t)} = o(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 3. *Let the condition* (2) *hold, and let:*

(1) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ $\Big($*or* $\dfrac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)\Big)$, $\dfrac{\lambda_1(t)}{\lambda_2(t)} = O(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 4. *Let the following conditions be fulfilled:*

(1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) \to -\infty$, $\lambda_2(t) < 0$ *at* I;

(2) $\lambda_1'(t)$ *is bounded at* $t \to +\infty$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 5. *Let the condition* (2) *hold for* $i = 1$, *and let:*

(1) $\lambda_1(t) = o(1)$, $\lambda_2(+\infty) = -\infty$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 6. *Let the following conditions hold:*

(1) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *or* $\dfrac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1))$, $\dfrac{\lambda_1(t)}{\lambda_2(t)} = O(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

In all the above cases we have also obtained estimates for the solutions of the equation (1).

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A Discussion of Some Sharp Constants in Oscillation and Stability Theory of Delay Differential Equations

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For the equation with variable delays $x'(t) = -\sum_{j=1}^{m} a_j x(t-h_j(t))$, where $a_j > 0$, $0 \le h_j(t) \le q_j$, the inequalities $\sum_{j=1}^{m} a_j q_j \le 3/2$ and $\sum_{j=1}^{m} a_j q_j < 3/2$ are sufficient for uniform and exponential stability, respectively [1]. If all $h_j(t)$ are constant, then $3/2$ can be replaced by $\pi/2$ [2].

However, these results are not valid if the coefficients are not constant. For the equation $x'(t) = -\sum_{j=1}^{m} a_j(t)x(t-h_j(t))$, where $0 \le a_j(t) < \alpha_j$, $0 \le h_j(t) \le q_j$, the inequality $\sum_{j=1}^{m} \alpha_j q_j \le 1$ is sufficient for uniform stability, and constant 1 is sharp, as proved in the paper [1].

In this talk, we answer the following question: does there exist a number $A > 0$ such that the inequality

$$\limsup_{t\to\infty} \int_{t-h(t)}^{t} a(s)\, ds \ge A \tag{1}$$

implies instability of equation

$$x'(t) = -a(t)x(t-h(t)) \tag{2}$$

with one variable delay and a positive coefficient?

Another object of the talk is to discuss constants which lead to either oscillation or non-oscillation. For delay differential equations the following result is well known [3]:

If

$$\limsup_{t\to\infty} \int_{t-\max_k h_k(t)}^{t} \sum_{j=1}^{m} a_j^{+}(s)\, ds < \frac{1}{e}, \tag{3}$$

then there exists an eventually positive solution of the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t-h_k(t)) = 0. \tag{4}$$

Here $1/e$ is the best possible constant since the equation $\dot{x}(t) + x(t-\tau) = 0$ is oscillatory for $\tau > 1/e$.

In the monograph [3] for the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \quad a(t) \ge 0, \quad \tau > 0, \tag{5}$$

the authors constructed a counterexample which shows that condition (3) is not necessary for non-oscillation of equation (4).

By [3, Theorem 3.4.3], the inequality

$$\limsup_{t\to\infty} \int\limits_{t-\min_k h_k(t)}^{t} \sum_{j=1}^{m} a_j(s)\, ds < 1 \tag{6}$$

is necessary for non-oscillation of equation (4), where $a_j(t) \geq 0$, however the result is valid for $h(t)$ monotone only (in particular, $h(t) = t - \tau$).

Let us start with oscillation. First, we present sufficient non-oscillation conditions for equation (4) when the number $\limsup\limits_{t\to\infty} \int\limits_{t-\min_k h_k(t)}^{t} \sum\limits_{j=1}^{m} a_j(s)\, ds$ is between $1/e$ and 1.

Consider equation (4) with constant delays

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t-\tau_k) = 0, \quad \tau_k > 0. \tag{7}$$

Theorem 1 ([4])**.** *Suppose that there exists $n_0 \geq 0$ and a sequence $\{\lambda_j\}_{j=n_0-1}^{\infty}$ of positive numbers such that $\sum\limits_{k=1}^{m} a_k^+(t) \leq \lambda_n e^{-[\lambda_{n-1}(n\tau - t)+\lambda_n(t-(n-1)\tau)]}$, $(n-1)\tau < t \leq n\tau$, $n \geq n_0$, where $\tau = \max_k \tau_k$. Then equation* (7) *is non-oscillatory.*

Remark 1. By applying comparison theorems [3], Theorem 1 can be extended to equations with variable delays $0 \leq h_k(t) \leq \tau_k$.

Theorem 2 ([4])**.** *For any $\alpha \in (1/e, 1)$ there exists a non-oscillatory equation* (5) *with $a(t) \geq 0$ such that $\limsup\limits_{t\to\infty} \int\limits_{t-\tau}^{t} a(s)\, ds = \alpha$.*

Second, we give a negative answer to the following question: does there exist constants A and B such that

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} a(u)\, du > A \tag{8}$$

implies oscillation of equation (2).

Theorem 3 ([5])**.** *There is no constant $A > 0$ such that* (8) *implies oscillation of* (2) *for arbitrary $h(t) \leq t$.*

Third, we present an explicit oscillation test in the terms of the maximal value of the deviated argument

$$g(t) := \sup_{s\leq t} h(s), \quad t \geq 0. \tag{9}$$

Clearly, $g(t)$ is nondecreasing, and $h(t) \leq g(t)$ for all $t \geq 0$.

Theorem 4 ([5])**.** *Assume that $\sup\Big\{t \geq 0 : \int\limits_{g(t)}^{t} a(u) \exp\Big\{\int\limits_{h(u)}^{g(t)} a(v)\, dv\Big\}\, du \geq 1\Big\} = \infty$ for $g(t)$ defined by* (9)*. Then every solution of* (2) *is oscillatory.*

Theorem 4 immediately implies the following result.

Theorem 5 ([5])**.** *If* $\limsup\limits_{t\to\infty} \int\limits_{g(t)}^{t} a(u)\exp\Big\{\int\limits_{h(u)}^{g(t)} a(v)\,dv\Big\}\,\mathrm{d}u > 1$, *then every solution of* (2) *is oscillatory.*

The following result allows to expand the set of constants in (6) such that equation (2) may be stable, up to the limit of 2.

Lemma 1 ([6])**.** *Suppose* $a(t) \ge 0$, $\liminf\limits_{t\to\infty} a(t) > 0$, $\limsup\limits_{t\to\infty} h(t) < \infty$, *and there exists* $r(t) \ge 0$ *such that the equation*

$$\dot{x}(t) + a(t)x(t - r(t)) = 0 \tag{10}$$

is non-oscillatory. If $\limsup\limits_{t\to\infty} \Big|\int\limits_{t-h(t)}^{t-r(t)} a(s)\,ds\Big| < 1$, *then equation* (2) *is exponentially stable.*

By choosing an appropriate $r(t) = \tau$ and the same coefficient as in the proof of Theorem 2, we can for each $\alpha < 1$ construct non-oscillatory equation (10) such that

$$\limsup_{t\to\infty} \int\limits_{t-\tau}^{t} a(s)\,ds = \alpha.$$

Further, applying Lemma 1, for any $\beta < 2$ we can construct an exponentially stable equation with $\limsup\limits_{t\to\infty} \int\limits_{t-\tau}^{t} a(s)\,ds = \beta$, see [4].

However, this is still an open problem whether there exists $A > 0$ such that the inequality $\liminf\limits_{t\to\infty} \int\limits_{t-h(t)}^{t} a(s)\,ds > A$ implies instability of equation (2) with $a(t) \ge 0$ or not.

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Difference Schemes for One Fully Nonlocal Boundary-Value Problem

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We study fully nonlocal problem for the Poisson equation when the classical boundary condition is not given on any part of the boundary of the integration domain. The paper represents generalization of investigation carried out by the authors in [1, 2].

We investigate finite difference scheme for the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=-f(x),\ \ x\in\Omega=(0,l_1)\times(0,l_2), \tag{1}$$

together with the integral constraints

$$\int\limits_0^{\xi} u(x)\,dx_\alpha=0,\quad \int\limits_{l_\beta-\xi}^{l_\beta} u(x)\,dx_\alpha=0,\quad 0\le x_\beta\le l_\beta,\ \ \xi\le l_\alpha/2,\ \ \beta=3-\alpha,\ \ \alpha=1,2. \tag{2}$$

We assume that the solution u of the nonlocal boundary-value problem (1), (2) belongs to the Sobolev space $W_2^m(\Omega)$, $m>1$.

Consider the grid domains: $\overline{\omega}_\alpha=\{x_{\alpha,i_\alpha}=i_\alpha h;\ i_\alpha=0,1,\dots,n_\alpha,\ h=l_\alpha/n_\alpha\}$, $\alpha=1,2$, $\overline{\omega}=\overline{\omega}_1\times\overline{\omega}_2$. Let $\xi=(m+\theta)h$, $0\le\theta<1$.

Let H be the set of all discrete functions $v=v(x)$, defined on the grid $\overline{\omega}$ and satisfying conditions

$$\check{\mathcal{P}}_j^{(1)}(v)=0,\ \ \hat{\mathcal{P}}_j^{(1)}(v)=0,\ \ 0\le j\le n_2,\quad \check{\mathcal{P}}_i^{(2)}(v)=0,\ \ \hat{\mathcal{P}}_i^{(2)}(v)=0,\ \ 1\le i\le n_1-1,$$

where

$$\begin{aligned}
\check{\mathcal{P}}_j^{(1)}(v)&:=\sum_{k=0}^{m}hv_{kj}-\frac{h}{2}\,(y_{0j}+v_{mj})+\frac{\theta h}{2}\,\big((2-\theta)v_{mj}+\theta v_{m+1,j}\big),\\
\check{\mathcal{P}}_i^{(2)}(v)&:=\sum_{k=0}^{m}hv_{ik}-\frac{h}{2}\,(v_{i0}+v_{im})+\frac{\theta h}{2}\,\big((2-\theta)v_{im}+\theta v_{i,m+1}\big),\\
\hat{\mathcal{P}}_j^{(1)}(v)&:=\sum_{k=n_1-m}^{n_1}hv_{kj}-\frac{h}{2}\,(v_{n_1-m,j}+v_{n_1j})+\frac{\theta h}{2}\,\big((2-\theta)v_{n_1-m,j}+\theta v_{n_1-m-1,j}\big),\\
\hat{\mathcal{P}}_i^{2)}(v)&:=\sum_{k=n_2-m}^{n_2}hv_{ik}-\frac{h}{2}\,(v_{i,n_2-m}+v_{in_2})+\frac{\theta h}{2}\,\big((2-\theta)v_{i,n_2-m}+\theta v_{i,n_2-m-1}\big).
\end{aligned}$$

We approximate the problem (1), (2) by the difference scheme

$$U_{\overline{x}_1x_1}+U_{\overline{x}_2x_2}=-\varphi(x),\ \ x\in\omega,\ \ U\in H,\ \ \varphi=T_1T_2f, \tag{3}$$

where U_{x_α}, $U_{\overline{x}_\alpha}$ denote forward and backward difference quotients in x_α directions respectively, and T_1, T_2 – some averaging operators.

An a priori estimate of the solution of the difference scheme (3) is obtained with the help of energy inequality method, from which it follows the unique solvability of the scheme.

To estimate the truncation error , we apply the well-known technique [3], which uses the generalized Bramble–Hilbert lemma.

It is proved that the discretization error of the difference scheme (3) in the discrete weighted W_2^1-norm is determined by the estimate

$$\|U-u\|_{W_2^1(\omega,\rho)} \le ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \le 3.$$

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Asymptotic Behavior of Solutions of Essentially Nonlinear Differential Equations of the n-th Order

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The differential equation

$$y^{(n)} = \alpha_0 p(t) f\big(t, y, y', \dots, y^{(n-1)}\big) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\,$[1] $\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, \dots, n)$ are continuous functions, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$[2], or the interval $]Y_i, y_i^0]$, $f : [a, \omega[\, \times \Delta_{Y_0} \times \dots \times \Delta_{Y_{n-1}} \to]0, +\infty[$ is continuously differentiable function, that satisfies the conditions

$$\lim_{\substack{z_i \to Y_i \\ z_i \in \Delta_{Y_i}}} \frac{z_i \frac{\partial f}{\partial z_i}(t, z_1, \dots, z_{n-1})}{f(t, z_1, \dots, z_{n-1})} = 0 \ \ (i = \overline{1, n-1}),$$

$$\text{uniformly for } \ t \in [a, \omega[\,, \ \ z_j \in \Delta_{Y_j} \ \ (j \in \{1, \dots, n-1\} \setminus \{i\}),$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \frac{\partial f}{\partial t}(t, z_1, \dots, z_{n-1})}{f(t, z_1, \dots, z_{n-1})} = 0 \ \text{ uniformly for } \ z_j \in \Delta_{Y_j} \ \ (j = \overline{1, n-1}),$$

is considered. We suppose also that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum\limits_{i=0}^{n-1} \sigma_i \ne 1$.

According to the type of the functions $\varphi_0, \dots, \varphi_{n-1}$ it is clear that the equation (1) is in some sense similar to the well-known differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} |y^{(i)}|^{\sigma_i}. \tag{2}$$

We call the solution y of the equation (1) the $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \le \lambda_{n-1}^0 \le +\infty$, if the next conditions take place

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, \dots, n-1), \quad \lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)\, y^{(n-2)}(t)} = \lambda_{n-1}^0.$$

All $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the equation (2) were investigated in [2, 3]. Then for all $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) the necessary and sufficient conditions of existence and asymptotic representations as $t \uparrow \omega$ were found in case $f(t, z_1, \dots, z_{n-1}) \equiv 1$ (see, for example, [4]). But it is clear that even slowly varying nonlinearities can not be represented as the product of functions of one variable. For equations of the type (1), that contain for example functions like $\exp(\sqrt{|\ln |tyy'||})$ or $\exp(\sqrt[m]{|\ln |t| |y|^\mu y''||})$, the asymptotic representations of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions were not before established. For the general case of equation (1) there are used some methods of investigations, that were perviously developed for the case

[1] If $\omega > 0$, we will take $a > 0$.
[2] If $Y_i = +\infty (Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

$f(t, z_1, \ldots, z_{n-1}) \equiv 1$. It is clear that equation (1) there may contain in the right part functions that are described before and many other slowly varying functions of many variables.

The cases $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are singular by the studying of $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1). $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions, where $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are regularly varying functions as $t \uparrow \omega$ of indexes $\{0, 1, \ldots, n-1\}$. To investigate such solutions we must put additional conditions on the functions $\varphi_0, \ldots, \varphi_{n-1}$ and the function p. The $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) in regular cases $\lambda_{n-1}^0 \in R \setminus \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are established in this work. The $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) are regularly varying functions as $t \uparrow \omega$ of indexes different from $\{0, 1, \ldots, n-1\}$ if $\lambda_{n-1}^0 \in R \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$. If $\lambda_0 = 1$, the $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) are rapidly varying functions as $t \uparrow \omega$.

Now we need the next subsidiary notations.

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-1} (n-j-1)\sigma_j, \quad \pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \quad a_{0i} = (n-i)\lambda_{n-1}^0 - (n-i-1) \ \ (i = 1, \ldots, n),$$

$$C = \alpha_0 |\lambda_{n-1}^0 - 1|^{\mu_n} \prod_{k=0}^{n-2} \Big| \prod_{j=k+1}^{n-1} a_{0j} \Big|^{-\sigma_k} \operatorname{sign} y_{n-1}^0,$$

$$I_0(t) = \int_{A_\omega^0}^t C p(\tau) |\pi_\omega(\tau)|^{\mu_n} \, d\tau, \quad I_1(t) = \int_{A_\omega^1}^t \alpha_0 p(\tau) \, d\tau,$$

$$A_\omega^0 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, d\tau < +\infty, \end{cases} \qquad A_\omega^1 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau < +\infty, \end{cases}$$

$$J(t) = \int_{B_\omega}^t |\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau, \quad B_\omega = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau < +\infty. \end{cases}$$

The following conclusions take place for equation (1).

Theorem 1. *The next conditions are necessary for the existence of* $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$*-solutions* ($\lambda_{n-1}^0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$) *of equation* (1)*:*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = \frac{\gamma_0}{\lambda_{n-1}^0 - 1}, \quad y_i^0 \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_{n-1}^0 - 1}} = Y_i, \tag{3}$$

$$y_i^0 y_{i+1}^0 a_{0i+1} (\lambda_{n-1}^0 - 1) \pi_\omega(t) > 0 \ \ \text{as} \ \ t \in [a, \omega[\,, \tag{4}$$

where $y_n^0 = \alpha_0$, $i = \overline{0, n-1}$.

If the equation

$$\sum_{k=0}^{n-1} \sigma_k \prod_{i=k+1}^{n-1} a_{0i} \prod_{i=1}^{k} (a_{0i} + \lambda) = (1 + \lambda) \prod_{i=1}^{n-1} (a_{0i} + \lambda)$$

has no roots with zero real part, then conditions (3) *and* (4) *are sufficient for the existence of* $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$*-solutions of equation* (1)*. For any such solution the next asymptotic rep-*

resentations as $t \uparrow \omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{f(t,y,y',\dots,y^{(n-1)})\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))} = \gamma_0 I_0(t)[1+o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \frac{[(\lambda^0_{n-1}-1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1} a_{0j}}[1+o(1)],$$

where $i=\overline{0,n-2}$, take place.

Theorem 2. *The next conditions are necessary for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega}\frac{I_1'(t)J(t)}{I_1(t)J'(t)} = \gamma_0, \quad y_i^0 \lim_{t\uparrow\omega}|I_1(t)|^{\frac{1}{\gamma_0}} = Y_i, \tag{5}$$

$$\alpha_0 y^0_{n-2} > 0, \quad y_i^0 y_{i+1}^0 J(t) > 0 \ \ as \ \ t\in[a,\omega[\,, \tag{6}$$

where $i=\overline{0,n-1}$.

If the equation $\sum\limits_{k=0}^{n-1}\sigma_k(1+\lambda)^k = (1+\lambda)^n$ has no roots with zero real part, then conditions (5) *and* (6) *are sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solutions of equation* (1)*. For any such solution the asymptotic representations as $t \uparrow \omega$*

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{f(t,y,y',\dots,y^{(n-1)})\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))} = \gamma_0 I_1(t)\Big|\frac{J(t)}{J'(t)}\Big|^{\mu_n}\operatorname{sign} y^0_{n-1}[1+o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \Big(\frac{J(t)}{J'(t)}\Big)^{n-i-1}[1+o(1)],$$

where $i=\overline{0,n-2}$, take place.

Let us notice that if $\sum\limits_{i=0}^{n-2}|\sigma_i| < |1-\sigma_{n-1}|$, the conditions (3) and (4) are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions ($\lambda^0_{n-1}\in\mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$) of equation (1) and the conditions (5) and (6) are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solutions of (1).

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The Connection Between the Delayed Hopfield Network Model and the Wilson–Cowan Neural Field Model with Delay

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We consider the following delayed Hopfield network model (see e.g. [1])

$$\dot{z}_i(t,N) = -\alpha z_i(t,N) + \sum_{j=1}^{N} \omega_{ij}(N) f\big(z_j\big(t-\tau_{ij}(t,N),N\big)\big) + \mathrm{J}_i(t,N), \quad t>a, \quad i=1,\dots,N, \tag{1}$$

parameterized by a natural parameter N. Here at each natural N, $z_i(\,\cdot\,,N)$ are n-dimensional vector functions, $\omega_{ij}(N)$ are real $n \times n$-matrices (connectivities), $\tau_{ij}(\,\cdot\,,N)$ are nonnegative real-valued continuous functions (axonal delays), $f: R^n \to R^n$ are firing rate functions which are Lipschitz and bounded on R and $\mathrm{J}_i(\,\cdot\,,N)$ are continuous external input n-dimensional vector functions.

The initial conditions for (1) are given as

$$z_i(\xi,N) = \varphi_i(\xi,N), \quad \xi \le a, \quad i=1,\dots,N. \tag{2}$$

We use the general well-posedness result from the paper [2] to justify the convergence of a sequence of the delayed Hopfield equations (1) (with the initial conditions (2)) to the following Volterra integral equation involving spatio-temporal delay

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega(x,y) f\big(u\big(s-\tau(t,x,y),y\big)\big)\, d_\nu(y) + J(t,x), \quad t>a, \quad x\in\Omega, \tag{3}$$

with the initial (prehistory) condition

$$u(\xi,x) = \varphi(\xi,x), \quad \xi \le a, \quad x \in \Omega. \tag{4}$$

The equation (3) generalizes the well-known neural field models introduced by Wilson and Cowan in [3, 4]. Here the function u represents the activity of a neural element at time t and position x. The connectivity kernel ω determines the coupling between elements at positions x and y. The non-negative activation function f gives the firing rate of a neuron with activity u. The non-negative function τ represents the time-dependent axonal delay effects in the neural field, which require a prehistory condition given by the function φ. The function $I(t,x)$ represents a variable external input.

The following assumptions will be imposed on the functions involved in (3) and (4):

(**A1**) The function $f: R^n \to R^n$ is continuous, bounded and Lipschitz.

(**A2**) The spatio-temporal delay $\tau: R \times \Omega \times \Omega \to [0,\infty)$ is a continuous function.

(**A3**) The initial (prehistory) function $\varphi: (-\infty, a] \times \Omega \to R^n$ is continuous.

(**A4**) For any $b>a$, the external input function $J: [a,b] \times \Omega \to R^n$ is uniform continuous.

(**A5**) The kernel function $\omega: \Omega \times \Omega \to R^n$ is continuous.

(**A6**) $\nu(\,\cdot\,)$ is the Lebesgue measure on Ω.

(**A7**) For any $b > a$,

$$\sup_{x\in\Omega} \int\limits_{\Omega} |\omega(x,y)|\, d_\nu(y) < \infty.$$

(**A8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{x\in\Omega} \int\limits_{\Omega-\Omega_r} |\omega(x,y)|\, d_\nu(y) = 0.$$

The following theorem shows the connection between the delayed Hopfield network model and the Wilson–Cowan neural field model with delay:

Theorem. *For each natural number N let $\{\Delta_i(N),\ i = 1,\dots,N\}$ be a finite family of open subsets of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ be arbitrary points in $\Delta_i(N)$. Finally, let the assumptions (**A1**) – (**A8**) *be fulfilled.*

Then the sequence of the solutions $z_i(t,N)$ $(t \in R)$ of the initial value problem (1), (2)*, where the coefficients are defined by*

$$\omega_{ij}(N) = \beta_i(N)\omega\big(y_i(N), y_j(N)\big), \quad \textit{where} \ \beta_i(N) = \nu(\Delta_i(N)),$$
$$\tau_{ij}(t,N) = \tau(t, y_i(N), y_j(N)), \quad \mathrm{J}_i(t,N) = J(t, y_i(N)),$$

converges for any $b > a$ to the solution $u(t,x)$ $(t \in R,\ x \in \Omega)$ of the initial value problem (3), (4)*, as $N \to \infty$, in the following sense:*

$$\lim_{N\to\infty} \Big\{ \sup_{t\in[a,b]} \Big\{ \sup_{1\le i\le N} \Big\{ \sup_{x\in\Delta_i(N)} \big|u(t,x) - z_i(t,N)\big| \Big\}\Big\}\Big\} = 0.$$

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Averaging Method in Optimal Control Problems for Systems of Ordinary Differential Equations

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The object of our investigation is an optimal control problem for the system of ordinary differential equations in standard Bogolyubov form:

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t, x, u), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ – small parameter, $x \in D$ – phase vector, D – domain in R^n, $u(t) \in U \subset R^m$ – control vector, $t \geq 0$, $T > 0$ – some constant, X – continuous of set of variables vector-function.

The idea of investigation consists in replacement of initial object (1) by simpler averaging object.

Our results are divided into two types:

(1) controls on asymptotically finite intervals (the order of $1/\varepsilon$);

(2) controls on semiaxis.

Let us denote by $x(t, u)$ a solution of system (1), corresponding to control $u(t)$.

We say that $u(t)$ are admissible controls, if

(1) $u(t) \in U$, where $t \geq 0$, $u(t)$ are measurable, locally Lebesgue integrable functions, where $t \geq 0$ and for every $u(t)$ there exists a constant $u_0 \in U$ such that $|u(t) - u_0| \leq \varphi(t)$, where $\varphi(t)$ doesn't depend on $u(t)$ and $\int\limits_0^\infty \varphi(t)dt < \infty$;

(2) a solution $x(t, u)$ of the Cauchy problem (1) is defined for every $t \in [0, \frac{T}{\varepsilon}]$.

The set of such controls is denoted by F.

The following problem is considered:

$$J_\varepsilon(u) = \Phi\Big(x\Big(\frac{T}{\varepsilon}, u\Big)\Big) \longrightarrow \inf, \tag{2}$$

where $\Phi(x)$ is some function.

Denote $J_\varepsilon = \inf\limits_{u(t) \in F} J_\varepsilon(u)$.

Let us consider the following optimal control problem, which is averaged to (1), (2):

$$\begin{aligned} \dot{y} &= \varepsilon X_0(y, u), \\ y(0) &= x_0, \end{aligned} \tag{3}$$

where

$$X_0(x,u) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T X(t,x,u)\,dt \tag{4}$$

and

$$\overline{J}_\varepsilon(u) = \Phi(y(\frac{T}{\varepsilon}, u)) \longrightarrow \inf\,. \tag{5}$$

Theorem 1. *Let us suppose that in the domain Q the following conditions are satisfied:*

(1) *$X(t,x,u)$ is measurable with respect to t and satisfies the Lipschitz condition with respect to x and u;*

(2) *solution $y = y(t,u_0)$, $y(0,u_0) = x_0$ is definite for every $t \geq 0$ and for every $u_0 \in U$ belongs to the domain D with some ρ-neighborhood;*

(3) *the limit* (4) *exists uniformly with respect to $x \in D$ and $u \in U$;*

(4) *function $\Phi(x)$ satisfies the Lipschitz condition with respect to $x \in D$;*

(5) *there exist optimal control of averaged problem $u^*(t,\varepsilon)$.*

Then for every $\eta > 0$ there exists $\varepsilon_0(\eta) > 0$ such that

(a) *for every $0 < \varepsilon < \varepsilon_0$, $J_\varepsilon > -\infty$;*

(b) *the following inequality is true*

$$\big|J_\varepsilon(u^*(t,\varepsilon)) - J_\varepsilon)\big| \leq \eta,$$

i.e. optimal control $u^*(t,\varepsilon)$ of averaged system is η-optimal control of precise system.

Consider the optimal control problem of the system of differential equations on semiaxis:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x,u), \\ x(0) &= x_0 \end{aligned} \tag{6}$$

with the control function

$$J(u) = \int\limits_0^\infty L(t,x,u)\,dt \longrightarrow \inf,$$

where $\varepsilon > 0$ – small parameter, $t \geq 0$, $x \in D$ – phase vector, D - domain in R^n, $u \in U \subset R^m$ – control vector, L satisfies the condition

$$\big|L(t,x,u) - L(t,y,u)\big| \leq \alpha(t)|x-y|, \text{ where } \int\limits_0^\infty \alpha(t)\,dt < \infty. \tag{7}$$

Let us consider that $u(t)$ are admissible controls, if

(a_1) $u(t)$ are measurable, locally integrable functions, where $t \geq 0$, $u(t) \in U$;

(b_1) for every $u(t) \in U$ there exists a constant $u_0 \in U$ such that $|u(t) - u_0| \leq \varphi(t)$, where $\varphi(t)$ doesn't depend on $u(t)$ and $\int\limits_0^\infty \varphi(t)dt < \infty$;

(c_1) there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ a solution of the Cauchy problem $x(t,u)$ is defined and unique for every $t \geq 0$;

(d_1) $|J(u)| < \infty$.

Let us consider the following optimal control problem which is averaged to (6):

$$\begin{gathered} \dot{y} = \varepsilon X_0(y, \overline{u}), \\ y(0) = x_0, \end{gathered} \tag{8}$$

with the control functional

$$\overline{J}(\overline{u}) = \int\limits_0^\infty L(t, y, \overline{u})\, dt,$$

where

$$X_0(x, u) = \lim_{T \to \infty} \frac{1}{T} \int\limits_t^{t+T} X(s, x, u)\, dt, \quad t \geq 0. \tag{9}$$

The admissible controls for averaged system satisfy the same conditions (a_1)–(d_1). Let's suppose that the following conditions are performed for averaged system:

$(A1)$ solution $\overline{y}(\tau) = \overline{y}(\tau, u_0)$ of the averaged system

$$\begin{gathered} \frac{d\overline{y}}{d\tau} = X_0(\overline{y}, u_0), \\ \overline{y}(0) = x_0, \ \tau = \varepsilon t \end{gathered} \tag{10}$$

is defined for every $\tau \geq 0$ and belongs to the domain D with some ρ - neighborhood, where ρ doesn't depend on u_0 for arbitrary constant control $u_0 \in U$;

$(A2)$ solution $\overline{y}(\tau)$ is equiasymptotically stable with respect to τ_0 and u_0.

The existence of η-optimal controls for precise system is proved in the following theorem with using of Lemma 3.

Theorem 2. *Let us suppose that in the domain $Q = \{t \geq 0, \ x \in D \subset R^n, \ u \in U \subset R^m\}$ the conditions of Lemma 3 are satisfied and there exists an optimal control $\overline{u}^*$ of averaged problem (8). Then for every $\eta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $|J^*| < \infty$ and the following inequality is true*

$$|J(\overline{u}^*) - J^*)| \leq \eta.$$

The Control Problem of the Frequency Spectrum of Irregular Oscillations for Linear Systems

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For a long time, until the end of the 1940s, the investigation of periodic solutions of periodic differential systems was based on the conjecture of commensurability of periods of the solution and the system. Apparently, Massera was the first to indicate that this conjecture is wrong. In 1950, he showed that periodic differential system can have periodic solutions with irrational ratio of periods of the solution and the system [1]. Later, such periodic solutions were considered by J. Kurzweil and O. Vejvoda [2], N. P. Erugin [3], E. I. Grudo [4] and other authors. In what follows, such periodic solutions and the oscillations described by them are said to be strongly irregular [5].

Note that Mandelshtam and Papaleksi [6] studied the parametric influence on two-circuit parametric systems in the mid-1930s. In particular, for the case in which some capacity is included in the feed circuit of an electric motor to compensate for variable inductance, the following facts were justified: the rotation velocity of the electric motor is not synchronous with the supply current frequency; this velocity varies smoothly with the fundamental frequency of the oscillation contour. Unlike the ordinary parametric excitation, which takes place only for an integer frequency ratio, they obtained a new peculiar transformation of the motor frequency with practically arbitrary ratio to the circuit frequency. Thus, the possibility of excitation of oscillations at frequencies incommensurable with the frequency of the system parameter variation was shown.

Devices transforming the energy of a high-frequency oscillation source into low-frequency oscillations whose frequency is almost independent of the source frequency were developed at the beginning of the 1970s. For example, the paper [7] deals with the case in which a harmonic force with which the field in a capacitor acts on a flying charge has a frequency incommensurable with the frequency of fundamental oscillations of the charge. In this case, there can appear stable undamped oscillations at the natural frequency, i.e., strongly irregular oscillations. The conditions of a process in which the oscillations of a system are described by strongly irregular oscillations are referred to as an asynchronous mode [8]. Asynchronous modes in particular occur in linear differential systems. We state the problem of synthesis of asynchronous modes of linear systems as a control problem for the spectrum of irregular oscillations.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \;\; x \in \mathbb{R}^n, \;\; n \geq 2, \tag{1}$$

where $A(t)$ is a continuous ω-periodic $n \times n$-matrix, B is a constant $n \times n$-matrix. We assume that the control is given in the form of a linear state feedback

$$u = U(t)x \tag{2}$$

with ω-periodic $n \times n$-matrix $U(t)$. The problem of finding the matrix $U(t)$ (the feedback coefficient) such that the closed system

$$\dot{x} = (A(t) + BU(t))x \tag{3}$$

has strongly irregular periodic solutions with a given frequency spectrum L (the objective set) will be called the problem of control of the frequency spectrum of irregular oscillations (asynchronous spectrum) with objective set L.

This problem is a version of the generalization of the spectrum assignment problem in the nonstationary case, but essentially differs from the problems of control [9, 10]. Note that, for regular oscillations, the choice of frequencies other than multiples of the frequencies of the right-hand side of system (1) is impossible.

Let $L = \{\lambda_1, \dots, \lambda_r'\}$ be an objective set of frequencies whose elements are pairwise distinct, commensurable with each other, and incommensurable with $2\pi/\omega$. Then there exists a maximum positive real number λ such that $\lambda_1, \dots, \lambda_r'$ are multiples of λ. Set $\Omega = 2\pi/\lambda$, then the ratio ω/Ω is irrational.

Consider the case of a singular matrix B, i.e. $\operatorname{rank} B = r < n$ $(n - r = d)$. One can assume that the first d rows of B are zero (otherwise it can always be achieved by a linear nonsingular time-independent transformation). Suppose that right upper $d \times r$ blocks of the averaged matrix $\widehat{A}$ are zero.

Let $\widetilde{A}_{d,d}^{(11)}$ and $\widetilde{A}_{d,r}^{(12)}$ be upper left and right blocks of the matrix $\widetilde{A}(t) = A(t) - \widehat{A}$ (the subscripts indicate the dimension). Assume that column bases of these blocks form a linearly independent set

$$\operatorname{rank}_{\rm col} \{\widetilde{A}_{d,d}^{(11)}, \widetilde{A}_{d,r}^{(12)}\} = r_1 + r_2 \quad \big(\operatorname{rank}_{\rm col} \widetilde{A}_{d,d}^{(11)} = r_1, \ \operatorname{rank}_{\rm col} \widetilde{A}_{d,r}^{(12)} = r_2\big).$$

By Q we denote a constant nonsingular $d \times d$ matrix such that the first $d_1 = d - r_1$ columns of the matrix $\widetilde{A}_{d,d}(t)Q$ are zero and the remaining columns are linearly independent. Let the left upper $d \times d$ block of the matrix $Q^{-1}\widehat{A}_{d,d}Q$ have p pairs of pure imaginary eigenvalues $\pm i\lambda_j$, $\lambda_j \in L$.

Then the following assertion holds.

Theorem. *Let the above assumptions are satisfied. The problem of control of the asynchronous spectrum with objective set L for system* (1) *with feedback* (2) *is solvable if and only if* $r_1 + r_2 < n$ *and* $|L| < p + [(r - r_2)/2]$.

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Nonoscillation and Exponential Stability of Second Order Delay Differential Equations without Damping Term

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Delays, arising in nonoscillatory and stable ordinary differential equations, can induce oscillation and instability of their solutions. That is why the traditional direction in the study of nonoscillation and stability of delay equations is to establish a smallness of delay, allowing delay differential equations to preserve these convenient properties of ordinary differential equations with the same coefficients. In this paper, we find cases in which delays, arising in oscillatory and asymptotically unstable ordinary differential equations, induce nonoscillation and stability of delay equations. We demonstrate that, although the ordinary differential equation

$$x''(t)+c(t)x(t)=0$$

can be oscillatiting and asymptoticaly unstable, the delay equation

$$x''(t)+a(t)x(t-h(t))-b(t)x(t-g(t))=0, \ \text{ where } \ c(t)=a(t)-b(t)$$

can be nonoscillating and exponentially stable.

Let us consider the equation

$$\begin{gathered} x''(t)+a(t)x(t-\tau(t))-b(t)x(t-\theta(t))=0, \ \ t\in[0,+\infty), \\ x(\xi)=0 \ \text{ for } \ \xi<0, \end{gathered} \tag{1}$$

where $a(t)$, $b(t)$, $\tau(t)$ and $\theta(t)$ are measurable essentially bounded nonnegative functions. We denote

$$q_*=\operatorname*{ess\,inf}_{t\geq 0} q(t), \quad q^*=\operatorname*{ess\,sup}_{t\geq 0} q(t).$$

Theorem. *Assume that* $0\leq\tau(t)\leq\theta(t)$, $0\leq b(t)\leq a(t)$,

$$\begin{gathered} 4\big\{a(t)-b(t)\big\}\leq\big[b(\theta-\tau)\big]_*^2, \ \ t\in[0,+\infty), \\ 0<\big[b(\theta-\tau)\big]^*\theta^*\leq\frac{1}{e}\,. \end{gathered} \tag{2}$$

Then

(1) *the Cauchy function* $C(t,s)$ *of equation* (1) *is positive for* $0\leq s<t<+\infty$*;*

(2) *if there exists such positive* ε *that*
$$a(t)-b(t)\geq\varepsilon,$$
then the Cauchy function $C(t,s)$ *of equation* (1) *satisfies the exponential estimate and the integral estimate*
$$\sup_{t\geq 0}\int\limits_0^t C(t,s)\,ds\leq\frac{1}{\varepsilon}\,;$$

(3) *if there exists* $\lim_{t\to\infty}\{a(t)-b(t)\}=k$, *with* $k>0$, *then*

$$\lim_{t\to\infty}\int_0^t C(t,s)\,ds=\frac{1}{k}\,.$$

Consider the equation

$$x''(t)+a(t)x(t-\tau)=f(t),\ \ t\in[0,+\infty),\ \ a(t)\ge a_*>0, \tag{3}$$

which is unstable. To stabilize its solution to the given "trajectory" $y(t)$ satisfying this equation, we choose the control in the form

$$u(t)=b(t)\big[x(t-\theta)-y(t-\theta)\big]. \tag{4}$$

Example 1. Stabilizing equation (3), where $a(t)\equiv a$, let us choose the control in the form (4) with constant coefficient $b(t)\equiv b$. We come to the study of the exponential stability of the equation

$$x''(t)+ax(t-\tau)-bx(t-\theta)=g(t),\ \ t\in[0,+\infty),$$

with constant coefficient and delays and $g(t)=f(t)+by(t-\theta)$. We can choose $\theta-\tau=\frac{1}{eb\theta}$ and

$$0<4\{a-b\}\le\frac{1}{e^2\theta^2}\,.$$

Example 2. The equation

$$x''(t)+a(t)x(t-\tau)=0,\ \ a(t)\to+\infty,\ \ t\in[0,+\infty),\ \ \tau=const,$$

where $a(t)\ge a_*>0$, possesses oscillating solutions with amplitudes tending to infinity that leads to the chaos in behavior of its solutions. This equation can also be stabilized by the control in form (4). Consider, for example, the equation

$$x''(t)+tx(t-\tau)=0,\ \ t\in[1,+\infty),\ \ \tau=const.$$

If we choose $b(t)=t-\Delta$, $\theta(t)=\tau+\frac{\gamma}{t}$, then the stabilization can be achieved by the control (4) with the parameters satisfying the inequalities

$$0<2\sqrt{\Delta}<\gamma<\frac{1}{\tau e}\,.$$

Example 3. Consider the equation

$$x''(t)+x(t)-bx(t-\theta)=0,\ \ t\in[0,+\infty),$$
$$x(\xi)=0,\ \ \xi<0.$$

It is clear from the definition of the Cauchy function that for $t<\theta$, this equation is equivalent to the ordinary differential equation

$$x''(t)+x(t)=0,\ \ t\in[0,\theta],$$

whose Cauchy function is $C(t,s)=\sin(t-s)$. Let us demonstrate that condition (2) is essential for positivity of the Cauchy function $C(t,s)$ in theorem, assuming that the numbers b and θ are chosen such that all other conditions of theorem are fulfilled. If $\pi<\theta$, then in the triangle $0\le s\le t<\theta$ its Cauchy function $C(t,s)=\sin(t-s)$ changes its sign.

