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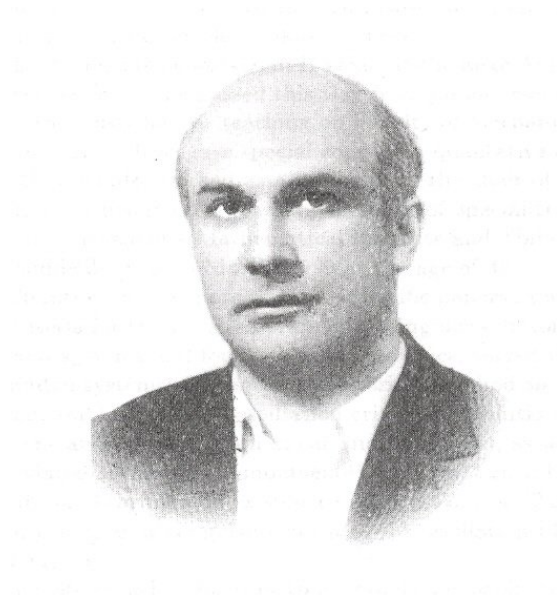
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**ABSTRACTS**

*Dedicated to the 70<sup>th</sup> birthday anniversary  
of Professor T. Chanturia*



**03.02.1941 – 06.05.1990**

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# On the Two Point Boundary Value Problem for Systems of Linear Generalized Differential Equations with Singularities

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For the system of linear generalized ordinary differential equations with singularities

$$dx(t) = dA(t) \cdot x(t) + df(t) \text{ for } t \in ]a, b[, \quad (1)$$

the two point boundary value problem

$$x_i(a+) = 0 \quad (i = 1, \dots, n_0), \quad x_i(b-) = 0 \quad (i = n_0 + 1, \dots, n), \quad (2)$$

is considered, where  $-\infty < a < b < +\infty$ ,  $x(t) = (x_i(t))_{i=1}^n$ ,  $n_0 \in \{1, \dots, n\}$ ,  $f = (f_l)_{l=1}^n : ]a, b[ \rightarrow \mathbb{R}^n$  is a vector-function whose components have bounded variations, and  $A = (a_{il})_{i,l=1}^n : ]a, b[ \rightarrow \mathbb{R}^{n \times n}$  is a matrix-function such that the functions  $a_{i1}, \dots, a_{in}$  have bounded variations on every closed interval from  $]a, b[$  for  $i \in \{1, \dots, n_0\}$  and on every closed interval from  $[a, b[$  for  $i \in \{n_0 + 1, \dots, n\}$ .

There are established sufficient conditions for the unique solvability of this problem in the case when considered system is singular, i. e., the components of the matrix-function  $A$  may have unbounded variation on the interval  $[a, b[$ .

By  $BV_{loc}(]a, b[, \mathbb{R}^{n \times m})$  we denote the set of all matrix-functions  $X : ]a, b[ \rightarrow \mathbb{R}^{n \times m}$  with bounded variation on every closed interval from  $]a, b[$ .

By a solution of the problem (1), (2) we mean a vector-function  $x = (x_i)_{i=1}^n \in BV_{loc}(]a, b[, \mathbb{R}^n)$  satisfying the condition (2) and the system (1), i.e., such that  $x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s)$  for  $a < s \leq t < b$ , where integral is considered in the Lebesgue–Stieltjes sense.

Let  $\det(I_n + (-1)^j d_j A(t)) \neq 0$  for  $t \in ]a, b[$  ( $j = 1, 2$ ) and let  $\gamma_\alpha(\cdot, s)$  be a unique solution of the Cauchy problem  $d\gamma(t) = \gamma(t) d\alpha(t)$ ,  $\gamma(s) = 1$ .

**Definition.** Let  $n_0 \in \{1, \dots, n\}$ . We say that a matrix-function  $C = (c_{il})_{i,l=1}^n \in BV([a, b], \mathbb{R}^{n \times n})$  belongs to the set  $\mathcal{U}(a+, b-; n_0)$  if the functions  $c_{il}$  ( $i \neq l$ ;  $i, l = 1, \dots, n$ ) are nondecreasing on  $[a, b]$  and the system

$$\operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) \cdot dx_i(t) \leq \sum_{l=1}^n x_l(t) dc_{il}(t) \text{ for } t \in ]a, b[ \quad (i = 1, \dots, n)$$

has no nontrivial, nonnegative solution satisfying the condition (2).

**Theorem.** *Let the vector-function  $f$  have bounded variation, and let the matrix-function  $A = (a_{il})_{i,l=1}^n \in BV_{loc}(]a, b[, \mathbb{R}^{n \times n})$  be such that the conditions*

$$\begin{aligned} (s_0(a_{ii})(t) - s_0(a_{ii})(s)) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) &\leq s_0(c_{ii} - \alpha_i)(t) - s_0(c_{ii} - \alpha_i)(s), \\ (-1)^j (|1 + (-1)^j d_j a_{ii}(t)| - 1) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) &\leq d_j(c_{ii}(t) - \alpha_i(t)); \\ |s_0(a_{il})(t) - s_0(a_{il})(s)| &\leq s_0(c_{il})(t) - s_0(c_{il})(s) \quad (i \neq l), \\ |d_j a_{il}(t)| &\leq d_j c_{il}(t) \quad (i \neq l) \end{aligned}$$

hold for  $a < s < t < b$  ( $i, l = 1, \dots, n$ ), where  $C = (c_{il})_{i,l=1}^n \in \mathcal{U}(a+, b-; n_0)$ ,  $\alpha_i : ]a, b[ \rightarrow \mathbb{R}$  ( $i = 1, \dots, n$ ) are nondecreasing functions and

$$\lim_{t \rightarrow a+} d_1 \alpha_i(t) < 1 \quad (i = 1, \dots, n_0), \quad \lim_{t \rightarrow b-} d_2 \alpha_i(t) < 1 \quad (i = n_0 + 1, \dots, n_0); \quad (3)$$

$$\begin{aligned} \limsup_{t \rightarrow a+} \{\gamma_{\alpha_i}(t, a + 1/k) : k = 1, 2, \dots\} &= 0 \quad (i = 1, \dots, n_0), \\ \limsup_{t \rightarrow b-} \{\gamma_{\alpha_i}(t, b - 1/k) : k = 1, 2, \dots\} &= 0 \quad (i = n_0 + 1, \dots, n_0). \end{aligned} \quad (4)$$

Then the problem (1), (2) has one and only one solution.

**Corollary.** Let the vector-function  $f$  have bounded variation, and let the matrix-function  $A = (a_{il})_{i,l=1}^n \in \text{BV}_{loc}(]a, b[, \mathbb{R}^{n \times n})$  be such that the conditions

$$\begin{aligned} (s_0(a_{ii})(t) - s_0(a_{ii})(s)) \operatorname{sgn} \left( n_0 + \frac{1}{2} - i \right) &\leq h_{ii} (s_0(\beta)(t) - s_0(\beta)(s)) \\ &\quad - (s_0(\alpha)(t) - s_0(\alpha)(s)), \\ (-1)^j (|1 + (-1)^j d_j a_{ii}(t)| - 1) \operatorname{sgn} \left( n_0 + \frac{1}{2} - i \right) &\leq h_{ii} d_j \beta(t) - d_j \alpha(t), \\ |s_0(a_{il})(t) - s_0(a_{il})(s)| &\leq h_{il} (s_0(\beta)(t) - s_0(\beta)(s)) \quad (i \neq l), \\ |d_j a_{il}(t)| &\leq h_{il} d_j \beta(t) \quad (i \neq l) \end{aligned}$$

hold for  $a < s < t < b$  ( $i, l = 1, \dots, n$ ), where  $\alpha$  is a nondecreasing on  $]a, b[$  function satisfying the conditions (3) and (4);  $\beta$  is a nondecreasing on  $[a, b]$  function having no more than a finite number of discontinuity points,  $h_{ii} \in \mathbb{R}$ ,  $h_{il} \in \mathbb{R}_+$  ( $i \neq l$ ;  $i, l = 1, \dots, n$ ). Let, moreover,  $\rho_0 r(\mathcal{H}) < 1$ , where  $\mathcal{H} = (h_{ik})_{i,k=1}^n$ ,

$$\begin{aligned} \rho_0 &= \max \left\{ \sum_{j=0}^2 \lambda_{mj} : m = 0, 1, 2 \right\}, \quad \lambda_{00} = \frac{2}{\pi} (s_0(\beta)(b) - s_0(\beta)(a)), \\ \lambda_{0j} &= \lambda_{j0} = (s_0(\beta)(b) - s_0(\alpha)(a))^{\frac{1}{2}} (s_j(\beta)(b) - s_j(\beta)(a))^{\frac{1}{2}} \quad (j = 1, 2), \\ \lambda_{mj} &= \frac{1}{2} (\mu_{\alpha m} \mu_{\alpha j})^{\frac{1}{2}} \sin^{-1} \frac{\pi}{4n_{\alpha m} + 2} \quad (m, j = 1, 2), \\ \mu_{\alpha m} &= \max \{ d_m \alpha(t) : t \in [a, b] \} \quad (m = 1, 2), \end{aligned}$$

and  $n_{\alpha m}$  is a number of points from  $[a, b]$  for which  $d_m \alpha(t) \neq 0$  for every  $m \in \{1, 2\}$ . Then the problem (1), (2) has one and only one solution.