On the Dimension of the Solution Set to the Homogeneous Linear Functional Differential Equation of the First-Order

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1 Introduction

In many applications, there is of a great importance to know whether the linear boundary value problem has a unique solution. Thus a lot of papers recently published are devoted to study the unique solvability of the general boundary value problem

$$u'(t) = \ell(u)(t) + q(t) \quad \text{for a.e. } t \in [a,b], \tag{1}$$
$$h(u) = c, \tag{2}$$

where $\ell : C\big([a,b];R\big) \to L\big([a,b];R\big)$ and $h : C\big([a,b];R\big) \to R$ are linear bounded operators, $q \in L\big([a,b];R\big)$, and $c \in R$. A well-known result, so-called Fredholm Theorem, describes the relation between the unique solvability of the inhomogeneous and the corresponding homogeneous linear boundary value problems. To be more precise, the following theorem is well-known from a general theory of boundary value problems for functional differential equations.

Theorem 1.1. *The problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t) \quad \textit{for a. e. } t \in [a,b], \tag{1_0}$$
$$h(u) = 0 \tag{2_0}$$

has only the trivial solution.

It is also known that the dimension of the solution set to the homogeneous equation (1_0) plays an important role in the theory. In spite of the first-order linear ordinary differential equation, the dimension of the solution set U to the equation (1_0) can be of any natural number. More precisely, it is known that dim $U \geq 1$ (see Section 4 in [3]), and if dim $U \geq 2$, then for every linear bounded operator $h : C\big([a,b];R\big) \to R$, the problem (1_0), (2_0) has a nontrivial solution (see [3, Remark 4.7]). Therefore, it is of a great importance to find conditions guaranteeing the relation dim $U = 1$, and, in general, to study the structure of the solution set U.

1.1 Basic Notation and Definitions

The following notation is used throughout the paper.

R is a set of all real numbers; $R_+ = [0, +\infty[$.

$C([a,b];R)$ is a Banach space of continuous functions $u:[a,b]\to R$ with the norm

$$\|u\|_C = \max\bigl\{|u(t)|:\ t\in[a,b]\bigr\}.$$

$L([a,b];R)$ is a Banach space of Lebesgue-integrable functions $p:[a,b]\to R$ with the norm

$$\|u\|_L = \int_a^b |p(t)|\,dt.$$

$AC([a,b];R)$ is a set of absolutely continuous functions $u:[a,b]\to R$.

$C([a,b];R_+) = \bigl\{u\in C([a,b];R):\ u(t)\ge 0 \quad \text{for } t\in[a,b]\bigr\}$.

$L([a,b];R_+) = \bigl\{p\in L([a,b];R):\ p(t)\ge 0 \quad \text{for a. e. } t\in[a,b]\bigr\}$.

$\mathcal{L}_{ab}$ is a set of all linear bounded operators $\ell: C([a,b];R)\to L([a,b];R)$.

$\mathcal{P}_{ab}$ is a set of all positive operators $\ell\in\mathcal{L}_{ab}$, i.e., such operators that transform a set $C([a,b];R_+)$ into a set $L([a,b];R_+)$.

An operator $\ell\in\mathcal{L}_{ab}$ is called an a-Volterra operator, resp. a b-Volterra operator, if for arbitrary $c\in\,]a,b]$, resp. $c\in[a,b[$, and $v\in C([a,b];R)$ such that

$$v(t)=0 \quad \text{for } t\in[a,c], \quad \text{resp. } v(t)=0 \quad \text{for } t\in[c,b],$$

the equality

$$\ell(v)(t)=0 \quad \text{for a. e. } t\in[a,c], \quad \text{resp. } \ell(v)(t)=0 \quad \text{for a. e. } t\in[c,b]$$

is fulfilled.

By a solution to the equation (1), resp. (1_0), we understand a function $u\in AC([a,b];R)$ satisfying (1), resp. (1_0), almost everywhere on $[a,b]$. By a solution to the problem (1), (2), resp. (1_0), (2_0), we understand a solution u to (1), resp. (1_0), satisfying (2), resp. (2_0).

Notation 1.1. Throughout the paper, by U we denote the set of all solutions u to the equation (1_0). Obviously, U is a linear vector space.

To formulate the main results it is convenient to introduce the following definitions.

Definition 1.1. An operator $\ell\in\mathcal{L}_{ab}$ belongs to the set $\mathcal{S}_{ab}(a)$ if every function $u\in AC([a,b];R)$ satisfying

$$u'(t)\ge \ell(u)(t) \quad \text{for a. e. } t\in[a,b], \tag{3}$$
$$u(a)\ge 0, \tag{4}$$

admits the inequality

$$u(t)\ge 0 \quad \text{for } t\in[a,b]. \tag{5}$$

Definition 1.2. An operator $\ell\in\mathcal{L}_{ab}$ belongs to the set $\mathcal{S}_{ab}(b)$ if every function $u\in AC([a,b];R)$ satisfying

$$u'(t)\le \ell(u)(t) \quad \text{for a. e. } t\in[a,b], \tag{6}$$
$$u(b)\ge 0, \tag{7}$$

admits the inequality (5).

Definition 1.3. An operator $\ell\in\mathcal{L}_{ab}$ belongs to the set $\mathcal{S}'_{ab}(a)$ if every function $u\in AC([a,b];R)$ satisfying (3) and (4) admits the inequalities (5) and

$$u'(t)\ge 0 \quad \text{for a. e. } t\in[a,b]. \tag{8}$$

Definition 1.4. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{S}'_{ab}(b)$ if every function $u \in AC\big([a,b];R\big)$ satisfying (6) and (7) admits the inequalities (5) and

$$u'(t) \leq 0 \quad \text{for a. e. } t \in [a,b]. \tag{9}$$

Definition 1.5. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{P}^+_{ab}$ (resp. $\mathcal{P}^-_{ab}$) if it transforms the non-negative non-decreasing (resp. non-increasing) absolutely continuous functions to the non-negative functions.

Similarly, we say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{N}^+_{ab}$ (resp. $\mathcal{N}^-_{ab}$) if it transforms the non-negative non-decreasing (resp. non-increasing) absolutely continuous functions to the non-positive functions.

Remark 1.1. Define an operator $\varphi : C\big([a,b];R\big) \to C\big([a,b];R\big)$ as follows:

$$\varphi(v)(t) = v(a+b-t) \quad \text{for } t \in [a,b], \quad v \in C\big([a,b];R\big).$$

Put

$$\widetilde{\ell}(v)(t) = -\ell(\varphi(v))(a+b-t) \quad \text{for a. e. } t \in [a,b], \;\; v \in C\big([a,b];R\big).$$

Then it can be easily verified that $\widetilde{\ell} \in \mathcal{S}_{ab}(a)$, resp. $\widetilde{\ell} \in \mathcal{S}'_{ab}(a)$, if and only if $\ell \in \mathcal{S}_{ab}(b)$, resp. $\ell \in \mathcal{S}'_{ab}(b)$. Furthermore, $\widetilde{\ell} \in \mathcal{P}^+_{ab}$, resp. $\widetilde{\ell} \in \mathcal{P}^-_{ab}$, if and only if $\ell \in \mathcal{N}^-_{ab}$, resp. $\ell \in \mathcal{N}^+_{ab}$.

2 Main Results

In this section, the main results are formulated using the general terms that the operator ℓ, respectively its positive or negative part, belongs to one of the sets $\mathcal{S}_{ab}(a)$, $\mathcal{S}_{ab}(b)$, $\mathcal{S}'_{ab}(a)$, and $\mathcal{S}'_{ab}(b)$. The effective criteria guaranteeing such an inclusion can be found in Section 3 of the paper. For more conditions guaranteeing the above-mentioned inclusions one can see [1, 2] or the monograph [4] where also the detailed introduction to the problem can be found.

Proposition 2.1. *Let $\ell \in \mathcal{S}'_{ab}(a)$. Then* dim $U = 1$ *and the set U is generated by a positive non-decreasing function.*

Theorem 2.1. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then* dim $U = 1$ *and the set U is generated by a positive non-decreasing function.*

Theorem 2.2. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC\big([a,b];R\big)$ satisfying*

$$\gamma(t) > 0 \quad \textit{for } t \in [a,b], \tag{10}$$

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \textit{for a. e. } t \in [a,b]. \tag{11}$$

Then dim $U = 1$ *and the set U is generated by a positive non-decreasing function.*

Theorem 2.3. *Let $\ell \in \mathcal{N}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then* dim $U = 1$ *and the set U is generated by a positive function u with the following property: the relation*

$$u(a) = \max\big\{u(t) : \; t \in [a,b]\big\} \tag{12}$$

holds and, in addition, if there exists $c \in \,]a,b]$ such that $u(c) = u(a)$, then

$$u(t) = u(c) \quad \textit{for } t \in [a,c]. \tag{13}$$

Theorem 2.4. *Let $\ell \in \mathcal{P}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, there exist $\gamma \in AC\big([a,b];R\big)$ satisfying* (10) *and* (11). *Then* dim $U = 1$ *and the set U is generated by a positive function u with the following property: the relation*

$$u(a) = \min\big\{u(t) :\ t \in [a,b]\big\} \tag{14}$$

holds and, in addition, if there exists $c \in\,]a,b]$ such that $u(c) = u(a)$, then (13) *is fulfilled.*

According to Remark 1.1, the following assertions immediately follows from Proposition 2.1 and Theorems 2.1–2.4:

Proposition 2.2. *Let $\ell \in \mathcal{S}'_{ab}(b)$. Then* dim $U = 1$ *and the set U is generated by a positive non-increasing function.*

Theorem 2.5. *Let $\ell \in \mathcal{N}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Then* dim $U = 1$ *and the set U is generated by a positive non-increasing function.*

Theorem 2.6. *Let $\ell \in \mathcal{N}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $\gamma \in AC\big([a,b];R\big)$ satisfying* (10) *and*

$$\gamma'(t) \le \ell(\gamma)(t) \quad \text{for a. e. } t \in [a,b]. \tag{15}$$

Then dim $U = 1$ *and the set U is generated by a positive non-increasing function.*

Theorem 2.7. *Let $\ell \in \mathcal{P}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Then* dim $U = 1$ *and the set U is generated by a positive function u with the following property: the relation*

$$u(b) = \max\big\{u(t) :\ t \in [a,b]\big\} \tag{16}$$

holds and, in addition, if there exists $c \in [a,b[$ such that $u(c) = u(b)$, then

$$u(t) = u(c) \quad \text{for } t \in [c,b]. \tag{17}$$

Theorem 2.8. *Let $\ell \in \mathcal{N}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Let, moreover, there exist $\gamma \in AC\big([a,b];R\big)$ satisfying* (10) *and* (15). *Then* dim $U = 1$ *and the set U is generated by a positive function u with the following property: the relation*

$$u(b) = \min\big\{u(t) :\ t \in [a,b]\big\} \tag{18}$$

holds and, in addition, if there exists $c \in [a,b[$ such that $u(c) = u(b)$, then (17) *is fulfilled.*

3 On the Sets $\mathcal{S}_{ab}(a)$, $\mathcal{S}_{ab}(b)$, $\mathcal{S}'_{ab}(a)$, and $\mathcal{S}'_{ab}(b)$

Theorem 3.1. *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then $\ell \in \mathcal{S}'_{ab}(a)$.*

Theorem 3.2. *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Then $\ell \in \mathcal{S}'_{ab}(a)$ if and only if there exists $\gamma \in AC\big([a,b];R\big)$ satisfying* (10) *and* (11).

Theorem 3.3. *Let $\ell \in \mathcal{N}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then $\ell \in \mathcal{S}_{ab}(a)$.*

Theorem 3.4. *Let $\ell \in \mathcal{P}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then $\ell \in \mathcal{S}_{ab}(a)$ if and only if there exists $\gamma \in AC\big([a,b];R\big)$ satisfying* (10) *and* (11).

4 Application

The following general theorem is a simple consequence of Theorem 1.1.

Theorem 4.1. *Let for every* $u \in U$ *the following implication hold: if* $h(u) = 0$*, then* $u \equiv 0$*. Then the problem* (1), (2) *is uniquely solvable.*

From Theorem 4.1 it immediately follows that the knowledge of the structure of the solution set U allows us to find effective criteria guaranteeing the unique solvability of the problem (1), (2). In particular, the following consequence is true.

Corollary 4.1. *Let* $\dim U = 1$ *and let the set* U *be generated by a positive function. Let, moreover, the operator* h *have the following property: if* $h(u) = 0$*, then* u *has a zero. Then the problem* (1), (2) *is uniquely solvable.*

According to the results obtained in Section 2, Theorem 4.1, respectively Corollary 4.1, one can easily derive the statements dealing with the solvability of the special cases of the problem (1), (2). As an illustration, we give the results dealing with the initial, anti-periodic, and periodic boundary value problems.

Theorem 4.2. *The assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution* u *to the equation* (1) *satisfying*

$$u(t_0) = c,$$

where $t_0 \in [a, b]$ *is arbitrary but fixed and* $c \in R$*.*

Theorem 4.3. *The assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution* u *to the equation* (1) *satisfying*

$$u(b) + u(a) = c,$$

where $c \in R$*.*

The previous theorems immediately follows from Corollary 4.1 and the results established in the Section 2. Applying Theorem 4.1 and the statements of Section 2 we obtain

Theorem 4.4. *Let* $\ell(1) \not\equiv 0$*. Then the assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution* u *to the equation* (1) *satisfying*

$$u(b) - u(a) = c,$$

where $c \in R$*.*

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Modeling and Optimal Control of One Commodity Production and Supply System

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If some company produces and supplies its commodities to the marker, the management of the company must carefully keep up with demands of the market. It is clear that overproduction and the necessity to store commodities lead to obvious losses for the company. Losses will be even bigger in the case of shortage, aking into consideration the unmet demand and a smaller profit. It is obvious that the fact that shortage impairs the company's reputation should also be take into account.

Let T be the planning period, $t = 0, 1, 2, \dots, T$, be discrete moments of time of commodity supplies to the market, $\tau(t)$, $t = 0, 1, 2, \dots, T$, be the demand function. It is assumed that $\tau(t)$ is the known function.

Denote by $x(t)$, $t = 0, 1, 2, \dots, T$, the supply volume. If the supply $x(t)$ and the demand $\tau(t)$ do not coincide, then the company bears a loss.

Let us introduce the notation $y(t) = x(t) - \tau(t)$. If $y(t) < 0$, then shortage takes place. The company loses the profit which could have been received if the commodity had been sold in a quantity $y(t)$. If $y(t) > 0$, then losses are caused by the necessity to search for new consumers and by unsold commodity storage.

Depending on practical situations, to evaluate losses the function $f_1(y)$ can be written as follows:

$$f_1(y) = \begin{cases} \varphi_1(y), & y < 0, \\ 0, & y = 0, \\ \varphi_2(y), & y > 0, \end{cases}$$

where φ_1, φ_2 increase when the absolute value $|y|$ grows. We can actually assume that losses caused shortage ($y < 0$) exceed losses incurred in case the supply volume exceeds the demand ($y > 0$). Therefore it can be assumed that the derivatives of the functions φ_1, φ_2 satisfy the condition $\varphi_1'(|y|) > \varphi_2'(|y|)$.

As an example let us consider the function

$$f_1(y) = \begin{cases} a_1 y^2, & y < 0, \\ 0, & y = 0, \quad a_1 > b_1 > 0, \\ b_1 y^2, & y > 0. \end{cases} \tag{1}$$

Let $u(t) = x(t+1) - x(t)$. The situation in which the production level is constant, i.e. $x(t) = const$ is the most preferable one. In that case, $u(t) = 0$. In the case of an increase of the production output ($u(t) > 0$) and its decrease ($u(t) < 0$), the producer bears losses because of the necessity to reorganize the production.

We call the function $u(t)$ the change dynamics function of the production volume. By analogy with (1), let us introduce the production loss function $f_2(u)$

$$f_2(u) = \begin{cases} a_2 u^2, & u > 0, \\ 0, & u = 0, \quad a_2, b_2 > 0, \\ b_2 u^2, & u < 0. \end{cases}$$

Which of the values a_2 and b_2 is greater depends on a concrete production situation.

Let us now formulate the discrete problem of optimal control: find a change dynamics function $u(t) \in R$, $t = 0, 1, \dots, T$ such that the total loss during the planning period T take a minimal value i.e.

$$\sum_{t=0}^{T-1} \Big(f_1\big(x(t) - \tau(t)\big) + f_2\big(u(t)\big) \Big) + f_1\big(x(T) - \tau(T)\big) \to \min, \tag{2}$$

where $x(t)$ is solution of the discrete equation

$$x(t+1) - x(t) = u(t), \;\; t = 0, 1, \dots, T-1 \tag{3}$$

with the initial condition

$$x(0) = x_0. \tag{4}$$

Let us consider the continuous version of the problem (2)–(4): in the set of piecewise-continuous functions $u(t) \in R$, $t \in [0, T]$, with finitely many discontinuities of the first kind, find a function $u_0(t)$ such that the total loss take a minimal value i.e.,

$$\int_0^T \Big(f_1\big(x(t) - \tau(t)\big) + f_2\big(u(t)\big) \Big)\, dt + f_1\big(x(T) - \tau(T)\big) \to \min,$$

where $x(t)$ is solution of the differential equation

$$\dot{x}(t) = u(t), \;\; t \in [0, T]$$

with the initial condition (4).

On the basis of the maximum principle [1], it is obtained that if $(\psi(t), x(t))$ is a solution of the following boundary value problem

$$\dot{\psi}(t) = \begin{cases} 2a_1\big(x(t) - \tau(t)\big), & x(t) \geq \tau(t), \\ 2b_1\big(x(t) - \tau(t)\big), & x(t) \leq \tau(t), \end{cases}$$

$$\dot{x}(t) = \begin{cases} \psi(t)/2\,a_2, & \psi(t) \geq 0, \\ \psi(t)/2\,b_2, & \psi(t) \leq 0, \end{cases}$$

$$\psi(T) = \begin{cases} -2a_1\big(x(T) - \tau(T)\big), & x(T) \geq \tau(T), \\ -2b_1\big(x(T) - \tau(T)\big), & x(T) \leq \tau(T), \end{cases}$$

$$x(0) = x_0,$$

then

$$u_0(t) = \begin{cases} \psi(t)/2\,a_2, & \psi(t) \geq 0, \\ \psi(t)/2\,b_2, & \psi(t) \leq 0. \end{cases}$$

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Asymptotic Behaviour of Solutions of Nonautonomous Ordinary Differential Equations with Rapidly Varying Nonlinearities

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The monograph by I. T. Kiguradze, T. A. Chanturiya [1] summarizes numerous studies of asymptotic properties of solutions of nonautonomous differential equations by 1990. In particular, the asymptotic behavior of solutions of binomial differential equations with power nonlinearities (Equations of Emden-Fowler type) is sufficiently well described there.

The following differential equation

$$y^{(n)} = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ – continuous function, $-\infty < a < \omega \leq +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ – twice continuously differentiable function satisfying the conditions

$$\varphi'(y) \neq 0 \quad \text{if } \; y \in \Delta_{Y_0}, \qquad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} \quad 0, \\ \text{or} \quad +\infty, \end{cases} \tag{2}$$

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{[\varphi'(y)]^2}{\varphi''(y)\varphi(y)} = 1, \tag{3}$$

Y_0 is zero, or $\pm\infty$, Δ_{Y_0}– one-sided neighborhood of Y_0, is considered in this paper.

By (2) and (3), the function $\varphi''(y)$ is nonzero in some neighborhood of Y_0, that is contained in Δ_{Y_0}. For definiteness, without loss of generality, we assume that

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ ]Y_0, y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

where $y_0 \in \mathbf{R}$ satisfies the inequality $|y_0| > 1$ ($|y_0| < 1$), if $Y_0 = \pm\infty$ ($Y_0 = 0$) and $\varphi''(y) \neq 0$, if $y \in \Delta_{Y_0}$.

Moreover, from (2) and (3) it follows that

$$\lim_{y\to Y_0} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty, \quad \lim_{y\to Y_0} \frac{y\varphi''(y)}{\varphi'(y)} = \pm\infty.$$

Hence the functions $\varphi(y)$ and $\varphi'(y)$ are rapidly varying if $y \to Y_0$ in the sense of definition from monograph Bingham N. H., Goldie C. M., Teugels J. L. [2, Ch. 2, § 2.4, p. 83]. Assuming

$$\mu_0 = \operatorname{sign} \varphi'(y) \quad \text{if } \; y \in \Delta_{Y_0},$$

let us notice that $\varphi(y)$ and $\varphi'(y)$ are rapidly tending to zero if $y \to Y_0$ in the cases

$$\mu_0 y_0 > 0, \;\; Y_0 = 0 \;\text{ or }\; \mu_0 y_0 < 0, \;\; Y_0 = \pm\infty,$$

and rapidly tending to infinity in the cases

$$\mu_0 y_0 < 0, \;\; Y_0 = 0 \;\text{ or }\; \mu_0 y_0 > 0, \;\; Y_0 = \pm\infty.$$

In the case $\varphi(y) = e^{\sigma y}$ ($\sigma \neq 0$) and $Y_0 = +\infty$, in works [3, 4] the asymptotic behavior of the solutions of differential equations (1) with rapidly varying functions φ was researched earlier for $n = 2$ and in works [5–7] for $n \geq 2$. The case $n = 2$ was also considered for an arbitrary function $\varphi : \Delta_{Y_0} \to]0, +\infty[$, that satisfies the conditions (2) and (3) in monograph V. Maric [8] and in work [9]. The case $Y_0 = 0$ and $\omega = +\infty$ was researched in [8]. The case $n = 2$, arbitrary $Y_0 \in \{0, \pm\infty\}$ and $\omega \leq +\infty$ was considered in [9]. It should be noted, that in [9] the class of solutions defined after the function φ was studied.

In this work we leave the class of solutions the same as it was researched earlier (for instance in [10]) for equations with regularly varying as $y \to Y_0$ functions φ.

Definition. Solution y of differential equation (1), defined on $[t_0, \omega[\subset \Delta_{Y_0}$, is called $P_\omega(Y_0, \lambda^0_{n-1})$ – solution, where $-\infty \leq \lambda^0_{n-1} \leq +\infty$, if it satisfies the conditions

$$y(t) \in \Delta_{Y_0} \ \text{ if } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \qquad (k = \overline{1, n-1}),$$

$$\lim_{t \uparrow \omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda^0_{n-1}.$$

Let us put the following subsidiary notations.

$$a_{0k} = (n-k)\lambda_0 - (n-k-1) \ \ (k = \overline{1, n}) \ \text{ if } \ \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty, \end{cases}, \quad J(t) = \int_A^t \pi_\omega^{n-1}(\tau) p(\tau)\, d\tau, \quad \Phi(y) = \int_B^y \frac{ds}{\varphi(s)},$$

$$q_1(t) = \frac{\prod\limits_{j=1}^{n-1} a_{0j}}{\alpha_0(\lambda^0_{n-1} - 1)^n p(t) \pi_\omega^n(t)} \frac{\Phi^{-1}\Big(\frac{\alpha_0(\lambda^0_{n-1}-1)^{n-1}}{\prod\limits_{j=2}^{n} a_{0j}} J(t)\Big)}{\varphi\Big(\Phi^{-1}\Big(\frac{\alpha_0(\lambda^0_{n-1}-1)^{n-1}}{\prod\limits_{j=2}^{n} a_{0j}} J(t)\Big)\Big)},$$

where the limit of integration $A \in \{a, \omega\}$ ($B \in \{y_0, Y_0\}$) is chosen so that the integral tends to zero or to $\pm\infty$ if $t \uparrow \omega$ ($y \to Y_0$), and Φ^{-1} – the inverse function to Φ.

The following result that is related to the not singular case was found for the differential equation (1).

Theorem. *Let $\lambda^0_{n-1} \in \mathbf{R} \setminus \{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1\}$. Then for the existence of the $P_\omega(Y_0, \lambda^0_{n-1})$ – solutions of the differential equation* (1) *it is necessary that*

$$\mu_0 \alpha_0 [\lambda^0_{n-1} - 1]^{n-1} \Big(\prod_{j=2}^{n} a_{0j}\Big) J(t) < 0, \quad \mu_0 y_0 (\lambda^0_{n-1} - 1) a_{01} \pi_\omega^n(t) J(t) < 0 \quad \text{if } \ t \in]a, \omega[\,, \tag{4}$$

$$\frac{\alpha_0(\lambda^0_{n-1} - 1)^{n-1}}{\prod\limits_{j=2}^{n} a_{0j}} \lim_{t \uparrow \omega} J(t) = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi(y), \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'(t)}{J(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q_1(t) = 1. \tag{5}$$

Moreover, for each such solution if $t \uparrow \omega$ the following asymptotic representations take place:

$$\varphi'(y(t)) = -\frac{\alpha_0 \prod\limits_{j=2}^{n} a_{0j}}{(\lambda^0_{n-1} - 1)^{n-1}} \frac{1 + o(1)}{J(t)},$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda^0_{n-1} - 1)\pi_\omega(t)} [1 + o(1)] \ \ (k = \overline{1, n-1}),$$

If for some $\lambda_{n-1}^0 \in \mathbf{R} \setminus \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\right\}$ *the following condition*

$$\lim_{t \uparrow \omega} [1 - q_1(t)] \ln |\pi_\omega(t)| = 0$$

is observed together with (4)–(5) *and algebraic about* ρ *equation*

$$\sum_{k=0}^{n-3} \prod_{i=k+2}^{n-1} a_{0i} \prod_{i=1}^{k} (a_{0i} + \rho) = (\lambda_{n-1}^0 - 1 - \rho) \prod_{i=1}^{n-2} (a_{0i} + \rho) \tag{6}$$

has no roots with zero real part, then (1) *has at least one* $P_\omega(Y_0, \lambda_{n-1}^0)$ *– solution. If (6) has* m *roots, real parts of which have the sign that is opposite to the sign of the function* $(\lambda_{n-1}^0 - 1)\pi_\omega(t)$ *on* $[t_0, \omega[$ *, then if* $CJ'(t)J(t) > 0$ *the equation* (1) *has* m*-parametric family, and if* $CJ'(t)J(t) < 0$ *–* $m+1$*-parametric family of such solutions, where* C *is defined from the equation*

$$\frac{1}{C} = 1 + \frac{a_{01}}{a_{02}} + \frac{a_{01}}{a_{03}} + \dots + \frac{a_{01}}{a_{0n-2}} + \frac{a_{01}}{a_{0n-1}} - a_{01}.$$

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Existence of 2π-Periodic Solutions for the Brillouin Electron Beam Focusing Equation

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1 Introduction

This note concerns the problem of existence of periodic solutions of the equation

$$\ddot{x} + b(1 + \cos t)x = \frac{1}{x}\,, \tag{1}$$

where b is a positive constant. Throughout the paper, we will not take into account solutions with collisions, but we will always search for positive 2π-periodic solutions of (1).

The physical meaning of equation (1) arises in the context of Electronics, since it governs the motion of a magnetically focused axially symmetric electron beam under the influence of the Brillouin flow, as shown in [1]. From a mathematical point of view, (1) is a singular perturbation of a Mathieu equation.

Motivated by some numerical experiments realized in [1], where it was conjectured that, if $b \in (0, 1/4)$, equation (1) should have a 2π-periodic solution, in the last fifty years the work of many mathematicians has given birth to an extensive literature about this topic. Although at the moment the conjecture has not been correctly proven yet, many advances in this line have been obtained, allowing to understand that the problem of existence of 2π-periodic solutions of (1) when $b \in (0, 1/4)$ can be really delicate, and arising doubts on the validity of the result conjectured in [1].

To the best of our knowledge, M. Zhang determined in [7] the best range of b actually known for the solvability of (1), using a non-resonance hypothesis for the associated Mathieu equation. He proves that whether $b \in (0, 0.16448)$, then (1) has at least one 2π-periodic solution. This last result has been extended to equations where the singularity may be of weak type (see [6]).

An important result to understand the difficulty of showing the validity of the conjecture proposed in [1] was proven in [8]. In that paper, it was established an unanimous relation between the stability intervals for the Mathieu equation and the existence of periodic solutions for the Yermakov–Pinney equation. Notice that the stability intervals of the Mathieu equation $(\lambda_0, \lambda_1'), (\lambda_2', \lambda_1), (\lambda_2, \lambda_3'), \dots$, are defined approximately by $\lambda = 0$, $\lambda_1' \approx 1/6$; $\lambda_2' \approx 0.4$, $\lambda_1 \approx 0.95$, … (see [5, Theorem 2.1] and [2, Figure 1]). This suggests that, in order to obtain a correct proof of the conjecture by V. Bevc, J. L. Palmer and C. Süsskind, one has to take into account some property of equation (1) which is not verified for the Yermakov–Pinney equation.

Quite unexpectedly with respect to the numerical and analytical results found in literature, we establish a new range for the existence of $2\pi-$periodic solutions of the Brillouin focusing beam equation. This is possible thanks to suitable nonresonance conditions acting on the rotation number of the solutions in the phase plane. Our main result is

Theorem 1. *If $b \in [0.4705, 0.59165]$, then (1) has at least one 2π-periodic solution.*

2 A Non-Resonance Theorem for Singular Perturbations of a Mathieu Equation

The proof of Theorem 1 is based on a non-resonance result which involves nonlinearities with “atypical” linear growing type, and could have interest by itself.

Theorem 2. *Let us assume that there exist positive constants A_+, B_+ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{B_+}, 1\right\} dt > \frac{n}{2\sqrt{B_+}}\,, \tag{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{A_+}, 1\right\} dt < \frac{n+1}{2\sqrt{A_+}}\,, \tag{3}$$

for some natural number n. Then (1) *has at least one 2π-periodic solution.*

A couple of remarks are in order.

Remark 1. With the aim of keeping the exposition at a rather simple level, and taking into account that our main goal will be to study the existence of 2π-periodic solutions of (1), we will always consider equation (1) as a starting point. However, the result can be extended, with the same approach and similar computations, to more general equations like

$$\ddot{x} + q(t)x - g(t,x) = 0, \tag{4}$$

where q is continuous and 2π-periodic and positive on almost interval $[0, 2\pi]$, and $g : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ has a similar behavior as $1/x^\gamma$, with $\gamma \geq 1$, being allowed to grow at most sublinearly at infinity. Of course, in this case $q(t)$ will replace $b(1 + \cos t)$.

Remark 2. Conditions (2) and (3) were introduced by Fabry in [3] for the equation

$$\ddot{x} + g(t,x) = 0,$$

with

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \leq q(t),$$

asking that

$$\sqrt{\lambda_j} < \sup_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \min\{p(t),\xi\}\,dt}{\sqrt{\xi}}\,, \qquad \inf_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \max\{q(t),\xi\}\,dt}{\sqrt{\xi}} < \sqrt{\lambda_{j+1}}\,,$$

where λ_j is the j-th eigenvalue of the considered 2π-periodic problem. Such conditions are usually coupled with the sign assumption $\liminf_{|x|\to+\infty} \operatorname{sgn} x f(t,x) > 0$, which, however, in our model, is not satisfied.

As it is easy to see, (2) and (3) are the counterpart of such conditions for the Dirichlet spectrum (which is the natural one to consider when dealing with problems with a singularity).

3 Application

In order to prove Theorem 1 it will be convenient, for any $n \in \mathbb{N}$, to define the absolutely continuous functions $F_n, G_n : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ by

$$F_n(b,x) = \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n}{2}\,,$$

$$G_n(b,x) = \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n+1}{2}\,.$$

Both functions are non-decreasing with respect to the variable b. Moreover, if there exists $n \in \mathbb{N}$ such that $\inf_{x>0} G_n(b,x) < 0$ and $\sup_{x>0} F_n(b,x) > 0$, then Theorem 2 implies that (1) has at least one periodic solution. Therefore, we have the following proposition.

Proposition 1. *Assume that there exists $n \in \mathbb{N}$ such that*

$$b \in \left(\inf\left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\}, \sup\left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} \right). \tag{5}$$

Then (1) *has at least one 2π-periodic solution.*

Let us first observe that, in view of the continuity and the monotonicity of the functions F_n, G_n in the variable b, there exist b_0^n and b_1^n such that

$$\left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\} = (b_0^n, +\infty) \text{ and } \left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} = (0, b_1^n).$$

The point is to prove that these two intervals contain common points, i.e., $b_0^n < b_1^n$. We will show this in the case when $n = 0$ and $n = 1$, and the estimates performed in this last case will allow to achieve the new result consisting in Theorem 1.

Remark 3. To see the details of the proofs, this results are published in [4].

Acknowledgements

M. Zamora is supported by the Ministerio de Educación y Ciencia, Spain, project MTM2011-23652, and by the Junta de Andalucía, Spain, Project FQM2216.

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Variation Formulas of Solution for a Nonlinear Functional Differential Equation Taking into Account Two Delay Parameters Perturbation and the Continuous Initial Condition

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Let $I=[a,b]$ be a finite interval and let $0<\tau_1<\tau_2$, $0<\sigma_1<\sigma_2$ be given numbers; suppose that $O\subset\mathbb{R}^n$ is an open set and E is a set of n-dimensional functions $f(t,x,y,z)$ satisfying the following conditions: for almost all $t\in I$ the function $f(t,\cdot\,):O^3\to\mathbb{R}^n$ is continuously differentiable; for any fixed $(x,y,z)\in O^3$ the functions $f(t,x,y,z)$, $f_x(\,\cdot\,)$, $f_y(\,\cdot\,)$, $f_z(\,\cdot\,)$ are measurable on I; for each $f\in E$ and compact set $K\subset O$ there exists a function $m_{f,K}(t)\in L(I,[0,\infty))$ such that

$$|f(t,x,y,z)|+|f_x(t,x,y,z)|+|f_y(t,x,y,z)|+|f_z(t,x,y,z)|\le m_{f,K}(t)$$

for all $(x,y,z)\in K^3$ and for almost all $t\in I$. Further, let Φ be the set of continuous functions $\varphi:I_1\to O$, where $I_1=[\widehat{\tau},b]$, $\widehat{\tau}=a-\max\{\tau_2,\sigma_2\}$.

To each element $\mu=(t_0,\tau,\sigma,\varphi,f)\in\Lambda=[a,b)\times[\tau_1,\tau_2]\times[\sigma_1,\sigma_2]\times\Phi\times E$ we assign the functional differential equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau),x(t-\sigma)\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0]. \tag{2}$$

Definition. Let $w=(t_0,\tau,\sigma,\varphi,f)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat{\tau},t_1]$, $t_0<t_1\le b$, is called the solution of equation (1) with the initial condition (2) or the solution corresponding to the element μ and defined on the interval $[\widehat{\tau},t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let us introduce the set of variations

$$V=\Big\{\delta\mu=(\delta t_0,\delta\tau,\delta\sigma,\delta\varphi,\delta f):\ |\delta t_0|\le\alpha,\ |\delta\tau|\le\alpha,\ |\delta\sigma|\le\alpha,\\ \delta\varphi=\sum_{i=1}^k\lambda_i\delta\varphi_i,\ \delta f=\sum_{i=1}^k\lambda_i\delta f_i,\ |\lambda_i|\le\alpha,\ i=\overline{1,k}\Big\},$$

where $\delta\varphi_i\in\Phi-\varphi_0$ $\delta f_i\in E-f_0$, $i=\overline{1,k}$, $\varphi_0\in\Phi$, $f_0\in E$ are fixed functions, $\alpha>0$ is a fixed number. Let $x_0(t)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,\sigma_0,\varphi_0,f_0)\in\Lambda$ with $t_{00}\in(a,b),\tau_0\in(\tau_1,\tau_2),\sigma_0\in(\sigma_1,\sigma_2)$ and defined on the interval $[\widehat{\tau},t_{10}],t_{10}\in(t_{00},b)$. There exist numbers $\delta_1>0$ and $\varepsilon>0$ such that for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_1]\times V$ we have $\mu_0+\varepsilon\delta\mu\in V$ and to the element $\mu_0+\varepsilon\delta\mu$ there corresponds the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$.

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Let us define the increment of the solution $x_0(t):=x(t;\mu_0)$:

$$\delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\ \ \forall\,(t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times[0,\varepsilon_1]\times V.$$

Theorem 1. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot\varphi_0(t)$ and $f_0(t,x,y,z),(t,x,y,z)\in I\times O^3$ be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^- = \lim_{t\to t_{00}-}\dot\varphi_0(t), \quad \lim_{w\to w_0} f_0(w)=f_0^-, \ \ w=(t,x,y,z)\in(a,t_{00}]\times O^3,$$

where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),\varphi_0(t_{00}-\sigma_0))$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that

$$\delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^-$, where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^- - f_0^-)\delta t_0+\beta(t;\delta\mu),$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta\varphi(t_{00})+\int\limits_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds+$$

$$+\int\limits_{t_{00}-\sigma_0}^{t_{00}} Y(s+\sigma_0;t)f_{0z}[s+\sigma_0]\delta\varphi(s)\,ds-\Bigg[\int\limits_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot x_0(s)\,ds\Bigg]\delta\tau-$$

$$-\Bigg[\int\limits_{t_{00}}^{t} Y(s;t)f_{0z}[s]\dot x_0(s)\,ds\Bigg]\delta\sigma+\int\limits_{t_{00}}^{t} Y(s;t)\delta f[s]\,ds.$$

Here $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_s(s;t)=-Y(s;t)f_{0x}[s]-Y(s+\tau_0;t)f_{0y}[s+\tau_0]-Y(s+\sigma_0;t)f_{0z}[s+\sigma_0], \quad s\in[t_{00},t]$$

and the condition $Y(s;t)=\begin{cases} H, & s=t,\\ \Theta, & s>t,\end{cases}$ *where H is the identity matrix,Θ is the zero matrix;*

$$f_{0y}[t]=f_{0y}\big(t,x_0(t),x_0(t-\tau_0),x_0(t-\sigma)\big), \quad \delta f[t]=\delta f\big(t,x_0(t),x_0(t-\tau_0),x_0(t-\sigma)\big).$$

The Theorem 1 is proved by the scheme described in [1].

Theorem 2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot\varphi_0(t)$ and $f_0(t,x,y,z),(t,x,y,z)\in I\times O^3$ be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^+ = \lim_{t\to t_{00}+}\dot\varphi_0(t), \quad \lim_{w\to w_0} f_0(w)=f_0^+, \ \ w\in[t_{00},t_{10}]\times O^3.$$

Then for each $\widehat t_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for any $(t,\varepsilon,\delta\mu)\in[\widehat t_0,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+$, where $V^+=\{\delta\mu\in V:\delta t_0\ge 0\}$, formula (3) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^+ - f_0^+)\delta t_0+\beta(t;\delta\mu)$.*

Theorem 3. *Let the assumptions of Theorems* 1 *and* 2 *be fulfilled. Moreover, $\dot\varphi_0^- - f_0^- = \dot\varphi_0^+ - f_0^+ := \widehat f_0$. Then for each $\widehat t_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for any $(t,\varepsilon,\delta\mu)\in[\widehat t_0,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$, formula* (3) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)\widehat f_0\delta t_0+\beta(t;\delta\mu)$.*

All assumptions of Theorem 3 are satisfied if the function $f_0(t,x,y,z)$ is continuous and bounded, and the function $\varphi_0(t)$ is continuously differentiable. It is clear that in this case

$$\widehat f_0=\dot\varphi_0(t_{00})-f_0\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),\varphi_0(t_{00}-\sigma_0)\big).$$

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Lipschitz Property of the Lower Sigma-Exponent of Linear Differential System

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Consider the linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \;\; t \geqslant 0, \tag{1_A}$$

with bounded piecewise continuous coefficients on the half-line $[0, +\infty)$, with the Cauchy matrix $X_A(t, \tau)$ and with characteristic exponents $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$.

Let $\lambda[Q]$ denote Lyapunov exponent of piecewice continuous nth-order matrix $Q(t)$. For the higher sigma-exponent [1; see also 2; 3, p. 225]

$$\nabla_\sigma(A) \equiv \sup_{\lambda[Q] \leqslant -\sigma} \lambda_n(A + Q), \;\; \sigma > 0, \tag{2}$$

of the linear system (1_A) the algorithm for its computation on the basis of the sequence $\{\xi_k(\sigma)\}$ was suggested in the above-mentioned papers:

$$\nabla_\sigma(A) = \varlimsup_{k \to \infty} \frac{\xi_k(\sigma)}{k}\,, \;\; \sigma > 0. \tag{3_1}$$

The elements of this sequence are defined recursively on the basis of the Cauchy matrix $X_A(t, \tau)$ of the system (1_A):

$$\xi_k(\sigma) = \max_{i<k} \big\{ \ln \|X_A q(k,i)\| + \xi_i(\sigma) - \sigma i \big\}, \;\; \xi_0(\sigma) = 0. \tag{3_2}$$

The higher sigma-exponent $\nabla_\sigma(A)$ as a function of the parameter $\sigma > 0$ is bounded [4, p. 31] non-increasing on the half-line $[0, +\infty)$ and identically equal to [5] the highest characteristic exponent $\lambda_n(A)$ of the system (1_A) on the interval $(\sigma_0, +\infty)$, where $\sigma_0 = \sigma_0(A) \geqslant 0$ is the Grobman irregularity coefficient of the system (1_A). It was shown in [1] that the higher sigma-exponent $\nabla_\sigma(A)$ is continuous function in parameter σ and, moreover, it has the Lipschits property with Lipschitz constant $L(\varepsilon) > 0$ in any interval $[\varepsilon, +\infty)$, $\varepsilon > 0$.

The analysis of properties of the higher sigma-exponent $\nabla_\sigma(A)$ as a function of the parameter $\sigma > 0$ in [6, 7] led to the following its main property–concavity. Finally it was proved [8] on the basis of the algorithm (3_1)–(3_2) that the properties 1) boundedness; 2) concavity; 3) coincidence with a constant for σ exceeding some $\sigma_0 \geqslant 0$ completely determine $\nabla_\sigma(A)$. In particular, for any function $f : (0, +\infty) \to R^1$ with the properties 1)–3) it was proved the existence of the linear system such that its higher sigma-exponent coincides with this function $f(\sigma)$ for all $\sigma > 0$.

As applications of the higher sigma-exponent we will indicate its use in the investigation of the exponential stability of the linear perturbed systems (1_{A+Q}) with exponentially decreasing perturbations Q and nonlinear systems with a linear approximation (1_A) and perturbations of the higher order of smallness.

The lower sigma-exponent [9,10]

$$\Delta_\sigma(A) \equiv \inf_{\lambda[Q] \leqslant -\sigma} \lambda_1(A + Q), \;\; \sigma > 0, \tag{4}$$

is defined by the lowest characteristic exponents $\lambda_1(A+Q)$ of the perturbed linear systems (1_{A+Q}). It is used in the investigation of both the instability of the zero solution ($\Delta_\sigma(A) > 0$) and in any case the one-dimensional conditional stabilization ($\Delta_\sigma(A) < 0$) of the linear differential system (1_{A+Q}) with the exponentially decreasing perturbation Q. It is also possible to use it in the investigation of the analogous properties of the zero solution of nonlinear differential systems with perturbations of the higher order of smallness in a neighborhood of the origin of coordinates.

For the lower sigma-exponent it is absent any algorithm for its computation on the basis of the Cauchy matrix $X_A(t,\tau)$ of linear system or of its solutions. At the same time such algorithm has been constructed [9; 3, p. 52] for its limit value $\Delta_0(A) \equiv \Delta(A)$ of this exponent when $\sigma \to +0$. From definition (4) of the lower sigma-exponent $\Delta_\sigma(A)$ of the system (1_A) as a function of the parameter $\sigma > 0$ we have the following its properties: 1) boundedness on the half-line $(0,+\infty)$ by virtue of boundedness of the matrix $A(t) + Q(t)$ of coefficients of the perturbed linear system (1_{A+Q}) (see [4, p. 31]); 2) nondecreasing on this half-line by virtue of the restriction of the range of matrix Q when σ increases, $\sigma > 0$, 3) identical coincidence of the exponent $\Delta_\sigma(A)$ with the lowest characteristic exponent $\lambda_1(A)$ of the initial system (1_A) for all $\sigma > \sigma_0(A)$ on the basis of the property of the Grobman irregularity coefficient $\sigma_0(A)$. However, even such fundamental property of continuity or, on the contrary, discontinuity of the lower sigma-exponent $\Delta_\sigma(A)$ as a function of $\sigma \in (0,+\infty)$ is not established in general case. Hence the problem [10] about complete description of the properties of this exponent is unsolved.

In our opinion, the sufficiently significant advance in the investigation of properties of the lower sigma-exponent is the proof of the existense of Lipschitz in $\sigma \in (0,+\infty)$ exponents $\Delta_\sigma(A)$ which has been obtained in [11]. It is self-evident that they have the mentioned above necessary properties 1)–3). At first for this purpose we construct piecewise linear lower sigma-exponents and the corresponding linear differential systems.

Lemma ([11])**.** *For any function*

$$\varphi(\sigma) = \begin{cases} \Delta_0 + (\Delta_1 - \Delta_0)\,\dfrac{\sigma}{\sigma_0} & \text{for } \sigma \in (0,\sigma_0], \\ \Delta_1 & \text{for } \sigma \in (\sigma_0,+\infty), \end{cases}$$

with arbitrary parameters

$$\Delta_1 - \Delta_0 > \sigma_0 > 0,$$

there exists a two-dimensional system (1_A) *with bounded piecewise constant on the half-line* $[0,+\infty)$ *coefficients and with lower sigma-exponent* $\Delta_\sigma(A) \equiv \varphi(\sigma)$, $\sigma \in (0,+\infty)$.

With the help of this lemma and stages of its proof we establish the validity of the necessary assertion.

Theorem 1 ([11])**.** *For any nondecreasing function* $f : (0,+\infty) \to [c_0, d_0] \subset (-\infty + \infty)$ *that coincides with a constant* d_0 *on some interval* $(\sigma_0,+\infty)$ *and satisfies the Lipschitz condition*

$$0 \leqslant f(\sigma_2) - f(\sigma_1) < L(\sigma_2 - \sigma_1), \quad 0 < \sigma_1 < \sigma_2 < +\infty,$$

with a finite Lipschitz constant $L > 1$, *there exists a linear system* (1_A) *with bounded piecewise constant coefficients such that its lower sigma-exponent* $\Delta_\sigma(A)$ *coincides with the function* $f(\sigma)$ *for all* $\sigma > 0$.

In connection with the assertion of this theorem the following question is interesting: will the lower sigma-exponent $\Delta_\sigma(A)$ be Lipschitz function in the whole range of definition $(0,+\infty)$? The negative answer to it is given by

Theorem 2. *For any nondecreasing bounded on the half-line* $[0,+\infty)$ *function* $f(\sigma)$ *coincident with a constant* d_0 *on some interval* $(\sigma_0,+\infty)$, $\sigma_0 > 0$, *and satisfying on any interval* $[\varepsilon,+\infty) \subset (0,+\infty)$ *the Lipschitz condition with Lipschitz constant* $L(\varepsilon) < +\infty$, $\varepsilon > 0$, *there exists a linear system* (1_A) *with bounded piecewise continuous coefficients and the lower sigma-exponent* $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$.

Remark. Evidently there is a function $f(\sigma)$ satisfying all the conditions of Theorem 2 and not satisfying the Lipschitz condition with finite Lipschitz constant on the whole interval.

We note that at the investigation of lower sigma-exponents the following problems remains unsolved.

Problem 1. *To determine whether the sufficient properties of the lower sigma-exponent obtained in Theorem* 2 *are also necessery.*

Problem 2. *To construct some algorithm of the calculation of the lower sigma-exponent of the system* (1_A) *on the basis of its Cauchy matrix or of its solutions.*

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On Some Properties of Irreducibility Sets of Linear Differential Systems

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We consider linear systems of the form

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in I = [0, +\infty), \tag{1}$$

with piecewise continuous bounded coefficients ($\|A(t)\| \leq a$ for $t \in I$). Along with original systems (1) we will consider the perturbed systems

$$\dot{y} = (A(t) + Q(t))y, \ \ y \in \mathbb{R}^n, \ \ t \in I, \tag{2}$$

with piecewise continuous perturbations Q defined on I and satisfying either the condition

$$\|Q(t)\| \leq C_Q e^{-\sigma t}, \ \ \sigma \geq 0, \ \ t \in I, \tag{3}$$

or more general condition

$$\|Q(t)\| \leq C_Q^{\varepsilon} e^{(\varepsilon - \sigma)t}, \ \ \sigma \geq 0, \ \ \forall\, \varepsilon > 0, \ \ t \in I, \tag{4}$$

which is equivalent to the inequality $\lambda[Q] \equiv \overline{\lim\limits_{t \to +\infty}}\, t^{-1} \ln \|Q(t)\| \leq -\sigma \leq 0$.

If $\sigma = 0$ in (3), (4), then we additionally suppose that $Q(t) \to 0$ as $t \to +\infty$.

Following Yu. S. Bogdanov [1] we say that systems (1) and (2) are asymptotically equivalent (Lyapunov's equivalent, reducible) if there exists a Lyapunov transformation

$$x = L(t)y, \quad \max\Big\{ \sup_{t \in I} \|L(t)\|, \sup_{t \in I} \|L^{-1}(t)\|, \sup_{t \in I} \|\dot{L}(t)\| \Big\} < +\infty,$$

reducing one of them to the other.

The sets $N_r(a, \sigma)$, $N_\rho(a, \sigma)$, $a > 0$, $\sigma \geq 0$, are said to be the irreducibility sets if they consist of all systems (1) with the following properties [2]:

(1) the norm of the coefficient matrix A is less than or equal to a on I;

(2) for each system (1) there exists a system (2) with the matrix Q satisfying either the condition (3) or the more general condition (4), respectively, which cannot be reduced to system (1).

If Q satisfies (3) or (4) with $\sigma > 2a$, then $\Big\| \int\limits_t^{+\infty} Q(u)\, du \Big\| \leq C e^{-\sigma_1 t}$ for some $C > 0$ and $\sigma_1 > 2a$, so [3] systems (1) and (2) are asymptotically equivalent, and, therefore, the sets $N_r(a, \sigma)$, $N_\rho(a, \sigma)$ are empty for all $\sigma > 2a$.

From the properties of perturbation matrix Q it follows that $N_r(a, 0) = N_\rho(a, 0)$ for all $a > 0$, and these sets are nonempty and related by the inclusion $N_r(a, \sigma) \subseteq N_\rho(a, \sigma)$ for $\sigma \in (0, 2a]$. Moreover [4], $N_r(a, \sigma) \neq N_\rho(a, \sigma)$ for $\sigma \in (0, 2a]$.

It is obvious that $N_r(a,\sigma_2) \subseteq N_r(a,\sigma_1)$, $N_\rho(a,\sigma_2) \subseteq N_\rho(a,\sigma_1)$ for $0 < \sigma_1 < \sigma_2 \le 2a$, and [4] these irreducibility sets strictly decrease with the increasing parameter σ, i.e.

$$N_r(a,\sigma_2) \subset N_r(a,\sigma_1), \quad N_\rho(a,\sigma_2) \subset N_\rho(a,\sigma_1) \ \ \forall\, 0 < \sigma_1 < \sigma_2 \le 2a.$$

We introduced [2, 5] the reducibility coefficient (the reducibility exponent) of the linear system (1) as the greatest lower bound of the set of all values of the parameter $\sigma > 0$ for which the perturbed system (2) with an arbitrary perturbation Q satisfying condition (3) (respectively, condition (4)) can be reduced to the original system (1). The properties of these reducibility coefficients and exponents allows us to investigate certain properties of the irreducibility sets as the functions of the parameters a and σ. The following statements are hold.

Theorem 1. *For all $a_0 > 0$, $\sigma \in (0, 2a_0)$ we have*

$$\operatorname*{Lim}_{a\to a_0-0} N_r(a,\sigma) \ne N_r(a_0,\sigma), \quad \operatorname*{Lim}_{a\to a_0+0} N_r(a,\sigma) = N_r(a_0,\sigma),$$
$$\operatorname*{Lim}_{a\to a_0-0} N_\rho(a,\sigma) \ne N_\rho(a_0,\sigma), \quad \operatorname*{Lim}_{a\to a_0+0} N_\rho(a,\sigma) = N_\rho(a_0,\sigma).$$

Theorem 2. *For all $a > 0$, $\sigma_0 \in (0, 2a]$ we have*

$$\operatorname*{Lim}_{\sigma\to \sigma_0-0} N_r(a,\sigma) \ne N_r(a,\sigma_0), \quad \operatorname*{Lim}_{\sigma\to \sigma_0+0} N_r(a,\sigma) \ne N_r(a,\sigma_0).$$

Theorem 3. *For all $a > 0$, $\sigma_0 \in (0, 2a]$ we have*

$$\operatorname*{Lim}_{\sigma\to \sigma_0-0} N_\rho(a,\sigma) = N_\rho(a,\sigma_0), \quad \operatorname*{Lim}_{\sigma\to \sigma_0+0} N_\rho(a,\sigma) \ne N_\rho(a,\sigma_0).$$

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On Some Properties and Approximate Solution of One System of Nonlinear Partial Differential Equations

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Many applied problems leads to the following nonlinear system

$$\frac{\partial U}{\partial t}=\sum_{\alpha=1}^{p}\frac{\partial}{\partial x_\alpha}\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big), \tag{1}$$

$$\frac{\partial V_\alpha}{\partial t}=f_\alpha\Big(V_\alpha,\frac{\partial U}{\partial x_\alpha}\Big),\ \ \alpha=1,\dots,p, \tag{2}$$

where f_α are given functions. If $p=2$ and

$$f_\alpha\Big(V_\alpha,\frac{\partial U}{\partial x_\alpha}\Big)=-V_\alpha+g_\alpha\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big),\ \ 0<\gamma_0<g_\alpha(\xi_\alpha)\le G_0,\ \ \alpha=1,2, \tag{3}$$

where g_α are given sufficiently smooth functions and γ_0, G_0 are constants, the system (1)–(3) describes the vein formation in meristematic tissues of young leaves [1]. Investigations for one-dimensional analog of model (1)–(3) are carried out in [2]. The large theoretical and practical importance of the investigation and construction of approximate solutions of the boundary value problems for systems (1)–(3) is pointed out in the above-mentioned works [1, 2]. In the direction of biological modeling it is necessary to note work [3], where many mathematical models of similar diffusion processes are also presented and discussed.

There are some effective algorithms for solving the multi-dimensional problems (see, for example, [4] and references therein). These algorithms mainly belong to the methods of splitting-up or sum approximation according to their approximate properties. Some schemes of the variable directions are constructed and studied in [5] and in a number of other works by this author too. Some questions of construction and investigation of the discrete analogs for systems of the type (1)–(3) are discussed in [6–9] and in a number of other works as well.

Let us consider the following two-dimensional initial-boundary value problem:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x_1}\Big(V_1\,\frac{\partial U}{\partial x_1}\Big)+\frac{\partial}{\partial x_2}\Big(V_2\,\frac{\partial U}{\partial x_2}\Big), \tag{4}$$

$$\frac{\partial V_\alpha}{\partial t}=-V_\alpha+g_\alpha\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big),\ \ \alpha=1,2, \tag{5}$$

$$U(x,0)=U_0(x),\quad V_\alpha(x,0)=V_{\alpha0}(x),\ \ x\in\overline{\Omega},\ \ \alpha=1,2, \tag{6}$$

$$U(x,t)=0,\ \ (x,t)\in\partial\Omega\times[0,T]. \tag{7}$$

Here $x=(x_1,x_2)$, $\Omega=(0,1)\times(0,1)$, $\partial\Omega$ is the boundary of the domain Ω, T is some fixed positive number, U_0, $V_{\alpha0}$, g_α are given sufficiently smooth functions, such that

$$V_{\alpha0}(x)\ge\delta_0,\ \ x\in\overline{\Omega}, \tag{8}$$

$$\gamma_0\le g_\alpha(\xi_\alpha)\le G_0,\quad \big|g'_\alpha(\xi_\alpha)\big|\le G_1,\ \ \xi_\alpha\in R,\ \ \alpha=1,2, \tag{9}$$

where δ_0, γ_0, G_0, G_1 are some positive constants.

Suppose that all necessary consistence conditions are satisfied and there exists a sufficiently smooth solution of the problem (4) - (7). It should be noted that the uniqueness of a solution of the problem (4)–(7) is studied in [6].

Under the conditions (8), (9) from (5), (6) in $\overline{Q} = \overline{\Omega} \times [0, T]$ we have

$$\delta_0 \le V_\alpha(x,t) \le \Delta_0, \quad \alpha = 1, 2, \tag{10}$$

where Δ_0 is a positive constant. Using (5), (9), (10) we get the estimates

$$\Big|\frac{\partial V_\alpha}{\partial t}\Big| \le C, \quad (x,t) \in \overline{Q}, \quad \alpha = 1, 2.$$

Here and below C is a positive constant.

Introduce on $\overline{Q}$ the grids $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $\overline{\omega}_{\alpha h\tau} = \overline{\omega}_{\alpha h} \times \omega_\tau$, $\alpha = 1, 2$, where

$$\overline{\omega}_h = \Big\{x_{i_1 i_2} = (i_1 h_1, i_2 h_2),\ i_\beta = 0, \ldots, M_\beta, M_\beta h_\beta = 1,\ \beta = 1, 2\Big\},$$
$$\overline{\omega}_{1h} = \Big\{x_{i_1 i_2} = ((i_1 - 1/2)h_1, i_2 h_2),\ i_1 = 1, \ldots, M_1,\ i_2 = 0, \ldots, M_2\Big\},$$
$$\overline{\omega}_{2h} = \Big\{x_{i_1 i_2} = (i_1 h_1, (i_2 - 1/2)h_2),\ i_1 = 0, \ldots, M_1,\ i_2 = 1, \ldots, M_2\Big\},$$
$$\omega_h = \Omega \cap \overline{\omega}_h, \quad \gamma_h = \overline{\omega}_h \setminus \omega_h, \quad \overline{\omega}_h = \omega_h \cup \gamma_h, \quad \omega_\tau = \big\{t_j = j\tau,\ j = 0, \ldots, N,\ N\tau = T\big\}.$$

Here h_α is the space step in direction x_α and τ is the time step on $[0, T]$.

Define the following inner products and the norms for the discrete functions y and z given on $\overline{\omega}_h$

$$(y, z) = \sum_{i_1=1}^{M_1-1} \sum_{i_2=1}^{M_2-1} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2, \quad (y, z]_1 = \sum_{i_1=1}^{M_1} \sum_{i_2=1}^{M_2-1} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2,$$
$$(y, z]_2 = \sum_{i_1=1}^{M_1-1} \sum_{i_2=1}^{M_2} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2, \quad \|y\| = (y, y)^{1/2}, \quad \|y]|_1 = (y, y]_1^{1/2}, \quad \|y]|_2 = (y, y]_2^{1/2}.$$

The inner products and the norms on $\overline{\omega}_{\alpha h}$ are defined in a similar way.

Using known notations let us correspond to the problem (4)–(7) the difference scheme of the type of variable directions:

$$u_{1t} = (\widehat{v}_1 \widehat{u}_{1\overline{x}_1})_{x_1} + (v_2 u_{2\overline{x}_2})_{x_2}, \quad u_{2t} = (\widehat{v}_1 \widehat{u}_{1\overline{x}_1})_{x_1} + (\widehat{v}_2 \widehat{u}_{2\overline{x}_2})_{x_2}, \tag{11}$$
$$v_{\alpha t} = -\widehat{v}_\alpha + g_\alpha(v_\alpha u_{\alpha \overline{x}_\alpha}), \tag{12}$$
$$u_\alpha(x, 0) = U_0(x), \quad x \in \overline{\omega}_h, \quad v_\alpha(x, 0) = V_{\alpha 0}(x), \quad x \in \overline{\omega}_{\alpha h}, \tag{13}$$
$$u_\alpha(x, t) = 0, \quad (x, t) \in \gamma_h \times \omega_\tau, \quad \alpha = 1, 2. \tag{14}$$

In (11), (12) the discrete functions u_1, u_2 are defined on $\overline{\omega}_{h\tau}$ and v_α on $\overline{\omega}_{\alpha h\tau}$.

Theorem 1. *If the differential problem* (4)–(7) *has the sufficiently smooth solution* U, V_1, V_2, *then there exist* $\tau_0 > 0$ *such that for all* $\tau < \tau_0$ *the scheme* (11)–(14) *is absolutely stable with respect to initial data and the following estimates hold*

$$\|u_{1\overline{x}_1}]|_1^2 + \|u_{2\overline{x}_2}]|_2^2 \le e^{CT}\Big\{\big\|(V_{10} U_{0\overline{x}_1})_{x_1}\big\|^2 + \|U_{0\overline{x}_1}]|_1^2 + \big\|(V_{20} U_{0\overline{x}_2})_{x_2}\big\|^2 + \|U_{0\overline{x}_2}]|_2^2\Big\},$$
$$0 < c \le v_\alpha(x, t) \le C, \quad (x, t) \in \overline{\omega}_{\alpha h\tau}, \quad \alpha = 1, 2.$$

Theorem 2. *If the differential problem* (4)–(7) *has the sufficiently smooth solution* U, V_1, V_2, *then the solution of the scheme* (11)–(14) *converges to the exact solution of problem* (4)–(7) *as* $\tau \to 0$, $h_1 \to 0$, $h_2 \to 0$, *and the following inequality holds*

$$\|z_{1\overline{x}_1}]|_1 + \|z_{2\overline{x}_2}]|_2 + \|s_1]|_1 + \|s_2]|_2 \le C(\tau + h_1^2 + h_2^2).$$

Here $z_\alpha = u_\alpha - U$, $s_\alpha = v_\alpha - V_\alpha$, $\alpha = 1, 2$. The statements analogous to Theorems 1 and 2 are true for multi-dimensional (1)–(3) case as well.

On each segment $\Delta_k = [k\tau, (k+1)\tau]$, $k = 1, 2, \dots, N$ for the system (1)–(3) with (6), (7) type initial-boundary conditions let us consider the following averaged model of sum approximation:

$$\eta_i \frac{\partial u_i^k}{\partial t} = \frac{\partial}{\partial x_i}\Big(v_i^k \frac{\partial u_i^k}{\partial x_i}\Big), \quad \frac{\partial v_i^k}{\partial t} = -v_i^k + g_i\Big(v_i^k \frac{\partial u_i^k}{\partial x_i}\Big), \tag{15}$$

$$u_i^k\big|_{x_i=0} = u_i^k\big|_{x_i=1} = 0, \quad u_i^0(x,0) = U_0(x), \quad v_i^0(x,0) = V_{i,0}(x), \tag{16}$$

$$u_i^k(x,t_k) = u^{k-1}(x,t_k), \quad v_i^k(x,t_k) = v_i^{k-1}(x,\, t_k), \tag{17}$$

$$u^k(x,t) = \sum_{i=1}^p \eta_i u_i^k(x,t), \quad \eta_i > 0, \quad \sum_{i=1}^p \eta_i = 1. \tag{18}$$

Theorem 3. *If the differential problem has the sufficiently smooth solution, then the solution of the averaged model* (15)–(18) *converges to the exact solution as* $\tau \to 0$ *and the following estimate holds*

$$\big\|u^k(t) - U(t)\big\| + \sum_{i=1}^p \big\|v_i^k(t) - V_i(t)\big\| = O(\tau^{1/4}).$$

Numerous numerical computations are carried out for two-dimensional biological problem (4)–(7) using (11)–(14) and (15)–(18) models. Numerical experiments agree with theoretical researches.

Acknowledgement

This research was supported by the Shota Rustaveli National Science Foundation (Project # FR/30/5-101/12, Agreement # 31/32).

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Asymptotic Analysis of Positive Decreasing Solutions of First Order Nonlinear Functional Differential Systems in the Framework of Regular Variation

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We consider the system of first order nonlinear functional differential equations of the type

$$\begin{cases} x'(t) + p_1(t)x(g_1(t))^{\alpha_1} + q_1(t)y(h_1(t))^{\beta_1} = 0, \\ y'(t) + p_2(t)x(g_2(t))^{\alpha_2} + q_2(t)y(h_2(t))^{\beta_2} = 0, \end{cases} \tag{A}$$

where α_i, β_i are positive constants, $p_i(t)$, $q_i(t)$ are positive continuous functions on $[a,\infty)$, $a>0$, and $g_i(t)$, $h_i(t)$ are positive continuous functions on $[a,\infty)$ such that $\lim\limits_{t\to\infty} g_i(t) = \lim\limits_{t\to\infty} h_i(t) = \infty$, $i=1,2$. By a positive solution of (A) we mean a vector function $(x(t),y(t))$ whose components are positive on an interval of the form $[T,\infty)$ and satisfy the system (A) there. A positive solution $(x(t),y(t))$ is said to be *strongly decreasing* if $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$.

Our aim is to derive nontrivial information about the existence and the asymptotic behavior of strongly decreasing solutions of (A) by making an analysis of the problem in the framework of regular variation (in the sense of Karamata). By definition a measurable function $f:[a,\infty)\to(0,\infty)$ is called *regularly varying of index* $\rho\in\mathbf{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for } \forall\,\lambda>0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. If in particular $\rho=0$, then the symbol SV is often used for $\mathrm{RV}(0)$, and its members are referred to as *slowly varying functions.*

The simplest cases of (A) are the diagonal system

$$x'(t) + p_1(t)x(g_1(t))^{\alpha_1} = 0, \quad y'(t) + q_2(t)y(h_2(t))^{\beta_2} = 0, \tag{$\mathrm{A_d}$}$$

and the cyclic system

$$x'(t) + q_1(t)y(h_1(t))^{\beta_1} = 0, \quad y'(t) + p_2(t)x(g_2(t))^{\alpha_2} = 0. \tag{$\mathrm{A_c}$}$$

Some of the main results for these systems follow.

Theorem 1. *Suppose that* $\alpha_1<1$ *and* $\beta_2<1$, $p_1\in\mathrm{RV}(\lambda_1)$ *and* $q_2\in\mathrm{RV}(\mu_2)$, *and*

$$\lim_{t\to\infty}\frac{g_1(t)}{t} = \lim_{t\to\infty}\frac{h_2(t)}{t} = 1. \tag{1}$$

System $(\mathrm{A_d})$ *possesses strongly decreasing solutions* $(x,y)\in\mathrm{RV}(\rho)\times\mathrm{RV}(\sigma)$ *with* $\rho<0$ *and* $\sigma<0$ *if and only if*

$$\lambda_1+1<0 \ \ \textit{and} \ \ \mu_2+1<0, \tag{2}$$

in which case ρ and σ are given by

$$\rho = \frac{\lambda_1 + 1}{1 - \alpha_1}, \quad \sigma = \frac{\mu_2 + 1}{1 - \beta_2}, \tag{3}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the unique decay law

$$x(t) \sim X_1(t) = \Big(\frac{tp_1(t)}{-\rho}\Big)^{\frac{1}{1-\alpha_1}}, \quad y(t) \sim Y_1(t) = \Big(\frac{tq_2(t)}{-\sigma}\Big)^{\frac{1}{1-\beta_2}}, \ \ t \to \infty. \tag{4}$$

Theorem 2. *Suppose that $\alpha_2\beta_1 < 1$, $q_1 \in \mathrm{RV}(\mu_1)$ and $p_2 \in \mathrm{RV}(\lambda_2)$, and*

$$\lim_{t\to\infty} \frac{g_2(t)}{t} = \lim_{t\to\infty} \frac{h_1(t)}{t} = 1. \tag{5}$$

System $(\mathrm{A_c})$ possesses strongly decreasing solutions $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ $\rho < 0$ and $\sigma < 0$ if and only if

$$\beta_1(\lambda_2 + 1) + \mu_1 + 1 < 0 \ \ \text{and} \ \ \lambda_2 + 1 + \alpha_2(\mu_1 + 1) < 0, \tag{6}$$

in which case ρ and σ are given by

$$\rho = \frac{\beta_1(\lambda_2 + 1) + \mu_1 + 1}{1 - \alpha_2\beta_1}, \quad \sigma = \frac{\lambda_2 + 1 + \alpha_2(\mu_1 + 1)}{1 - \alpha_2\beta_1}, \tag{7}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the unique decay law

$$x(t) \sim X_2(t) = \Big[\frac{tq_1(t)}{-\rho}\Big(\frac{tp_2(t)}{-\sigma}\Big)^{\beta_1}\Big]^{\frac{1}{1-\alpha_2\beta_1}}, \quad y(t) \sim Y_2(t) = \Big[\frac{tp_2(t)}{-\sigma}\Big(\frac{tq_1(t)}{-\rho}\Big)^{\alpha_2}\Big]^{\frac{1}{1-\alpha_2\beta_1}}, \tag{8}$$

as $t \to \infty$.

Here the symbol $\sim$ is used to mean the asymptotic equivalence of two positive functions:

$$f(t) \sim g(t), \ \ t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Turning to system (A), it is expected that if (A) can be regarded as a "small" perturbation of the diagonal system $(\mathrm{A_d})$ (resp. $(\mathrm{A_c})$), then it may possess strongly decreasing regularly varying solutions $(x(t), y(t))$ which behave like (4) (resp. (8)) as $t \to \infty$. The correctness of this expectation is shown by the following results.

Theorem 3. *Suppose that $\alpha_1 < 1$ and $\beta_2 < 1$, that $p_1 \in \mathrm{RV}(\lambda_1)$ and $q_2 \in \mathrm{RV}(\mu_2)$, and that $g_1(t)$ and $h_2(t)$ satisfy (1). Suppose moreover that λ_1 and μ_2 satisfy (2) and define ρ and σ by (3). If*

$$\lim_{t\to\infty} \frac{q_1(t)Y_1(h_1(t))^{\beta_1}}{p_1(t)X_1(g_1(t))^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_1(g_2(t))^{\alpha_2}}{q_2(t)Y_1(h_2(t))^{\beta_2}} = 0, \tag{9}$$

then system (A) possesses a strongly decreasing regularly varying solution $(x(t), y(t))$ belonging to $\mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ and enjoying the asymptotic behavior (4).

Theorem 4. *Suppose that $\alpha_2\beta_1 < 1$, that $q_1 \in \mathrm{RV}(\mu_1)$ and $p_2 \in \mathrm{RV}(\lambda_2)$, and that $g_2(t)$ and $h_1(t)$ satisfy (5). Suppose moreover that the inequalities (6) hold, and define ρ and σ by (7). If*

$$\lim_{t\to\infty} \frac{p_1(t)X_2(g_1(t))^{\alpha_1}}{q_1(t)Y_2(h_1(t))^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_2(h_2(t))^{\beta_2}}{p_2(t)X_2(g_2(t))^{\alpha_2}} = 0, \tag{10}$$

then system (A) possesses a strongly decreasing regularly varying solution $(x(t), y(t))$ belonging to $\mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ and enjoying the asymptotic behavior (8).

Remark 1. There are three possible types of strongly decreasing regularly varying solutions $(x(t), y(t))$ of (A):

(I) $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ with $\rho < 0$ and $\sigma < 0$;

(II) $(x, y) \in \mathrm{SV} \times \mathrm{RV}(\sigma)$ with $\sigma < 0$, or $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{SV}$ with $\rho < 0$;

(III) $(x, y) \in \mathrm{SV} \times \mathrm{SV}$.

All of the above theorems are concerned with solutions of type (I). It should be noticed that for system ($\mathrm{A_d}$) with $\alpha_1 < 1$ and $\beta_2 < 1$ (resp. system ($\mathrm{A_c}$) with $\alpha_2\beta_1 < 1$) one can characterize the existence of solutions of types (II) and (III) (resp. solutions of type (II)) and determine their asymptotic behavior at infinity precisely.

Remark 2. We note that in Theorem 3 neither $p_2(t)$ nor $q_1(t)$ is assumed to be regularly varying, and neither $g_2(t)$ nor $h_1(t)$ is required to be asymptotic to t as $t \to \infty$, and the same is true of $p_1(t)$, $q_2(t)$, $g_1(t)$ and $h_2(t)$ in Theorem 4.

Remark 3. Assume in Theorems 3 and 4 that all $p_i(t)$, $q_i(t)$, $g_i(t)$ and $h_i(t)$ are regularly varying: $p_i \in \mathrm{RV}(\lambda_i)$, $q_i \in \mathrm{RV}(\mu_i)$, $g_i \in \mathrm{RV}(\gamma_i)$, $h_i \in \mathrm{RV}(\delta_i)$, $i = 1, 2$. Then, in the language of regular variation, condition (9) is interpreted as

$$\lim_{t\to\infty} t^{\mu_1+\beta_1\sigma\delta_1-(\lambda_1+\alpha_1\rho)} L(t) = 0, \quad \lim_{t\to\infty} t^{\lambda_2+\alpha_2\rho\gamma_2-(\mu_2+\beta_2\sigma)} M(t) = 0, \tag{11}$$

where $L(t)$ and $M(t)$ are slowly varying functions which can easily be computed from the slowly varying parts of $p_i(t)$ and $q_i(t)$. It is clear that the inequalities

$$\mu_1 + \beta_1\sigma\delta_1 < \lambda_1 + \alpha_1\rho, \quad \lambda_2 + \alpha_2\rho\gamma_2 < \mu_2 + \beta_2\sigma, \tag{12}$$

are sufficient for (11) to be satisfied regardless of $L(t)$ and $M(t)$, and hence (12) is often utilized as a convenient criterion for the existence of strongly decreasing solutions of system (A) which are regularly varying of negative indices. A similar interpretation of (10) in Theorem 4 can be made in terms of regular variation.

The Mixed Problem for the Semilinear Wave Equation with a Nonlinear Boundary Condition

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In the plane of independent variables x and t in the domain $D_T := \{(x,t) \in \mathbb{R}^2 :\ 0 < x < l,\ 0 < t < T\}$ we consider a mixed problem of finding a solution $u(x,t)$ for the semilinear wave equation of the type

$$u_{tt} - u_{xx} + g(u) = f(x,t), \ \ (x,t) \in D_T, \tag{1}$$

satisfying the following initial

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \ \ 0 \leq x \leq l, \tag{2}$$

and boundary conditions

$$u_x(0,t) = F[u(0,t)] + \beta(t), \quad u(l,t) = \nu(t), \ \ 0 \leq t \leq T, \tag{3}$$

where f, φ, ψ, β, ν, g and F are the given and u is an unknown real functions.

Let the conditions of smoothness

$$\begin{gathered} f \in C^1(\overline{D}_T), \ \ g, \ F \in C^1(\mathbb{R}), \\ \varphi \in C^2([0,l]), \ \ \psi \in C^1([0,l]), \ \ \beta \in C^1([0,l]), \ \ \nu \in C^2([0,l]) \end{gathered} \tag{4}$$

and the agreement

$$\begin{gathered} \varphi(0) = 0, \quad \varphi(l) = 0, \quad \psi(0) = 0, \quad \varphi'(0) = F(0) + \beta(0), \quad \psi'(0) = \beta'(0), \\ \varphi(l) = \nu(0), \quad \psi(l) = \nu'(0), \quad \nu''(0) - \varphi''(l) + g(0) = f(l,0) \end{gathered} \tag{5}$$

be fulfilled.

Note that the nonlinear boundary condition of the type (3) arises, for example, in describing the process of longitudinal string oscillations if one of its ends is elastically fixed, when tension on that end is a nonlinear function of displacement, and also in processes taking place in distributed self-oscillating systems.