# Uniform Estimates and Existence of Solution with Prescribed Domain to a Nonlinear Third-Order Differential Equation

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## 1 Introduction

In 1990, after my lecture at Enlarged Sessions of the Seminar of I.Vekua Institute of Applied Mathematics, professor T. A. Chanturia posed me a question about existence for any finite  $x_*$ ,  $x^*$ ,  $x_* < x^*$  of a non-extensible solution  $y(x)$  with domain  $(x_*, x^*)$  to the equation  $y''' + p(x)|y|^{k-1}y = 0$ . For such equation of the second order some related result was obtained in [1]. I've got an answer for the third-order equation in 1992 ([3]), but unfortunately T. A. Chanturia could not see it... The proof of this result was very complicated, but with the help of uniform estimates of solutions ([8]) it became much better.

## 2 Uniform Estimates of Solutions

Consider the differential equation

$$y''' + p(x, y, y', y'')|y|^{k-1}y = 0, \quad k > 1, \quad (1)$$

the function  $p(x, y_0, y_1, y_2)$  is continuous in  $x$  and Lipschitz continuous in  $y_0, y_1, y_2$  with

$$0 < p_* \leq p(x, y_0, y_1, y_2) \leq p^*, \quad (2)$$

where  $p_*, p^*$  are positive constants.

Put  $\beta = \frac{k-1}{3} > 0$ .

**Theorem 1.** *For any  $k > 1, p_* > 0, p^* > p_*, h > 0$  there exists a constant  $C > 0$  such that for any  $p(x, y, y', y'')$  satisfying (2), any solution  $y(x)$  to (1) satisfying the condition  $|y(x_0)| = h > 0$  in some point  $x_0 \in \mathbb{R}$  cannot be extended to the interval  $(x_0 - Ch^{-\beta}, x_0 + Ch^{-\beta})$ .*

**Theorem 2.** *For any  $k > 1, p_* > 0, p^* > p_*$  there exists a constant  $C > 0$  such that for any  $p(x, y, y', y'')$  satisfying (2) and any solution  $y(x)$  to (1) defined on  $[-a, a]$  it holds  $|y(0)| \leq (\frac{C}{a}ig)^{1/\beta}$ .*

**Theorem 3.** *For any  $k > 1, p_* > 0, p^* > p_*$  there exists a constant  $C > 0$  such that for any  $p(x, y, y', y'')$  satisfying (2) and any solution  $y(x)$  to (1) defined on  $[a, b]$  it holds*

$$|y(x)| \leq C \min(x - a, b - x)^{-1/\beta}. \quad (3)$$

*Remark 1.* In [5] uniform estimates for positive solutions with the same domain to the equation

$$y^{(n)} + \sum_{j=0}^{n-1} a_j(x)y^{(j)} + p(x)|y|^{k-1}y = 0$$

with continuous functions  $p(x)$  and  $a_j(x)$ ,  $n \geq 1, k > 1$  were obtained. In [6] similar uniform estimates for absolute values of all solutions to the equation

$$y^{(n)} + \sum_{j=0}^{n-1} a_j(x)y^{(j)} + p(x)|y|^k = 0$$

were proved.

### 3 Existence of Solution with Prescribed Domain

Consider the differential equation (1) with the same propositions about the function  $p(x, y_0, y_1, y_2)$ .

**Definition.** A solution  $y(x)$  has a *resonance asymptote*  $x = x^*$  if

$$\overline{\lim}_{x \rightarrow x^*} y(x) = +\infty, \quad \underline{\lim}_{x \rightarrow x^*} y(x) = -\infty.$$

**Theorem 4.** Suppose that condition (2) holds. Let  $y(x)$  be a solution to (1) defined on  $[x_0, x^*)$  with the resonance asymptote  $x = x^*$ . Then the position of the asymptote  $x = x^*$  depends continuously on  $y(x_0)$ ,  $y'(x_0)$ ,  $y''(x_0)$ .

**Theorem 5.** Suppose that condition (2) holds. Then for any finite  $x_* < x^*$  there exists a non-extensible solution  $y(x)$  to (1) defined on  $(x_*, x^*)$  with the vertical asymptote  $x = x_*$  and the resonance asymptote  $x = x^*$ .

**Corollary 1.** Suppose that condition (2) holds. Then for any  $x_* \in \mathbb{R}$  there exists a Kneser solution of (1) with the vertical asymptote  $x = x_*$  defined on the interval  $(x_*, +\infty)$  and tending to 0 as  $x \rightarrow +\infty$ .

**Corollary 2.** Suppose that condition (2) holds. Then for any  $x^* \in \mathbb{R}$  there exists a non-extensible solution  $y(x)$  of (1) with the resonance asymptote  $x = x^*$  defined on the interval  $(-\infty, x_*)$  and tending to 0 as  $x \rightarrow -\infty$ .

**Theorem 6.** Suppose that condition (2) holds. Then for any finite or infinite  $x_* < x^*$  there exists a non-extensible solution  $y(x)$  of (1) with domain  $(x_*, x^*)$ .

*Remark 2.* In [4], [7] asymptotic behavior of all possible solutions to (1) is described.

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# On the Convergence of Difference Schemes for Generalized BBM Equation

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We consider the initial boundary value problem for generalized Benjamin–Bona–Mahony (BBM) equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \gamma \frac{\partial(u)^m}{\partial x} - \mu \frac{\partial^3 u}{\partial x^2 \partial t} = 0, \quad (x, t) \in Q_T, \quad (1)$$

$$u(a, t) = u(b, t) = 0, \quad t \in [0, T], \quad u(x, 0) = \varphi(x), \quad x \in [a, b], \quad (2)$$

where  $\gamma$  and  $\mu$  are positive constants,  $m \geq 2$  is a positive integer, and  $Q_T := (a, b) \times (0, T)$ . In the cases  $m=2,3$  (1) represents the BBM (or regularized long-wave) and modified BBM equations, respectively.

For convenience we introduce the notation

$$x_i = a + ih, \quad t_j = j\tau, \quad i = 0, 1, 2, \dots, n, \quad j = 0, 1, 2, \dots, J,$$

where  $h = (b - a)/n$  and  $\tau = T/J$  denote the spatial and the temporal mesh size, respectively. Let  $w_i^j := u(x_i, t_j)$ ,  $U_i^j \sim u(x_i, t_j)$ ,

$$(U_i^j)_x := \frac{U_{i+1}^j - U_i^j}{h}, \quad (U_i^j)_{\bar{x}} := \frac{U_i^j - U_{i-1}^j}{h}, \quad (U_i^j)_{\circlearrowleft} := \frac{1}{2}((U_i^j)_x + (U_i^j)_{\bar{x}}),$$

$$(U_i^j)_t := \frac{U_i^{j+1} - U_i^j}{\tau}, \quad (U_i^j)_{\bar{t}} := \frac{U_i^j - U_i^{j-1}}{\tau}, \quad (U_i^j)_{\circlearrowright} := \frac{1}{2}((U_i^j)_t + (U_i^j)_{\bar{t}}),$$

$$(U^j, V^j) := \sum_{i=1}^{n-1} h U_i^j V_i^j, \quad (U^j, V^j] := \sum_{i=1}^n h U_i^j V_i^j,$$

$$\|U^j\|^2 := (U^j, U^j), \quad \|U^j\|^2 := (U^j, U^j], \quad \|U^j\|_{W_2^1}^2 := \|U_{\bar{x}}^j\|^2 + \|U^j\|^2.$$

We approximate the problem (1), (2) with the help of the difference scheme:

$$\mathcal{L}U_i^j = 0, \quad i = 1, 2, \dots, n-1, \quad j = 0, 1, \dots, J-1, \quad (3)$$

$$U_0^j = U_n^j = 0, \quad j = 0, 1, \dots, J, \quad U_i^0 = \varphi(x_i), \quad i = 0, 1, \dots, n, \quad (4)$$

where

$$\begin{aligned} \mathcal{L}U_i^j &:= (U_i^j)_{\circlearrowright} + \frac{1}{2}(U_i^{j+1} + U_i^{j-1})_{\circlearrowleft} + \\ &+ \frac{\gamma m}{2(m+1)} \Lambda U_i^j - \mu (U_i^j)_{\bar{x}xt}, \quad i = \overline{1, n-1}, \quad j = 1, 2, \dots, J-1, \end{aligned}$$

$$\begin{aligned} \mathcal{L}U_i^0 &:= (U_i^0)_t + \frac{1}{2}(U_i^1 + U_i^0)_{\circlearrowleft} + \\ &+ \frac{\gamma m}{2(m+1)} \Lambda U_i^0 - \mu (U_i^0)_{\bar{x}xt}, \quad i = \overline{1, n-1}, \quad j = 0, \end{aligned}$$

$$\Lambda U_i^j := (U_i^j)^{m-1} (U_i^{j+1} + U_i^{j-1})_{\circlearrowleft} + ((U_i^j)^{m-1} (U_i^{j+1} + U_i^{j-1}))_{\circlearrowleft}, \quad j = 1, 2, \dots, J-1,$$

$$\Lambda U_i^0 := (U_i^0)^{m-1} (U_i^1 + U_i^0)_{\circlearrowleft} + ((U_i^0)^{m-1} (U_i^1 + U_i^0))_{\circlearrowleft}.$$

The obtained algebraic equations are linear with respect to the values of desired function for each new level.