Consider the conditions

$$G(g;s) := \int\limits_0^s g(s_1)\, ds_1 \geq -M_1 s^2 - M_2, \quad \int\limits_0^s F(s_1)\, ds_1 \geq -M_3 \ \ \forall\, s \in \mathbb{R}, \tag{6}$$

where $M_i := const \geq 0$, $1 \leq i \leq 3$.

The following theorem is valid.

Theorem 1. *Let the conditions* (4)–(6) *be fulfilled. Then there exists a unique classical solution of the problem* (1)–(3).

Remark. In the case if at least one of the conditions (6) imposed on the nonlinear functions g and F is violated, then relying on the comparison theorem, we can distinguish those data classes of the problem (1)–(3) for which the problem is globally solvable in one case or has blow up solution in the other case.

Before formulating the comparison theorem, we consider nonlinear mixed problems in the following statement:

$$\begin{gathered} u_{tt} - u_{xx} + g_i(u) = f_i(x,t), \quad (x,t) \in D_T, \\ u(x,0) = \varphi_i(x), \quad u_t(x,0) = \psi_i(x), \quad 0 \le x \le l, \\ u_x(0,t) = F_i[u(0,t)] + \beta_i(t), \quad u(l,t) = \nu_i(t), \quad 0 \le t \le T, \end{gathered} \tag{7}$$

where the data of the problem satisfy the corresponding smoothness and agreement conditions.

Theorem 2. *Let u_1 and u_2 be classical solutions of the problem* (7) *for $i = 1$ and $i = 2$, respectively. Then, if*

$$\begin{gathered} g_1'(s) \le 0 \ \text{ or } \ g_2'(s) \le 0; \quad g_1(s) \ge g_2(s), \quad s \in \mathbb{R}, \\ f_1(x,t) \le f_2(x,t), \quad (x,t) \in \overline{D}_T, \\ \varphi_1(x) \le \varphi_2(x), \quad \varphi_1'(x) \ge \varphi_2'(x), \quad \psi_1(x) \le \psi_2(x), \quad 0 \le x \le l, \\ F_1'(s) \le 0 \ \text{ or } \ F_2'(s) \le 0; \quad F_1(s) \ge F_2(s), \quad s \in \mathbb{R}, \\ \beta_1(t) \ge \beta_2(t), \quad \nu_1(t) \le \nu_2(t), \quad 0 \le t \le T, \end{gathered}$$

then

$$u_1(x,t) \le u_2(x,t), \quad (x,t) \in \overline{D}_T.$$

Let the functions g, f, φ, ψ, F, β and ν satisfy the conditions

$$\begin{gathered} g \ge 0, \ \ g(s) = 0, \ s \ge 0; \ \ f = 0; \ \ \varphi \ge 0, \ \varphi' \le 0, \ \psi \ge 0, \ \psi \not\equiv 0; \ \ \beta = 0, \ \nu \ge 0, \\ F' \le 0, \quad F(0) = 0, \quad F(s) \ge -\delta s^\alpha, \quad s \ge 0, \quad \delta > 0, \quad \alpha > 1, \\ \frac{1}{\delta(\alpha-1)[\varphi(0) + k_1 l]^{\alpha-1}} > T, \end{gathered} \tag{8}$$

where $k_1 := \|\psi\|_{C([0,l])}$.

Theorem 3. *Let the conditions* (8) *be fulfilled. Then the problem* (1)–(3) *has a unique classical solution.*

Let now the conditions

$$\begin{gathered} g \le 0; \quad f \ge 0, \quad \beta \le 0, \\ F(s) \le -\delta |s|^\alpha s, \quad \delta := const > 0, \quad \alpha := const > 0, \quad s \in \mathbb{R}, \\ \varphi(0) > k_2 l, \quad T^* := \frac{1}{\delta\alpha[\varphi(0) - k_2 l]^{\alpha-1}} \le T \end{gathered} \tag{9}$$

be fulfilled, where $k_2 := \max\limits_{0 \le x \le l} |\varphi'(x) + \psi(x)|$.

Theorem 4. *Let the conditions* (9) *be fulfilled. Then there is $T_* \in (0, T^*]$ such that in the domain D_{T_*} there exist a unique solution of the problem* (1)–(3) *of the class $C^2(\overline{D}_{T_*} \setminus t = T_*)$ which blows up, i.e., satisfies the condition*

$$\lim_{T \to T_* - 0} \Big(\|u\|_{C(\overline{D}_T)} + \|u_t\|_{C(\overline{D}_T)} \Big) = \infty.$$

Note that if the above conditions for the problem data are violated, the local solvability with respect to t remains valid. In this case it is sufficient to require of the problem data that $f \in C^1(\overline{D}_T)$, $g, F \in C^1(\mathbb{R})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, $\beta \in C^1([0,T])$, $\nu \in C^2([0,T])$.

Exponential Stability of Linear Itô Equations with Delay and Azbelev's W-Transform

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By $\mathbb{E}$ we denote the expectation on this probability space. The scalar stochastic process $\mathcal{B}$ is a Brownian motion on $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. [6]).

Below we always assume that the real number p satisfies the conditions $1 \leq p < \infty$.

Consider the following scalar stochastic delay differential equation

$$dx(t) = \big(-a(t)x(t) - b(t)x(h(t))\big)\,dt + c(t)x(H(t))\,d\mathcal{B}(t) \ \ (t \geq 0), \tag{1}$$

equipped with the prehistory condition

$$x(\nu) = \varphi(\nu) \ \ (\nu < 0), \tag{2}$$

where φ is a $\mathcal{F}_0$-measurable stochastic process which almost surely (a. s.) has trajectories from L^∞. The functions a, b, c, h, H in (1) are all Lebesgue-measurable, a, b are, in addition, locally integrable, c is locally square-integrable, $h(t) \leq t$, $H(t) \leq t$ for $t \in [0,\infty)$ a.s., $\operatorname*{vraisup}_{t\geq 0} (t - h(t)) < \infty$, $\operatorname*{vraisup}_{t\geq 0} (t - H(t)) < \infty$.

The initial condition for the equation (1) is given by

$$x(0) = x_0, \tag{3}$$

where x_0 is a $\mathcal{F}_0$-measurable scalar random variable.

Definition. The zero solution of the equation (1) is called exponentially Lyapunov $2p$-stable (with respect to the prehistory data (2) and the initial data (3)) if there exist positive numbers $\bar{c}$ and β such that

$$\mathbb{E}|x(t, x_0, \varphi)|^{2p} \leq \bar{c}\Big(\mathbb{E}|x_0|^{2p} + \operatorname*{vraisup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p}\Big) \exp\{-\beta t\} \ \ (t \geq 0) \tag{4}$$

for any $\mathcal{F}_0$-measurable scalar random variable x_0 and any $\mathcal{F}_0$-measurable stochastic process φ, which a.s. has trajectories belonging to L^∞.

In the theorems below we use the universal constants c_p $(1 \leq p < \infty)$ from the celebrated Burkholder–Davis–Gandy inequalities to estimate stochastic integrals. From [7] it is e.g. known that $c_p = 2\sqrt{12}p$. However, other sources give other values (see e.g. [6]). All these values are not optimal for our purposes. For instance, in our theorems we may assume that $c_1 = 1$, as we estimate $\sup_t \mathbb{E}|x(t)|^2$, and not $\mathbb{E}\sup_t |x(t)|^2$, as in the Burkholder–Davis–Gandy inequalities.

Given a function $h(t)$ $(t \in [T,\infty))$, we construct the function $h^T(t)$ $(t \in [T,\infty))$ in the following way:

$$h^T(t) = \begin{cases} h(t) & \text{if } \ h(t) \ge T, \\ T & \text{if } \ h(t) < T. \end{cases}$$

Theorem 1. *Assume that there exist numbers $a_0 > 0$, $\gamma_i > 0$, $i = 1,2$, $T \in [0,\infty)$ such that one of the following conditions holds:*

(1) $a(t) \ge a_0$, $|b(t)| \le \gamma_1 a(t)$, $(c(t))^2 \le 2\gamma_2 a(t)$ $(t \ge T)$ *a.s.*,

(2) $a(t) + b(t) \ge a_0$, $|b(t)| \Big[\int\limits_{h^T(t)}^{t} \big(|a(s)| + |b(s)|\big)\, ds + c_p \Big(\int\limits_{h^T(t)}^{t} (c(s))^2\, ds \Big)^{0.5} \Big] \le$

$$\le \gamma_1 (a(t) + b(t)), \quad (c(t))^2 \le 2\gamma_2 (a(t) + b(t)) \ \ (t \ge T) \ \ a.s.$$

If, in addition, $\gamma_1 + c_p\sqrt{\gamma_2} < 1$, then the solutions of the equation (1) *satisfy the estimate* (4) *for some $\beta > 0$.*

Theorem 2. *Assume that there exists a number $T \in [0,\infty)$ such that the coefficients in* (1) *satisfy*

$$a(t) = Ar(t), \ \ b(t) = Br(t), \ \ c(t) = C\sqrt{r(t)}\,, \ \ r(t) \ge r_0 > 0 \ \ (t \in [T,\infty)) \ \ a.s.$$

Assume also that one of the following conditions holds:

(1) $A > 0$, $|B|/A + c_p|C|/\sqrt{2A} < 1$,

(2) $A + B > 0$, $\lim\limits_{t\to\infty} \sup\limits_{T\le\tau\le t} |B| \Big[\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds(|A| + |B|) + c_p \Big(\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds \Big)^{0.5} |C| \Big] \Big/ (A+B) +$

$$+ c_p|C|/\sqrt{2(A+B)} < 1,$$

(3) $A > 0$, $B > 0$, $\lim\limits_{t\to\infty} \sup\limits_{T\le\tau\le t} B \Big[\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds + c_p \Big(\int\limits_{h^T(\tau)}^{\tau} r(s) ds \Big)^{0.5} |C| \Big/ (A+B) \Big] +$

$$+ c_p|C|/\sqrt{2(A+B)} < 1.$$

Then the solutions of the equation (1) *satisfy the estimate* (4) *for some $\beta > 0$.*

The proof of the results is based on Azbelev's W-transform of the equation (1) (see e.g. [1, 2, 5]), which uses the so-called 'reference equation' and which in our case is defined as follows:

$$dx(t) = \big[a(t)x(t) + g_1(t)\big]\, dt + g_2(t)\, d\mathcal{B}(t) \ \ (t \ge 0),$$

where g_1 is an $(\mathcal{F}_t)_{t\ge0}$-adapted stochastic process with a.s. locally integrable trajectories, g_2 is an $(\mathcal{F}_t)_{t\ge0}$-adapted stochastic process with a.s. locally square-integrable trajectories, and the other parameters are defined in (1). This is an ordinary differential equation with stochastic perturbations. The W-method works if the reference equation is exponentially stable and if a certain integral operator, which combines the equation (1) with the reference equation, is invertible. To describe this idea in more detail, we introduce the notation $Z(t) = (t, \mathcal{B}(t))'$ and unify the equation (1) with the prehistory condition (2) into a single stochastic functional differential equation

$$dx(t) = \left[(Vx)(t) + f(t)\right] dZ(t) \ \ (t \geq 0), \tag{5}$$

where

$$(Vx)(t) = \big((V_1x)(t), (V_2x)(t)\big), \quad (V_1x)(t) = a(t)x(t) + b(t)(S_hx)(t), \quad (V_1x)(t) = c(t)(S_Hx)(t),$$
$$f(t) = \big(f_1(t), f_2(t)\big), \quad f_1(t) = b(t)\varphi_h(t), \quad f_2(t) = c(t)\varphi_H(t),$$
$$(S_\gamma x)(t) = \begin{cases} x(\gamma(t)) & \text{if } \gamma(t) \geq 0, \\ 0 & \text{if } \gamma(t) < 0, \end{cases} \qquad \varphi_\gamma(t) = \begin{cases} 0 & \text{if } \gamma(t) \geq 0, \\ \varphi(\gamma(t)) & \text{if } \gamma(t) < 0. \end{cases}$$

Similarly, we can rewrite the reference equation as follows:

$$dx(t) = \left[(Qx)(t) + g(t)\right] dZ(t) \ \ (t \geq 0),$$

where $(Qx)(t) = (ax(t), 0)$, $g(t) = (g_1(t), g_2(t))$.

It is well-known (see e.g. [5]) that the latter equation admits the integral representation $x(t) = U(t)x(0) + (Wg)(t)$ $(t \geq 0)$, where $U(t)$ is the fundamental matrix of the associated homogeneous equation, W is the integral operator

$$(Wg)(t) = \int_0^t C(t,s)g(s)\, dZ(s) \ \ (t \geq 0),$$

and the function $C(t,s)$ is defined on $G := \{(t,s) : \ t \in [0, \infty), \ 0 \leq s \leq t\}$.

Assume that U and W satisfy the following conditions:

R1. $|U(t)| \leq \overline{c}$, where $\overline{c} \in \mathbf{R}_+$.

R2. $|C(t,s)| \leq \overline{\overline{c}} \exp\{-\alpha(t-s)\}$ for some $\alpha > 0$, $\overline{\overline{c}} > 0$.

The integral representation gives then rise to the W-transform, which is applied to the equation (5) in the following manner:

$$dx(t) = \left[(Qx)(t) + ((V-Q)x)(t) + f(t)\right] dZ(t) \ \ (t \geq 0),$$

or, alternatively,

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t) \ \ (t \geq 0).$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation

$$((I - \Theta)x)(t) = U(t)x(0) + (Wf)(t) \ \ (t \geq 0).$$

Finally, we put

$$M_{2p,T} \equiv \left\{x : \ x \in C, \ \|x\|_{M_{2p,T}} := \Big(\sup_{t \geq T} \mathbb{E}|x(t)|^{2p}\Big)^{1/2p} < \infty\right\},$$

where C denotes the set of all $(\mathcal{F}_t)_{t \geq T}$-adapted stochastic processes with a. s. continuous trajectories.

The main idea of the W-transform approach is to prove invertibility of the operator $I - \Theta$ in the space $M_{2p,T}$, which would imply the exponential $2p$-stability of the solutions of the equation (1). This is summarized in the following lemma:

Lemma. *Let the reference equation satisfy the conditions* **R1**–**R2**. *Assume that the operator* $(I - \Theta) : M_{2p,T} \to M_{2p,T}$ *has a bounded inverse for some* $T \geq 0$. *Then the solutions of the equation* (1) *satisfy the estimate* (4) *for some* $\beta > 0$.

The proof of the Theorems 1 and 2 consists then in checking the assumptions of the lemma.

Remark. More details about the theory and applications of the W-transform to deterministic functional differential equations can be found in [1, 2]. The stochastic version of this transform, an outline of which is presented above, was comprehensively studied in [3]. An alternative yet similar version of the stochastic W-transform, where the integral substitution is performed in a different manner, was suggested in [4] on the basis of the deterministic results obtained in [1, 2].

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Precise Baire Characterization of the Lyapunov Exponents of Families of Morphisms of Metrized Vector Bundles with a Given Base

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In [1] V. M. Millionschikov introduced the concept of the Lyapunov exponents of families of morphisms of metrized vector bundles.

Let (E, p, B) be a vector bundle with fixed Riemann metric, fiber $\mathbb{R}^n$ and let the base B be a complete metric space. Let $|\cdot|$ denote norm in the fibers induced by the Riemann metric of a bundle. Suppose that for each $m \in \mathbb{N}$ a morphism $(X(m), \chi(m)) : (E, p, B) \to (E, p, B)$ is defined such that mappings $X(m)$ are non-degenerate on fibers. In other words, for each $m \in \mathbb{N}$ continuous mappings $X(m) : E \to E$ and $\chi(m) : B \to B$ are given such that $X(m) \circ p = p \circ \chi(m)$ and the restriction $X(m)|_{p^{-1}(b)}$ of mapping $X(m)$ to the fiber $p^{-1}(b)$ is a non-singular linear mapping (denote this restriction by $X(m, b)$). Suppose further that there is a function $a(\cdot) : B \to [0, +\infty)$ such that for each $m \in \mathbb{N}$ the inequality $\max\{\|X(m, b)\|, \|X^{-1}(m, b)\|\} \leqslant \exp(m \cdot a(b))$ holds, where $\|\cdot\|$ is an operator norm. Such a vector bundle is called n-dimensional Millionschikov bundle.

By $\mathfrak{M}_n$ denote the collection of all n-dimensional Millionschikov bundles, and by $\mathfrak{M}_n(B)$ denote the collection of all n-dimensional Millionschikov bundles with base B.

Then k-th Lyapunov exponent λ_k, $k = 1, \dots, n$, of family of morphisms of Millionschikov bundle is defined [1] by:

$$\lambda_k(b) \stackrel{\text{def}}{=} \min_{V \in G_{n-k+1}(\mathbb{R}^n)} \max_{\xi \in V, |\xi|=1} \varlimsup_{m \to +\infty} m^{-1} \ln |X(m, b)\xi|,$$

where $\mathbb{R}^n$ is a fiber $p^{-1}(b)$ and $G_{n-k+1}(\mathbb{R}^n)$ is a Grassmannian manifold of $(n-k+1)$-dimensional lineals in $\mathbb{R}^n$. It follows from the definition that exponent $\lambda_k(\cdot)$ is a function $B \to \mathbb{R}$ and $\lambda_1(b) \geqslant \dots \geqslant \lambda_n(b)$ for all $b \in B$.

Consider the question: what is the precise characterization of Lyapunov exponents of families of morphisms of Millionschikov bundles as functions on the base of the bundle? V. M. Millionschikov [1] proved that every function $\lambda_k(\cdot) : B \to \mathbb{R}$ is a function of the second Baire class.

M. I. Rakhimberdiev [2] proved that the number of Baire class in the statement above cannot be reduced. A. N. Vetokhin [3, 4] proved that the Lyapunov exponents are functions of class $({}^*, G_\delta)$ in spaces $\mathcal{M}_n^{\mathrm{c}}$ and $\mathcal{M}_n^{\mathrm{u}}$. Here $\mathcal{M}_n^{\mathrm{c}}$ and $\mathcal{M}_n^{\mathrm{u}}$ are the spaces of linear differential systems with piecewise continuous coefficients bounded on the half-line and with the topology of compact and uniform convergence, respectively.

A complete answer to the question formulated for the class $\mathfrak{M}_n$ of all Millionschikov bundles is given in the article [5]. In this article it is shown that all exponents $\lambda_k(\cdot) : B \to \mathbb{R}$ belong to a class $({}^*, G_\delta)$, have upper semi-continuous minorant and satisfy inequalities $\lambda_1(b) \geqslant \dots \geqslant \lambda_n(b)$ for all $b \in B$; and also there is shown sufficiency of this three conditions. The theorem below shows that a similar result holds for each class $\mathfrak{M}_n(B)$ for any $m \in \mathbb{N}$ and a complete metric space B.

Recall that a real function belongs to a class $({}^*, G_\delta)$ if the inverse image of any interval $[r, +\infty)$, $r \in \mathbb{R}$ is a G_δ-set. Let $m(\cdot)$ and $\lambda(\cdot)$ be functions $B \to \mathbb{R}$, then function $m(\cdot)$ is called minorant of a function $\lambda(\cdot)$ if $\lambda(b) \geqslant m(b)$ for all $b \in B$.

Theorem. *Every Lyapunov exponent λ_k, $k = 1, \dots, n$, of a family of morphisms of Millionschikov bundle is a function of class $({}^*, G_\delta)$ having upper semi-continuous minorant and satisfying $\lambda_1(b) \geqslant \dots \geqslant \lambda_n(b)$ for all $b \in B$.*

Conversely for every $n \in \mathbb{N}$ and a complete metric space B there exists an n-dimensional vector bundle $\mathfrak{R}$ with the base B and a fiber $\mathbb{R}^n$ such that for any set $(f_1(\cdot), \dots, f_n(\cdot))$ of functions $B \to \mathbb{R}$

of a class $({}^{*}, G_\delta)$ *having upper semi-continuous minorant and satisfying* $f_1(b) \geqslant \cdots \geqslant f_n(b)$ *for all* $b \in B$*, there exists such a family of morphisms of* $\mathfrak{R}$ *that* $\mathfrak{R}$ *together with this family of morphisms is a Millionschikov bundle and set of Lyapunov exponents* $\lambda_1, \ldots, \lambda_k$ *of this bundle coincide with the set* $(f_1, \ldots, f_n)$ *i.e.* $\lambda_k(b) = f_k(b)$ *for all* $b \in B$ *and* $k \in \{1, \ldots, n\}$.

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Nonlocal in Time Problems for Semilinear Multidimensional Wave Equations

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In the space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t, in the cylindrical domain $D_T=\Omega\times(0,T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, consider a nonlocal problem on finding a solution $u(x,t)$ of the following equation

$$L_\lambda u:=\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\lambda f(x,t,u)=F(x,t),\ \ (x,t)\in D_T, \tag{1}$$

satisfying the homogeneous boundary condition for the side boundary $\Gamma:=\partial\Omega\times(0,T)$ of the cylinder D_T

$$u\big|_\Gamma=0, \tag{2}$$

the initial condition

$$u(x,0)=\varphi(x),\ \ x\in\Omega, \tag{3}$$

and the nonlocal condition

$$K_\mu u_t:=u_t(x,0)-\sum_{i=1}^{l}\mu_i u_t(x,t_i)=\psi(x),\ \ x\in\Omega, \tag{4}$$

where f, F, φ and ψ are given functions; $t_i=const$, $0<t_1<t_2<\dots<t_l\le T$; λ and μ are given constants and $n\ge 2$, $l\ge 1$.

Definition. Let $f\in C(\overline{D}_T\times\mathbb{R})$, $F\in L_2(D_T)$, $\varphi\in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi\in L_2(\Omega)$. We call function u a strong generalized solution of the problem (1)–(4) of the class W_2^1, if $u\in \overset{\circ}{W}{}_2^1(D_T,\Gamma):=\{w\in W_2^1(D_T): w|_\Gamma=0\}$ and there exists a sequence of functions $u_m\in \overset{\circ}{C}{}^2(\overline{D}_T,\Gamma):=\{w\in C^2(\overline{D}_T): w|_\Gamma=0\}$ such that $u_m\to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$, $L_\lambda u_m\to F$ in the space $L_2(D_T)$, $u_m|_{t=0}\to\varphi$ in the space $\overset{\circ}{W}{}_2^1(\Omega)$, and $K_\mu u_{mt}\to\psi$ in the space $L_2(\Omega)$.

It is obvious that a classical solution of the problem (1)–(4) of the space $C^2(\overline{D}_T)$ represents a strong generalized solution of this problem of the class W_2^1.

Let

$$g(x,t,u)=\int_0^u f(x,t,s)\,ds,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}. \tag{5}$$

Consider the following conditions imposed on functions $f=f(x,t,u)$ and $g=g(x,t,u)$ from (1) and (5)

$$f\in C(\overline{D}_T\times\mathbb{R}),\ \ |f(x,t,u)|\le M_1+M_2|u|^\alpha,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{6}$$

$$g(x,t,u)\ge -M_3,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{7}$$

$$g_t\in C(\overline{D}_T\times\mathbb{R}),\ \ g_t(x,t,u)\le M_4,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{8}$$

where $M_i=const\ge 0$, $1\le i\le 4$; $\alpha=const\ge 0$.

Remark 1. Let us consider some classes of functions $f=f(x,t,u)$ frequently encountered in applications and which satisfy the conditions (6), (7) and (8):

1. $f(x,t,u) = f_0(x,t)\beta(u)$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{D}_T)$ and $\beta \in C(\mathbb{R})$, $|\beta(u)| \le \widetilde{M}_1 + \widetilde{M}_2|u|^\alpha$, $\widetilde{M}_i = const \ge 0$, $\alpha = const \ge 0$. In this case $g(x,t,u) = f_0(x,t)\int\limits_0^u \beta(s)\,ds$ and when $f_0 \ge 0$, $\frac{\partial}{\partial t} f_0 \le 0$, $\int\limits_0^u \beta(s)\,ds \ge -M$, $M = const \ge 0$, the conditions (6), (7) and (8) will be fulfilled.

2. $f(x,t,u) = f_0(x,t)|u|^\alpha \operatorname{sign} u$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{D}_T)$ and $\alpha > 1$. In this case $g(x,t,u) = f_0(x,t)\frac{|u|^{\alpha+1}}{\alpha+1}$ and when $f_0 \ge 0$, $\frac{\partial}{\partial t} f_0 \le 0$, the conditions (6), (7) and (8) will be also fulfilled.

Theorem 1. *Let $\lambda > 0$, $\sum\limits_{i=1}^{l} |\mu_i| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$; the conditions* (6), (7), *and* (8) *be fulfilled. Then, if the exponent of nonlinearity α in the condition* (6) *satisfies the inequality $\alpha < \frac{n+1}{n-1}$, then the problem* (1)–(4) *has at least one strong generalized solution of the class W_2^1.*

On the function f in the equation (1) let us impose the following additional requirements

$$f, f'_u \in C(\overline{D}_T \times \mathbb{R}), \ \ |f'_u(x,t,u)| \le a + b|u|^\gamma, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{9}$$

where $a, b, \gamma = const \ge 0$.

It is obvious that from (9) we have the condition (6) for $\alpha = \gamma + 1$, and when $\gamma < \frac{2}{n-1}$, we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 2. *Let $\lambda > 0$, $\sum\limits_{i=1}^{l} |\mu_i| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ and the condition* (9) *be fulfilled for $\gamma < \frac{2}{n-1}$, and also hold the conditions* (7), (8). *Then there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ such that for $0 < \lambda < \lambda_0$ the problem* (1)–(4) *can not have more than one strong generalized solution of the class W_2^1.*

Remark 2. Note that if condition $\sum\limits_{i=1}^{l} |\mu_i| < 1$ is violated, as shown by specific examples, even in the linear case, i.e. for $f = 0$, the homogeneous problem corresponding to (1)–(4) may have finite or even an infinite set of independent solutions.

Remark 3. In the case, when the condition (7) is violated, the problem (1)–(4) may have no strong generalized solution of the class W_2^1. Indeed, let us consider the following condition imposed on function f

$$f(x,t,u) \le -|u|^p, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}; \ \ p = const > 1. \tag{10}$$

Theorem 3. *Let $f \in C(\overline{D}_T \times \mathbb{R})$ satisfy the condition* (6), *when $0 \le \alpha < \frac{n+1}{n-1}$ and the condition* (10); *$\lambda > 0$, function $F^0 \in L_2(D_T)$, $F^0 \ge 0$, $\|F^0\|_{L_2(D_T)} \ne 0$ and $F = \gamma F^0$, $\gamma = const > 0$. Then there exists the number $\gamma_0 = \gamma_0(F^0, \alpha, p, \lambda) > 0$ such that for $\gamma > \gamma_0$ the problem* (1)–(4) *does not have a strong generalized solution of the class W_2^1.*

Positive Solutions of Nonlinear Nonlocal Problems for Second Order Singular Differential Equations

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In a finite interval $]a,b[$ we consider the nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with the nonlinear nonlocal boundary conditions of one of the following two types:

$$u(a) = \ell_1(u), \quad u(b) = \ell_2(u) \tag{2}$$

and

$$u(a) = \ell_1(u), \quad u'(b) = \ell_2(u). \tag{3}$$

Here, $f :]a,b[\times]0,+\infty[\to \mathbb{R}_+$ is a measurable in the first and continuous in the second argument function, $\mathbb{R}_+ = [0,+\infty[$, and $\ell_i : C([a,b];\mathbb{R}_+) \to \mathbb{R}_+$ $(i=1,2)$ are continuous functionals.

Let $C([a,b];\mathbb{R})$ be a space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm

$$\|u\| = \max\big\{|u(t)| : \ a \le t \le b\big\};$$

$C([a,b];\mathbb{R}_+) = \big\{u \in C([a,b];\mathbb{R}) : \ u(t) \ge 0 \text{ for } a \le t \le b\big\}$, and let $\widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ be a space of continuously differentiable functions $u :]a,b[\to \mathbb{R}$ whose first derivative is absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

A function $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ is said to be a **positive solution** of the equation (1) if

$$u(t) > 0 \ \text{ for } \ a < t < b$$

and

$$u''(t) = f(t,u(t)) \ \text{ for almost all } \ t \in]a,b[\,.$$

A positive solution u of the equation (1) is said to be a **positive solution of the problem** $(1),(2)$ (**of the problem** $(1),(3)$) if it satisfies the equalities (2) (has a finite limit $u'(b) = \lim\limits_{t\to b} u'(t)$ and satisfies the equalities (3)).

The theorems below on the existence of a positive solution of the problem $(1),(2)$ (of the problem $(1),(3)$) deal with the cases where the function f in the domain $]a,b[\times]0,+\infty[$ satisfies the inequality

$$p_0(t) \le -q(x) f(t,x) \le p_1(t) + p_2(t)(1+x), \tag{4}$$

and the functionals ℓ_i $(i=1,2)$ on the set $C([a,b];\mathbb{R}_+)$ satisfy one of the following three conditions:

$$\ell_i(u) \le r\|u\| + r_0 \ \ (i=1,2), \tag{5}$$

$$\ell_1(u) \le r\|u\| + r_0, \quad \ell_2(u) \le \|u\|_{[a,t_0]}, \tag{6}$$

and

$$\ell_1(u) + (b-a)\ell_2(u) \le r\|u\| + r_0, \tag{7}$$

where $p_i :]a,b[\to \mathbb{R}_+$ $(i=0,1,2)$ are measurable functions, $q :]0,+\infty[\to]0,+\infty[$ is a continuous, nondecreasing function, r and r_0 are nonnegative constants, $t_0 \in]a,b[$ and

$$\|u\|_{[a,t_0]} = \max\big\{u(t) : \ a \le t \le t_0\big\}.$$

We are, mainly, interested in the case

$$\lim_{x\to 0} q(x) = 0 \ \text{ and } \ p_0(t) > 0 \ \text{ for } \ t \in I,$$

where $I \subset [a, b]$ is a set of positive measure. In this case,

$$\lim_{x\to 0} f(t, x) = -\infty \ \text{ for } \ t \in I,$$

i.e., the equation (1) is singular with respect to the phase variable.

Let

$$q(+\infty) = \lim_{x\to +\infty} q(x).$$

The following theorems are valid.

Theorem 1. *If along with* (4) *and* (5) *the conditions*

$$0 < \int_a^b (t-a)(b-t)p_i(t)\,dt < +\infty \ \ (i = 0, 1), \tag{8}$$

$$r < 1, \quad \int_a^b (t-a)(b-t)p_2(t)\,dt < (1-r)(b-a)q(+\infty)$$

are fulfilled, then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If along with* (4), (6) *and* (8) *the conditions*

$$r < 1, \quad \int_a^b (t-a)(b-t)p_2(t)\,dt < (1-r)(b-t_0)q(+\infty)$$

are fulfilled, then the problem (1), (2) *has at least one positive solution.*

Theorem 3. *If along with* (4) *and* (7) *the conditions*

$$0 < \int_a^b (t-a)p_i(t)\,dt < +\infty \ \ (i = 0, 1),$$

$$r < 1, \quad \int_a^b (t-a)p_2(t)\,dt < (1-r)(b-a)q(+\infty)$$

are fulfilled, then the problem (1), (3) *has at least one positive solution.*

Note that if the conditions of Theorem 1 or 2 (of Theorem 3) are fulfilled, but

$$\int_a^b p_i(t)\,dt = +\infty \ \ (i = 0, 1),$$

then the equation (1) has a nonintegrable singularity in the time variable at the points $t = a$ and $t = b$ (at the point $t = a$).

As an example, we consider the differential equation

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q(u)} \tag{9}$$

with the nonlocal boundary conditions of one of the following three types:

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u(b) = \int_a^b h_2(u(s))\, d\psi(s); \tag{10}$$

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u(b) = \int_a^{t_0} h_2(u(s))\, d\sigma(s); \tag{11}$$

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u'(b) = \int_a^b h_2(u(s))\, d\psi(s), \tag{12}$$

where

$$0 < \lambda_i \le 1 \ \ (i = 1, \dots, n), \ \ t_0 \in \,]a, b[\,,$$

$f_k :]a, b[\to \mathbb{R}_+$ $(k = 0, 1, \dots, n)$ are measurable functions, $q :]0, +\infty[\to]0, +\infty[$ is a continuous, nondecreasing function, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ are continuous functions, $\varphi : [a, b] \to [0, 1]$, $\psi : [a, b] \to [0, 1]$ and $\sigma : [a, t_0] \to [0, 1]$ are nondecreasing functions.

Theorems 1–3 result in the following corollaries.

Corollary 1. *If*

$$0 < \int_a^b (t-a)(b-t) f_0(t)\, dt < +\infty, \quad \int_a^b (t-a)(b-t) f_i(t)\, dt < +\infty \ \ (i = 1, \dots, n), \tag{13}$$

$$\lim_{x \to +\infty} \frac{h_i(x)}{x} < 1 \ \ (i = 1, 2),$$

and

$$q(+\infty) = +\infty, \tag{14}$$

then the problem (9), (10) *has at least one positive solution.*

Corollary 2. *If along with* (13) *and* (14) *the conditions*

$$\lim_{x \to +\infty} \frac{h_1(x)}{x} < 1, \ \ h_2(x) \le x \ \textit{for} \ x > 0$$

are fulfilled, then the problem (9), (11) *has at least one positive solution.*

Corollary 3. *If*

$$0 < \int_a^b (t-a) f_0(t)\, dt < +\infty, \quad \int_a^b (t-a) f_i(t)\, dt < +\infty \ \ (i = 1, 2),$$

$$\limsup_{x \to +\infty} \frac{h_1(x)}{x} + (b-a) \limsup_{x \to +\infty} \frac{h_2(x)}{x} < 1,$$

and the condition (14) *is fulfilled, then the problem* (9), (12) *has at least one positive solution.*

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

Initial Value Problems for Nonlinear Singular Hyperbolic Equations of Higher Order with Two Independent Variables

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For the hyperbolic partial differential equation

$$u^{(m,n)} = f(x,y,u) \tag{1}$$

the Darboux problem

$$\begin{aligned} u^{(i,0)}(x,0) = 0 \quad \text{for} \;\; 0 \le x \le a \;\; (i = 0,\dots,m-1),\\ u^{(0,k)}(\gamma(y),y) = 0 \quad \text{for} \;\; 0 \le y \le b \;\; (k = 0,\dots,n-1), \end{aligned} \tag{2}$$

and the Cauchy problem

$$u^{(i,n)}(x,\varphi(x)) = 0, \quad u^{(0,k)}(x,\varphi(x)) = 0 \quad \text{for} \;\; 0 \le x \le a \;\; (i = 0,\dots,m-1; \;\; k = 0,\dots,n-1) \tag{3}$$

are studied. Here m and n are positive integers,

$$u^{(i,k)}(x,y) = \frac{\partial^{i+k}u(x,y)}{\partial x^i \partial y^k} \;\; (i = 0,\dots,m; \;\; k = 0,\dots,n),$$

and $\gamma : [0,b] \to [0,a]$ and $\varphi : [0,a] \to [0,b]$ $(i,\, k = 1,2)$ are continuously differentiable functions such that

$$\begin{gathered} \gamma(0) = 0, \;\; \gamma(y) < a, \;\; \gamma'(y) \ge 0 \quad \text{for} \;\; 0 < y < b, \\ \varphi(0) = b, \;\; \varphi'(x) < 0 \quad \text{for} \;\; 0 \le x \le a. \end{gathered}$$

Problem (1), (2) (problem (1), (3)) is considered in the domain $D_0 = \{(x,y) \in \mathbb{R}^2 : \;\gamma(y) < x < a, \; 0 < y < b\}$ (in the domain $D = \{(x,y) \in \mathbb{R}^2 : \; 0 < x < a, \; \varphi(x) < y < b\}$). By $\overline{D}_0$ and $\overline{D}$ we denote the closure of the domains D_0 and D.

The existence theorems formulated below concern the case where f is a nonnegative function defined in the domain $D_0 \times (0,+\infty)$ (in the domain $D \times (0,+\infty)$) and satisfies the inequality

$$p(x,y)z^{-\lambda} \le f(x,y,z) \le q(x,y)z^{-\lambda} + r(x,y)(1+z), \tag{4}$$

where p, q and r are nonnegative Lebesgue integrable functions in D_0 (in D).

Furthermore, $f(\,\cdot\,,\,\cdot\,,z) : D_0 \to [0,+\infty)$ $(f(\,\cdot\,,\,\cdot\,,z) : D \to [0,+\infty))$ is measurable for every $z \in (0,+\infty)$, and $f(x,y,\,\cdot\,) : (0,+\infty) \to [0,+\infty)$ is continuous almost for every $(x,y) \in D_0$ (almost for every $(x,y) \in D$).

Introduce the functions

$$\mathcal{P}_0(x,y) = \int\limits_0^y (y-t)^{n-1}\bigg(\int\limits_{\gamma(y)}^x (x-s)^{m-1}p(s,t)\,ds\bigg)\,dt \quad \text{for} \;\; (x,y) \in D_0$$

and

$$\mathcal{P}(x,y)=\int\limits_{\varphi(x)}^{y}(y-t)^{n-1}\bigg(\int\limits_{\psi(t)}^{x}(x-s)^{m-1}p(s,t)\,ds\bigg)\,dt\quad\text{for }\ (x,y)\in D,$$

where ψ is the function inverse to φ.

If

$$\mathcal{P}_0(x,y)>0\quad\text{for }\ (x,y)\in D_0, \tag{5}$$

then by conditions (4), there exists a set $D_0'\subset D_0$ of a positive measure such that

$$\lim_{z\to 0+}f(x,y,z)=+\infty\quad\text{for }\ (x,y)\in D_0'.$$

Similarly, if

$$\mathcal{P}_0(x,y)>0\quad\text{for }\ (x,y)\in D, \tag{6}$$

then there exists a set $D'\subset D$ of a positive measure such that

$$\lim_{z\to 0+}f(x,y,z)=+\infty\quad\text{for }\ (x,y)\in D'.$$

Consequently equation (1) has singularity with resect to the phase variable. For such case problems $(1),(2)$ and $(1),(3)$ have not been studied before.