**Theorem 1.** *The finite difference scheme (3), (4) is uniquely solvable and possesses the following invariant*

$$E^j := \|U^j\|^2 + \mu \|U_{\bar{x}}^j\|^2 = \|\varphi\|^2 + \mu \|\varphi_{\bar{x}}\|^2 := E^0, \quad j = 1, 2, \dots$$

**Definition.** Let  $U, V$  are solutions of a difference scheme respectively with any initial data  $U^0, V^0$ . If there exists a constant  $c(T) > 0$ , independent of mesh sizes  $\tau$  and  $h$ , such that

$$\|U^j - V^j\|_{1_h} \leq c(T) \|U^0 - V^0\|_{2_h}, \quad j \geq 1,$$

where  $\|\cdot\|_{1_h}$  and  $\|\cdot\|_{2_h}$  are suitable norms on the set of grid functions, then we say that difference scheme is stable with respect to initial data. A difference scheme is said to be absolutely stable if it is stable for any  $\tau$  and  $h$ .

**Theorem 2.** *Difference scheme (3), (4) is absolutely stable with respect to initial data.*

Using procedure proposed by R. D. Lazarov, V. L. Makarov and A. A. Samarskii [1] and developed in [2], we obtain convergence rate estimates that are compatible with the smoothness of the desired solution.

**Theorem 3.** *Let the exact solution of the initial-boundary value problem (1), (2) belong to  $W_2^k(Q_T)$ ,  $k > 1$ . Then the convergence rate of the finite difference scheme (3), (4) is determined by the estimate*

$$\|U^j - u^j\|_{W_2^1} \leq c(\tau^{k-1} + h^{k-1}) \|u\|_{W_2^k(Q_T)}, \quad 1 < k \leq 3,$$

where  $c = c(u)$  denotes positive constant, independent of  $h$  and  $\tau$ .

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# Asymptotic Behavior of Solutions of Differential Equations of the

$$\text{Type } y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)})$$

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The differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \quad (1)$$

where  $\alpha_0 \in \{-1, 1\}$ ,  $p : [a, \omega[ \rightarrow ]0, +\infty[$  ( $-\infty < a < \omega \leq +\infty$ ),  $\varphi_i : \Delta_{Y_i} \rightarrow ]0, +\infty[$  ( $i = 0, \dots, n$ ) are the continuous functions,  $Y_i \in \{0, \pm\infty\}$ ,  $\Delta_{Y_i}$  is either the interval  $[y_i^0, Y_i[$ <sup>2</sup> or the interval  $]Y_i, y_i^0]$ , is considered. We suppose also that every  $\varphi_i(z)$  is regularly varying as  $z \rightarrow Y_i$  ( $z \in \Delta_{Y_i}$ ) of index  $\sigma_i$  and  $\sum_{i=0}^{n-1} \sigma_i \neq 1$ .

According to the type of the functions  $\varphi_0, \dots, \varphi_{n-1}$  it is clear that the equation (1) is in some sense similar to the well known differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} |y^{(i)}|^{\sigma_i}. \quad (2)$$

We call the solution  $y$  of the equation (1) the  $P_\omega(\lambda_{n-1}^0)$ -solution, where  $-\infty \leq \lambda_{n-1}^0 \leq +\infty$ , if the following conditions take place

$$y^{(i)} : [t_0, \omega[ \rightarrow \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \quad (i = 0, \dots, n-1), \quad \lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_{n-1}^0.$$

All  $P_\omega(\lambda_{n-1}^0)$ -solutions of the equation (2) were investigated in [2], [3]. In case  $n = 2$  for all  $P_\omega(\lambda_{n-1}^0)$ -solutions of the equation (1) the necessary and sufficient conditions of existence and asymptotic representations as  $t \uparrow \omega$  were found later [4]–[7]. The methods of this investigations are used in this work for the equation (1), where  $n \geq 2$ .

The cases  $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}\}$  are singular by the studying of  $P_\omega(\lambda_{n-1}^0)$ -solutions of the equation (1).  $P_\omega(\lambda_{n-1}^0)$ -solutions, where  $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}\}$  are regularly varying functions as  $t \uparrow \omega$  of indexes  $\{0, 1, \dots, n-1\}$ . To investigate such solutions we have to put additional conditions on the functions  $\varphi_0, \dots, \varphi_{n-1}$  and the function  $p$ . The necessary and sufficient conditions of the existence of  $P_\omega(\lambda_{n-1}^0)$ -solutions of the equation (1) for  $\lambda_{n-1}^0 \in \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}\}$  are found in this work. The asymptotic representations as  $t \uparrow \omega$  for such solutions and their derivatives from the first to  $(n-1)$ -th order are found too. We will illustrate this results for the case  $\lambda_0 = 0$ , that is one of the most difficult for investigation.

We call the slowly varying as  $z \rightarrow Y$  ( $z \in \Delta$ ) function  $\theta$  satisfies the condition  $S$  if for every continuously differentiable function  $L : \Delta \rightarrow ]0, +\infty[$  such that  $\lim_{\substack{z \rightarrow Y \\ z \in \Delta}} \frac{zL'(z)}{L(z)} = 0$ , the next representation takes place

$$\theta(zL(z)) = \theta(z)[1 + o(1)] \quad \text{as } z \rightarrow Y \quad (z \in \Delta).$$

<sup>1</sup>If  $\omega > 0$  we will take  $a > 0$ .

<sup>2</sup>If  $Y_i = +\infty$  ( $Y_i = -\infty$ ) we take  $y_i^0 > 0$  ( $y_i^0 < 0$ ) correspondingly.

Let us introduce subsidiary notations.

$$C = \frac{1}{1 - \sigma_{n-1}}, \quad \mu = \sum_{i=0}^{n-3} (i + 2 - n)\sigma_i,$$

$$\pi_\omega(t) = \begin{cases} t & \text{for } \omega = +\infty, \\ t - \omega & \text{for } \omega < +\infty, \end{cases} \quad \theta_{n-1}(z) = \varphi_{n-1}(z)|z|^{-\sigma_{n-1}},$$

$$I_0(t) = \int_{A_\omega^0}^t p(\tau) d\tau, \quad I_1(t) = \int_{A_\omega^1}^t \left| \frac{I_0(\tau)}{C|\pi_\omega(\tau)|^\mu} \theta_{n-1} \left( \frac{|I_0(\tau)|^C}{y_{n-1}^0} \right) \right|^C d\tau,$$

$$A_\omega^0 = \begin{cases} a, & \text{if } \int_a^\omega p(\tau) d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega p(\tau) d\tau < +\infty, \end{cases} \quad A_\omega^1 = \begin{cases} a, & \text{if } \int_a^\omega \left| \frac{I_0(\tau)}{|\pi_\omega(\tau)|^\mu} \theta_{n-1} \left( \frac{|I_0(\tau)|^C}{y_{n-1}^0} \right) \right|^C d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega \left| \frac{I_0(\tau)}{|\pi_\omega(\tau)|^\mu} \theta_{n-1} \left( \frac{|I_0(\tau)|^C}{y_{n-1}^0} \right) \right|^C d\tau < +\infty. \end{cases}$$

The following conclusion takes place for the equation (1).

**Theorem.** *Let the function  $\theta_{n-1}$  satisfy the condition S and  $\sigma_{n-1} \neq 1$ . Then the following conditions are necessary for the existence of  $P_\omega(0)$ -solutions of the equation (1):*

$$\lim_{t \uparrow \omega} \frac{I_1'(t)I_0(t)}{p(t)I_1(t)} = 0, \quad \lim_{t \uparrow \omega} y_{n-1}^0 |I_0(t)|^C = Y_{n-1}, \quad (3)$$

$$\lim_{t \uparrow \omega} y_{n-2}^0 |I_1(t)|^{(1-\sigma_{n-1})/(1-\sum_{j=0}^{n-1} \sigma_j)} = Y_{n-2}, \quad \lim_{t \uparrow \omega} y_i^0 |\pi_\omega(t)|^{n-i-2} = Y_i, \quad (4)$$

$$\alpha_0 y_{n-1}^0 C I_0(t) > 0, \quad y_{n-2}^0 y_{n-1}^0 I_1(t) > 0, \quad y_i^0 y_{i+1}^0 (n-i-1)(n-i-2) > 0 \text{ if } t \in [a, \omega[, \quad (5)$$

where  $i = 0, \dots, n-3$ . If there exists a finite or an infinite limit  $\lim_{t \uparrow \omega} \frac{\pi_\omega(t)p(t)}{I_0(t)}$ , then the conditions (3)–(5) are sufficient for the existence of such solutions of the equation (1). Moreover, for any  $P_\omega(0)$ -solution of (1) the following asymptotic representations

$$\frac{y^{(n-1)}(t)}{\prod_{j=0}^{n-1} \varphi_j(y^{(j)}(t))} = \alpha_0 (1 - \sigma_{n-1}) I_0(t) [1 + o(1)],$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{I_1'(t)}{I_1(t)} [1 + o(1)], \quad \frac{y^{(i)}(t)}{y^{(n-2)}(t)} = \frac{[\pi_\omega(t)]^{n-i-2}}{(n-i-2)!} [1 + o(1)]$$

hold as  $t \uparrow \omega$ , where  $i = 0, \dots, n-3$ .

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# On Stability of Delay Systems

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Consider the system

$$\begin{aligned} x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(t - \tau_{ij}(t)) &= f_i(t), \quad i = 1, \dots, n, \quad t \in [0, +\infty), \\ x_i(\theta) &= 0 \quad \text{for } \theta < 0. \end{aligned} \tag{1}$$

Its general solution has the following representation

$$x(t) = \int_0^t C(t, s)f(s) ds + X(t)\alpha,$$

where the  $n \times n$  matrix  $C(t, s)$  is called the Cauchy matrix of equation (1),  $X(t)$  is a  $n \times n$  fundamental matrix of the system

$$\begin{aligned} (M_i x)(t) \equiv x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(t - \tau_{ij}(t)) &= 0, \quad i = 1, \dots, n, \quad t \in [0, +\infty), \\ x_i(\theta) &= 0 \quad \text{for } \theta < 0, \end{aligned} \tag{10}$$

such that  $X(0) = I$  ( $I$  is the unit  $n \times n$  matrix),  $f = \text{col}(f_1, \dots, f_n)$ ,  $\alpha = \text{col}(\alpha_1, \dots, \alpha_n)$

**Definition 1.** Let us say that system (1) is exponentially stable if the Cauchy and fundamental matrices  $X(t) = \{X_{ij}(t, s)\}_{i,j=1,\dots,n}$  and  $C(t, s) = \{C_{ij}(t, s)\}_{i,j=1,\dots,n}$  satisfy the exponential estimate, i.e. there exist  $N$  and  $\gamma$  such that

$$|C_{ij}(t, s)| \leq Ne^{-\gamma(t-s)}, \quad |X_{ij}(t)| \leq Ne^{-\gamma t},$$

for  $0 \leq s \leq t < +\infty$ ,  $i, j = 1, \dots, n$ .

**Theorem 1.** *Let the following conditions be fulfilled:*

- 1)  $p_{ij} \leq 0$  for  $i \neq j$ ,  $i, j = 1, \dots, n - 1$ ;
- 2)  $\int_{t-\tau_{ii}(t)}^t p_{ii}(s)ds \leq \frac{1}{e}$  for  $i = 1, \dots, n - 1$ ;
- 3)  $p_{jn} \geq 0$ ,  $p_{nj} \leq 0$  for  $j = 1, \dots, n - 1$ ,  $p_{nn} \geq 0$ ;
- 4) *there exist positive  $\alpha$  and  $\beta_i$  such that*

$$\begin{aligned} p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j e^{\alpha\tau_{nj}(t)} &\leq \alpha \leq \\ &\leq \min_{1 \leq i \leq n-1} \left\{ -p_{in}(t) \frac{1}{\beta_i} e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t) \frac{\beta_j}{\beta_i} e^{\alpha\tau_{jn}(t)} \right\}, \quad t \in [0, +\infty). \end{aligned}$$

Then the elements of the  $n$ -th row of the Cauchy matrix of system (1) satisfy the inequalities:  $C_{nn}(t, s) > 0$ ,  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n-1$ ,  $0 \leq s \leq t < +\infty$ . If in addition there exist positive  $\varepsilon$  and  $z_1, \dots, z_{n-1}$  such that

$$\begin{aligned} \sum_{j=1}^{n-1} p_{ij}(t) \geq \varepsilon > 0, \quad p_{in}(t) - \sum_{j=1}^{n-1} p_{ij}(t)z_j \geq \varepsilon > 0, \quad i = 1, \dots, n-1, \\ p_{nn}(t) + \sum_{j=1}^{n-1} |p_{nj}(t)|z_j \geq \varepsilon > 0, \end{aligned} \quad t \in [0, +\infty), \quad (2)$$

then system (1) is exponentially stable.

*Remark 1.* It was assumed in condition 3) that  $p_{jn} \geq 0$ , for  $j = 1, \dots, n-1$ , and Wazewskii's condition, generally speaking, is not fulfilled.

*Remark 2.* A possible case is  $p_{nn} = 0$  and the principle of main diagonal dominance (even in its generalized form, assuming, for example, that the matrix  $\{p_{ij}\}_{i,j=1,\dots,n}$  is  $M$ -matrix), is not fulfilled.

*Remark 3.* The inequality

$$p_{nn}(t) + \sum_{j=1}^{n-1} |p_{nj}(t)|\beta_j > 0, \quad t \in [0, +\infty)$$

cannot be set instead of the inequality

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t)\beta_j \geq \varepsilon > 0, \quad t \in [0, +\infty)$$

in the condition (2) of Theorem 1. Actually in the case  $p_{nj}(t) \equiv 0$  for  $j = 1, \dots, n-1$  and  $p_{nn}(t) = \frac{1}{t^2}$ , the component  $x_n(t)$  of the solution vector of the homogeneous system (1<sub>0</sub>) does not tend to zero when  $t \rightarrow +\infty$ .

*Remark 4.* Conditions 1) and 2) imply that all elements of the Cauchy matrix  $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1,\dots,n-1}$  of auxiliary system

$$x'_i(t) + \sum_{j=1}^{n-1} p_{ij}(t)x_j(t - \tau_{ij}(t)) = f_i(t), \quad i = 1, \dots, n-1, \quad t \in [0, +\infty), \quad (3)$$

are nonnegative.

*Remark 5.* The inequality

$$\sum_{j=1}^{n-1} p_{ij}(t) \geq \varepsilon > 0, \quad t \in [0, +\infty)$$

implies that the Cauchy matrix of the auxiliary system (3) satisfies the exponential estimate and this is essential in the case of separated  $n$ -th equation, i.e. when  $p_{nj}(t) \equiv 0$ ,  $p_{jn}(t) \equiv 0$  for  $j = 1, \dots, n-1$ .

# On Well-Posedness with Respect to Functional for an Optimal Control Problem with Distributed Delays

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Let  $R_x^n$  be an  $n$ -dimensional vector space of points  $x = (x^1, \dots, x^n)^T$ , where  $T$  means transpose;  $O \subset R_x^n$  and  $V \subset R_u^r$  be open sets; the  $(n + 1)$ -dimensional function  $F(t, x, u) = (f^0, f)^T$  be continuous on the set  $I \times O \times V$  and continuously differentiable with respect to  $x$ , where  $I = [t_0, t_1]$ ; next,  $\tau > 0$ ,  $\theta > 0$  given numbers, let  $\Phi$  and  $\Omega$  be sets of continuous initial functions  $\varphi(t) \in O, t \in [t_0 - \tau, t_0]$  and measurable control functions  $u(t) \in U, t \in [t_0 - \theta, t_1]$ , respectively, where  $U \subset V$  is a compact set.

Let  $x_0 \in O$  and  $\varphi \in \Phi$  be fixed initial vector and function. To each control function  $u \in \Omega$  we assign the differential equation with distributed delays

$$\dot{x}(t) = \int_{-\tau}^0 \int_{-\theta}^0 f(t, x(t+s), u(t+\xi)) ds d\xi, \quad t \in I \quad (1)$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0), \quad x(t_0) = x_0. \quad (2)$$

**Definition 1.** Let  $u \in \Omega$  be a given control. A function  $x(t) = x(t; u) \in O, t \in [t_0 - \tau, t_1]$ , is called a solution of Eq. (1) with the initial condition (2) or a solution corresponding to  $u$  and defined on  $[t_0 - \tau, t_1]$ , if it satisfies condition (2) and is absolutely continuous on the interval  $[t_0, t_1]$  and satisfies Eq. (1) almost everywhere on  $[t_0, t_1]$ .