By a solution of problem $(1),(2)$ we understand a continuous function $u:\overline{D}_0\to[0,+\infty)$ which is positive in the domain D_0 and satisfies in that domain the integral equation

$$u(x,y)=\frac{1}{(m-1)!(n-1)!}\int\limits_{\gamma(y)}^{x}(x-s)^{m-1}\bigg(\int\limits_{0}^{y}(y-t)^{n-1}f(s,t,u(s,t))\,ds\bigg)\,dt.$$

By a solution of problem $(1),(3)$ we understand a continuous function $u:\overline{D}\to[0,+\infty)$ which is positive in the domain D and satisfies in that domain the integral equation

$$u(x,y)=\frac{1}{(m-1)!(n-1)!}\int\limits_{\varphi(x)}^{y}(y-t)^{n-1}\bigg(\int\limits_{\psi(t)}^{x}(x-s)^{m-1}f(s,t,u(s,t))\,ds\bigg)\,dt.$$

Theorem 1. *If along with* (4) *and* (5) *the condition*

$$\iint\limits_{D_0}q(x,y)(\mathcal{P}_0(x,y))^{-\frac{\lambda}{\lambda+1}}\,dx\,dy<+\infty$$

holds, then problem $(1),(2)$ *has at least one solution.*

Corollary 1. *Let the condition*

$$l_1(x-\gamma(y))^{\alpha}y^{\beta}z^{-\lambda}\le f(x,y,z)\le l_2(x-\gamma(y))^{\alpha}y^{\beta}z^{-\lambda}+r(x,y)(1+z)$$

hold in $D_0\times(0,+\infty)$*, where* $l_1>0$*,* $l_2>0$*,* $\lambda>0$*,* α *and* β *are some constants. Then problem* $(1),(2)$ *is solvable if and only if*

$$\alpha>(m-1)\lambda-1,\quad \beta>(n-1)\lambda-1.$$

Remark. If $\gamma(y)\equiv 0$, then problem $(1),(2)$ is a characteristic value problem. Thus Theorem 1 and Corollary 1 contain optimal sufficient conditions of solvability of a characteristic value problem.

Theorem 2. *If along with* (4) *and* (6) *the condition*

$$\iint\limits_D q(x,y)(\mathcal{P}(x,y))^{-\frac{\lambda}{\lambda+1}}\,dx\,dy < +\infty$$

holds, then problem $(1),(3)$ *has at least one solution.*

Corollary 2. *Let the condition*

$$l_1(y-\varphi(x))^{\mu}z^{-\lambda} \le f(x,y,z) \le l_2(y-\varphi(x))^{\mu}z^{-\lambda} + r(x,y)(1+z)$$

hold in $D\times(0,+\infty)$, *where* $l_1>0$, $l_2>0$, $\lambda>0$ *and* μ *are some constants. Then problem* $(1),(3)$ *is solvable if and only if*

$$\mu > (m+n-1)\lambda - 1.$$

Asymptotic Property and Semi-Discrete Scheme for One System of Nonlinear Partial Integro-Differential Equations

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Based on Maxwell's system [1] the following kind of nonlinear integro-differential model arises for mathematical modeling of the process of penetrating of magnetic field in the substance [2]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t |\operatorname{rot} H|^2 \, d\tau\right) \operatorname{rot} H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the system of the integro-differential equations (1) is complex. Equations and systems of type (1) still yield to the investigation for special cases (see, for example, [2]–[11] and references therein).

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, then we have

$$\operatorname{rot}(a(S) \operatorname{rot} H) = \left(0, -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right), -\frac{\partial}{\partial x}\left(a(S)\,\frac{\partial V}{\partial x}\right)\right),$$

and from (1) we obtain the following system of nonlinear integro-differential equations

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\,\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\,\frac{\partial V}{\partial x}\right], \tag{2}$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau. \tag{3}$$

In [6] the asymptotic behavior of solutions as $t \to \infty$ of initial-boundary value problem for system (2), (3) with the homogeneous boundary conditions in the norm of the space H^1 is given. Here and below we use usual Sobolev spaces $H^k(0,1)$.

In [8] some generalization of the system of type (2), (3) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by following averaged integro-differential model:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad \frac{\partial V}{\partial t} = a(S)\,\frac{\partial^2 V}{\partial x^2}\,, \tag{4}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\, d\tau. \tag{5}$$

The existence and uniqueness of solutions of the initial-boundary value problems for the models of type (2), (3) and (4), (5) are studied in many works (see, for example, [2]–[6], [8] and reference therein).

Our aim is to study the asymptotic behavior of solutions as $t \to \infty$ and semi-discrete scheme for the initial-boundary value problem for system (4), (5) for the case $a(S) = (1+S)^p$, $0 < p \le 1$.

In the domain $[0,1] \times [0,\infty)$ for the system (4), (5) we consider the following initial-boundary value problem

$$\begin{gathered} U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \ge 0, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \end{gathered} \tag{6}$$

where U_0 and V_0 are given functions.

The following statement is valid.

Theorem 1. *If $a(S) = (1+S)^p$, $0 < p \le 1$, U_0, $V_0 \in H^3(0,1) \cap H_0^1(0,1)$, then for the solution of problem* (4)–(6) *the following asymptotic relations hold as $t \to \infty$:*

$$\begin{gathered} \Big|\frac{\partial U(x,t)}{\partial x}\Big| \le C \exp\Big(-\frac{t}{2}\Big), \quad \Big|\frac{\partial V(x,t)}{\partial x}\Big| \le C \exp\Big(-\frac{t}{2}\Big), \\ \Big|\frac{\partial U(x,t)}{\partial t}\Big| \le C \exp\Big(-\frac{t}{2}\Big), \quad \Big|\frac{\partial V(x,t)}{\partial t}\Big| \le C \exp\Big(-\frac{t}{2}\Big). \end{gathered}$$

Here and below C denotes a positive constant.

Now let us consider the semi-discrete scheme for (4)–(6) problem. On [0,1] let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \ldots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) are designed by $u_i = u_i(t)$ and $v_i = v_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$ and $V_i = V_i(t)$. At points $i = 1, 2, \ldots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences.

Let us correspond to problem (4)–(6) the following semi-discrete scheme:

$$\begin{gathered} \frac{du_i}{dt} = \Big(1 + h\int_0^t \sum_{k=1}^M \big[(u_{\overline{x},k})^2 + (v_{\overline{x},k})^2\big]\, d\tau\Big)^p u_{\overline{x}x,\,i}, \\ \frac{dv_i}{dt} = \Big(1 + h\int_0^t \sum_{k=1}^M \big[(u_{\overline{x},k})^2 + (v_{\overline{x},k})^2\big]\, d\tau\Big)^p v_{\overline{x}x,\,i}, \\ i = 1, 2, \ldots, M-1, \end{gathered} \tag{7}$$

$$u_0(t) = u_M(t) = v_0(t) = v_M(t) = 0, \tag{8}$$

$$u_i(0) = U_{0,i}, \quad v_i(0) = V_{0,i}, \quad i = 0, 1, \ldots, M. \tag{9}$$

So, we obtained the Cauchy problem (7)–(9) for nonlinear system of ordinary integro-differential equations.

It is not difficult to obtain the following estimates:

$$\|u(t)\|^2 + \int_0^t \|u_{\overline{x}}]|^2\, d\tau \le C, \quad \|v(t)\|^2 + \int_0^t \|v_{\overline{x}}]|^2\, d\tau \le C, \tag{10}$$

where

$$\|w(t)\|^2 = \sum_{i=1}^{M-1} w_i^2(t) h, \quad \|w_{\overline{x}}]|^2 = \sum_{i=1}^{M} w_{\overline{x},\,i}^2(t) h.$$

The a priori estimates (10) guarantee the global solvability of the problem (7)–(9).

The following statement is true.

Theorem 2. *If $a(S) = (1+S)^p$, $0 < p \le 1$, and problem* (4)–(6) *has a sufficiently smooth solution $U(x,t)$, $V(x,t)$, then the solution of problem* (7)-(9) *$u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$, $v = v(t) = (v_1(t), v_2(t), \dots, v_{M-1}(t))$ tends to $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$, $V = V(t) = (V_1(t), V_2(t), \dots, V_{M-1}(t))$ as $h \to 0$ and the following estimates are true*

$$\|u(t) - U(t)\| \le Ch, \quad \|v(t) - V(t)\| \le Ch.$$

Acknowledgement

This research was supported by the Shota Rustaveli National Science Foundation (Project # FR/30/5-101/12, Agreement # 31/32).

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The Existence of o-Solutions of Quasi-Linear Two-Dimensional System of Differential Equations in the Case when the Roots of the Characteristic Equation are $0 \neq \lambda_1(+\infty) \in \mathbb{R},\ \lambda_2(+\infty) = 0$

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We study the asymptotic for $t \to +\infty$ of o-solutions of real system of differential equations:

$$\begin{cases} y_1' = \alpha_1(t) f_1(t, y_1, y_2), \\ y_2' = \alpha_2(t) f_2(t, y_1, y_2), \end{cases} \qquad (t, y_1, y_2) \in G, \tag{1}$$

$G = \{t \in \Delta = \Delta(t_0) = [t_0, +\infty) \subset \mathbb{R};\ |y_k| \leqslant b,\ b \in (0, +\infty)\ (k = 1, 2)\}$, $0 < \alpha_k(t) \in C(\Delta)$, $\int\limits_{t_0}^{+\infty} \alpha_k(t)\, dt = +\infty$ $(k = 1, 2)$; $\exists \lim\limits_{t \to +\infty} \frac{\alpha_2(t)}{\alpha_1(t)} \in [0, +\infty)$; $f_k(t, y_1, y_2) \in C^{1\,2\,2}_{t\,y_1 y_2}(G)$, $\exists\, f_k(+\infty, 0, 0) = 0$, $\exists\, \frac{\partial^{i+j} f_k}{\partial y_1^i y_2^j}(+\infty, 0, 0) \in \mathbb{R}$ $(k = 1, 2;\ i, j \in \{0, 1, 2\};\ (i + j) \in \{1, 2\})$; $0 \neq \lambda_k(t) \in C^1_\Delta$ $(k = 1, 2)$ – the real roots of the characteristic equation

$$\begin{bmatrix} \dfrac{\partial f_1(t,0,0)}{\partial y_1} - \lambda & \dfrac{\partial f_1(t,0,0)}{\partial y_2} \\ \dfrac{\alpha_2(t)}{\alpha_1(t)} \dfrac{\partial f_2(t,0,0)}{\partial y_1} & \dfrac{\alpha_2(t)}{\alpha_1(t)} \dfrac{\partial f_2(t,0,0)}{\partial y_2} - \lambda \end{bmatrix} = 0,$$

where $\lambda_1(t) \to \lambda_1^0 \in \mathbb{R}$, $\lambda_1^0 \neq 0$, $\lambda_2(t) = o(1)$ for $t \to +\infty$, $(\frac{\partial f_1(+\infty,0,0)}{\partial y_2})^2 + (\lambda_1^0 - \frac{\partial f_1(+\infty,0,0)}{\partial y_1})^2 > 0$.

The case $\lambda_k(+\infty) \neq 0$ $(k = 1, 2)$ was discussed in an article [1] for systems of dimension $n \geq 1$. The results obtained in the article [1] are not applicable for system (1) with predetermined conditions.

We introduce the following notation.

$$p_{ij}(t) = \frac{\partial f_i(t,0,0)}{\partial y_j} \ (i, j = 1, 2);$$

$$q_1(t) = \frac{f_1(t,0,0) p_{12}(t) + \frac{\alpha_2(t)}{\alpha_1(t)} f_2(t,0,0)(\lambda_1(t) - p_{11}(t))}{\sqrt{p_{12}^2(t) + (\lambda_1(t) - p_{11}(t))^2}},$$

$$q_2(t) = \frac{-f_1(t,0,0)(\lambda_1(t) - p_{11}(t)) + \frac{\alpha_2(t)}{\alpha_1(t)} f_2(t,0,0) p_{12}(t)}{\sqrt{p_{12}^2(t) + (\lambda_1(t) - p_{11}(t))^2}}.$$

The functions $p_{ij}(t)$, $\tilde{q}_i(t) \in C^1_\Delta$, $p_{ij}(+\infty) = p^0_{ij}$, $\tilde{q}_i(+\infty) = 0$ $(i, j = 1, 2)$ in view of the conditions of system (1).

We formulate some theorems.

Theorem 1. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2^2(t)} \to \ell_{11} \in \mathbb{R}$ *for* $t \to +\infty$, $\int\limits_{t_0}^{+\infty} \alpha_1(t)\lambda_2(t) dt = \infty$;

(2) $q_1(t)=o(\frac{q_2(t)}{\lambda_2(t)})$, $\frac{q_2(t)}{\lambda_2^2(t)}\to\ell_{12}\in\mathbb{R}$, $\frac{q_2'(t)}{\alpha_1(t)\lambda_2(t)\tilde{q}_2(t)}\to\ell_{13}\in\mathbb{R}$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(\alpha_1(t)\lambda_2(t))$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in \mathrm{C}^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot{\sim}\,\frac{q_2(t)}{\lambda_2(t)},\quad y_2(t)\,\dot{\sim}\,\frac{q_2(t)}{\lambda_2(t)}.^{*}\tag{2}$$

Theorem 2. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2^2(t)}\to\ell_{21}\in\mathbb{R}$ *for* $t\to+\infty$, $\int\limits_{t_0}^{+\infty}\alpha_1(t)\lambda_2(t)dt=\infty$;

(2) $q_1(t)=o(\lambda_2(t))$, $\frac{q_2(t)}{\lambda_2^2(t)}\to\ell_{22}\in\mathbb{R}$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(\alpha_1(t)\lambda_2(t))$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in \mathrm{C}^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot{\sim}\,\lambda_2(t),\quad y_2(t)\,\dot{\sim}\,\lambda_2(t).\tag{3}$$

Theorem 3. *Let the system* (1) *meet the additional conditions:*

(1) $\alpha_1(t)\lambda_2(t)\to\ell_{31}\in\mathbb{R}$, $\ell_{31}\neq0$ *for* $t\to+\infty$;

(2) $\frac{q_2(t)}{\lambda_2(t)}=o(q_1(t))$, $\alpha_1(t)q_1(t)=o(1)$, $q_1^2(t)=o(q_2(t))$ *for* $t\to+\infty$;

(3) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)}\to\ell_{32}\in\mathbb{R}$, $\ell_{32}\neq\lambda_1^0$, $(\frac{q_2(t)}{\lambda_2(t)})'=o(\frac{q_2(t)}{\lambda_2(t)})$ *for* $t\to+\infty$;

(4) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(1)$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in \mathrm{C}^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot{\sim}\,q_1(t),\quad y_2(t)\,\dot{\sim}\,\frac{q_2(t)}{\lambda_2(t)}.\tag{4}$$

Theorem 4. *Let the system* (1) *meet the additional conditions:*

(1) $\alpha_1(t)\lambda_2(t)\to\ell_{41}\in\mathbb{R}$, $\ell_{41}\neq0$, $\lambda_2(t)=o(q_1(t))$, $\frac{\alpha_1(t)q_2(t)}{\lambda_2(t)}\to\ell_{42}\in\mathbb{R}$ *for* $t\to+\infty$;

(2) $\lambda_2'(t)=o(\lambda_2(t))$, $\frac{q_1'(t)}{\alpha_1(t)q_1(t)}\to\ell_{43}\in\mathbb{R}$, $\ell_{43}\neq\lambda_1^0$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(1)$*for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in \mathrm{C}^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot{\sim}\,q_1(t),\quad y_2(t)\,\dot{\sim}\,\lambda_2(t).\tag{5}$$

Theorem 5. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{q_2(t)}{\lambda_2(t)}=o(q_1(t))$, $q_1^2(t)=o(\frac{q_2(t)}{\lambda_2(t)})$ *for* $t\to+\infty$;

$^{*}f_i\,\dot{\sim}\,f_j$ $(i\neq j)$ means that $\exists\lim\limits_{t\to+\infty}\frac{f_i}{f_j}\neq0,\pm\infty$.

(2) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)} \to \ell_{51} \in \mathbb{R}$, $\ell_{51} \neq \lambda_1^0$, $[(\frac{q_2(t)}{\lambda_2(t)})' \frac{\lambda_2(t)}{\alpha_1(t)q_2(t)}] \to \ell_{52} \in \mathbb{R}$, $\ell_{52} \neq 0$ *for* $t \to +\infty$;

(3) $|p_{12}'(t)| + |\lambda_1'(t) - p_{11}'(t)| = o(\alpha_1(t))$ *for* $t \to +\infty$.

Then in $\Delta(t_1) \subset \Delta$ *the system* (1) *for* $t \to +\infty$ *exists at least one o-solution* $y_k \in \mathrm{C}^1_{\Delta(t_1)}$ $(k = 1, 2)$ *of the form* (4).

Theorem 6. *Let the system* (1) *meet the additional conditions:*

(1) $\lambda_2(t) = o(q_1(t))$, $q_1^2(t) = o(\lambda_2(t))$, $\frac{q_2(t)}{\lambda_2(t)} \to \ell_{61} \in \mathbb{R}$ *for* $t \to +\infty$;

(2) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)} \to \ell_{62} \in \mathbb{R}$, $\ell_{62} \neq \lambda_1^0$, $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2(t)} \to \ell_{63} \in \mathbb{R}$, $\ell_{63} \neq 0$ *for* $t \to +\infty$;

(3) $|p_{12}'(t)| + |\lambda_1'(t) - p_{11}'(t)| = o(\alpha_1(t))$ *for* $t \to +\infty$.

Then in $\Delta(t_1) \subset \Delta$ *the system* (1) *for* $t \to +\infty$ *exists at least one o-solution* $y_k \in \mathrm{C}^1_{\Delta(t_1)}$ $(k = 1, 2)$ *of the form* (5).

The constants $\ell_{ij} \in \mathbb{R}$ $(i = \overline{1,6},\ j \in \{1, 2, 3\})$ take part in finding the exact asymptotic of o-solutions.

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On the Properties of Regular Linear Differential Systems with Unbounded Coefficients

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Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1}$$

where $n \geqslant 2$, with the piecewise continuous coefficient matrix. We denote the class of all such systems (1) by $\mathcal{M}_n^*$, and by $\mathcal{M}_n^0$ we denote its subclass, consisting of the systems, any nonzero solution of which has a finite Lyapunov exponent. Also we denote by $\mathcal{M}_n$ a subclass of $\mathcal{M}_n^*$, consisting of the systems with bounded on time semiaxis coefficient matrix. We will identify system (1) with its coefficient matrix and write thereby $A \in \mathcal{M}_n^*$. Lyapunov exponents of the system $A \in \mathcal{M}_n^0$ are denoted by $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$.

The class of the regular differential systems is defined by A. M. Lyapunov [1]. Let us formulate this definition as well as classical criteria of regularity, since we will need it in order to formulate Theorem 1. The system $A \in \mathcal{M}_n$ is called regular if and only if any of the following conditions holds:

(i) the equality $\lambda_1(A) + \cdots + \lambda_n(A) = \varliminf\limits_{t\to+\infty} t^{-1} \int\limits_0^t \mathrm{Sp}A(\tau)\,\mathrm{d}\tau$ is valid;

(ii) the Lyapunov exponents $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ of the system A are symmetrical with respect to zero to the Lyapunov exponents $\lambda_1(-A^\mathsf{T}) \leqslant \cdots \leqslant \lambda_n(-A^\mathsf{T})$ of the conjugate system, i.e., the equality $\lambda_k(A) = -\lambda_{n-k+1}(-A^\mathsf{T})$ holds for every $k = 1, \dots, n$;

(iii) there exists a generalized Lyapunov transformation reducing the system A to the diagonal system with constant coefficients;

(iv) for every normal basis $\{x_1(\,\cdot\,), \dots, x_n(\,\cdot\,)\}$ of the solutions to the system A their Lyapunov exponents are exact (i.e., for every $k \in \{1, \dots, n\}$ there exists $\lim\limits_{t\to\infty} t^{-1}\ln\|x_k(t)\|$), and for every $k = 1, \dots, n-1$ an angle $\gamma_k(t) = \angle(x_k(t), \mathrm{span}\{x_{k+1}(t), \dots, x_n(t)\})$, $t \geqslant 0$, has the exact and zero Lyapunov exponent (i.e., $\lim\limits_{t\to+\infty} t^{-1}\ln\gamma_k(t) = 0$, $k \in \{1, \dots, n-1\}$);

(v) there exists such fundamental system $\{x_1(\,\cdot\,), \dots, x_n(\,\cdot\,)\}$ of the solutions to the system A, the Lyapunov exponents of which are exact, and the angle $\angle(x_k(t), \mathrm{span}\{x_{k+1}(t), \dots, x_n(t)\})$, $t \geqslant 0$, has the exact and zero Lyapunov exponent for every $k = 1, \dots, n-1$.

The condition (i) represents the definition of the regular system [1] given by A. M. Lyapunov, and the conditions (ii) and (iii) are Perron [2] and Basov–Bogdanov–Grobman [3–5] criteria of regularity, respectively. The conditions (iv) and (v) give two different forms of Vinograd criterion [6] of regularity of a system.

We denote by $\mathcal{R}_n$ the class of the regular according to Lyapunov linear differential n-dimensional systems. In [3] the notion of regularity is extended to systems with unbounded on the semiaxis coefficients: a system is called regular [3] if it can be transformed to a diagonal system with constant coefficients by some generalized Lyapunov transformation (let us recall, that linear invertible for every $t \geqslant 0$ transformation $x = L(t)y$ is called generalized Lyapunov transformation if $\lim\limits_{t\to+\infty} L(t) = \lim\limits_{t\to+\infty} L^{-1}(t) = 0$). We denote by $\mathcal{R}_n^0$ the class of regular n-dimentional systems (in general, with unbounded coefficients). In particular, as it follows from the definition, the inclusion $\mathcal{R}_n^0 \subset \mathcal{M}_n^0$ holds.

The question naturally arises, which properties of the regular systems of $\mathcal{R}_n$ inherit systems of $\mathcal{R}_n^0$. It turns out that for the systems of $\mathcal{R}_n^0$ all properties (i)–(v) hold. Moreover, each of these properties can be taken as the definition of the regular system of class $\mathcal{R}_n^0$.

Theorem 1. *The system A belongs to the class $\mathcal{R}_n^0$ if and only if any of the conditions* (i)–(v) *is valid for it.*

However, Lyapunov criterion of the regularity of the system of $\mathcal{R}_n$ is well-known: the system is regular if and only if under some (and hence under any) Lyapunov transformation, reducing system to a triangular form, diagonal elements of the obtained coefficient matrix have exact integral mean values (moreover, the set of those mean values, taking into account their multiplicities, coincides with the set of the Lyapunov exponents of the system).

The above condition is necessary for the regularity of the system of $\mathcal{M}_n^0$ as well, as the following theorem shows.

Theorem 2. *For any reduction by generalized Lyapunov transformation* (*and, in particular, Lyapunov transformation*) *of the system $A \in \mathcal{R}_n^0$ to the triangular form, its diagonal coefficients have finite exact mean values, the set of which, taking into account their multiplicities, coincides with the set of the Lyapunov exponents of the system A.*

It appears that for the systems from the class $\mathcal{R}_n^0$ Lyapunov criterion of regularity, generally speaking, does not hold (i.e., the above condition is not sufficient – Theorem 2 is irreversible).

Theorem 3. *There exists an irregular system, Lyapunov exponents of any nonzero solutions of which are finite and exact, such that for any its reduction by generalized Lyapunov transformation* (*and, in particular, the Lyapunov transformation*) *to a triangular form, its diagonal coefficients have finite exact integral mean values, the set of which, taking into account their multiplicities, is the same for any such transformation.*

Nevertheless, as the following Theorem 4 shows, the basic property of the systems of $\mathcal{R}_n$ – to save conditional exponential stability, as well as the dimension of the exponentially stable manifold and asymptotic indicators of its solutions under perturbations of higher order of smallness – is also valid for the systems of $\mathcal{R}_n^0$.

We denote by $P_n^{(0)}$ a class of continuous vector-valued functions $f(\,\cdot\,,\,\cdot\,) : [0,+\infty) \times B_f \to \mathbb{R}^n$, where B_f is a closed ball in $\mathbb{R}^n$, centered at the origin (dependent on f; its radius is denoted by r_f), such that $f(t,0) = 0$ for all $t \geqslant 0$ and $\|f(t,x_1) - f(t,x_2)\| \leqslant F(t)N(r)\|x_1 - x_2\|$ for all $x_1, x_2 \in B_f$ and $t \geqslant 0$, where $r = \max\{\|x_1\|, \|x_2\|\}$, and $F(\,\cdot\,)$ and $N(\,\cdot\,)$ are continuous (dependent on f) functions, defined respectively on $[0,+\infty)$ and $[0,r_f]$, satisfying the following conditions: Lyapunov exponent of the function $F(\,\cdot\,)$ is nonpositive and $N(r) = O(r^\varepsilon)$ as $r \to +0$ for some $\varepsilon > 0$ (dependent on f).

Theorem 4. *Let the system* (1) *belong to $\mathcal{R}_n^0$ and have exactly k distinct negative Lyapunov exponents: $\Lambda_1(A) < \cdots < \Lambda_k(A) < 0$ of multiplicities $n_1, \ldots, n_k$, respectively. Then for solutions of the system*

$$\dot{x} = A(t)x + f(t,x), \quad (t,x) \in [0,+\infty) \times B_f, \tag{2}$$

where $f \in P_n^{(0)}$, the following properties hold:

(1) *there exists a ball $B \subset \mathbb{R}^n$ and a sequence of enclosed manifolds $\mathbf{0} = M_0 \subset M_1 \subset \cdots \subset M_k \subset B$, such that $\dim M_i = n_i$, $i = 1, \ldots, k$, and every solution $x(\,\cdot\,)$ to the system* (2)*, beginning at $t = 0$ on $M_i \setminus M_{i-1}$, is extendable on* $[0,+\infty)$ *and for any $\delta > 0$ and for all $t \geqslant 0$ satisfies the two-sided estimate*

$$C_{\delta,x} \exp\big\{(\Lambda_i(A) - \delta)t\big\}\|x(0)\| \leqslant \|x(t)\| \leqslant C_\delta \exp\big\{(\Lambda_i(A) + \delta)t\big\}\|x(0)\|,$$

where the constant C_δ depends only on δ, and the constant $C_{\delta,x}$ – only on δ and on the selection of solution $x(\,\cdot\,)$;

(2) *any solution to the system* (2), *beginning at* $t = 0$ *in* $B \setminus M_k$, *leaves the ball* B *in finite time if* $\lambda_{n_1+\dots+n_k+1}(A) > 0$, *and if* $\lambda_{n_1+\dots+n_k+1}(A) = 0$, *then any such solution, continued inside* B *on* $[0, +\infty)$, *has zero exponent;*

(3) *for every* $t \geqslant 0$ *and* $i = 1, \dots, k$ *an orthogonal projection of manifold* $M_i(t) = \{x(t) \in \mathbb{R}^n : x(0) \in M_i\}$ *on lineal* $V_i(t)$, *formed at time* t *by the values of the solutions of* (1), *Lyapunov exponent of which does not exceed* $\Lambda_i(A)$, *is a continuous bijection onto its image, and the manifold* $M_i(t)$ *is tangent to lineal* $V_i(t)$ *at the origin;*

(4) *if the vector function* $f(\cdot\,,\,\cdot)$ *is continuously differentiable in* x, *then for every* $t \geqslant 0$ *and* $i = 1, \dots, k$ *the manifold* $M_i(t)$ *belongs to the class* C^1.

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Properties of Stable and Attracting Sets of L-Systems

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Many of the current problems of predicting of physical and mechanical processes are reduced to study of solutions of evolution equations on large intervals of time in infinite-dimensional phase space. In recent decades, significant progress has been received in studying of such problems. This is connected to the fact that the attractors (compact attracting invariant sets) of many evolution equations generated by partial differential equations have been found. The phase space is not locally compact for most of evolution equations generated by partial differential equations, which prevents applying existing results of the theory of dynamical systems in locally compact spaces for studying of these equations, although the most profound results in studying of neighborhoods of attractors are obtained for dynamic systems in locally compact spaces. In [1] the class of dynamical systems which is called L-systems was introduced. These systems satisfy many properties of dynamical systems in locally compact spaces but their phase space is not necessarily locally compact.

Let $f(t,x)$ be a semidynamical system which is specified in a metric space X. Let Φ be the set of all the motions of semidynamical system $f(t,x)$.

Semidynamical system $f(t,x)$ is said to be L-system if it satisfies the following property: there exists a constant $\omega \geq 0$ such that if the sequence $\varphi_n(t) \in \Phi$ is bounded on any segment $[a-\omega; b]$, then there exists a subsequence of the sequence $\varphi_n(t)$ converging to some motion $\varphi(t) \in \Phi : \varphi_{n_k}(t) \to \varphi(t)$, $t \in [a;b]$.

Differential equations and inclusions, functional-differential inclusions, evolution parabolic equations generate L-systems in corresponding metric spaces.

We will use the following definitions and notations.

$D(x) = \big\{y \in X : \exists\, x_n \to x,\ \exists\, \varphi_n(t),\ \varphi_n(0) = x_n,\ \exists\, t_n \geq 0, \text{ such that } \varphi_n(t_n) \to y \text{ as } n \to \infty\big\}$;

$D(M) = \bigcup\limits_{x \in M} D(x)$;

The set M is weakly semiattracting, if $\exists\, \delta > 0$ such that for all $x \in S(M,\delta)$, for all motions $\varphi(t)$, $\varphi(0) = x$, there exists a sequence of the moments of time $t_n \to +\infty$ such that $\rho(\varphi(t_n), M) \to 0$ as $n \to \infty$;

$A_2(M) = \big\{x \in X : \forall\, \varphi_x,\ \rho(\varphi_x(t), M) \to 0 \text{ as } t \to +\infty\big\}$;

$A_\omega(M) = \big\{x \in X : \forall\, \varphi_x\ \exists\, t_n \to +\infty \text{ such that } \rho(\varphi_x(t_n), M) \to 0 \text{ as } n \to \infty\big\}$.

We say that a subset M of the space X satisfies condition A if there exist constants $\varepsilon > 0$, $L > 0$, $a > \omega$ such that for all $x \in S(M,\varepsilon)$, for all $\varphi_x(t)$ such that $\varphi_x(-T) \in S(M,\varepsilon)$ for some moment $T > a$, the inequality holds $\rho(\varphi_x(t), M) < L\ \forall\, t \in [-a;0]$.

The following theorems are generalizations of well-known properties of semidynamical systems in locally compact metric spaces for L-systems [2].

Theorem 1. *Compact set M is stable if and only if $D(M) = M$.*

Theorem 2. *Let $f(t,x)$ be L-system, a compact set M be weakly semiattracting and satisfy the condition A. Then the set $D(M)$ is the least compact asymptotically stable set containing M and $A_\omega(M) = A_\omega(D(M)) = A(D(M))$.*

The set $B_0 \subseteq X$ is said to be an absorbing set of semidynamical system $f(t,x)$, if for each bounded set $B \subseteq X$ there exists $T \geq 0$ such that $f(t,B) \subseteq B_0$, $\forall\, t \geq T$.

In the following theorem which is a generalization of the well-known Yosidzava theorem ([2]) sufficient conditions of existence of the bounded absorbing set for L-system are given.

Theorem 3. *Let $f(t,x)$ be an L-system, and let there exist a continuous function $V(u), V : X \to R$, satisfying the following properties outside of some sphere $S(O,r_0)$, $O \in X$, $r_0 > 0$:*

(1) $V(u) \le a(\rho(u,O))$, $\forall\, u \notin S(O,r_0)$, $a(r)$ – *positive continuous increasing function for* $r \ge 0$*;*

(2) $V(u) \to +\infty$ *as* $\rho(u,O) \to \infty$*;*

(3) *for every* $x \in X$ *and every* $\varphi_x(t) \notin S(O,r_0)$, $\forall\, t \in [t_1,t_2]$ *the inequality* $V(\varphi_x(t_1)) \ge V(\varphi_x(t_2))$ *holds;*

(4) *there does not exist a full motion* $\varphi_x^\infty(t)$ *such that* $V(\varphi_x^\infty(t)) = V(x) = const$ $\forall\, t \in R$ *and* $V(\varphi_x^\infty(R)) \cap S(0,r_0) = \varnothing$.

Then there exists a bounded absorbing set for L-system $f(t,x)$.

Nonempty set $U \subseteq X$ is said to be a global attractor of semidynamical system $f(t,x)$ if it satisfies the following properties:

(1) U is a compact;

(2) for each bounded set $B \subseteq X$, $\beta(f(t,B),U) \to 0$ as $t \to 0$;

(3) U is strict invariant, i. e. $f(U,t) = U$ $\forall\, t \ge 0$.

In the work [4] it is proved that if there exists a compact absorbing set $f(t,x)$, then there exists a global attractor $f(t,x)$. If $f(t,x)$ is L-system, then the last conditions could be weakened.

Theorem 4. *If there exists a bounded absorbing set for L-system* $f(t,x)$*, then there exists a global attractor for L-system* $f(t,x)$.

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Carathéodory Solutions to a Hyperbolic Differential Inequality with a Non-Positive Coefficient and Delayed Arguments

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On the rectangle $\mathcal{D} = [a,b] \times [c,d]$ we consider the Darboux problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = p(t,x)u\big(\tau(t,x), \mu(t,x)\big) + q(t,x), \tag{1}$$

$$u(t,c) = \varphi(t) \quad \text{for } \ t \in [a,b], \quad u(a,x) = \psi(x) \quad \text{for } \ x \in [c,d], \tag{2}$$

where $p, q\colon \mathcal{D} \to \mathbb{R}$ are Lebesgue integrable functions, $\tau\colon \mathcal{D} \to [a,b]$ and $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions, and $\varphi\colon [a,b] \to \mathbb{R}$, $\psi\colon [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$. By a *solution* to problem (1), (2) we mean a function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory[1] which satisfies equality (1) almost everywhere in $\mathcal{D}$ and verifies initial conditions (2).

We have introduced the following definition in [2].

Definition 1. Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions. We say that a theorem on differential inequalities (maximum principle) holds for equation (1) and we write $(p, \tau, \mu) \in \mathcal{S}_{ac}(\mathcal{D})$ if for any function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory satisfying the inequalities

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} \geq p(t,x)u\big(\tau(t,x), \mu(t,x)\big) \quad \text{for a.e. } \ (t,x) \in \mathcal{D},$$

$$u(a,c) \geq 0, \quad \frac{\partial u(t,c)}{\partial t} \geq 0 \quad \text{for a.e. } \ t \in [a,b], \quad \frac{\partial u(a,x)}{\partial x} \geq 0 \quad \text{for a.e. } \ x \in [c,d],$$

the relation

$$u(t,x) \geq 0 \quad \text{for } \ (t,x) \in \mathcal{D} \tag{3}$$

holds.

It is also mentioned in [2] that under the assumption $(p, \tau, \mu) \in \mathcal{S}_{ac}(\mathcal{D})$, problem (1), (2) has a unique (Carathéodory) solution and this solution satisfies relation (3) provided $q(t,x) \geq 0$ for a.e. $(t,x) \in \mathcal{D}$, $\varphi(a) = \psi(c) \geq 0$ $\varphi'(t) \geq 0$ for a.e. $t \in [a,b]$, and $\psi'(x) \geq 0$ for a.e. $x \in [c,d]$. Moreover, some efficient conditions are given in [2] for the validity of the inclusion $(p, \tau, \mu) \in \mathcal{S}_{ac}(\mathcal{D})$ in the case, where $p(t,x) \geq 0$ for a.e. $(t,x) \in \mathcal{D}$. For the case where

$$p(t,x) \leq 0 \quad \text{for a.e. } \ (t,x) \in \mathcal{D}, \tag{4}$$

[1] This notion is introduced in [1].

we have presented a general sufficient condition in [2] guaranteeing the validity of the inclusion $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ under the assumption that equation (1) is delayed in both arguments, i. e., if the inequalities

$$|p(t,x)|(\tau(t,x)-t)\le 0,\quad |p(t,x)|(\mu(t,x)-x)\le 0\quad \text{for a. e. } (t,x)\in\mathcal{D} \tag{5}$$

hold. Using that general result, we have also proved in [2] that if p, μ, and τ satisfy conditions (4) and (5), then $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ provided

$$\iint\limits_{\mathcal{D}} |p(t,x)|\mathrm{d}t\mathrm{d}x\le 1. \tag{6}$$

Observe that assumption (5) is not restrictive in the case (4) because it is necessary as it is shown in [4]. Moreover, the number 1 on the right-hand side of inequality (6) is, in general, optimal (see [2, Example 6.2]). However, it does not mean that inequality (6) is necessary and cannot be weakened. Below, we give an efficient criteria for the validity of the inclusion $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ in the case (4) which are optimal for equations “close” to the equation without argument deviations

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x}=p(t,x)u(t,x)+q(t,x).$$

For this equation with a constant non-positive coefficient p the following proposition holds (see, e. g., [5, § 3.4] or [3, Example 8.1]).

Proposition 1. *Let $k\le 0$ and $p(t,x):=k$, $\tau(t,x):=t$, $\mu(t,x):=x$ for $(t,x)\in\mathcal{D}$. Then $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ if and only if*

$$|k|\le\frac{j_0^2}{4(b-a)(d-c)}\,, \tag{7}$$

where j_0 denotes the first positive zero of the Bessel function J_0.

Now for any $\nu>-1$, we denote by J_ν the Bessel function of the first kind and order ν and let j_ν be the first positive zero of the function J_ν. Moreover, we put

$$E_\nu(s):=\begin{cases} s^{-\nu}J_\nu(s) & \text{for } s>0,\\ 2^{-\nu}\dfrac{1}{\Gamma(1+\nu)} & \text{for } s=0,\end{cases}$$

where Γ is the gamma function of Euler.