**Definition 2.** A control  $u \in \Omega$  is said to be admissible if the corresponding solution  $x(t) = x(t; u)$  is defined on the interval  $[t_0 - \tau, t_1]$ .

We denote the set of admissible controls by  $\Omega_0$ .

**Definition 3.** A control  $u_0 \in \Omega_0$  is said to be optimal if for any  $u \in \Omega_0$  we have

$$J(u_0) \leq J(u),$$

where

$$J(u) = \int_{t_0}^{t_1} \left\{ \int_{-\tau}^0 \int_{-\theta}^0 f^0(t, x(t+s), u(t+\xi)) ds d\xi \right\} dt. \quad (3)$$

The optimal control  $u_0$  is called solution of the problem (1)–(3).

Introduce the following notations: By  $Y$  we denote the set of continuous functions  $y(t) \in O, t \in I_1 = [t_0 - \tau, t_0) \cup (t_0, t_1]$ , with  $cl y(I_1) \subset O$  is the compact set;

$$F(t, y(\cdot), u) = \int_{-\tau}^0 F(t, y(t+s), u) ds, \quad y \in Y, P(t, y(\cdot)) = \{F(t, y(\cdot), u) : u \in U\}.$$

**Theorem 1.** *Let the following conditions hold:  $\Omega_0 \neq \emptyset$ ; there exists a compact set  $K_0 \subset O$  such that  $x(t; u) \in K_0, t \in [t_0 - \tau, t_1], \forall u \in \Omega_0$ ; for any  $(t, y) \in I \times Y$  the set  $P(t, y(\cdot))$  is convex. Then the problem (1)–(3) has a solution  $u_0$ .*



Let  $K_1 \subset O$  be a compact set containing a certain neighborhood of the set  $K_0$ . By  $E$  we denote the set of functions  $G(t, x, y) = (g^0, g)^T \in R_x^{n+1}$  continuous on the set  $I \times O$  and continuously differentiable with respect to  $x$  and satisfying the condition

$$\int_I |G_\delta(t, x)| dt \leq \text{const}, \quad \forall x \in K_1.$$

**Theorem 2.** *Let the conditions of Theorem 1 hold. Then for any  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that for an arbitrary vector  $x_{0\delta} \in O$  and functions  $\varphi_\delta \in \Phi$ ,  $G_\delta \in E$  satisfying the conditions*

$$|x_0 - x_{0\delta}| + \|\varphi - \varphi_\delta\| + \|G_\delta\| \leq \delta,$$

the perturbed optimal problem

$$\begin{aligned} \dot{x}(t) &= \int_{-\tau}^0 \int_{-\theta}^0 \left\{ f(t, x(t+s), u(t+\xi)) + g(t, x(t+s)) \right\} ds d\xi, \quad t \in I_1, \\ x(t) &= \varphi_\delta(t), \quad t \in [t_0 - \tau, t_0), \quad x(t_0) = x_{0\delta}, \\ J(u, \delta) &= \int_{t_0}^{t_1} \left\{ \int_{-\tau}^0 \int_{-\theta}^0 \left[ f^0(t, x(t+s), u(t+\xi)) + g^0(t, x(t+s)) \right] ds d\xi \right\} dt \end{aligned}$$

has a solution  $u_{0\delta}$  and the following inequality

$$|J(u_0) - J(u_{0\delta})| \leq \varepsilon$$

is fulfilled. Here

$$\begin{aligned} \|\varphi - \varphi_\delta\| &= \sup \{ |\varphi(t) - \varphi_\delta(t)| : t \in I_1 \}, \\ \|G_\delta\| &= \sup \left\{ \left| \int_{t'}^{t''} g(t, x) dt \right| : t', t'' \in I, x \in K_1 \right\}. \end{aligned}$$

Theorems 1 and 2 are proved by scheme given in [1], [2].

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# Asymptotic Representations of Solutions of Ordinary Differential Equations of $N$ -th Order with Regularly Varying Nonlinearities

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We consider the differential equation

$$y^{(n)} = \sum_{k=1}^m \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}), \quad (1)$$

where  $\alpha_k \in \{-1; 1\}$  ( $k = \overline{1, m}$ ),  $p_k : [a, \omega[ \rightarrow ]0, +\infty[$  ( $k = \overline{1, m}$ ) are continuous functions,  $\varphi_{kj} : \Delta_{Y_j} \rightarrow ]0, +\infty[$  ( $k = \overline{1, m}; j = \overline{0, n-1}$ ) are continuous and regularly varying at  $y^{(j)} \rightarrow Y_j$  functions of orders  $\sigma_{kj}$ ,  $-\infty < a < \omega \leq +\infty$ ,  $\Delta_{Y_j}$  one-sided neighbourhood  $Y_j$ ,  $Y_j$  is either 0, or  $\pm\infty$ .

Continuous function  $\varphi : \Delta_Y \rightarrow ]0, +\infty[$ , where  $Y$  is either 0, or  $\pm\infty$  and  $\Delta_Y$  one-sided neighbourhood  $Y$ , is regularly varying at  $y \rightarrow Y$ , if there exists a number  $\sigma \in \mathbb{R}$  such that  $\lim_{\substack{y \rightarrow Y \\ y \in \Delta_Y}} \frac{\varphi(\lambda y)}{\varphi(y)} = \lambda^\sigma$  for any  $\lambda > 0$ . In this case the number  $\sigma$  is called the order of regularly varying function. Regularly varying at  $y \rightarrow Y$  zero-order function is called slowly changing function.

Since each regularly varying at  $y \rightarrow Y$  function  $\varphi$  of  $\sigma$  order is represented as  $\varphi(y) = |y|^\sigma L(y)$ , where  $L$ - slowly changing function at  $y \rightarrow Y$ , then the differential equation (1) is asymptotically close at  $y^{(j)} \rightarrow Y_j$  ( $j = \overline{0, n-1}$ ) to the equation

$$y^{(n)} = \sum_{k=1}^m \alpha_k p_k(t) \prod_{j=0}^{n-1} |y^{(j)}|^{\sigma_{kj}}.$$

Asymptotic behavior of certain classes of solutions of those equation is investigated in works [1], [2].

A solution  $y$  of the equation (1) is called  $P_\omega(Y_1, \dots, Y_{n-1}, \lambda_0)$ - solution, where  $-\infty \leq \lambda_0 \leq +\infty$ , if it is defined on an interval  $[t_0, \omega[ \subset [a, \omega[$  and satisfies the following conditions

$$y^{(j)}(t) \in \Delta_{Y_j} \text{ at } t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \quad (j = \overline{0, n-1}), \quad \lim_{t \uparrow \omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0.$$

Choose the numbers  $b_j \in \Delta_{Y_j}$  ( $j = \overline{0, n-1}$ ) such that  $|b_j| < 1$  at  $Y_j = 0$ ,  $b_j > 1$  ( $b_j < -1$ ) at  $Y_j = +\infty$  ( $Y_j = -\infty$ ), and introduce the numbers

$$\nu_{0j} = \text{sign } b_j, \quad \nu_{1j} = \begin{cases} 1, & \text{if } \Delta_{Y_j} \text{ - left neighborhood } Y_j, \\ -1, & \text{if } \Delta_{Y_j} \text{ - right neighborhood } Y_j, \end{cases} \quad (j = \overline{0, n-1}).$$

Note that  $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$  solution of equation (1) satisfies the following conditions

$$\nu_{0j}\nu_{1j} < 0, \text{ if } Y_j = 0, \quad \nu_{0j}\nu_{1j} > 0, \text{ if } Y_j = \pm\infty \quad (j = \overline{0, n-1}). \quad (2)$$

Next, let

$$a_{0i} = (n-i)\lambda_0 - (n-i-1) \quad (i = 1, \dots, n) \text{ at } \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases}, \quad \beta = \begin{cases} 1, & \text{if } \omega = +\infty, \\ -1, & \text{if } \omega < +\infty, \end{cases},$$

$$\gamma_k = 1 - \sum_{j=0}^{n-1} \sigma_{kj}, \quad \mu_{kn} = \sum_{j=0}^{n-2} \sigma_{kj}(n-j-1), \quad C_k = \prod_{j=0}^{n-2} \left| \frac{(\lambda_0 - 1)^{n-j-1}}{\prod_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_{kj}} \quad (k = \overline{1, m}),$$

$$J_{kn}(t) = \int_{A_{kn}}^t p_k(\tau) |\pi_\omega(\tau)|^{\mu_{kn}} d\tau, \quad A_{kn} = \begin{cases} a, & \text{if } \int_a^\omega p_k(t) |\pi_\omega(t)|^{\mu_{kn}} dt = +\infty, \\ \omega, & \text{if } \int_a^\omega p_k(t) |\pi_\omega(t)|^{\mu_{kn}} dt < +\infty \end{cases} \quad (k = \overline{1, m}).$$