Theorem 1. *Let $p\colon\mathcal{D}\to\mathbb{R}$ be a Lebesgue integrable function and $\tau\colon\mathcal{D}\to[a,b]$, $\mu\colon\mathcal{D}\to[c,d]$ be measurable functions satisfying conditions (4) and (5). Moreover, let there exist numbers $\lambda\in\,]0,1]$, $\alpha\in[0,1[$, and $\beta\in[0,\alpha]$ such that the inequalities*

$$\big[(t-a)(x-c)\big]^{1-\lambda}|p(t,x)|\le\frac{\lambda^2}{4}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}}\,, \tag{8}$$

$$\begin{aligned}\big[(t-a)(x-c)\big]^{1-\lambda}\Big(E_{-\alpha}\big(z(\tau(t,x),x)\big)-E_{-\alpha}\big(z(t,x)\big)\Big)|p(t,x)|\le\\ \le\frac{\lambda^2\beta}{2}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}}\,E_{1-\alpha}\big(z(t,x)\big),\end{aligned} \tag{9}$$

$$\begin{aligned}\big[(t-a)(x-c)\big]^{1-\lambda}\Big(E_{-\alpha}\big(z(t,\mu(t,x))\big)-E_{-\alpha}\big(z(t,x)\big)\Big)|p(t,x)|\le\\ \le\frac{\lambda^2(\alpha-\beta)}{2}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}}\,E_{1-\alpha}\big(z(t,x)\big)\end{aligned} \tag{10}$$

are fulfilled a. e. on $\mathcal{D}$, where

$$z(t,x) := j_{-\alpha}\Big[\frac{(t-a)(x-c)}{(b-a)(d-c)}\Big]^{\frac{\lambda}{2}} \quad \textit{for } (t,x) \in \mathcal{D}.$$

Then a theorem on differential inequalities holds for equation (1)*, i. e.,* $(p,\tau,\mu) \in \mathcal{S}_{ac}(\mathcal{D})$*.*

Observe that if $\tau(t,x) = t$ for a. e. $(t,x) \in \mathcal{D}$, then the left-hand side of inequality (9) is equal to zero. Therefore, assumption (9) of Theorem 1 describes how "close" must $\tau(t,x)$ be to t and this "closeness" is understood through the composition of the functions $E_{-\alpha}$ and z. Similarly, "closeness" of $\mu(t,x)$ to x is required in assumption (10). A meaning of assumptions (9) and (10) of Theorem 1 is more transparent in the following corollary.

Corollary 1. *Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions satisfying conditions* (4) *and* (5)*. Moreover, let there exist numbers $\alpha \in [0,1[$ and $\beta \in [0,\alpha]$ such that the inequalities*

$$|p(t,x)| \leq \frac{j_{-\alpha}^2}{4(b-a)(d-c)}\,,$$
$$(x-c)\big(t-\tau(t,x)\big)|p(t,x)| \leq \beta j_{-\alpha}^*, \quad (t-a)\big(x-\mu(t,x)\big)|p(t,x)| \leq (\alpha-\beta) j_{-\alpha}^*$$

are fulfilled a. e. on $\mathcal{D}$, where

$$j_{-\alpha}^* := \frac{E_{1-\alpha}(j_{-\alpha})}{E_{1-\alpha}(0)}\,.$$

Then $(p,\tau,\mu) \in \mathcal{S}_{ac}(\mathcal{D})$*.*

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An Axiomatic Definition for Smallness Classes in Lyapunov Exponents Theory

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Consider the linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with a piecewise continuous and bounded coefficient matrix A. Together with the system (1) consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with a piecewise continuous and bounded perturbation matrix Q. Denote the Cauchy matrix of (1) by X_A and the higher exponent of (2) – by $\lambda_n(A+Q)$.

One of the basic problem of Lyapunov exponents theory is to evaluate the quantity $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : Q \in \mathfrak{M}\}$ where $\mathfrak{M}$ is a *smallness class* of perturbations (see [1] and the references therein). The notion of smallness class is not precisely defined in general. The following classes are commonly used in this problem:

infinitesimal perturbations

$$Q(t) \to 0, \ \ t \to +\infty; \tag{3}$$

exponentially small perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)t), \quad C(Q) > 0, \ \ \sigma(Q) > 0; \tag{4}$$

σ-perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma t), \quad C(Q) > 0, \ \ \sigma > 0; \tag{5}$$

power perturbations

$$\|Q(t)\| \leq C(Q)t^{-\gamma}, \quad C(Q) > 0, \ \ \gamma > 0; \tag{6}$$

generalized power perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma\theta(t)), \quad C(Q) > 0, \ \ \sigma > 0, \tag{7}$$

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)\theta(t)), \quad C(Q) > 0, \ \ \sigma(Q) > 0, \tag{8}$$

where θ is a positive function satisfying some additional conditions;

infinitesimal average and integrable perturbations

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \|Q(t)\|\,dt = 0, \quad \int_0^{+\infty} \|Q(t)\|\,dt < +\infty, \tag{9}$$

and their modifications with some positive weights φ and powers $p \geq 1$

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \varphi(\tau)\|Q(\tau)\|^p\,d\tau = 0; \quad \int_0^{+\infty} \varphi(\tau)\|Q(t)\|^p\,dt < +\infty. \tag{10}$$

Generally to calculate $\Lambda(\mathfrak{M})$ we can construct an algorithm analogous to a famous Izobov algorithm for σ-exponent. Alternatively, in some cases (e.g., for classes (3), (4), (8)) we have formulas like the following Millionshcikov formula

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{k\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \left\| X_A(kT, kT-T) \right\|, \tag{11}$$

for the central exponent. A smallness class $\mathfrak{M}$ is said to be a limit class if $\Lambda(\mathfrak{M})$ has a representation similar to (11). Any other general criteria for a class of perturbation to be a limit class are not known. In order to find such criteria or conditions (necessary or sufficient) we need to have a general definition of a smallness class.

Let $\mathbb{R}^{n\times n}$ be the normed vector space of all real $n\times n$-matrices with some norm $\|\cdot\|$ and $\mathrm{PC}_n(\mathbb{R}^+)$ be the space of bounded piecewise continuous functions $Q:\mathbb{R}^+\to\mathbb{R}^{n\times n}$ where $\mathbb{R}^+ := [0,+\infty[$. The space $\mathrm{PC}_n(\mathbb{R}^+)$ with the operation of pointwise multiplication forms an algebra. We denote it by $\mathcal{K}_n$.

All commonly used classes (3)–(10) satisfy the following natural conditions.

(A0) $\mathfrak{M}\neq\varnothing$ and $\mathfrak{M}\neq\mathrm{PC}_n(\mathbb{R}^+)$.

(A1) The set $\mathfrak{M}$ is invariant with respect to Lyapunov transformations.

(A2) If $Q\in\mathfrak{M}$, $P\in\mathrm{PC}_n(\mathbb{R}^+)$, and $\|P(t)\|\le\|Q(t)\|$ for all $t\ge 0$, then $P\in\mathfrak{M}$.

The axioms (A0), (A1), and (A2) express the obvious necessary requirements to a smallness class. Unfortunately, these requirements are not sufficient. E.g., for an arbitrary nonempty $S\subset\mathbb{R}^+$ the set $I(S) := \{Q\in\mathrm{PC}_n(\mathbb{R}^+) : Q(t)=0\ \forall\, t\in S\}$ satisfies (A0), (A1), and (A2). It seems to be clear that $I(S)$ is not a smallness class in any sense. In order to eliminate these and some other improper classes, we propose two additional conditions.

(A3) There exists $Q\in\mathfrak{M}$ such that $Q(t)\neq 0$ for all $t\ge 0$.

(A4) $Q+P\in\mathfrak{M}$ for each $P,Q\in\mathfrak{M}$.

These conditions are valid for (3)–(10) too.

Definition 1. A set $\mathfrak{b}\subset\mathrm{PC}_1(\mathbb{R}^+)$, $\mathfrak{b}\neq\mathrm{PC}_1(\mathbb{R}^+)$, is said to be a one-dimensional smallness class if $\mathfrak{b}$ is an ideal of $\mathcal{K}_1$ and contains some strictly positive function.

Definition 2. A set $\mathfrak{M}\subset\mathrm{PC}_n(\mathbb{R}^+)$, $\mathfrak{M}\neq\mathrm{PC}_n(\mathbb{R}^+)$, is said to be a smallness class if $\mathfrak{M}=\mathfrak{b}^{n\times n}$ for some one-dimensional smallness class $\mathfrak{b}$.

Theorem. *Let $n\ge 2$. A set $\mathfrak{M}\subset\mathrm{PC}_n(\mathbb{R}^+)$ is a smallness class iff $\mathfrak{M}$ satisfies the axioms* (A0)–(A4).

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On the Solvability of Linear Overdetermined Boundary Value Problems for a Class of Functional Differential Equations

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1 Introduction

We consider here a system of functional differential equations (FDE, FDS) that, formally speaking, is a concrete realization of the so-called abstract functional differential equation (AFDE). Theory of AFDE is thoroughly treated in [1, 4, 3]. On the other hand, the system under consideration covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference) [6, 7, 8].

First we descript in detail a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. We concerned with the representation of general solution and basic relationships for the Cauchy operator. Next the setting of the general linear boundary value problem (BVP) is given and discussed, and conditions for the solvability of BVP are obtained in the case that it is not everywhere solvable. As for linear overdetermined BVP's for AFDE in general, the principal results by L. F. Rakhmatullina are given in detail in [1, 4, 3]. Here we propose a somewhat different approach without recourse to the adjoint BVP and an extension of the original BVP. Our approach is based, in essence, on the assumption that the space to which the derivative of the solution does belong is a Hilbert space.

Let us introduce the Banach spaces where operators and equations are considered.

Fix a segment $[0,T] \subset R$. By $L_2^n = L_2^n[0,T]$ we denote the Hilbert space of square summable functions $v : [0,T] \to R^n$ with the inner product $(u,v) = \int\limits_0^T u'(t)v(t)\,dt$ ($\cdot'$ is the symbol of transposition). The space $AC_2^n = AC_2^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ such that $\dot{x} \in L_2^n$ with the norm $\|x\|_{AC_2^n} = |x(0)| + \sqrt{(\dot{x},\dot{x})}$, where $|\cdot|$ stands for the norm of R^n.

In what follows we will use some results from [2, 5, 4] concerning the equation

$$\mathcal{L}x \equiv \dot{x} - \mathcal{K}\dot{x} - A(\cdot)x(0) = f, \tag{1}$$

where the linear bounded operator $\mathcal{K} : L_2^n \to L_2^n$ is defined by $(\mathcal{K}z)(t) = \int\limits_0^t K(t,s)z(s)\,ds$, $t \in [0,T]$, the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}(t,s)| \le u(t)v(s)$, $i,j = 1,\dots,n$, $u,v \in L_2^1[0,T]$, $n \times n$-matrix A has elements square summable on $[0,T]$.

Recall that the homogeneous equation (1) ($f(t) = 0$, $t \in [0,T]$) has the fundamental matrix $X(t)$ of dimension $n \times n$:

$$X(t) = E_n + V(t),$$

where E_n is the identity $n \times n$-matrix, each column $\nu_i(t)$ of the $n \times (n+mn)$-matrix $V(t)$ is a unique solution to the Cauchy problem

$$\dot{\nu}(t) = \int\limits_0^t K(t,s)\dot{\nu}(s)\,ds + a_i(t), \quad \nu(0) = 0, \quad t \in [0,T],$$

$a_i(t)$ is the i-th column of the matrix A.

The solution of (1) with the initial condition $x(0) = 0$ has the representation $x(t) = (Cf)(t) = \int_0^t C(t,s)f(s)\,ds$, where $C(t,s)$ is the Cauchy matrix of the operator $\mathcal{L}$. This matrix can be defined (and constructed) as the solution to

$$\frac{\partial}{\partial t} C(t,s) = \int_s^t K(t,\tau) \frac{\partial}{\partial \tau} C(\tau,s)\,d\tau + K(t,s), \;\; 0 \le s \le t \le T,$$

under the condition $C(s,s) = E_n$.

The matrix $C(t,s)$ is expressed in terms of the resolvent kernel $R(t,s)$ of the kernel $K(t,s)$. Namely, $C(t,s) = E_n + \int_s^t R(\tau,s)\,d\tau$. The general solution of (1) has the form $x(t) = X(t)\alpha + \int_0^t C(t,s)f(s)\,ds$, with arbitrary $\alpha \in R^{n+mn}$.

2 General Linear Boundary Value Problem

The general linear BVP is the system (1) supplemented by linear boundary conditions

$$\ell x = \gamma, \;\; \gamma \in R^N, \tag{2}$$

where $\ell : AC^n \to R^N$ is the linear bounded vector functional. Let us recall the representation of ℓ:

$$\ell x = \int_0^T \Phi(s)\dot{x}(s)\,ds + \Psi x(0). \tag{3}$$

Here Ψ is a constant $N \times n$-matrix, Φ is $N \times n$ matrix with square summable on $[0,T]$ elements. We assume that the components $\ell^i : AC_2^n \to R$, $i = 1, \ldots, N$ are linearly independent.

BVP $(1),(2)$ is well-defined if $N = n$. In such a situation, BVP $(1),(2)$ is uniquely solvable for any f, γ if and any if the matrix

$$\ell X = (\ell X^1, \ldots, \ell X^n),$$

where X^j is the j-th column of X, is nonsingular, i.e. $\det \ell X \neq 0$. It should be noted that this condition cannot be verified immediately because the fundamental matrix X cannot be (as a rule) evaluated explicitly. In addition, even if X were known, then the elements of ℓX, generally speaking, could not be evaluated explicitly. By the theorem about inverse operators, the matrix ℓX is invertible if one can find an invertible matrix Γ such that $\|\ell X - \Gamma\| < 1/\|\Gamma^{-1}\|$. As it has been shown in [9], such a matrix Γ for the invertible matrix ℓX always can be found among the matrices $\Gamma = \overline{\ell X}$, where $\bar{\ell} : AC_2^n \to R^n$ is a vector-functional near to ℓ, and $\overline{X}$ is an approximation of X. That is why the basis of the so-called constructive study of linear BVP's includes a special technique of approximate constructing the solutions to FDE with guaranteed explicit error bounds as well as the reliable computing experiment (RCE), whose theory has been worked out in [7, 9, 8].

We assume in the sequel that $N > n$ and the system $\ell^i : AC_2^n \to R$, $i = 1, \ldots, N$ can be splitted into two subsystems $\ell_1 : AC_2^n \to R^n$ and $\ell_2 : AC_2^n \to R^{N-n}$ such that the BVP

$$\mathcal{L}x = f, \;\; \ell_1 x = \gamma_1 \tag{4}$$

is uniquely solvable. Without loss of generality we will consider that ℓ_1 is formed by first n components of ℓ and the elements of γ_1 in (4) are the corresponding components of γ. Thus ℓ_2 will stand for the final $(N-n)$ components of ℓ, and elements of $\gamma_2 \in R^{N-n}$ are defined as the final $(N-n)$ components of γ.

Define the $(N-n)\times n$-matrix $B(s)$ with square summable elements by the representation

$$\ell_2 Cf - (\ell_2 X)(\ell_1 X)^{-1}\ell_1 Cf = \int\limits_0^T B(s)f(s)\,ds \tag{5}$$

for all $f \in L_2^n$. An explicit form of B is simple to derive by elementary transformations taking into account (3) and the properties of the Cauchy matrix $C(t,s)$.

Theorem. *Let the matrix* $W = \int\limits_0^T B(s)B'(s)\,ds$*, where* B *is defined by* (5)*, be nonsingular. Then BVP* (1), (2) *is solvable for all* f *of the form*

$$f(t) = f_0(t) + \varphi(t),$$

where

$$f_0(t) = B'(t)\big[W^{-1}\gamma_2 - W^{-1}(\ell_2 X)(\ell_1 X)^{-1}\gamma_1\big],$$

$\varphi(\,\cdot\,) \in L_2^n$ *is arbitrary function orthogonal to each column of* $B'(\,\cdot\,)$: $\int\limits_0^T B(s)\varphi(s)\,ds = 0.$

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Generalized Linear Differential Equations in a Banach Space (Extension of the Opial Continuous Dependence Result)

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Our aim is to present new conditions ensuring the continuous dependence on a parameter k of solutions to linear integral equations of the form

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]\, x + f_k(t) - f_k(a), \quad t \in [a,b], k \in \mathbb{N}\,, \tag{eq_k}$$

where $-\infty < a < b < \infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ have bounded variations on $[a,b]$, $f_k : [a,b] \to X$ are regulated on $[a,b]$. The integrals are understood as the abstract Kurzweil–Stieltjes integrals and the studied equations are usually called generalized linear differential equations (in the sense of J. Kurzweil, cf. [3] or [4]). Basic results on the theory of Kurzweil–Stieltjes integral in abstract spaces can be found, for example, in [5] and [10].

Continuing in our research from [6], here we focus our attention on the case when the variations $\operatorname{var}_a^b A_k$ need not be uniformly bounded. More precisely, here we extend Theorem 4.2 from [6], which is an analogy of the Opial's result [7] for ODEs. The new result reads as follows:

Main Theorem. *Assume:* $A_k \in \mathrm{BV}([a,b], L(X))$, $f_k \in G([a,b], X)$, $\widetilde{x}_k \in X$ *for* $k \in \mathbb{N}$,

- $A \in \mathrm{BV}([a,b], L(X))$, $f \in \mathrm{BV}([a,b], X)$, $\widetilde{x} \in X$,
- $\big[I - \Delta^- A(t)\big]^{-1} \in L(X)$ *for* $t \in (a,b]$,
- $\widetilde{x}_k \to \widetilde{x}$,
- $\lim\limits_{k\to\infty} \big(1 + \operatorname{var}_a^b A_k\big)\|A_k - A\|_\infty = 0$,
- $\lim\limits_{k\to\infty} \big(1 + \operatorname{var}_a^b A_k\big)\|f_k - f\|_\infty = 0$.

Then the equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]x + f(t) - f(a), \quad t \in [a,b], \tag{eq}$$

has a unique solution $x \in \mathrm{BV}([a,b], X)$, *equation* (eq_k) *has a unique solution and* $x_k \in G([a,b], X)$ *for* $k \in \mathbb{N}$ *sufficiently large and* $x_k \rightrightarrows x$ *on* $[a,b]$.

The proof relies on the following lemma which is an analogue of the assertion formulated for ODEs by Kiguradze in [2, Lemma 2.5]. Its variant was used also for FDEs by Hakl, Lomtatidze and Stavrolaukis in [1, Lemma 3.5].

Lemma. *Assume:* $A_k \in \mathrm{BV}([a,b], L(X))$ *for* $k \in \mathbb{N}$,

- $A \in \mathrm{BV}([a,b], L(X))$ *with* $\big[I - \Delta^- A(t)\big]^{-1} \in L(X)$ *for* $t \in (a,b]$,
- $\lim\limits_{k\to\infty} \big(1 + \operatorname{var}_a^b A_k\big)\|A_k - A\|_\infty = 0$.

Then there exist $r^*>0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\|y\|_\infty \le r^* \Big(\|y(a)\|_X + \big(1+\operatorname{var}_a^b A_k\big) \sup_{t\in[a,b]} \Big\| y(t)-y(a)-\int_a^t \mathrm{d}[A_k]y \Big\|_X \Big)$$

for all $y \in G([a,b],X)$ *and* $k \ge k_0$.

In the sequel, we present the example that shows that the condition $f \in \mathrm{BV}([a,b],X)$ in Main Theorem can not be avoided.

Example. Let $[a,b]=[0,1]$. For $k \in \mathbb{N}$ put

$$n_k = [k^{3/2}]+1, \quad \tau_{m,k} = \frac{1}{2^{n_k-m}} \quad \text{for } m \in \{0,1,\dots,n_k\}$$

(where $[\lambda]$ stands, as usual, for the integer part of the nonnegative real number λ),

$$\begin{aligned} a_{0,k} &= (-1)^{n_k} \frac{2^{n_k}}{k}, \quad b_{0,k} = (-1)^{n_k-1} \frac{1}{k}, \\ a_{m,k} &= (-1)^{n_k-m} \frac{2^{n_k-m+1}}{k}, \quad b_{m,k} = (-1)^{n_k-m+1} \frac{3}{k} \quad \text{for } m \in \{1,2,\dots,n_k-1\}, \end{aligned}$$

and define

$$\begin{aligned} A_k(t) &= \begin{cases} 0 & \text{for } t \in [0,\tau_0^k], \\ a_{m,k}t+b_{m,k} & \text{for } t \in [\tau_{m,k},\tau_{m+1,k}] \text{ and } m \in \{0,1,\dots,n_k-1\}, \end{cases} \\ A(t) &= 0 \quad \text{for } t \in [0,1]. \end{aligned}$$

We can verify that $\operatorname{var}_0^1 A_k<\infty$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big(1+\operatorname{var}_a^b A_k\big)\|A_k-A\|_\infty \le \lim_{k\to\infty} \Big(\frac{1}{k}+\frac{2}{\sqrt{k}}\Big) = 0.$$

Consider the function

$$f(t) = \begin{cases} \dfrac{(-1)^k}{\sqrt[4]{n}} & \text{if } t \in (2^{-n},2^{-(n-1)}] \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } t=0, \end{cases}$$

and define $f_k(t)=f(t)$ for $t \in [0,1]$ and $k \in \mathbb{N}$. Note that, the conditions of the Main Theorem are satisfied, except for the fact that $\operatorname{var}_0^1 f=\infty$. However, it is possible to prove that the sequence of solutions x_k of (eq_k) does not converge to the solution of (eq). Roughly speaking, we can observe that, for each $k \in \mathbb{N}$, $x_k(1)$ involves partial sums of divergent series $\sum\limits_{m=2}^{\infty} \frac{1}{\sqrt[4]{m}}$. Therefore, $x_k(1)$ cannot have a finite limit for $k\to\infty$.

Applications to linear dynamic equations on time scales are then enabled by their relationship with generalized differential equations as disclosed by A. Slavík in [8].

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Nonlocal Boundary Value Problem for Strongly Singular Higher-Order Linear Functional-Differential Equations

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Consider the differential equations with deviating arguments

$$u^{(2m+1)}(t)=\sum_{j=0}^{m}p_j(t)u^{(j)}(\tau_j(t))+q(t)\quad\text{for }\ a<t<b, \tag{1}$$

with the boundary conditions

$$\begin{gathered}\int_a^b u(s)\,d\varphi(s)=0\quad\text{where }\ \varphi(b)-\varphi(a)\neq 0,\\ u^{(i)}(a)=0,\quad u^{(i)}(b)=0\ \ (i=1,\dots,m).\end{gathered} \tag{2}$$

Here $m\in N$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=0,\dots,m)$, $\varphi:[a,b]\to R$ is a function of bounded variation, and $\tau_j:]a,b[\to]a,b[$ are measurable functions. By $u^{(i)}(a)$ (resp., $u^{(i)}(b)$), we denote the right (resp., left) limit of the function $u^{(i)}$ at the point a (resp., b).

The Agarwal–Kiguradze type theorems [1] are obtained by us, which contains unimprovable in a certain sense conditions guaranteeing the unique solvability of problem (1), (2).

We use the following notations.

$R^+=[0,+\infty[$;

$[x]_+$ is the positive part of a number x, that is $[x]_+=\frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ is the space of functions $y:]a,b[\to R$, which are integrable on $[a+\varepsilon,b-\varepsilon]$ for arbitrary small $\varepsilon>0$;

$L_{\alpha,\beta}(]a,b[)$ ($L^2_{\alpha,\beta}(]a,b[)$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a,b[\to R$, with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int_a^b(s-a)^\alpha(b-s)^\beta|y(s)|\,ds\quad\Big(|y\|_{L^2_{\alpha,\beta}}=\Big(\int_a^b(s-a)^\alpha(b-s)^\beta y^2(s)\,ds\Big)^{1/2}\Big);$$

$L([a,b])=L_{0,0}(]a,b[)$, $L^2([a,b])=L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau:]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ is the Banach space of functions $y\in L_{loc}(]a,b[)$ such that

$$\begin{aligned}\|y\|_{\widetilde{L}^2_{\alpha,\beta}}:=\max\bigg\{\bigg[\int_a^t(s-a)^\alpha\Big(\int_s^t y(\xi)\,d\xi\Big)^2ds\bigg]^{1/2}:\ a\le t\le\frac{a+b}{2}\bigg\}+\\ +\max\bigg\{\bigg[\int_t^b(b-s)^\beta\Big(\int_t^s y(\xi)\,d\xi\Big)^2ds\bigg]^{1/2}:\ \frac{a+b}{2}\le t\le b\bigg\}<+\infty.\end{aligned}$$

$\widetilde{C}^n_{loc}(]a,b[)$ is the space of functions $y :]a,b[\to R$ which are absolutely continuous together with $y', y'', \dots, y^{(n)}$ on $[a+\varepsilon, b-\varepsilon]$ for an arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{\,n,m}(]a,b[)$ $(m \le n)$ is the space of functions $y \in \widetilde{C}^n_{loc}(]a,b[)$, satisfying

$$\int\limits_a^b |y^{(m)}(s)|^2\, ds < +\infty. \tag{3}$$

When problem (1), (2) is discussed, we assume that the conditions

$$p_j \in L_{loc}(]a,b[) \ \ (j = 0, \dots, m) \tag{4}$$

are fulfilled.

A solution of problem (1), (2) is sought for in the space $\widetilde{C}^{\,2m,m+1}(]a,b[)$.

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1, \dots, m)$ we denote the functions and, respectively, the operators defined by the equalities

$$\begin{aligned} h_1(t,s) &= \bigg| \int\limits_s^t \big[(-1)^m p_1(\xi)\big]_+ \, d\xi \bigg|, \\ h_j(t,s) &= \bigg| \int\limits_s^t p_j(\xi)\, d\xi \bigg| \ \ (j = 2, \dots, m), \end{aligned} \tag{5}$$

and,

$$f_j(c,\tau_j)(t,s) = \bigg| \int\limits_s^t |p_j(\xi)| \, \bigg| \int\limits_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}\, d\xi_1 \bigg|^{1/2} d\xi \bigg| \ \ (j = 1, \dots, m), \tag{6}$$

and also we put that

$$f_0(t,s) = \bigg| \int\limits_s^t |p_0(\xi)|\, d\xi \bigg|.$$

Let $m = 2k+1$, then

$$m!! = \begin{cases} 1 & \text{for } \ m \le 0 \\ 1 \cdot 3 \cdot 5 \cdots m & \text{for } \ m \ge 1 \end{cases}.$$

Along with (1), we consider the homogeneous equation

$$v^{(2m+1)}(t) = \sum_{j=0}^m p_j(t) v^{(j)}(\tau_j(t)) \quad \text{for} \quad a < t < b. \tag{1_0}$$

Theorem 1. *Let there exist $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, numbers $l_{kj} > 0$, $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ $(k = 0,1;$ $j = 1, \dots, m)$ such that*

$$\begin{gathered} (t-a)^{2m-j} h_j(t,s) \le l_{0j} \quad (j = 1, \dots, m) \ \ \textit{for} \ \ a < t \le s \le a_0, \\ \limsup_{t \to a} \, (t-a)^{m-\frac12-\gamma_{00}} f_0(t,s) < +\infty, \\ \limsup_{t \to a} \, (t-a)^{m-\frac12-\gamma_{0j}} f_j(a,\tau_j)(t,s) < +\infty \ \ (j = 1, \dots, m), \end{gathered} \tag{7}$$

$$\begin{gathered} (b-t)^{2m-j} h_j(t,s) \le l_{1j} \quad (j = 1, \dots, m) \ \ \textit{for} \ \ b_0 \le s \le t < b, \\ \limsup_{t \to b} \, (b-t)^{m-\frac12-\gamma_{10}} f_0(t,s) < +\infty, \\ \limsup_{t \to b} \, (b-t)^{m-\frac12-\gamma_{1j}} f_j(b,\tau_j)(t,s) < +\infty \ \ (j = 1, \dots, m), \end{gathered} \tag{8}$$

and

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, l_{kj} < 1 \ \ (k=0,1). \tag{9}$$

Let, moreover, the homogeneous problem $(1_0),(2)$ *have only the trivial solution in the space* $\widetilde{C}^{2m,m+1}(]a,b[)$. *Then problem* $(1),(2)$ *has a unique solution* u *for an arbitrary* $q\in\widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, *and there exists a constant* r, *independent of* q, *such that*

$$\|u^{(m+1)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2m-2,2m-2}}. \tag{10}$$

Theorem 2. *Let there exist numbers* $t^* \in]a,b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \ge 0$, *and* $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ $(k=0,1;\ j=1,\dots,m)$ *such that along with*

$$B_0 \equiv \bar{l}_{00}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}}\, \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}\, d\xi +$$
$$+\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\,\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big) < \frac{1}{2}\,, \tag{11}$$

$$B_1 \equiv \bar{l}_{10}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}}\, \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}\, d\xi +$$
$$+\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1} l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\,\bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\Big) < \frac{1}{2}\,, \tag{12}$$

the conditions

$$\begin{gathered}(t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \le \bar{l}_{00},\\ (t-a)^{2m-j} h_j(t,s) \le l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j}\end{gathered} \tag{13}$$

for $a < t \le s \le t^*$ *and*

$$\begin{gathered}(b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \le \bar{l}_{10},\\ (b-t)^{2m-j} h_j(t,s) \le l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \le \bar{l}_{1j}\end{gathered} \tag{14}$$

for $t^* \le s \le t < b$ *hold with any* $j=1,\dots,m$. *Then problem* $(1),(2)$ *is uniquely solvable in the space* $\widetilde{C}^{2m,m+1}(]a,b[)$ *for every* $q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$.

Acknowledgement

This work is supported by the RVO: 67985840 and by the Shota Rustaveli National Science Foundation (Project # GNSF/ST09_175_3-101).

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Myshkis Type Oscillation Criteria for Second-Order Linear Delay Differential Equations

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On the half-line $\mathbb{R}_+ = [0,+\infty[$ we consider the second-order differential equation

$$u''(t) + p(t)u(\tau(t)) = 0, \tag{1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function and $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\tau(t) \le t \quad \text{for } t \ge 0, \qquad \lim_{t\to+\infty} \tau(t) = +\infty.$$

Solutions to equation (1) can be defined in various ways. Since we are interested in properties of solutions in a neighbourhood of $+\infty$, we introduce the following commonly used definition.

Definition 1. Let $t_0 \in \mathbb{R}_+$ and $a_0 = \min\{\tau(t) :\ t \ge t_0\}$. A continuous function $u\colon [a_0,+\infty[\to \mathbb{R}$ is said to be a *solution to equation* (1) *on the interval* $[t_0,+\infty[$ if it is absolutely continuous together with its first derivative on every compact interval contained in $[t_0,+\infty[$ and satisfies equality (1) almost everywhere in $[t_0,+\infty[$. A solution u to equation (1) on the interval $[t_0,+\infty[$ is called *proper* if the inequality $\sup\big\{|u(s)| :\ s \ge t\big\} > 0$ holds for $t \ge t_0$.

Definition 2. A proper solution to equation (1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

We have proved in [3, Proposition 2.1] that if $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s < +\infty$, then (1) has a proper non-oscillatory solution. Therefore, we assume in what follows that $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s = +\infty$. Moreover, we consider the case where

$$\int_0^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s < +\infty \tag{2}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t s\tau(s)p(s)\,\mathrm{d}s \le 1, \quad \limsup_{t\to+\infty} t \int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s \le 1, \tag{3}$$

because otherwise every proper solution to equation (1) is oscillatory (see, e. g., [2, Corollaries 3.4 and 3.5]).

In the paper [1], R. Koplatadze proved, among other things, the following oscillation criteria.

Criterion 1 ([1, Theorem 1])**.** *Let there exist a continuous non-decreasing function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequalities*

$$\tau(t) \le \sigma(t) \le t \quad \textit{for } t \ge 0 \tag{4}$$

and

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^{t} \tau(s)p(s)\,\mathrm{d}s > 1 \tag{5}$$

are fulfilled. Then every proper solution to equation (1) *is oscillatory.*

Criterion 2 ([1, Theorem 2])**.** *Let the inequality*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} \tau(s)p(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{6}$$

hold. Then every proper solution to equation (1) *is oscillatory.*

Remark 1. In Criterion 2, the constant $\frac{1}{\mathrm{e}}$ is optimal and can not be in general improved. A counterexample is constructed in [1] for equation (1) with a proportional delay.

Below we present new Myshkis type oscillation criteria for equation (1), which generalise known results of R. Koplatadze. In particular, under some natural additional assumptions, we improve constants on the right-hand side of inequalities (5) and (6).

Let

$$G_* := \liminf_{t\to+\infty} \frac{1}{t} \int_0^t s\tau(s)p(s)\,\mathrm{d}s, \quad F_* := \liminf_{t\to+\infty} t \int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s.$$

In view of assumption (2), the number F_* is well defined and, moreover, assumptions (3) yield that $G_* \le 1$ and $F_* \le 1$. We shall also assume in what follows that

$$\int_0^{+\infty} s^{\lambda} \Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s < +\infty \quad \text{for all } \lambda < 1,\ \ \varepsilon \in [0, 1[\,, \tag{7}$$

because otherwise every proper solution to equation (1) is oscillatory without any additional condition (see [3, Theorem 2.4]). It allows one to define, for any $\lambda < 1$ and $\varepsilon \in [0, 1[$, the function

$$Q(t; \lambda, \varepsilon) := t^{1-\lambda} \int_t^{+\infty} s^{\lambda} \Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

Moreover, for any $\mu > 1$ and $\varepsilon \in [0, 1[$, we put

$$H(t; \mu, \varepsilon) := \frac{1}{t^{\mu-1}} \int_0^t s^{\mu} \Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

Corollaries 2.11 and 2.12 stated in [3] claim that every proper solution to equation (1) is oscillatory provided that for some $\lambda < 1$, $\mu > 1$, and $\varepsilon \in [0, 1[$,

$$\text{either } Q_*(\lambda, \varepsilon) > \frac{1}{4(1-\lambda)} \ \text{ or } \ H_*(\mu, \varepsilon) > \frac{1}{4(\mu-1)}\,,$$

where
$$Q_*(\lambda,\varepsilon) := \liminf_{t\to+\infty} Q(t;\lambda,\varepsilon), \quad H_*(\mu,\varepsilon) := \liminf_{t\to+\infty} H(t;\mu,\varepsilon).$$
Therefore, it is natural to restrict ourself to the case where
$$Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}, \quad H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)} \quad \text{for all } \lambda < 1, \ \mu < 1, \ \varepsilon \in [0,1[\,.$$
Under these assumptions, we can improve Criteria 1 and 2, for example, as follows.

Theorem 1. *Let there exist numbers $\lambda < 1$, $\mu > 1$, $\varepsilon \in [0,1[$ and a non-decreasing function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that condition* (4) *is fulfilled,*
$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}\,, \quad \frac{\mu(2-\mu)}{4(\mu-1)} \le H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)}\,, \tag{8}$$
and
$$\limsup_{t\to+\infty} \int\limits_{\sigma(t)}^{t} \tau(s)p(s)\Big(\frac{\sigma(s)}{\tau(s)}\Big)^{\varepsilon G_*}\,\mathrm{d}s > R_0 - \alpha_* r_0,$$
where
$$r_0 := \frac{1}{2}\Big(1 - \sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)}\Big), \quad R_0 := \frac{1}{2}\Big(1 + \sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\Big), \tag{9}$$
and $\alpha_* := \liminf\limits_{t\to+\infty} (\frac{\sigma(t)}{t})^{1-\varepsilon F_*}$. Then every proper solution to equation (1) *is oscillatory.*

Remark 2. Observe that $0 \le \alpha_* \le 1$ and $\max\{\frac{\lambda}{2},0\} \le r_0 \le \frac{1}{2} \le R_0 \le \min\{\frac{\mu}{2},1\}$. Therefore, we have $R_0-\beta_* r_0 \le 1$ and thus Theorem 1 improves (under additional assumptions (8)) Criterion 1.

Theorem 2. *Let there exist numbers $\lambda < 1$, $\mu > 1$, and $\varepsilon \in [0,1[$ such that inequalities* (8) *are satisfied and*
$$\liminf_{t\to+\infty} \frac{t}{\tau(t)} < +\infty, \quad \liminf_{t\to+\infty} \tau^{\varepsilon G_*}(t) \int\limits_{\tau(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)\,\mathrm{d}s > R_0 - \beta_* r_0,$$
where the numbers r_0 and R_0 are given by relations (9) *and $\beta_* := \liminf\limits_{t\to+\infty} (\frac{\tau(t)}{t})^{\varepsilon G_*}$. Then every proper solution to equation* (1) *is oscillatory.*

Remark 3. Observe that $0 \le \beta_* \le 1$, the numbers r_0 and R_0 given by relations (9) satisfy
$$R_0 - r_0 = \frac{1}{2}\Big(\sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)} + \sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\Big),$$
and thus the difference $R_0 - r_0$ converges to zero if $Q_*(\lambda,\varepsilon) \to \frac{1}{4(1-\lambda)}$ and $H_*(\mu,\varepsilon) \to \frac{1}{4(\mu-1)}$. Consequently, it may happen that $R_0 - \beta_* r_0 < \frac{1}{\mathrm{e}}$ in which case Theorem 2 improves Criterion 2.

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A Priori Estimates of the Kneser Solutions of Singular in Time and Phase Variables Second Order Differential Inequalities

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In a positive semi-axis $]0,+\infty[$, we consider the second order differential inequalities

$$u''(t) \geq \frac{p(t)}{q(u(t))} \tag{1}$$

and

$$\frac{p(t)}{q(u(t))} \leq u''(t) \leq \frac{p_0(t)}{q_0(u(t))}\,, \tag{2}$$

where $p,\ p_0\ :]0,+\infty[\,\to [0,+\infty[$ are measurable and $q,\ q_0\ :\ [0,+\infty[\,\to [0,+\infty[$ are continuous nondecreasing functions such that

$$0 < \int\limits_t^{+\infty} (s-t)p(s)\,ds \leq \int\limits_t^{+\infty} (s-t)p_0(s)\,ds < +\infty \quad \text{for } t \geq 0,$$

$$q_0(x) \geq q(x) > 0 \quad \text{for } x > 0.$$

A nonincreasing function $u : [0,+\infty[\,\to]0,+\infty[$ is said to be the Kneser solution of the differential inequality (1) (of the differential inequality (2)) if it is absolutely continuous together with u' on every finite interval contained in $[0,+\infty[$, and satisfies this differential inequality almost everywhere on $]0,+\infty[$.

The Kneser solution u of the differential inequality (1) or (2) is said to be vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0.$$

Let

$$Q(x) = \int\limits_0^x q(y)\,dy \quad \text{for } x > 0,$$

and let Q^{-1} be the inverse function to Q. Suppose

$$r(t) = Q^{-1}\bigg(\int\limits_t^{+\infty} (s-t)q(s)\,ds\bigg) \quad \text{for } t \geq 0.$$

The following theorems are proved.

Theorem 1. *Every Kneser solution of the differential inequality* (1) *admits the estimate*

$$u(t) \geq r(t) \quad \textit{for } t \geq 0.$$

Theorem 2. *If*

$$r_0(t) = \int_t^{+\infty} (s-t)\,\frac{p_0(s)}{q_0(r(s))}\,ds < +\infty \quad \textit{for } \ t \geq 0,$$

then every vanishing at infinity Kneser solution of the differential inequality (2) *admits the estimates*

$$r(t) \leq u(t) \leq r_0(t) \quad \textit{for } \ t \geq 0.$$

Note that the above formulated theorems cover the case, where $q(0) = q_0(0) = 0$ and

$$\int_0^t p(s)\,ds = +\infty, \quad \int_0^t p_0(s)\,ds = +\infty \quad \text{for } \ t > 0,$$

i.e. the case, where the differential inequalities (1) and (2) have singularities both in time and phase variables.

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

On Kneser Solutions of Second Order Nonlinear Singular Differential Equations

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We consider the problem on the existence of a solution of the differential equation

$$u'' = f(t, u, u'), \tag{1}$$

satisfying the conditions

$$u(0) = c, \quad u(t) > 0, \quad u'(t) < 0 \ \text{ for } \ t > 0. \tag{2}$$

Here $f : D \to R_+$ is a continuous function,

$$D = \big\{(t, x, y) : \ t > 0, \ x > 0, \ y < 0\big\}, \quad R_+ = [0, +\infty[\,,$$

and c is a positive constant.

This problem is often called the Kneser problem since it was first studied by Kneser in the case where $f(t, x, y) \equiv f_0(t, x)$ and $f_0 : R_+ \times R_+ \to R_+$ is a continuous function such that $f_0(t, 0) \equiv 0$.

We are interested in the case where the inequality

$$g_0(t) \le x^\lambda |y|^\mu f(t, x, y) \le g_1(t) \tag{3}$$

is satisfied in the domain D, where λ and μ are positive constants, and $g_i : (0, +\infty) \to (0, +\infty)$ $(i = 0, 1)$ are continuous functions. In this case

$$\lim_{x\to 0} f(t, x, y) = +\infty, \quad \lim_{y\to 0} f(t, x, y) = +\infty,$$

i.e. equation (1) has singularities in phase variables.

The Kneser problem for differential equations with a singularity in one of the phase variables first was investigated by I. Kiguradze [1]. However, in [1] the Kneser problem is considered not for general but for the Emden–Fowler type higher order differential equation

$$u^{(n)} = p(t)u^{-\lambda}.$$

A continuous function $u : [0, +\infty) \to (0, +\infty)$ is said to be the Kneser solution of equation (1) if it is twice continuously differentiable on the open interval $(0, +\infty)$ and satisfies the inequality

$$u'(t) < 0$$

and equation (1) on that interval.

The Kneser solution is called vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0,$$

and it is called remote from zero if

$$\lim_{t\to+\infty} u(t) > 0.$$

The following theorems are valid.

Theorem 1. *Let inequality* (3) *be fulfilled. Then for the existence of at least one Kneser solution of equation* (1) *it is necessary the conditions*

$$\int\limits_t^{+\infty} g_0(s)\,ds < +\infty \ \textit{for}\ t > 0, \quad \int\limits_0^{+\infty}\bigg(\int\limits_t^{+\infty} g_0(s)\,ds\bigg)^{\frac{1}{\mu+1}} dt < +\infty \tag{4}$$

to be fulfilled. In addition, if conditions (4) *hold, then for any Kneser solution of equation* (1) *the estimate*

$$u(t) > v_0(t;\delta) \ \textit{ for } \ t \geq 0$$

is valid, where

$$v_0(t;\delta) = \bigg[\delta^\nu + (1+\mu)^{\frac{1}{1+\mu}}\nu \int\limits_t^{+\infty}\bigg(\int\limits_s^{+\infty} g_0(x)\,dx\bigg)^{\frac{1}{1+\mu}} ds\bigg]^{\frac{1}{\nu}},$$

$$\delta = u(+\infty), \quad \nu = \frac{1+\nu+\mu}{1+\mu}\,.$$

Theorem 2. *If*

$$\int\limits_t^{+\infty} g_1(s)\,ds < +\infty \ \textit{for}\ t > 0, \quad \int\limits_0^{+\infty}\bigg(\int\limits_t^{+\infty} g_1(s)\,ds\bigg)^{\frac{1}{\mu+1}} dt < +\infty, \tag{5}$$

then for each positive number δ *equation* (1) *has at least one Kneser solution satisfying the equality*

$$\lim_{t\to+\infty} u(t) = \delta. \tag{6}$$

Theorem 3. *If along with* (3) *and* (5) *the conditions*

$$\int\limits_t^{+\infty} \frac{g_1(s)}{v_0^\lambda(s,0)}\,ds < +\infty \ \textit{ for } \ t > 0, \quad \int\limits_0^{+\infty}\bigg(\int\limits_t^{+\infty} \frac{g_1(s)}{v_0^\lambda(s,0)}\,ds\bigg)^{\frac{1}{\mu+1}} dt < +\infty$$

are satisfied, then equation (1) *has at least one vanishing at infinity Kneser solution.*

Suppose conditions (5) hold. We introduce the function

$$v_1(t;\delta) = \delta + \bigg[(1+\mu)\int\limits_t^{+\infty}\bigg(\int\limits_s^{+\infty} \frac{g_1(x)}{v_0^\lambda(x;\delta)}\,dx\bigg)^{\frac{1}{\mu+1}} ds\bigg], \ \ t \geq 0, \ \ \delta > 0,$$

and the constant

$$c_0 = \inf\big\{v_1(0;\delta) : \ \delta > 0\big\}.$$

Theorem 4. *Let conditions* (3) *and* (5) *be fulfilled. If, moreover,*

$$c > c_0,$$

then problem (1), (2) *has at least one solution, and if*

$$c \leq v_0(0;0),$$

then problem (1), (2) *has no solution.*

Remark. In the case where conditions (5) hold and

$$c \in (v_0(0;0), c_0],$$

then the question on the solvability of problem (1), (2) remains open.

Consider now the case, where $g_1(t) \equiv \ell g_0(t)$, $\ell = const \geq 1$, i.e. the case where inequality (3) has the form

$$g_0(t) \leq x^\lambda |y|^\mu f(t,x,y) \leq \ell g_0(t), \tag{7}$$

where, as above, $g_0 : (0,+\infty) \to (0,+\infty)$ is a continuous function and λ, μ are positive constants.

From Theorems 1, 2, and 4 it follows

Corollary 1. *Let inequality* (7) *be fulfilled. Then the following assertions are equivalent:*

(A) *Conditions* (4) *are satisfied;*

(B) *Equation* (1) *has at least one remote from zero Kneser solution;*

(C) *Problem* (1), (6) *is solvable for each positive number* δ*;*

(D) *Problem* (1), (2) *is solvable for large* c*.*

Corollary 2. *Let inequality* (7) *be fulfilled, where*

$$g_0(t) = \begin{cases} \gamma\, t^{-\alpha} & \text{for } 0 < t \leq 1, \\ \gamma\, t^{-\beta} & \text{for } t > 1, \end{cases}$$

$\gamma > 0$, $\alpha = const$, $\beta = const$. *Then the following assertions are equivalent:*

(A) $\alpha < 2 + \mu$, $\beta > 2 + \mu$;

(B) *Equation* (1) *has at least one remote from zero Kneser solution;*

(C) *Equation* (1) *has at least one vanishing at infinity Kneser solution;*

(D) *Problem* (1), (6) *is solvable for each positive number* δ*;*

(E) *Problem* (1), (2) *is solvable for large* c*.*

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

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New Results on Preservation of Invariant Tori of Nonlinear Multi-Frequency Systems

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One of the important issues of a theory of multi-frequency oscillations is roughness of invariant manifold and its preservation under small perturbations [6]. In numerous papers (e.g. [1, § 10]) a direct Lyapunov method was utilized for the investigations of roughness of invariant toroidal manifold. It was proved that a sufficiently small perturbation of right-hand side of system do not ruin the invariant torus.

Here were have established new conditions for the preservation of the asymptotically stable invariant toroidal manifold that demand the perturbation to be small not on the whole surface of a torus $\mathcal{T}_m$, but only in non-wandering set of dynamical system on torus (see [4] for details).

Consider the perturbed system of differential equations defined in the direct product of a torus $\mathcal{T}_m$ an an Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = (A + B(\varphi))x + f(\varphi), \tag{1}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, A is a constant matrix, $B(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. Our goal is to establish new sufficient conditions for the existence of invariant torus of system (1) when an unperturbed system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = Ax + f(\varphi), \tag{2}$$

has an asymptotically stable invariant toroidal manifold. As it is known [1, § 10], system (1) has an invariant torus for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$, if a perturbation term $B(\varphi)$ is sufficiently small for every $\varphi \in \mathcal{T}_m$. We are weakening this condition and demand that $\|B(\varphi)\| \le \delta$ only for $\varphi \in \Omega$, where Ω is a non-wandering set of dynamical system $\frac{d\varphi}{dt} = a(\varphi)$.

Definition 1. A point φ is called non-wandering if there exist a neighbourhood $U(\varphi)$ and a positive constant T such that

$$U(\varphi) \cdot \varphi_t(U(\varphi)) = 0 \quad \text{for } \ t \ge T. \tag{3}$$

Denote by W and $\Omega = \mathcal{T}_m - W$ a set of wandering and non-wandering points respectively. From compactness of torus it follows that a set Ω is non-empty and compact.

Theorem 1. *Let in system* (1) *real parts of all eigenvalues of matrix A be negative:* $\operatorname{Re} \lambda_j(A) < 0$, $j = 1, \dots, n$. *Then there exists $\delta > 0$ such that for an arbitrary function $B(\varphi) \in C(\mathcal{T}_m)$ such that $\|B(\varphi)\| \le \delta$, $\varphi \in \Omega$ and for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$, system* (1) *has an asymptotically stable invariant toroidal manifold.*

From theorem 1 it follows an important corollary that allows to investigate a qualitative behavior of solutions of complex systems that have simple dynamics in non-wandering set Ω.

Corollary 1. *Consider the system*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{4}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, $A(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. Let matrix $A(\varphi)$ be constant in non-wandering set Ω and real parts of all eigenvalues of constant matrix be negative. Then for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$ system (4) *has an asymptotically stable invariant toroidal manifold.*

Utilizing a classical perturbation theory for multi-frequency systems it is easy to prove a sufficient conditions for the existence of asymptotically stable invariant toroidal manifold of nonlinear system of the form

$$\frac{d\varphi}{dt} = a(\varphi) + a_1(\varphi, x), \quad \frac{dx}{dt} = A(\varphi)x + F(\varphi, x), \tag{5}$$

where $\varphi \in \mathcal{T}_m,\ x \in \overline{J}_h,\ a(\varphi) \in C_{Lip}(\mathcal{T}_m),\ a_1(\varphi, x) \in C_{Lip}(\mathcal{T}_m, \overline{J}_h),\ F(\varphi, x) \in C^{(0,2)}(\mathcal{T}_m, \overline{J}_h)$, $\overline{J}_h = \{x \in \mathbb{R}^n,\ \|x\| \le h,\ h > 0\}$. System (5) may be rewritten in the form

$$\frac{d\varphi}{dt} = a(\varphi) + a_1(\varphi, x), \quad \frac{dx}{dt} = A(\varphi)x + A_1(\varphi, x)x + f(\varphi), \tag{6}$$

where $A_1(\varphi, x) = \int\limits_0^1 \frac{\partial F(\varphi, \tau x)}{\partial(\tau x)}\, d\tau,\ f(\varphi) = F(\varphi, 0)$.

Corollary 2. *Let in system* (5) *matrix $A(\varphi)$ be constant in non-wandering set Ω and real parts of all eigenvalues of constant matrix be negative:*

$$A(\varphi)|_{\varphi \in \Omega} = \widetilde{A}, \quad \operatorname{Re} \lambda_j(\widetilde{A}) < 0, \ \ j = 1, \ldots, n.$$

Then there exist sufficiently small constants η and δ and sufficiently small Lipschitz constants L_1 and L_2 such that for an arbitrary functions $a_1(\varphi, x) \in C_{Lip}(\mathcal{T}_m, \overline{J}_h)$, $F(\varphi, x) \in C^{(0,2)}(\mathcal{T}_m, \overline{J}_h)$ such that

$$\max_{\varphi \in \mathcal{T}_m,\, x \in \overline{J}_h} \|a_1(\varphi, x)\| \le \eta, \qquad \max_{\varphi \in \mathcal{T}_m,\, x \in \overline{J}_h} \|A_1(\varphi, x)\| \le \delta,$$
$$\big\|a_1(\varphi, x') - a_1(\varphi, x'')\big\| \le L_1 \|x' - x''\|, \quad \big\|A_1(\varphi, x') - A_1(\varphi, x'')\big\| \le L_2 \|x' - x''\|,$$

for any $x', x'' \in \overline{J}_h$, system (5) *has an asymptotically stable invariant toroidal manifold.*

Consider a case when function $A(\varphi_t(\varphi))$ is periodic with respect to t for $\varphi \in \Omega$. For example, such a situation appears when a set Ω consists only from a single trajectory that is a cycle.

Corollary 3. *Consider the system*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{7}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, $A(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. Let matrix $A(\varphi_t(\varphi))$ be a periodic with respect to t for $\varphi \in \Omega$ and all the multipliers of linear periodic system $\frac{dx}{dt} = A(\varphi_t(\varphi))x$, $\varphi \in \Omega$ lie inside the unit circle. Then for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$ system (7) *has an asymptotically stable invariant toroidal manifold.*

Note that one should built a fundamental matrix of periodic system to get the multipliers. In general case it could be very difficult problem, but in the set Ω matrix $A(\varphi_t(\varphi))$, $\varphi \in \Omega$ may be simpler and easier to investigate.

Generalizing corollaries 1 and 3, it is easy to formulate sufficient conditions for the existence of an asymptotically stable invariant torus of linear extension of dynamical system that has a simple structure of limit sets and recurrent trajectories.

Corollary 4. *Let non-wandering set Ω of dynamical system $\frac{d\varphi}{dt} = a(\varphi)$, $\varphi \in \mathcal{T}_m$ consist only from the finite number of stationery points $\{\overline{\varphi}_1, \ldots, \overline{\varphi}_k\} = \Phi$ and finite number of cycles $\{\mathbb{C}_1, \ldots, \mathbb{C}_l\}$ and the real parts of all eigenvalues of matrices $A(\varphi)$, $\varphi \in \Phi$ be negative and all the multipliers of linear periodic systems $\frac{dx}{dt} = A(\varphi_t(\varphi))x$, $\varphi \in \mathbb{C}_i$, $i = 1, \ldots, l$ lie inside the unit circle. Then system* (7) *has an asymptotically stable invariant toroidal manifold for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$.*

The proof of theorem 1 is sufficiently flexible. Utilizing the inequalities of Gronwall–Bellman type one can get similar results for equations of different types, for instance for impulsive differential equations [7, 5]. In papers [3, 2] the analog of corollary 1 is proved for a linear extension of dynamical system on torus with impulsive perturbations at non-fixed moments.

Theorem and corollaries stated here allow to investigate the behavior of sufficiently wide class of multi-frequency systems effectively. Linear extensions of dynamical systems on torus that have a simple structure of non-wandering set are suitable for qualitative analysis without finding fundamental matrices.

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On a Certain Discontinuous Dynamical System in the Plane

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In this article we investigate the behavior of solutions of a certain discontinuous dynamical system in the plane:

$$\dot{x} = Jx, \quad \langle a, x\rangle \neq 0; \quad \Delta x\big|_{\langle a,x\rangle=0} = Bx. \tag{1}$$

Here $x = col(x_1, x_2)$, $\langle a, x\rangle = a_1x_1 + a_2x_2 = 0$ is a given line in the plane, $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ are constant matrices.

The act of motion of the phase point $(x_1(t), x_2(t))$ is defined by the given system of differential equations $\dot{x} = Jx$, when this point is out of the line $x_2 = kx_1$, $k = -\frac{a_1}{a_2}$, and at the time $t = t^*$, when $x_2(t^*) = kx_1(t^*)$, phase point "instantly" jumps to a point

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^+ = \begin{pmatrix} 1+b_{11} & b_{12} \\ b_{21} & 1+b_{22} \end{pmatrix} \begin{pmatrix} x_1(t^*) \\ x_2(t^*) \end{pmatrix}.$$

Note that the linear homogeneous transformation

$$(E+B) : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1+b_{11} & b_{12} \\ b_{21} & 1+b_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

of the plane (x_1Ox_2) into itself maps the line $x_2 = kx_1$ into the line $x_2 = \mu x_1$, where the angular coefficients k and μ relate to equality:

$$\mu = \frac{k(1+b_{22}) + b_{21}}{1 + b_{11} + kb_{12}}.$$

The phase point will necessarily meet the line $x_2 = \mu x_1$ despite the initial position. So it is enough to investigate the behavior of solutions that begin on this line.

Without loss of generality, we consider only the case when $k > 0$. Other cases are investigated similarly.

Since the solutions of differential systems are explicitly written out, simple cases reduce the investigation of behavior of discontinuous trajectories to the study the Poincare mapping

$H : R \to R$ line into a line:

$$H : x_1 \to e^{\frac{\alpha}{\beta}(\operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, \quad \text{if } \mu > k > 0,$$

and

$$H : x_1 \to e^{\frac{\alpha}{\beta}(\pi + \operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, \quad \text{if } \mu \leq k.$$

Denote by h the relation

$$h = \begin{cases} e^{\frac{\alpha}{\beta}(\operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, & \text{if } \mu > k > 0 \\ e^{\frac{\alpha}{\beta}(\pi + \operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, & \text{if } \mu \leq k \end{cases}.$$

Theorem 1. *If the parameters of the original discontinuous dynamical system* (1) *are such that* $|h| < 1$*, then all its solutions over time tends to zero; if* $|h| > 1$*, then all solutions tends to infinity as* $t \to \infty$*. If* $h = 1$*, then all of the solutions which touch the line* $x_2 = \mu x_1$ *are periodic with one impulsive perturbations per period* (*all trajectories are one-impulsive cycles*)*. If* $h = -1$*, then the system has an one-parameter family of two-impulsive cycles, which were generated by discontinuous periodic solutions with two breaks per period.*

As an example, consider the possibility of undamped oscillations of a linear oscillator with friction

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0, \quad \alpha > 0, \quad \omega^2 > \alpha^2.$$

We assume that the perturbation of oscillator subjected to impulsive perturbations at the moment, when the instantaneous speed of the phase point is zero, and the value of the pulse action is proportional to the strength of the coefficients γ position of the phase point at this moment, namely,

$$\Delta\dot{x}\big|_{\dot{x}=0} = \gamma x, \quad \delta x\big|_{\dot{x}=0}.$$

Having the replacement

$$y = \frac{1}{\beta}(\dot{x} + \alpha x), \quad \beta = \sqrt{\omega^2 - \alpha^2}\,,$$

we obtain a system of the form (1)

$$\begin{cases} \dot{x} = -\alpha x + \beta y \\ \dot{y} = -\beta x - \alpha y, \quad y \neq \dfrac{\alpha}{\beta}\, x \end{cases},$$

$$\Delta y\big|_{y=\frac{\alpha}{\beta}x} = \frac{\gamma}{\beta}\, x.$$

Here

$$J = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} -\beta & -\alpha \\ -\dfrac{\gamma}{\beta} & 0 \end{pmatrix}, \quad \begin{cases} K = \dfrac{\alpha}{\beta} \\ \mu = \dfrac{\alpha+\gamma}{\beta} \end{cases}.$$

Direct calculations show that when $\mu > k$, the system has no one-impulsive and two-impulsive cycles.

If $\mu < k$, namely $\gamma < 0$, one-impulsive cycles in the system do not exist, but there are two-impulsive in case $\gamma = \gamma^*$, where γ^* is a negative root of the equation

$$e^{-2\frac{\alpha}{\beta}\left(\pi + \operatorname{arctg}\left(k + \frac{\gamma}{\beta}\right) - \operatorname{arctg}(k)\right)} = \frac{1+k^2}{1+(\frac{\gamma}{\beta})^2}\,.$$

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Boundary Value Problems with State-Dependent Impulses

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Impulsive differential equations have attracted lots of interest due to their important applications in many areas such as aircraft control, drug administration, and threshold theory in biology. A particular case of impulsive problems are *problems with impulses at fixed moments.* This occurs when the moments, at which impulses act in state variable, are known. Very different situation arises, when the impulses appear in evolutionary trajectories fulfilling a predetermined relation between state and time variables. This case, which is represented by state-dependent impulses, is discussed here. Studies of real-life problems with state-dependent impulses can be found e.g. in [1], [3]–[5].

In particular, here we investigate the solvability of *boundary value problems with state-dependent impulses.* As the methods used for problems with finitely many impulses acting at fixed points do not apply to problems with state-dependent impulses, only few paper dealing with boundary value problems in the state-dependent case may be found in the literature. Most of them consider periodic problems which can be transformed to fixed point problems of corresponding Poincaré maps. So, in the case of a periodic boundary conditions, difficulties with the construction of a proper function space and a proper operator representation have been cleared, see e.g. [2].

The main cause of difficulties in the investigation of boundary value problems with state-dependent impulses lies in the following fact: the operator, corresponding to the problem with state-dependent impulses which is constructed in a standard way (used for problems with fixed-time impulses), is not continuous. Therefore, in [6]–[9] we provide a new approach which makes possible to find sufficient conditions for solvability of the ordinary differential equation

$$z''(t) = f(t, z(t), z'(t)) \quad \text{for a.e. } t \in [a, b], \tag{1}$$

subject to the impulse conditions

$$z(\tau_i+) - z(\tau_i) = J_i(\tau_i, z(\tau_i)), \quad z'(\tau_i+) - z'(\tau_i-) = M_i(\tau_i, z(\tau_i)), \tag{2}$$

where the points $\tau_1, \dots, \tau_p$ depend on z through the equations

$$\tau_i = \gamma_i(z(\tau_i)), \quad i = 1, \dots, p, \ \ p \in N.$$

Problem (1), (2) is studied together with the general linear boundary conditions

$$\ell_1(z, z') = c_1, \quad \ell_2(z, z') = c_2. \tag{3}$$

Here f fulfils the Carathéodory conditions on $[a, b] \times R^2$, the impulse functions J_i, M_i, $i = 1, \dots, p$, are continuous on $[a, b] \times R$, $c = (c_1, c_2) \in R^2$, and ℓ_1, ℓ_2 are linear and bounded functionals in the space $G_L([a, b]; R^2)$ of left-continuous regulated vector functions. Consequently, $\ell = (\ell_1, \ell_2)$ is represented by the formula containing the Kurzweil–Stieltjes integral

$$\ell(x) = Kx(a) + \int_a^b V(t) d\,[x(t)], \ \ x = (x_1, x_2) \in G_L([a, b]; R^2),$$

where K is a constant matrix and elements of a function matrix V are functions of bounded variation on $[a, b]$. The barriers γ_i, $i = 1, \dots, p$, which determine the impulse points τ_i, $i = 1, \dots, p$, are ordered and differentiable on some compact real interval.

A function $u : [a, b] \to R$ is a solution of problem (1)–(3) if for each $i \in \{1, \dots, p\}$ there exists a unique $\tau_i \in (a, b)$ such that

$$\tau_i = \gamma_i(u(\tau_i)), \quad a < \tau_1 < \tau_2 < \cdots < \tau_p < b,$$

the restrictions $u|_{[a,\tau_1]}$, $u|_{(\tau_1,\tau_2]}$, $\dots$, $u|_{(\tau_p,b]}$ have absolutely continuous first derivatives, u satisfies (1) for a.e. $t \in [a, b]$ and fulfils conditions (2) and (3).

Provided the data functions f, J_i, M_i, $i = 1, \dots, p$ are bounded, transversality conditions which guarantee that each possible solution of equation (1) in a given region crosses each barrier γ_i at a unique impulse point τ_i are presented, and consequently the existence of a solution to problem (1)–(3) is proved. In order to do it, we choose the way which we have developed in our joint papers [6]–[9]. The main idea of our approach lies in the representation of the solution u of problem (1)–(3) by an ordered $(p+1)$-tuple of functions smooth on $[a, b]$. In particular, we define a Banach space $X = [C^1([a, b]; R)]^{p+1}$, a set $\Omega \subset X$ and an operator $F : \bar{\Omega} \to X$ which is compact. We prove the existence of a fixed point $(u_1, \dots, u_{p+1}) \in \bar{\Omega}$ of the operator F, and then we construct a solution u of problem (1)–(3) by means of this fixed point.

Such existence result can be extended to unbounded functions f, J_i, M_i, $i = 1, \dots, p$ by means of the method of a priori estimates. This can be found for the special case of ℓ in [9].

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Construction of Periodic Solutions and Interval Halving Procedure

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We are interested in a constructive numerical-analytic method of investigation of the periodic boundary value problem

$$u'(t) = f(t, u(t)), \quad t \in [a, b], \tag{1}$$
$$u(b) = u(a), \tag{2}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function Lipschitzian with respect to the second variable:

$$\big|f(t, z_1) - f(t, z_2)\big| \leq K|z_1 - z_2| \tag{3}$$

for all z_1, z_2 from a certain bounded set D. The inequality and the absolute value sign in (3) are understood componentwise. The numerical-analytic scheme based on the successive approximations (see, e.g., the references in [1])

$$u_m(t, z) := z + \int_a^t f\big(s, u_{m-1}(s, z)\big)\, ds - \frac{t-a}{b-a} \int_a^b f\big(s, u_{m-1}(s, z)\big)\, ds, \tag{4}$$

where $u_0(t, z) := z$, $t \in [a, b]$, $m = 1, 2, \ldots$, $z \in D$, allows one to both study the solvability of the problem and practically construct approximations to its solution. The idea of use of iterations (4) to study problem (1), (2) is based on the fact that, in the case of solvability, the free parameter z involved in (4) can always be chosen so that the limit of sequence (4) is a solution of the problem. The parameter z plays the role of an unknown initial value of the solution.

In order to guarantee the applicability of this approach, the previous works contained the assumptions that the eigenvalues of the Lipschitz matrix are sufficiently small and the domain D where (3) holds is, in a sense, wide enough. In particular, the method based on (4) is known to be convergent provided that

$$r(K) < \frac{10}{3(b-a)}\,. \tag{5}$$

We have shown recently in [2] that, at some computational expence, the scheme of the method can be modified so that the convergence condition becomes twice as weak as the original one. More precisely, the mentioned modified version converges provided that

$$r(K) < \frac{20}{3(b-a)}\,, \tag{6}$$

which is a considerably weaker assumption. The idea is to construct the iterations from suitable sequences defined on the half-intervals.

Let Ω be a closed convex subset of D, in which one looks for initial values of periodic solutions. We assume that f satisfies the Lipshitz (3) condition on D. Note that the main role is now played by Ω, and not D. Let ξ and η be arbitrary vectors from Ω. By analogy to [2], put

$$x_0(t,\xi,\eta) := \Big(1 - \frac{2(t-a)}{b-a}\Big)\xi + \frac{2(t-a)}{b-a}\,\eta, \quad t \in [a,(a+b)/2],$$
$$y_0(t,\xi,\eta) := \Big(1 - \frac{2(t-a-b)}{b-a}\Big)\xi + \frac{2t-a-b}{b-a}\,\xi, \quad t \in [(a+b)/2,b],$$

define the recurrence sequences of functions $x_m : [a,(a+b)/2] \times \Omega^2 \to \mathbb{R}^n$ and $y_m : [(a+b)/2,b] \times \Omega^2 \to \mathbb{R}^n$, $m = 0,1,\dots$, according to the formulas

$$x_m(t,\xi,\eta) := x_0(t,\xi,\eta) + \int_a^t f\big(s, x_{m-1}(s,\xi,\eta)\big)\,ds - \\ - \frac{2(t-a)}{b-a}\int_a^{\frac{a+b}{2}} f\big(s, x_{m-1}(s,\xi,\eta)\big)\,ds, \quad t \in [a,(a+b)/2], \tag{7}$$

$$y_m(t,\xi,\eta) := y_0(t,\xi,\eta) + \int_{\frac{a+b}{2}}^t f\big(s, y_{m-1}(s,\xi,\eta)\big)\,ds - \\ - \frac{2t-a-b}{b-a}\int_{\frac{a+b}{2}}^{b} f\big(s, y_{m-1}(s,\xi,\eta)\big)\,ds, \quad t \in [(a+b)/2,b], \tag{8}$$

where $m \geq 0$. Note the presence of the two parameter vectors, ξ and η, in (7), (8), in contrast to one appearing in (4).

Let

$$\delta_\Omega(f) := \max\Big\{\delta_{[a,(a+b)/2],\Omega}(f),\ \delta_{[(a+b)/2,b],\Omega}(f)\Big\}, \tag{9}$$

where $\delta_{J,\Omega}(f) := \max\limits_{(t,z)\in J\times\Omega} f(t,z) - \min\limits_{(t,z)\in J\times\Omega} f(t,z)$ for any closed interval $J \subseteq [a,b]$. Given a non-negative vector ϱ, put $\Omega_\varrho := \bigcup\limits_{\xi\in\Omega} B(\xi,\varrho)$, where $B(\xi,\varrho) := \{z \in \mathbb{R}^n : \ |\xi - z| \leq \varrho\}$.

Theorem 1. *If the spectral radius of K satisfies* (6) *and*

$$\exists\, \varrho : \quad \Omega_\varrho \subset D \ \text{ and } \ \varrho \geq \frac{b-a}{8}\,\delta_{\Omega_\varrho}(f) \tag{10}$$

then, for all fixed $(\xi,\eta) \in \Omega^2$, the sequence $\{x_m(\,\cdot\,,\xi,\eta) : m \geq 0\}$ (resp., $\{y_m(\cdot,\xi,\eta) : m \geq 0\}$) converges to a limit function $x_\infty(\,\cdot\,,\xi,\eta)$ (resp., $y_\infty(\,\cdot\,,\xi,\eta)$) uniformly in $t \in [a,(a+b)/2]$ (resp., $t \in [(a+b)/2,b]$).

Condition is twice as weak as the original inequality (5). Furthermore, comparing condition (10) with similar assumptions from earlier works, we find that (10) is easier to verify because in order to do so one has only to find the value $\delta_{\Omega_\varrho}(f)$, which is computed directly by estimating f. It is also clear from (9) that the value $\delta_{\Omega_\varrho}(f)$ does not change when D grows.

Theorem 2. *If* (6) *and* (10) *hold, then the function $u_\infty(\,\cdot\,,\xi,\eta) : [a,b] \to \mathbb{R}^n$ defined by the formula*

$$u_\infty(t,\xi,\eta) := \begin{cases} x_\infty(t,\xi,\eta) & \text{if } t \in [a,(a+b)/2], \\ y_\infty(t,\xi,\eta) & \text{if } t \in [(a+b)/2,b], \end{cases}$$

for $(\xi, \eta) \in \Omega^2$ *is a solution of problem* (1), (2) *if and only if*

$$\Xi(\xi, \eta) = 0, \quad H(\xi, \eta) = 0,$$

where

$$\Xi(\xi, \eta) := \eta - \xi - \int_a^{\frac{a+b}{2}} f\big(\tau, x_\infty(\tau, \xi, \eta)\big)\, d\tau,$$

$$H(\xi, \eta) := \xi - \eta - \int_{\frac{a+b}{2}}^{b} f\big(\tau, y_\infty(\tau, \xi, \eta)\big)\, d\tau.$$

Moreover, for every solution $u(\cdot)$ *of problem* (1), (2) *with* $(u(a), u(\frac{a+b}{2})) \in \Omega^2$*, there exists a pair* (ξ_0, η_0) *in* Ω^2 *such that* $u(\cdot) = u_\infty(\,\cdot\,, \xi_0, \eta_0)$.

Theorems 1 and 2 suggest a scheme of investigation of the periodic boundary value problem (1), (2), which can be realised on practice by using certain approximate determining functions considered for a finite number of step and, thus, computable explicitly. These approximate determining functions also allow one to obtain constructive conditions guaranteeing the solvability of problem (1), (2) (see [2]).

Multiple interval divisions can be carried out, which, at the price of increase of the number of determining equations to be solved numerically, proportionally diminishes the constants in the conditions. For example, with $[a, b]$ divided into 4 parts at once, we replace (6) by the weaker condition

$$r(K) < \frac{40}{3(b-a)}\,. \tag{11}$$

The condition on the neighbourhood of Ω also becomes proportionally weaker: instead of (10), one arrives at the assumption that

$$\exists\, \varrho: \quad \Omega_\varrho \subset D \ \text{ and } \ \varrho \geq \frac{b-a}{16}\, \delta_{\Omega_\varrho}(f). \tag{12}$$

Assumption (12) is, clearly, weaker than (10).

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Bursting Effect in Neuron Systems

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It is known that self-oscillation (in time) processes in neuron systems exhibit the phenomenon of alternation of pulse bursts (sets of several consecutive intensive spikes) with relatively smooth intervals of membrane potential variation. This phenomenon is referred as *bursting behavior.*

Bursting behavior was studied in numerous works (see, for example, [1–5]). As a rule, the mathematical modeling of this behavior is based on singularly perturbed systems of ordinary differential equations with one slow and two fast variables; under certain conditions such systems may possess stable bursting cycles (periodic motions with bursting behavior). We propose an different approach to the solution of this problem by introducing time delays.

For the mathematical model of an individual neuron we will take the difference-differential equation

$$u' = \lambda\big[f(u(t-h)) - g(u(t-1))\big]u. \qquad (*)$$

Here $u(t) > 0$ is the membrane potential of the neuron, the parameter $\lambda > 0$ (characterizing the time rate of change of electric processes in the system) is assumed to be large, and the parameter $h \in (0,1)$ is fixed. We assume that the functions

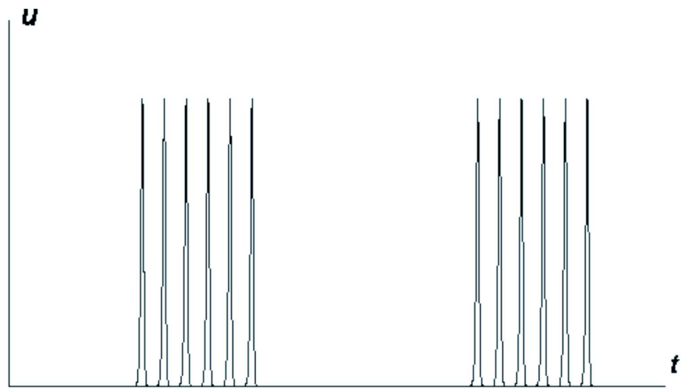
$$f(u), g(u) \in C^1(R^+), \quad R^+ = \big\{u \in R : \geq 0\big\},$$

possess the following properties: $f(0) = 1$, $g(0) = 0$, and as $u \to +\infty$

$$f(u) = -a_0 + O(1/u), \quad uf'(u) = O(1/u),$$
$$g(u) = b_0 + O(1/u), \quad ug'(u) = O(1/u),$$

where a_0 and b_0 are positive constants.

Our main result is as follows. For any fixed natural number n, one can choose the parameters h, a_0, b_0 so that, for all sufficiently large λ, the equation $(*)$ will have an exponentially orbitally stable cycle $u = u^*(t,\lambda)$ of period $T^*(\lambda)$, where $T^*(\lambda)$ tends to a finite limit $T^* > 0$ as $\lambda \to \infty$. On a closed time interval of period length, the function $u^*(t,\lambda)$ has exactly n consecutive asymptotically high (of order $\exp(\lambda h))$) spikes of duration $\Delta t = [1 + (1/a_0)]h$,while it is asymptotically small at other times. In other words, under such a choice of the parameters h, a_0, b_0 bursting behavior is realized.



The properties of the bursting cycle $u^*(t,\lambda)$ is illustrated on the graph in the plane (t,u) scaled $25:1$ for the case $h = 1/26$, $\lambda = 130$ and for the functions

$$f(u) = (1-u)/(1+0.5u), \quad g(u) = 4u/(1+u).$$

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On Existence of a Special Kind's Integral Manifold of the Nonlinear Differential System, Containing Slowly Varying Parameters

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Let $G(\varepsilon_0) = \{t, \varepsilon : \ t \in \mathbf{R}, \ \varepsilon \in [0, \ \varepsilon_0], \ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. We say that a function $f(t,\varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if

(1) $f : G(\varepsilon_0) \to \mathbf{C}$;

(2) $f(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect to t;

(3) $d^k f(t,\varepsilon)/dt^k = \varepsilon^k f_k^*(t,\varepsilon)$ $(0 \le k \le m)$,

$$\|f\|_{S_m(\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t,\varepsilon)| < +\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta)$ belongs to the class $F_m^\theta(\varepsilon_0,\alpha)$ $(m \in \mathbf{N} \cup \{0\}$, $\alpha \in (0,+\infty))$ if

(1) $t, \varepsilon \in G(\varepsilon_0)$, $\theta \in \mathbf{R}$;

(2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

(3)
$$f(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon) \exp(in\theta),$$

and

(a) $f_n(t,\varepsilon) \in S_m(\varepsilon_0)$, $f_{-n}(t,\varepsilon) \equiv \overline{f_n(t,\varepsilon)}$;

(b) $\exists K \in (0,+\infty)$: $\|f_n\|_{S_m(\varepsilon_0)} \le K \exp(-|n|\alpha)$, $n \in \mathbf{Z}$;

(c)
$$\|f\|_{F_m^\theta(\varepsilon_0,\alpha)} \stackrel{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S_m(\varepsilon_0)} < \frac{K(1+e^{-\alpha})}{1-e^{-\alpha}}\,.$$

So the function $f(t,\varepsilon,\theta)$ and its partial derivatives with respect to t up to m-th order inclusive are analytic with respect to $\theta \in \mathbf{R}$.

Definition 3. We say that a function $f(t,\varepsilon,x)$ belongs to the class $S_m^x(\varepsilon_0,x_0,d)$, if

(1) $t, \varepsilon \in G(\varepsilon_0)$, $x \in \mathbf{R}$;

(2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

(3)

$$f(t,\varepsilon,x)=\sum_{l=0}^{\infty} f_l(t,\varepsilon)(x-x_0)^l,$$

and

(a) $f_l : G(\varepsilon_0)\to \mathbf{R}$;

(b) $f_l(t,\varepsilon)\in S_m(\varepsilon_0)$;

(c) the series $\sum\limits_{l=0}^{\infty}\|f_l\|_{S_m(\varepsilon_0)}(x-x_0)^l$ is convergent if $|x-x_0|<d$.