**Theorem.** Let  $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\}$  and for some  $s \in \{1, \dots, m\}$  the inequality  $\gamma_s \neq 0$  and the conditions

$$\limsup_{t \uparrow \omega} \frac{\ln p_k(t) - \ln p_s(t)}{|\ln |\pi_\omega(t)||} < \frac{\beta}{\lambda_0 - 1} \sum_{j=0}^{n-1} (\sigma_{sj} - \sigma_{kj}) a_{0j+1} \quad \text{for all } k \in \{1, \dots, m\} \setminus \{s\}$$

be fulfilled. Then for  $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$  solutions of equation (1) to exist it is necessary and if algebraic relatively  $\rho$  equation

$$\sum_{j=0}^{n-1} \sigma_{sj} \prod_{i=j+1}^{n-1} a_{0i} \prod_{i=1}^j (a_{0i} + \rho) = (1 + \rho) \prod_{i=1}^{n-1} (a_{0i} + \rho)$$

does not have roots with zero real part, it is also sufficient that inequality (2), the inequalities

$$\nu_{0j} \nu_{0j+1} a_{0j+1} (\lambda_0 - 1) \pi_\omega(t) > 0 \quad (j = \overline{0, n-2}), \quad \alpha_s \nu_{0n-1} \gamma_s J_{sn}(t) > 0 \quad \text{at } t \in ]a, \omega[$$

and the condition

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'_{sn}(t)}{J_{sn}(t)} = \frac{\gamma_s}{\lambda_0 - 1}$$

are satisfied. Moreover, for each such a solution at  $t \uparrow \omega$  the asymptotic representations

$$y^{(j)}(t) = \frac{[(\lambda_0 - 1) \pi_\omega(t)]^{n-j-1}}{\prod_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t) [1 + o(1)] \quad (j = 0, 1, \dots, n-2),$$

$$\frac{|y^{(n-1)}(t)|^{\gamma_s}}{\prod_{j=0}^{n-1} L_{sj} \left( \frac{[(\lambda_0 - 1) \pi_\omega(t)]^{n-j-1}}{\prod_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t) \right)} = \alpha_s \nu_{0n-1} \gamma_s C_s J_{sn}(t) [1 + o(1)]$$

hold, where  $L_{sj}(y^{(j)}) = |y^{(j)}|^{-\sigma_{sj}} \varphi_{sj}(y^{(j)})$  ( $j = \overline{0, n-1}$ ).

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# On Mixed Type Quasi-Linear Equations with General Integrals, Represented by Superposition of Arbitrary Functions

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We discuss second-order quasi-linear equations with two independent variables  $x, t$ . The coefficients of the principal part of the equation are assumed to be polynomials with respect to unknown solutions and to their derivatives of the first order, while the characteristic roots expressed by these values are always real. Depending on the solutions, on some sets of points these roots may coincide. Therefore the equations under consideration should be referred to a class of equations of mixed type, in particular, to hyperbolic equations with admissible parabolic degeneration. For every particular equation of such a type, a structure of points of parabolic degeneration depends completely on the data of the initial, or of some other formulated problem. In the present report we will consider equations which admit explicit representation of general integrals in the form of superposition of arbitrary functions. A number of these arbitrary elements is equal to two, and their arguments, except independent variables, contain an unknown solution as well. Such is, for example, the equation

$$\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \left[ u \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} \right] \frac{\partial u^2}{\partial x \partial t} - u \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial t^2} = 0, \quad (1)$$

having certain practical application. A class of its hyperbolic solutions is defined by the condition

$$H \equiv \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial t} \neq 0. \quad (2)$$

If  $H$  is identical zero for some solution, then the solution itself is parabolic. If, however, the condition (2) is fulfilled not everywhere, then the corresponding solution is called mixed one. A general integral of the equation has the form

$$u f'(u) - f(u) + g[x - f(u)] = t, \quad (3)$$

where  $f, g \in C^2(R^1)$  are arbitrary functions. Consider now the Cauchy problem with the initial conditions

$$u|_{t=0} = \tau(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \nu(x), \quad (x, t) \in H, \quad \tau, \nu \in C^2(J) \quad (4)$$

on the data support

$$J := \{(x, t) : t = 0, \alpha \leq x \leq \beta\}.$$

The functional equations

$$\tau(x) = \zeta, \quad \int_{\alpha}^x \frac{\tau(z)\nu(z)}{\tau'(z) + \tau(z)\nu(z)} dz = \xi \quad (5)$$

with respect to  $x$  are assumed to be uniquely solvable and we denote their solutions by, respectively,  $x = T(\zeta)$  and  $x = G(\xi)$ . In this case, subjecting the general integral (3) to the conditions (4), one will be able to define arbitrary functions in terms of the initial perturbations

$$f[\zeta] = f[\tau(\alpha)] + f'[\tau(\alpha)][\tau(T(\zeta)) - \tau(\alpha)] + \int_{\alpha}^{T(\zeta)} \frac{[\tau(T(\zeta)) - \tau(\zeta)]\tau'(z)}{\tau'(z) + \tau(z)\nu(z)} dz,$$

$$g(\xi) = g[\alpha - f'(\alpha)] + \int_{\alpha}^{G(\xi)} \frac{\tau(z)\tau'(z)}{\tau'(z) + \tau(z)\nu(z)} dz.$$

Having defined in such a way arbitrary functions and substituting them into the representation (3), we construct the integral of the problem (1), (4). To describe a structure of the domain of definition of the integral of the problem, we use the properties of characteristic invariants, represented in the given case by the right-hand sides of the functional equations (5), in particular, the fact that along any characteristic of the corresponding family they are constant. A global character of that property allows one to construct all characteristics emanated from the points of the data support of the problem (1), (4) and, consequently, the domain of definition of the integral by means of a set of points of intersection of these characteristics.

# Fredholm Type Theorem for Systems of Functional-Differential Equations with Positively Homogeneous Operators

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Consider the system of functional-differential equations

$$u'_i(t) = p_i(u_1, \dots, u_n)(t) + f_i(u_1, \dots, u_n)(t) \text{ for a. e. } t \in [a, b] \quad (i = 1, \dots, n) \quad (1)$$

together with boundary conditions

$$\ell_i(u_1, \dots, u_n) = h_i(u_1, \dots, u_n) \quad (i = 1, \dots, n). \quad (2)$$

Here,  $p_i, f_i : [C([a, b]; \mathbb{R})]^n \rightarrow L([a, b]; \mathbb{R})$  are continuous operators satisfying Carathéodory condition, i.e., they are bounded on every ball by an integrable function, and  $\ell_i, h_i : [C([a, b]; \mathbb{R})]^n \rightarrow \mathbb{R}$  are continuous functionals which are bounded on every ball by a constant. Furthermore, we assume that  $p_i$  and  $\ell_i$  satisfy the following condition: there exist positive real numbers  $\lambda_{ij}$  and  $\mu_i$  such that  $\lambda_{ij}\lambda_{jm} = \lambda_{im}$  whenever  $i, j, m \in \{1, \dots, n\}$ , and for every  $c > 0$  and  $u_k \in C([a, b]; \mathbb{R})$  ( $k = 1, \dots, n$ ) we have

$$\begin{aligned} cp_i(u_1, \dots, u_n)(t) &= p_i(c^{\lambda_{i1}}u_1, \dots, c^{\lambda_{in}}u_n)(t) \text{ for a. e. } t \in [a, b], \\ c^{\mu_i}\ell_i(u_1, \dots, u_n) &= \ell_i(c^{\lambda_{i1}}u_1, \dots, c^{\lambda_{in}}u_n). \end{aligned}$$

By a solution to (1), (2) we understand an absolutely continuous vector-valued function  $(u_i)_{i=1}^n : [a, b] \rightarrow \mathbb{R}^n$  satisfying (1) almost everywhere in  $[a, b]$  and (2).

*Remark 1.* From the above-stated assumptions it follows that  $\lambda_{ii} = 1$ ,  $\lambda_{ij} = 1/\lambda_{ji}$  for every  $i, j \in \{1, \dots, n\}$ .