Thus the function $f(t,\varepsilon,x)$ is real, analytic with respect to x, if $|x-x_0|<d$ together with its partial derivatives up to m-th order inclusive. Moreover, $\forall\, x\in(x_0-d,x_0+d) : f(t,\varepsilon,x)\in S_m(\varepsilon_0)$.

Definition 4. We say that a function $f(t,\varepsilon,\theta,x)$ belongs to the class $F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$ ($m\in\mathbf{N}\cup\{0\}$, $\alpha\in(0,+\infty)$) if

(1) $t,\varepsilon\in G(\varepsilon_0)$, $\theta\in\mathbf{R}$, $x\in\mathbf{R}$;

(2) $f : G(\varepsilon_0)\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$;

(3)

$$f(t,\varepsilon,\theta,x)=\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty} f_{n,l}(t,\varepsilon)e^{in\theta}(x-x_0)^l,$$

and

(a) $f_{n,l}(t,\varepsilon)\in S_m(\varepsilon_0)$, $f_{-n,l}(t,\varepsilon)\equiv\overline{f_{n,l}(t,\varepsilon)}$,

(b) $\exists\, K\in(0,+\infty) :\ \forall\, n\in\mathbf{Z}$, $\forall\,\rho\in(0,d)$:

$$\|f_{n,l}(t,\varepsilon)\|_{S_m(\varepsilon)}\le\frac{Ke^{-|n|\alpha}}{\rho^l}\,.$$

We denote

$$X_0(t,\varepsilon,x)=\frac{1}{2\pi}\int_0^{2\pi}X(t,\varepsilon,\theta,x)\,d\theta.$$

Consider the following system of differential equations

$$\begin{aligned}\frac{dx}{dt}&=\mu X(t,\varepsilon,\theta,x)+\varepsilon a(t,\varepsilon,\theta,x),\\ \frac{d\theta}{dt}&=\omega(t,\varepsilon)+\mu\Theta(t,\varepsilon,\theta,x)+\varepsilon b(t,\varepsilon,\theta,x),\end{aligned}\tag{1}$$

where $t,\varepsilon\in G(\varepsilon_0)$, $\theta,x\in\mathbf{R}$; $X,\Theta\in F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $a,b\in F_{m-1}^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $\omega\in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\omega=\omega_0>0$, $\mu\in(0,\mu_0)$.

We study the question of the existence of the integral manifold $x=w(t,\varepsilon,\theta,\mu)\in F_k^{\theta}(\varepsilon_1,\alpha_1)$ ($k<m-1$, $\varepsilon_1<\varepsilon_0$, $\alpha_1<\alpha$) of the system (1).

Let us assume that the following conditions hold.

(A) There is a real function $x_0(t,\varepsilon)$ such that

(1) $X_0(t,\varepsilon,x_0(t,\varepsilon))\equiv 0$;

(2)

$$\inf_{G(\varepsilon_0)} \left| \frac{\partial X_0(t,\varepsilon,x_0(t,\varepsilon))}{\partial x} \right| = \gamma > 0; \tag{2}$$

(3) in system (1) a function $x_0(t,\varepsilon)$ is taken as a point x_0 and is taken as d – sufficiently small positive number in the d-neighborhood of the point x_0 is no other roots of the equation $X_0(t,\varepsilon,x) = 0$, than x_0. Owing to the condition (2) the number d are exists.

(B) Parameters μ and ε are related by inequalities

$$\mu^{r-2} \leq \varepsilon^{m_1-1}, \tag{3}$$

where $r, m_1 \in \mathbf{N}$, $r > 2m_1$, $m > 2m_1$, $m_1 \geq 1$,

$$\mu + \frac{\varepsilon}{\mu^2} < \delta, \tag{4}$$

where $\delta \in (0, +\infty)$.

Theorem. *Suppose that the system* (1) *satisfies conditions* (A), (B). *Then* $\exists\, \delta_0 \in (0, +\infty)$ *such that* $\forall\, \delta \in (0, \delta_0)$ (δ *– value in condition* (B)) *the system* (1) *has the integral manifold*

$$x = w(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m_1-1}(\varepsilon_1^*, \alpha^*),$$

where $\varepsilon_1^* \in (0, \varepsilon_0)$, $\alpha^* \in (0, \alpha)$, *and on this manifold the system* (1) *is reduced to the equation*

$$\frac{d\theta}{dt} = \omega(t,\varepsilon) + \mu\Theta(t,\varepsilon,\theta,w(t,\varepsilon,\theta,\mu)) + \varepsilon b(t,\varepsilon,\theta,w(t,\varepsilon,\theta,\mu)).$$

Differential and Fractional Boundary Value Problems with Strong Time Singularities

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Let $T \in (0,\infty)$ and let $\{\alpha_n\} \subset (0,1)$ be such that $\lim\limits_{n\to\infty} \alpha_n = 1$. We investigate the sequence of singular fractional boundary value problems

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^c\!D^{\alpha_n} u(t) = \rho p(t)\,{}^c\!D^{\alpha_n} u(t) + p(t)a(t)f(t,u(t)), \tag{1}$$

$$u(0) = {}^c\!D^{\alpha_n} u(t)\big|_{t=T}, \quad {}^c\!D^{\alpha_n} u(t)\big|_{t=0} = \frac{a(0)f(0,u(0))}{|\rho|}\,, \tag{2}$$

where $\rho \in (-\infty,0)$, $f \in C([0,T]\times\mathbb{R})$ and the functions p, a satisfy the condition

(H_1) $p \in C(0,T]$, $a \in C[0,T]$, $p > 0$, $a > 0$ on $(0,T]$ and $\int\limits_0^T p(t)\,\mathrm{d}t = \infty$.

Here, ${}^c\!D$ is the Caputo fractional derivative.

We say that a function $u : [0,T] \to \mathbb{R}$ is *a solution of problem* (1), (2) if ${}^c\!D^{\alpha_n} u \in C[0,T] \cap C^1(0,T]$, u satisfies the boundary conditions (2), and (1) is satisfied for $t \in (0,T]$.

The Caputo fractional derivative ${}^c\!D^{\gamma} x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : [0,T] \to \mathbb{R}$ is given as [1, 2]

$${}^c\!D^{\gamma} x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int\limits_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}\, s^k\Big)\,\mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of the fractional number γ.

Hence equation (1) can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int\limits_0^t \frac{(t-s)^{-\alpha_n}}{\Gamma(1-\alpha_n)}\,(u(s)-u(0))\,\mathrm{d}s = \rho p(t)\,\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\alpha_n}}{\Gamma(1-\alpha_n)}\,(u(s)-u(0))\,\mathrm{d}s + p(t)a(t)f(t,u(t)).$$

The special case of (1) is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^c\!D^{\alpha_n} u(t) = \frac{\rho}{t^{\gamma}}\,{}^c\!D^{\alpha_n} u(t) + \frac{a(t)f(t,u(t))}{t^{\mu}}\,,$$

where $\gamma \in [1,\infty)$ and $\mu \in [0,\gamma]$.

Along with problem (1), (2), we discuss the singular differential boundary value problem

$$u''(t) = \rho p(t)u'(t) + p(t)a(t)f(t,u(t)), \tag{3}$$

$$u(0) = u'(T), \quad u'(0) = \frac{a(0)f(0,u(0))}{|\rho|}\,. \tag{4}$$

A function $u \in C^1[0,T] \cap C^2(0,T]$ is called *a solution of problem* (3), (4) if u satisfies (4), and (3) holds for $t \in (0,T]$.

The following result is proved by the Rothe fixed point theorem [3] and gives the existence result for problem (1), (2).

Theorem 1. *Let* (H_1) *hold and let*

(H_2) $f \in C([0,T] \times \mathbb{R})$ *and there exist a positive constant* S *such that for* $t \in [0,T]$ *and* $|x| \le S$*, the estimate*

$$a(t)|f(t,x)| \le \Big(\frac{\max\{1,T\}}{\Delta} + 1\Big)^{-1} |\rho| S$$

is fulfilled, where $\Delta = \min\{\Gamma(\tau) : \ 1 \le \tau \le 2\}$ ($\doteq 0.88$).

Then for each $n \in \mathbb{N}$ *problem* (1), (2) *has at least one solution* u_n *and*

$$\|u_n\| \le S \quad \textit{for } n \in \mathbb{N}.$$

Remark. If f satisfies condition

(H_3) $f \in C([0,T] \times \mathbb{R})$ and for $(t,x) \in [0,T] \times \mathbb{R}$ the estimate

$$a(t)|f(t,x)| \le w(|x|)$$

holds, where $w \in C[0,\infty)$, w is nondecreasing and $\lim\limits_{v\to\infty} w(v)/v = 0$,

then f satisfies condition (H_2).

The relation between solutions of problems (1), (2) and (3), (4) is stated in the following theorem.

Theorem 2. *Let* (H_1) *and* (H_2) *hold. Let* u_n *be a solution of* (1), (2). *Then there exist a subsequence* $\{u_{\ell_n}\}$ *of* $\{u_n\}$ *and a solution* u *of* (3), (4) *such that*

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0, \quad \lim_{n\to\infty} \|{}^c\!D^{\alpha_{\ell_n}} u_{\ell_n} - u'\| = 0.$$

The following results deals with the uniqueness of solutions to problems (1), (2) and (3), (4).

Theorem 3. *Let* (H_1) *hold. Let* $f \in C([0,T] \times \mathbb{R})$ *and*

$$|f(t,x) - f(t,y)| \le K|x-y| \quad \textit{for } t \in [0,T] \textit{ and } x,y \in \mathbb{R},$$

where

$$K < \frac{|\rho|}{(T+1)\|a\|}\,.$$

Then for all sufficiently large n *problem* (1), (2) *has a unique solution* u_n *and*

$$\lim_{n\to\infty} \|u_n - u\| = 0, \quad \lim_{n\to\infty} \|{}^c\!D^{\alpha_n} u_n - u'\| = 0,$$

where u *is the unique solution of* (3), (4).

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Invariant Sets of Ito Stochastic Systems

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We study invariant sets of Ito stochastic systems

$$dx = a(t,x)dt + \sum_{r=1}^{k} b_r(t,x)dW_r(t), \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $a(t,x)$, $b_r(t,x)$ are in $\mathbb{R}^n$, and $W_1, \ldots, W_r$ are jointly independent scalar Wiener processes defined on a complete probability space (Ω, F, P).

We assume that functions $a(t,x)$ and $b_r(t,x)$ are Borel on the set of variables and Lipschitz in x for $\{t \geq 0\} \times \mathbb{R}^n$ and $a(t,0)$, $b_r(t,0)$ are bounded. It is well known that those conditions assure an existence and uniqueness of a solution of the Cauchy Problem for $t \geq 0$.

Let S be a Borel set in $\{t \geq 0\} \times \mathbb{R}^n$ and $S_t = \{x : \ (t,x) \in S\}$. Let $S_t \neq \varnothing$ for $t \geq 0$.

Definition 1. The set S is a positive invariant set for the system (1) for $t \geq 0$ if the equality

$$P\Big\{\big(t, x(t,t_0,x_0)\big) \in S, \ \forall\, t \geq t_0\Big\} = 1 \tag{2}$$

holds under the condition $(t_0, x_0(\omega)) \in S$ with $P1$, where $x(t,t_0,x_0)$ is a solution of the system (1) with an initial condition $x(t_0,t_0,x_0) = x_0$, $t_0 \geq 0$.

In other words, if a solution starts in an invariant set, then it remains in the same set.

Remark 1. Note that the set S from the definition (1) is nonrandom (deterministic). Thus we want to obtain the conditions ensuring that the random process "settles" on a deterministic set.

Definition 2. An invariant set S is stochastic stable if for all $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that for $\rho(x_0, S_{t_0}) < \delta$ the next inequality holds

$$P\Big\{\sup_{t \geq t_0} \rho\big(x(t,t_0,x_0), S_t\big) > \varepsilon_1\Big\} < \varepsilon_2. \tag{3}$$

Here is a distance from a point to a set $\rho(x, S_t) = \inf\limits_{y \in S_t} \|x - y\|$.

Let D be a bounded domain in $\mathbb{R}^n$, and a nonnegative Liapunov function $V(t,x)$ be defined for $\{t \geq 0\} \times \overline{D}$ and continuously differentiable twice in x and once in t.

Let N be a set of zeros of $V(t,x)$ in $\{t \geq 0\} \times D$ and $N_t = \{x \in D : \ V(t,x) = 0\}$. Assume that $N_t \neq \varnothing$ for $t \geq 0$ and let the projection of set N on $\mathbb{R}^n$ be closed in D.

We want to find conditions for the set $N = \{(t,x): V(t,x)=0\}$ to be an invariant set for the system (1). Consider the generating operator for the system (1)

$$LV = \frac{\partial V}{\partial t} + \sum \big(\nabla V, a(t,x)\big) + \frac{1}{2}\sum_{r=1}^{k} \big(\nabla, b_r(t,x)\big)^2 V, \tag{4}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $(\,\cdot\,,\,\cdot\,)$ is a scalar product.

Theorem 1. *If the inequality $LV(t,x) \le 0$ holds in domain $\{t \ge 0\} \times D$, then the set*

$$V(t,x) = 0, \;\; t \ge 0, \;\; x \in D \tag{5}$$

is positively invariant for (1)*. If, in addition,*

$$\inf_{t\ge 0,\; x\in D,\; \rho(N_t,x)>\delta} V(t,x) = V_\delta > 0$$

for $\delta > 0$, then the set (5) *is stochastically stable.*

Let the system (1) have positively-invariant set S which is a part of bigger invariant set N, $S \subset N$, and on set N system (1) degenerate into deterministic one.

Definition 3. A set S is stable on N for $t \ge t_0$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_0 \in N$ with $\rho(x_0, S) < \delta$ the next inequality holds

$$\rho(x(t,t_0,x_0), S) < \varepsilon \;\text{ for }\; t \ge t_0. \tag{6}$$

Theorem 2. *Let a positively-invariant set $N \subset D \subset \mathbb{R}^n$ of system* (1) *include a closed positively-invariant set S ($S \subset N$) which is asymptotically-stable on N.*

Then, if the set N is of the form $V(x) = 0$, $x \in D$, $V(x)$ is nonnegative-defined twice continuously-differentiable in $\mathbb{R}^n$ function and

$$LV \le -c_1 V, \tag{7}$$
$$V_r = \inf_{|x|>r} V(x) \to \infty, \;\; r \to \infty, \tag{8}$$
$$|\sigma_r(t,x)|^2 \le c_2 V(x), \tag{9}$$

where $c_1 > 0$, $c_2 > 0$ are constants, then the set S is uniformly stochastically stable for system (1)*. That is for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that for $\rho(x_0, S) < \delta$ the next inequality holds*

$$P\Big\{\sup_{t\ge t_0} \rho\big(x(t,t_0,x_0), S\big) > \varepsilon_1\Big\} < \varepsilon_2. \tag{10}$$

Remark 2. The condition (9) means that the system (1) degenerates into deterministic one on N.

Now we come to an analogue of the Pliss reduction principle.

Let for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \ge 0$ we have the Ito system

$$\begin{cases} dx = X(t,y)dt, \\ dy = A(t)ydt + \sigma(t,x,y)dW(t), \end{cases} \tag{11}$$

where X is n-dimensional vector, $A(t)$ is $m\times m$-dimensional matrix, σ is $m\times r$-dimensional matrix, $W(t)$ is r-dimensional Wiener process.

Functions X and σ are Lipschitz over x and y with constants L_1 and L_2, respectively.

Let the fundamental matrix $\Phi(t, s)$ of the system

$$\frac{dy}{dt} = A(t)y \tag{12}$$

satisfy the condition

$$\|\Phi(t, s)\| \leq K \exp\big\{-\rho(t - s)\big\} \tag{13}$$

for $t \geq s$, $K > 0$, $\rho > 0$.

Let $X(0, 0) = 0$ and $\sigma(t, x, 0) \equiv 0$. Consequently, $(0, 0)$ is a solution of the system (11) and the set $y = 0$ is an invariant set for the system (11). Also on the set $y = 0$ the original stochastic system degenerates into deterministic one

$$dx = X(x, 0)dt. \tag{14}$$

We study stability of a trivial solution of the stochastic system (11) using a fact that on the invariant set $y = 0$ the trivial solution is stable as a solution of the deterministic system (14).

Theorem 3. *Let a trivial solution of the system* (14) *be asymptotically stable and* $L_2 < \frac{(2\rho)^{\frac{1}{2}}}{K}$. *Then a trivial solution of the system* (11) *is stochastically stable.*

Initial Data Optimization Problem for One Class of Neutral Functional Differential Equation with the Continuous Initial Condition

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Let $I=[a,b]$ be a finite interval and let $\mathbb{R}^n$ be an n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T means transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set and $M\subset O$ is a convex set. Let the function $f(t,x,y)=(f^1(t,x,y),\dots,f^n(t,x,y))^T$ be defined on $I\times O^2$ and satisfy the following conditions: for almost all fixed $t\in I$ the function $f(t,x,y)$ is continuously differentiable with respect to $(x,y)\in O^2$; for any fixed $(x,y)\in O^2$ the functions $f(t,x,y)$, $f_x(t,x,y)$, $f_y(t,x,y)$ are measurable on I; for any compact set $K\subset O$ there exists a function $m_K(t)\in L(I,[0,\infty))$ such that

$$|f(t,x,y)|+|f_x(t,x,y)|+|f_y(t,x,y)|\le m_K(t)$$

for all $(x,y)\in K^2$ and for almost all $t\in I$. Further, let D be the set of continuous differentiable scalar functions (delay functions) $r(t)$, $t\in I$, satisfying the conditions: $\tau(t)<t$, $\dot\tau(t)>0$ with $\widehat\tau:=\inf\{\tau(a):\tau(t)\in D\}$ is finite. By Φ we denote the set of continuously differentiable initial functions $\varphi(t)\in M$, $t\in[\widehat\tau,b]$.

Let $a<t_{01}<t_{02}<t_{11}<t_{12}<b$ be given numbers and let the functions $q^i(t_0,t_1,x)$, $i=1,\dots,l$ be continuously differentiable with respect to all arguments $t_0,t_1\in I$ and $x\in O$.

The collection of initial moment $t_0\in[t_{01},t_{02}]$, finally moment $t_1\in[t_{11},t_{12}]$, delay function $\tau(t)\in D$ and initial function $\varphi(t)\in\Phi$ is said to be initial data and will be denoted by $w=(t_0,t_1,\tau(t),\varphi(t))$.

To each initial data $w=(t_0,t_1,\tau(t),\varphi(t))\in W=[t_0,t_1]\times[t_{11},t_{12}]\times D\times\Phi$ we assign the quasi-linear neutral functional differential equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+f\big(t,x(t),x(\tau(t))\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0]. \tag{2}$$

Here $A(t)$ is a given continuous matrix function with dimension $n\times n$ and $\sigma(t)\in D$ is a fixed delay function. The condition (2) is said to be continuous initial condition since always $x(t_0)=\varphi(t_0)$.

Definition 1. Let $w=(t_0,t_1,\tau(t),\varphi(t))\in W$. A function $x(t)=x(t;w)\in O$, $t\in[\widehat\tau,t_1]$, is called the solution of equation (1) with the continuous initial condition (2) or the solution corresponding to the element w and defined on the interval $[\dot\tau,t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Definition 2. An initial data $w=(t_0,t_1,\tau(t),\varphi(t))\in W$ is said to be admissible if the corresponding solution $x(t)$ is defined on the interval $[\widehat\tau,t_1]$ and the following conditions hold

$$q^i(t_0,t_1,x(t_1))=0,\ \ i=1,\dots,l.$$

The set of admissible initial data will be denoted by W_0.

Definition 3. An initial data $w_0 = (t_{00}, t_{10}, \tau_0(t), \varphi_0(t)) \in W_0$ is said to be optimal if for any $w = (t_0, t_1, \tau(t), \varphi(t)) \in W_0$ we have

$$q^0\big(t_{00}, t_{10}, x_0(t_{10})\big) \le q^0\big(t_0, t_1, x(t_1)\big),$$

where $x_0(t) = x(t; w_0)$, $x(t) = x(t; w)$.

The initial data optimization problem consists in finding an optimal initial data w_0.

Theorem. *Let $w_0 \in W_0$ be an optimal initial data and $t_{00} \in [t_{01}, t_{02}), t_{10} \in (t_{11}, t_{12}]$. Moreover, the function $f(t, x, y), (t, x, y) \in I \times O^2$ is bounded and there exist the finite limits $\dot{x}_0(\sigma(t_{10})-)$,*

$$\lim_{z \to z_0} f(z) = f^+, \quad z = (t, x, y) \in [t_{00}, t_{10}) \times O^2,$$
$$\lim_{z \to z_1} f(z) = f^-, \quad z \in (t_{00}, t_{10}] \times O^2,$$

where $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$, $z_1 = (t_{10}, x_0(t_{10}), x_0(\tau_0(t_{10})))$. Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \ne 0$, $\pi_0 \le 0$ and the solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t) f_x[t] - \psi(\gamma_0(t)) f_y[\gamma_0(t)]\dot{\gamma}_0(t), & \\ \psi(t) = \chi(t) + \psi(\rho(t)) A(\rho(t))\dot{\rho}(t), & t \in [t_{00}, t_{10}], \\ \chi(t) = \psi(t) = 0, & t > t_{10} \end{cases} \tag{3}$$

such that the conditions listed below hold:

- *the condition for $\chi(t)$ and $\psi(t)$*

$$\chi(t_{10}) = \psi(t_{10}) = \pi Q_{0x},$$

  where $Q = (q^0, \ldots, q^l)^T$, $Q_{0x} = Q_x(t_{00}, t_{10}, x_0(t_{10}))$;

- *the condition for the optimal initial function $\varphi_0(t)$*

$$\chi(t_{00})\varphi_0(t_{00}) + \int\limits_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_y[\gamma_0(t)]\dot{\gamma}_0(t)\varphi_0(t)\,dt + \int\limits_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}_0(t)\,dt =$$

$$= \max_{\varphi(t)\in\Phi}\left[\chi(t_{00})\varphi(t_{00}) + \int\limits_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_y[\gamma_0(t)]\dot{\gamma}_0(t)\varphi(t)\,dt + \int\limits_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}(t)\,dt\right];$$

- *the condition for the optimal delay function $\tau_0(t)$*

$$\int\limits_{t_{00}}^{t_{10}} \psi(t) f_y[t]\dot{x}_0(t)\tau_0(t)\,dt = \min_{\tau(t)\in D} \int\limits_{t_{00}}^{t_{10}} \psi(t) f_y[t]\dot{x}_0(t)\tau(t)\,dt;$$

- *the condition for the optimal initial moment t_{00}*

$$\pi Q_{0t_0} + \psi(t_{00})\Big[\dot{\varphi}_0(t_{10}) - A(t_{00})\dot{\varphi}_0(t_{00}) - f^+\Big] \le 0; \tag{4}$$

- *the condition for the optimal final moment t_{10}*

$$\pi Q_{0t_1} + \psi(t_{10})\Big[A(t_{10})\dot{x}_0(\sigma(t_{10})-) + f^-\Big] \ge 0. \tag{5}$$

Here $\gamma_0(t)$ is the inverse function of $\tau_0(t)$ and $\rho(t)$ is the inverse function of $\sigma(t)$;

$$f_x[t] = f_x(t, x_0(t), x_0(\tau_0(t))).$$

Some comments

The essential innovation in this work is necessary optimality condition for delay function. The above given theorem is proved by a scheme described in [1]. Let $f(t,x,y)$ be continuous at the points z_0 and z_1, and let $\dot{x}_0(t)$ be continuous at the point $\sigma(t_{10})$, then instead of inequalities (4) and (5) we have equalities

$$\pi Q_{0t_0} + \psi(t_{00})\Big[\dot{\varphi}_0(t_{10}) - A(t_{00})\dot{\varphi}_0(t_{00}) - f(z_0)\Big] = 0$$

and

$$\pi Q_{0t_1} + \psi(t_{10})\Big[A(t_{10})\dot{x}_0(\sigma(t_{10})) + f(z_1)\Big] = 0.$$

The function $\psi(t)$, generally, is discontinuous at points $\sigma(t_{10}), \sigma(\sigma(t_{10})), \dots$ (see system (3)). The initial data optimization problem for linear neutral functional differential equation with constant delays and the discontinuous initial condition is considered in [2].

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation, Grant No. 31/23.

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Asymptotic Representations of Solutions of Essentially Nonlinear Systems of Ordinary Differential Equations

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We consider the system of differential equations

$$\begin{cases} y_i' = f_i(t, y_1, \dots, y_n), \\ i = \overline{1, n}, \end{cases} \tag{1}$$

where $f_i : [a, \omega[\times \prod_{i=1}^{n} \Delta_{Y_i^0} \longrightarrow \mathbf{R}$ $(i = \overline{1, n})$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\Delta_{Y_i^0}$ $(i \in \{1, \dots, n\})$ is a one-sided neighborhood of the point Y_i^0 and Y_i^0 equals either zero or $\pm\infty$.

Definition 1. A solution $(y_i)_{i=1}^n$ of system (1), defined on an interval $[t_0, \omega[\subset [a, \omega[$, is called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution, where $-\infty \le \Lambda_i \le +\infty$ $(i = \overline{1, n-1})$ if it satisfies the following conditions

$$y_i(t) \in \Delta_{Y_i^0} \ \text{ while } \ t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0 \ \ (i = \overline{1, n}), \tag{2}$$

$$\lim_{t \uparrow \omega} \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} = \Lambda_i \ \ (i = \overline{1, n-1}). \tag{3}$$

Earlier the asymptotic behavior of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of cyclic systems of differential equations

$$\begin{cases} y_i' = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) \quad (i = \overline{1, n-1}), \\ y_n' = \alpha_n p_n(t) \varphi_1(y_1), \end{cases}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1, n})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1, n})$ are continuous functions, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = \overline{1, n})$ are continuous and regularly varying functions (see [1]) when $y_i \to Y_i^0$ of σ_i orders, which satisfies the conditions

$$\lim_{\substack{y_i \to Y_i^0 \\ y_i \in \Delta(Y_i^0)}} \frac{y_i \varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \ \ (i = \overline{1, n}), \ \ \prod_{i=1}^{n} \sigma_i \ne 1,$$

was investigated in [2–4].

The aim of the present paper is to derive necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of system (1) of a more general form and asymptotic formulas for such solutions as $t \uparrow \omega$ for the case in which the Λ_i $(i = \overline{1, n-1})$ are nonzero real constants.

In this care can be determined nonzero real constant Λ_n, which establishes relationship between the first and nth components of the $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution. We have

$$\Lambda_n = \lim_{t \uparrow \omega} \frac{y_n(t) y_1'(t)}{y_n'(t) y_1(t)} = \lim_{t \uparrow \omega} \Big[\frac{y_n(t) y_{n-1}'(t)}{y_n'(t) y_{n-1}(t)} \cdots \frac{y_2(t) y_1'(t)}{y_2'(t) y_1(t)} \Big] = \frac{1}{\Lambda_1 \cdots \Lambda_{n-1}}. \tag{4}$$

We set $\mu_i = 1$ if either $Y_i^0 = +\infty$ or $Y_i^0 = 0$ and $\Delta(Y_i^0)$ is a right neighborhood of the point 0 and $\mu_i = -1$ if either $Y_i^0 = -\infty$ or $Y_i^0 = 0$ and $\Delta(Y_i^0)$ is a left neighborhood of the point 0. Note that the numbers μ_i $(i = \overline{1,n})$ determine the signs of the components of the $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution in some left neighborhood of ω.

We examine the question of the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of system (1) with fixed values $\Lambda_i \in \mathbf{R}/\{0\}$ $(i = \overline{1, n-1})$ and the question of the asymptotic behavior of such solutions as $t \uparrow \omega$ for the case in which the system is in some sense close to the cyclic with regularly varying nonlinearities.

Definition 2. System of differential equations (1) satisfies the condition $N(\Lambda_1, \dots, \Lambda_{n-1})$, where $\Lambda_i \in \mathbf{R}/\{0\}$ $(i = \overline{1, n-1})$, if for each $k \in \{1, \dots, n\}$ there exist a number $\alpha_k \in \{-1, 1\}$, continuous function $p_k : [a, \omega[\longrightarrow]0, +\infty[$ and $\varphi_{k+1} : \Delta_{Y^0_{k+1}} \longrightarrow]0, +\infty[$ continuous function properly varying as $y_{k+1} \to Y^0_{k+1}$ of order σ_{k+1} such that for any functions $y_i : [a, \omega[\longrightarrow \Delta_{Y^0_i}$ $(i = \overline{1,n})$ with conditions (2), (3), we have the representation

$$f_k\big(t, y_1(t), \dots, y_n(t)\big) = \alpha_k p_k(t) \varphi_{k+1}(y_{k+1}(t))[1 + o(1)] \text{ as } t \uparrow \omega. \tag{5}$$

Let us introduce notation considerations while the system satisfies the condition $N(\Lambda_1, \dots, \Lambda_{n-1})$ for some Λ_i $(i \in \{1, \dots, n-1\})$ and the orders σ_k $(k = \overline{1,n})$ of the functions φ_k are such that the conditions

$$\prod_{k=1}^{n} \sigma_k \neq 1. \tag{6}$$

From (4) it follows that $\prod\limits_{i=1}^{n} \Lambda_i = 1$; therefore, by the condition (6), the expression $1 - \Lambda_i \sigma_{i+1}$ is nonzero for at least one $i \in \{1, \dots, n\}$. Let

$$\mathfrak{I} = \big\{ i \in \{1, \dots, n\} : \ 1 - \Lambda_i \sigma_{i+1} \neq 0 \big\}, \ \ \overline{\mathfrak{I}} = \{1, \dots, n\} \setminus \mathfrak{I}$$

and let l be the minimal element of the set $\mathfrak{I}$.

Taking into account the choice of l, we introduce auxiliary functions I_i $(i = \overline{1,n})$ and nonzero constants β_i $(i = \overline{1,n})$ by the relations

$$I_i(t) = \begin{cases} \displaystyle\int_{A_i}^{t} p_i(\tau)\, d\tau & \text{for } i \in \mathfrak{I}, \\ \displaystyle\int_{A_i}^{t} I_l(\tau) p_i(\tau)\, d\tau & \text{for } i \in \overline{\mathfrak{I}}, \end{cases}$$

$$\beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{for } i \in \mathfrak{I}, \\ \dfrac{\beta_l}{\Lambda_l \cdots \Lambda_{i-1}}, & \text{for } i \in \{l+1, \dots, n\} \setminus \mathfrak{I}, \\ \dfrac{\beta_l}{\Lambda_l \cdots \Lambda_n \Lambda_1 \cdots \Lambda_{i-1}}, & \text{for } i \in \{1, \dots, l-1\} \setminus \mathfrak{I}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such way that the corresponding integral I_i tends either to zero or to infinity as $t \uparrow \omega$.

In addition, we introduce the numbers

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \quad (i = \overline{1,n}).$$

It follows from the condition $N(\Lambda_1, \dots, \Lambda_{n-1})$ and (2) that for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$ – solutions of system (1), it is necessary for each $i \in \{1, \dots, n\}$

$$\alpha_i \mu_i > 0 \text{ for } Y_i = \pm\infty, \quad \alpha_i \mu_i < 0 \text{ for } Y_i = 0.$$

Theorem. *Let $\Lambda_i \in \mathbb{R} \setminus \{0\}$ $(i = \overline{1, n-1})$, system of ordinary differential equations* (1) *satisfy the condition $N(\Lambda_1, \dots, \Lambda_{n-1})$ and the orders of properly varying functions φ_k $(i = \overline{1, n})$ in the representations* (5) *be such that the conditions* (6) *hold. Let, moreover, $l = \min \mathfrak{I}$. Then, for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$ – solutions of system* (1), *it is necessary and, if the algebraic equation about ρ*

$$\prod_{i=1}^{n} \Big(\prod_{j=1}^{i-1} \Lambda_j + \rho \Big) - \prod_{i=1}^{n} \Big(\sigma_i \prod_{j=1}^{i-1} \Lambda_j \Big) = 0 \tag{7}$$

does not have roots with zero real part, it is also sufficient that for each $i \in \{1, \dots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \, \frac{\beta_{i+1}}{\beta_i}$$

and the following conditions to be satisfied

$$A_i^* \beta_i > 0 \;\; \textit{if} \;\; Y_i = \pm\infty, \quad A_i^* \beta_i < 0 \;\; \textit{if} \;\; Y_i = 0,$$
$$\operatorname{sign} \big[\alpha_i A_i^* \beta_i \big] = \mu_i.$$

Moreover, components of each solution of that type admit the following asymptotic representation as $t \uparrow \omega$:

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i I_i(t) [1 + o(1)] \;\; \textit{for} \;\; i \in \mathfrak{I},$$
$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i \, \frac{I_i(t)}{I_l(t)} \, [1 + o(1)] \;\; \textit{for} \;\; i \in \overline{\mathfrak{I}},$$

and also the asymptotic representations in plicit form

$$y_i(t) = \mu_i |I_i(t)|^{\frac{1}{\beta_i} + o(1)} \;\; (i = \overline{1, n}) \;\; \textit{for} \;\; t \uparrow \omega;$$

and there exists the whole k-parametric family of these solutions if there are k positive roots (*including multiple roots*) *among the solutions of algebraic equation* (7) *with signs of real parts different from those of the number $A_l^* \beta_l$.*

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Calculation of Exact Upper Bounds of Lyapunov Exponents of Linear Differential Systems with Exponentially Decreasing Perturbations of the Matrix Coefficients

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Let us consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

where $n \geqslant 2$ and $A(\,\cdot\,)$ is a sectionally continuous and bounded matrix-function. Denote the set of all such systems by $\mathcal{M}_n$. Identifying system (1) with its matrix, we use the notation $A(\,\cdot\,) \in \mathcal{M}_n$. Let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ be the Lyapunov exponents of system (1). Let us also consider the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \quad x \in \mathbb{R}^n, \quad t \geqslant 0, \tag{2}$$

with a sectionally continuous $n \times n$-matrix-perturbation $Q(\,\cdot\,)$, which belongs to E_n. By E_n we denote the class of exponentially decreasing perturbations (i.e. the Lyapunov exponent of $\|Q(\,\cdot\,)\|$ is negative: $\lambda[Q] < 0$). Following our notation, let $\lambda_1(A+Q) \leqslant \cdots \leqslant \lambda_n(A+Q)$ be the Lyapunov exponents of system (2). From article [1] it follows that the Lyapunov exponents of system (1) can be unstable under perturbations from E_n. Let $\Delta_k(A) = \inf\{\lambda_k(A+Q) : \ Q \in E_n\}$ be the exact lower bound of mobility of $\lambda_k(A)$ and $\nabla_k(A) = \sup\{\lambda_k(A+Q) : \ Q \in E_n\}$ be the exact upper bound of mobility of $\lambda_k(A)$ with $k = 1, \ldots, n$. Hence there arises a natural problem of calculating $\Delta_k(A)$ and $\nabla_k(A)$ by the Cauchy matrix of the initial system (1). The values of so-called Izobov exponents $\Delta_1(A)$ and $\nabla_n(A)$ were calculated in article [2]. In this paper, for any system $A(\,\cdot\,) \in \mathcal{M}_n$ and each $k = 1, \ldots, n$ we calculate the exact upper bound $\nabla_k(A)$ of mobility.

The formulated problem of calculation of the exact extreme bounds of mobility can be considered for any other class of perturbations. The exact upper bounds of mobility of the Lyapunov exponents were calculated in article [3] for the class of small perturbations. Although the given below theorem coincides with I.N. Sergeyev theorem by form and is proven by using the same ideas, there are still some substantial technical differences in the proofs. The fundamental difference is contained in the following definition.

Line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ of solutions of system (1) are called exponentially integrally divided if for any $\varepsilon > 0$ there exists a $T_\varepsilon \geqslant 0$ such that for all $t \geqslant \tau \geqslant T_\varepsilon$ inequality $(\|x_1(t)\|/\|x_1(\tau)\|) : (\|x_2(t)\|/\|x_2(\tau)\|) \geqslant \exp(-\varepsilon\, t)$ holds for any nonzero solutions $x_1(\,\cdot\,) \in N(\,\cdot\,)$ and $x_2(\,\cdot\,) \in L(\,\cdot\,)$. In such a case the line-element $N(\,\cdot\,)$ is called exponentially larger than line-element $L(\,\cdot\,)$. Moreover, the line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ are called strongly exponentially divided if they are exponentially integrally divided and the angle $\angle\,(N(\,\cdot\,), L(\,\cdot\,))$ between these line-elements has the exact zero the Lyapunov exponent. The notion of exponentially integrally divided line-elements was introduced in [4]; the definition of strongly exponentially divided line-elements was introduced in [5] implicitly.

The notion of strongly exponentially divided line-elements when $Q(\,\cdot\,) \in E_n$ is the exact analog of the notion of integrally divided line-elements. The following lemma is very important for the proof of the theorem.

Lemma. *If system* (1) *has strongly divided line-elements* $N(\,\cdot\,)$ *and* $L(\,\cdot\,)$ *such that* $\dim N + \dim L = n$ *and* $N(\,\cdot\,)$ *is exponentially larger than* $L(\,\cdot\,)$*, then every system* (2) *with* $Q(\,\cdot\,) \in E_n$ *has strongly exponentially divided line-elements* $N_Q(\,\cdot\,)$ *and* $L_Q(\,\cdot\,)$ *such that* $N_Q(\,\cdot\,)$ *is exponentially larger than* $L_Q(\,\cdot\,)$, $\dim N_Q = \dim N$ *and* $\dim L_Q = \dim L$.

Let us define the upper exponential exponent $\nabla|_L(A)$ of the line-element $L(\,\cdot\,)$ of solutions of system (1) as $\nabla|_L(A) = \lim\limits_{\theta\to 1+0} \varlimsup\limits_{\mathbb{N}\ni m\to+\infty} \theta^{-m} \sum\limits_{j=1}^{m} \ln \|X|_L(\theta^j, \theta^{j-1})\|$. By $X|_L(t,\tau)$ we denote the restriction of the Cauchy operator $X(t,\tau)$ of system (1) to $L(\tau)$.

Theorem. *Let i be the least number greater than or equal to k such that there exist strongly exponentially divided line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ for which $N(\,\cdot\,)$ is exponential greater than $L(\,\cdot\,)$ and $\dim N + \dim L = n$, $\dim L = i$ then $\nabla_k(A)$ is equal to $\nabla|_L(A)$.*

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