*Notation 1.* Define, for every  $i \in \{1, \dots, n\}$ , the following functions

$$\begin{aligned} q_i(t, \rho) &\stackrel{\text{def}}{=} \sup \left\{ |f_i(u_1, \dots, u_n)(t)| : \|u_k\|_C \leq \rho^{\lambda_{ik}}, k = 1, \dots, n \right\} \text{ for a. e. } t \in [a, b], \\ \eta_i(\rho) &\stackrel{\text{def}}{=} \sup \left\{ |h_i(u_1, \dots, u_n)| : \|u_k\|_C \leq \rho^{\frac{\lambda_{ik}}{\mu_i}}, k = 1, \dots, n \right\}. \end{aligned}$$

**Theorem 1.** *Let*

$$\lim_{\rho \rightarrow +\infty} \int_a^b \frac{q_i(s, \rho)}{\rho} ds = 0, \quad \lim_{\rho \rightarrow +\infty} \frac{\eta_i(\rho)}{\rho} = 0 \quad (i = 1, \dots, n). \quad (3)$$

*If the problem*

$$\begin{aligned} u'_i(t) &= (1 - \delta)p_i(u_1, \dots, u_n)(t) - \delta p_i(-u_1, \dots, -u_n)(t) \text{ for a. e. } t \in [a, b] \quad (i = 1, \dots, n), \\ (1 - \delta)\ell_i(u_1, \dots, u_n) - \delta\ell_i(-u_1, \dots, -u_n) &= 0 \quad (i = 1, \dots, n) \end{aligned}$$

*has only the trivial solution for every  $\delta \in [0, 1/2]$ , then the problem (1), (2) has at least one solution.*

*Sketch of proof.* There exists  $r > 0$  such that for any absolutely continuous vector-valued function  $(u_i)_{i=1}^n$  defined on  $[a, b]$  and every  $\delta \in [0, 1/2]$ , the a priori estimate

$$\sum_{k=1}^n \|u_k\|_C^{\lambda_{k1}} \leq r \sum_{i=1}^n \left( \|f_i\|_L^{\lambda_{i1}} + |\tilde{h}_i|^{\frac{\lambda_{i1}}{\mu_i}} \right)$$

holds, where

$$\begin{aligned} \tilde{f}_i(t) &\stackrel{def}{=} u_i'(t) - (1 - \delta)p_i(u_1, \dots, u_n)(t) + \delta p_i(-u_1, \dots, -u_n)(t) \text{ for a. e. } t \in [a, b] \quad (i = 1, \dots, n), \\ \tilde{h}_i &\stackrel{def}{=} (1 - \delta)\ell_i(u_1, \dots, u_n) - \delta\ell_i(-u_1, \dots, -u_n) \quad (i = 1, \dots, n). \end{aligned}$$

Put

$$\begin{aligned} x &= ((u_i)_{i=1}^n, (\alpha_i)_{i=1}^n) \in X = [C([a, b]; \mathbb{R})]^n \times \mathbb{R}^n \text{ with the norm } \|x\| = \sum_{k=1}^n (\|u_k\|_C + |\alpha_k|), \\ A(x) &\stackrel{def}{=} \left( \left( u_i(a) + \alpha_i + \int_a^t p_i(u_1, \dots, u_n)(s) + f_i(u_1, \dots, u_n)(s) ds \right)_{i=1}^n, \right. \\ &\quad \left. (\alpha_i + \ell_i(u_1, \dots, u_n) - h_i(u_1, \dots, u_n))_{i=1}^n \right), \\ \Omega &= \left\{ x \in X : \sum_{k=1}^n (\|u_k\|_C^{\lambda_{k1}} + |\alpha_k|) < \rho_0 \right\} \text{ for sufficiently large } \rho_0. \end{aligned}$$

Then using Krasnosel'skii theorem (see [1, Theorem 41.3, p. 325]):

$$\frac{A(x) - x}{\|A(x) - x\|} \neq \frac{A(-x) + x}{\|A(-x) + x\|} \quad (x \in \partial\Omega) \implies \exists x_0 \in \Omega \text{ such that } A(x_0) = x_0,$$

we prove the assertion of this theorem. For more detailed idea of the proof one can see [2].  $\square$

If the operators  $p_i$  and  $\ell_i$  are homogeneous, i.e., if

$$p_i(-u_1, \dots, -u_n)(t) = -p_i(u_1, \dots, u_n)(t) \text{ for a. e. } t \in [a, b], \quad u_j \in C([a, b]; \mathbb{R}) \quad (i, j = 1, \dots, n), \quad (4)$$

$$\ell_i(-u_1, \dots, -u_n) = -\ell_i(u_1, \dots, u_n) \quad u_j \in C([a, b]; \mathbb{R}) \quad (i, j = 1, \dots, n) \quad (5)$$

hold, then from Theorem 1 we obtain the following assertion.

**Corollary 1.** *Let (3), (4), and (5) be fulfilled. If the problem*

$$u_i'(t) = p_i(u_1, \dots, u_n)(t) \text{ for a. e. } t \in [a, b] \quad (i = 1, \dots, n), \quad (6)$$

$$\ell_i(u_1, \dots, u_n) = 0 \quad (i = 1, \dots, n) \quad (7)$$

*has only the trivial solution, then the problem (1), (2) has at least one solution.*

For a particular case when  $p_i$  are defined by

$$p_i(u_1, \dots, u_n)(t) \stackrel{def}{=} \tilde{p}_i(t) |u_{i+1}(t)|^{\lambda_i} \operatorname{sgn} u_{i+1}(t) \text{ for a. e. } t \in [a, b] \quad (i = 1, \dots, n-1), \quad (8)$$

$$p_n(u_1, \dots, u_n)(t) \stackrel{def}{=} \tilde{p}_n(t) |u_1(t)|^{\lambda_n} \operatorname{sgn} u_1(t) \text{ for a. e. } t \in [a, b], \quad (9)$$

where  $\tilde{p}_i \in L([a, b]; \mathbb{R})$ , we have the following assertion.

**Corollary 2.** *Let (3), (5), (8), and (9) be fulfilled. Let, moreover,*

$$\prod_{i=1}^n \lambda_i = 1,$$

*and let the problem (6), (7) have only the trivial solution. Then the problem (1), (2) has at least one solution.*

In [3], the problem (1), (2) is studied with  $n = 2$ ,  $p_1, p_2$  defined by (8), (9), and

$$\ell_1(u_1, \dots, u_n) \stackrel{\text{def}}{=} \int_a^{a_0} u_1(s) d\alpha_1(s), \quad \ell_2(u_1, \dots, u_n) \stackrel{\text{def}}{=} \int_{b_0}^b u_1(s) d\alpha_2(s), \quad (10)$$

resp.

$$\ell_1(u_1, \dots, u_n) \stackrel{\text{def}}{=} \int_a^{a_0} u_1(s) d\alpha_1(s), \quad \ell_2(u_1, \dots, u_n) \stackrel{\text{def}}{=} \int_{b_0}^b u_2(s) d\alpha_2(s), \quad (11)$$

where  $a < a_0 \leq b$ ,  $a \leq b_0 < b$ ,  $\alpha_1 : [a, a_0] \rightarrow \mathbb{R}$  and  $\alpha_2 : [b_0, b] \rightarrow \mathbb{R}$  are functions of bounded variation. For this particular case, Theorem 1 yields

**Corollary 3.** *Let  $n = 2$ , (3) be fulfilled,  $\lambda_1 \lambda_2 = 1$ ,  $k \in \{1, 2\}$ , and let*

$$\begin{aligned} u_1' &= \tilde{p}_1(t) |u_2|^{\lambda_1} \operatorname{sgn} u_2, & u_2' &= \tilde{p}_2(t) |u_1|^{\lambda_2} \operatorname{sgn} u_1, \\ \int_a^{a_0} u_1(s) d\alpha_1(s) &= 0, & \int_{b_0}^b u_k(s) d\alpha_1(s) &= 0 \end{aligned}$$

*have only the trivial solution. Then the problem (1), (2) with  $p_i$  and  $\ell_i$  defined by (8), (9), and (10), resp. (11) has at least one solution.*

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# Periodic Solution to Two-Dimensional Half-Linear System

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On the interval  $[0, \omega]$  we consider the system of differential equations

$$\begin{aligned} u_1' &= p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2 + f_1(t, u_1, u_2), \\ u_2' &= p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 + f_2(t, u_1, u_2), \end{aligned} \quad (1)$$

subjected to the periodic-type boundary conditions

$$u_1(0) - u_1(\omega) = h_1(u_1, u_2), \quad u_2(0) - u_2(\omega) = h_2(u_1, u_2). \quad (2)$$

Here,  $p_i \in L([0, \omega]; \mathbb{R})$ ,  $f_i \in \operatorname{Car}([0, \omega] \times \mathbb{R}^2; \mathbb{R})$ ,  $h_i : C([0, \omega]; \mathbb{R}) \times C([0, \omega]; \mathbb{R}) \rightarrow \mathbb{R}$  are continuous functionals bounded on every ball, and  $\lambda_i > 0$  such that  $\lambda_1 \lambda_2 = 1$ .

**Notation 1.** Define the following functions

$$\begin{aligned} q_i(t, \rho) &\stackrel{\text{def}}{=} \sup \left\{ |f_i(t, x_1, x_2)| : |x_i| \leq \rho, |x_{3-i}| \leq \rho^{\lambda_{3-i}} \right\} \text{ for a. e. } t \in [0, \omega] \quad (i = 1, 2), \\ \eta_i(\rho) &\stackrel{\text{def}}{=} \sup \left\{ |h_i(u_1, u_2)| : \|u_i\|_C \leq \rho, \|u_{3-i}\|_C \leq \rho^{\lambda_{3-i}} \right\} \quad (i = 1, 2). \end{aligned}$$

**Theorem 1.** *Let*

$$\lim_{\rho \rightarrow +\infty} \int_0^\omega \frac{q_i(s, \rho)}{\rho} ds = 0, \quad \lim_{\rho \rightarrow +\infty} \frac{\eta_i(\rho)}{\rho} = 0 \quad (i = 1, 2). \quad (3)$$

*Let, moreover,  $\sigma \in \{1, -1\}$  be such that*

$$\sigma p_1(t) \geq 0 \text{ for a. e. } t \in [0, \omega], \quad p_1 \not\equiv 0, \quad (4)$$

*and let there exist  $\alpha_i \in AC([0, \omega]; \mathbb{R})$  ( $i = 1, 2$ ) such that*

$$\begin{aligned} \alpha_1'(t) &= p_1(t)|\alpha_2(t)|^{\lambda_1} \operatorname{sgn} \alpha_2(t) \text{ for a. e. } t \in [0, \omega], \quad \alpha_1(0) = \alpha_1(\omega), \\ \alpha_2'(t) &\leq p_2(t)|\alpha_1(t)|^{\lambda_2} \operatorname{sgn} \alpha_1(t) \text{ for a. e. } t \in [0, \omega], \quad \alpha_2(0) \leq \alpha_2(\omega), \\ &\sigma \alpha_1(t) > 0 \text{ for } t \in [0, \omega], \\ \operatorname{meas} \left\{ t \in [0, \omega] : \alpha_2'(t) < p_2(t)|\alpha_1(t)|^{\lambda_2} \operatorname{sgn} \alpha_1(t) \right\} + \alpha_2(\omega) - \alpha_2(0) &> 0. \end{aligned}$$

*Then the problem (1), (2) has at least one solution.*

**Corollary 1.** *Let (3) and (4) be fulfilled with  $\sigma \in \{1, -1\}$ . Let, moreover,*

$$\begin{aligned} \int_0^\omega \sigma p_1(s) ds \left( \int_0^\omega [\sigma p_2(s)]_- ds \right)^{\lambda_1} &< 2^{1+\lambda_1}, \\ \int_0^\omega [\sigma p_2(s)]_- ds &< \int_0^\omega [\sigma p_2(s)]_+ ds \left( 1 - \frac{1}{2^{1+\lambda_1}} \int_0^\omega \sigma p_1(s) ds \left( \int_0^\omega [\sigma p_2(s)]_- ds \right)^{\lambda_1} \right)^{\lambda_2}. \end{aligned}$$

*Then the problem (1), (2) has at least one solution.*

*Remark 1.* Theorem 1 and Corollary 1 are applicable in the case when  $\int_0^\omega \sigma p_2(s) ds > 0$ . For the case when  $\int_0^\omega \sigma p_2(s) ds < 0$  one can use the following assertion.

**Theorem 2.** *Let (3) and (4) be fulfilled with  $\sigma \in \{1, -1\}$ . Let, moreover,*

$$\int_0^\omega \sigma p_2(s) ds < 0, \quad \int_0^\omega \sigma p_1(s) ds \left( \int_0^\omega [\sigma p_2(s)]_- ds \right)^{\lambda_1} < 4^{1+\lambda_1}.$$

*Then the problem (1), (2) has at least one solution.*

*Sketch of the proofs.* According to the general result established in [1] one can see that the following assertion holds:

**Proposition 1.** *Let (3) be fulfilled. If the problem*

$$u_1' = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \tag{5}$$

$$u_2' = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1,$$

$$u_1(0) - u_1(\omega) = 0, \quad u_2(0) - u_2(\omega) = 0 \tag{6}$$

*has only the trivial solution, then the problem (1), (2) has at least one solution.*

Then the conditions of Theorems 1 and 2 and Corollary 1 are obtained by direct analysing the non-trivial solutions of the problem (5), (6).  $\square$

*Remark 2.* Results obtained are unimprovable in that sense that neither of the strict inequalities established in Corollary 1 and Theorem 2 can be weakened.

*Remark 3.* When  $\lambda_i = 1$ ,  $p_1 \equiv 1$ ,  $h_i \equiv 0$ ,  $f_1 \equiv 0$ ,  $f_2(t, x, y) = f(t)$  for a. e.  $t \in [0, \omega]$ ,  $x, y \in \mathbb{R}$  with  $f \in L([0, \omega]; \mathbb{R})$ , then the problem (1), (2) becomes a periodic problem for the second-order linear equation

$$u'' = p_2(t)u + f(t), \quad u(0) = u(\omega), \quad u'(0) = u'(\omega).$$

In this case, Theorems 1 and 2, and Corollary 1 coincide with the results obtained in [2].

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# Necessary Conditions of Optimality for Optimal Problems with General Variable Delays and the Mixed Initial Condition

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Let  $R_x^n$  be an  $n$ -dimensional vector space of points  $x = (x^1, \dots, x^n)^T$ , where  $T$  means transpose;  $P \subset R_p^k$ ,  $Z \subset R_z^e$  and  $V \subset R_u^r$  be open sets and  $O = \{x = (p, z)^T \in R_x^n : p \in R_p^k, z \in R_z^e\}$ , with  $k + e = n$ ; the  $n$ -dimensional function  $f(t, p_1, \dots, p_s, z_1, \dots, z_m, u_1, \dots, u_\nu)$  be continuous on the set  $[a, b] \times P^s \times Z^m \times V^\nu$  and continuously differentiable with respect to  $p_i, i = \overline{1, s}$  and  $z_j, j = \overline{1, m}$ ; the functions  $q^i(t_0, t_1, p, z, x), i = \overline{0, l}$  be continuously differentiable on the set  $[a, b] \times [a, b] \times P \times Z \times O$ .

Let us consider the optimal control problem:

$$\begin{aligned} \dot{x}(t) &= f\left(t, p(\tau_1(t)), \dots, p(\tau_s(t)), z(\sigma_1(t)), \dots, z(\sigma_m(t)), u(\theta_1(t)), \dots, u(\theta_\nu(t))\right), \quad u(\cdot) \in \Omega, \\ x(t) &= (p(t), z(t))^T = (\varphi(t), g(t))^T, \quad t \in [\tau_0, t_0], \\ x(t_0) &= (p_0, g(t_0))^T, \quad p_0 \in P, \quad \varphi(\cdot) \in \Phi, \quad g(\cdot) \in G, \\ q^i(t_0, t_1, p_0, g(t_0), x(t_1)) &= 0, \quad i = \overline{1, l}, \\ q^0(t_0, t_1, p_0, g(t_0), x(t_1)) &\longrightarrow \min, \end{aligned}$$

where the functions  $\tau_i(t), i = \overline{1, s}$  are continuously differentiable and satisfying the conditions  $\tau_i(t) \leq t, \dot{\tau}_i(t) > 0$ ; the functions  $\sigma_i(t), i = \overline{1, m}, \theta_j(t), j = \overline{1, \nu}$  satisfy the similar conditions;  $\Phi$  and  $G$  are sets of continuous initial functions  $\varphi : [\tau_0, b] \rightarrow P_1$ , and  $g : [\tau_0, b] \rightarrow Z_1$ , where  $P_1 \subset P$  and  $Z_1 \subset Z$  are compact convex sets,  $\tau_0 = \min\{\tau_1(a), \dots, \tau_s(a), \sigma_1(a), \dots, \sigma_m(a)\}$ ;  $\Omega$  is the set of piecewise continuous control functions,  $u : [\theta, b] \rightarrow U$ , with finite number of points of discontinuity,  $\theta = \min\{\theta_1(a), \dots, \theta_\nu(a)\}, U \subset V$  is a compact convex set.

In the paper, on the basis of variation formulas of solution [1] and by a scheme given in [2], necessary conditions of optimality are obtained: in the form of linearized maximum principle for control and initial function, in the form of equalities and inequalities for initial and final moments.

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# Asymptotic Equivalence of Linear Systems for Small Linear Perturbations

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## 1 Tests of Lyapunov's Reducibility of Linear Systems

Consider the linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I = [0, +\infty), \quad (1)$$

with piecewise continuous and bounded (by the constant  $a \geq \|A(t)\|$  for  $t \in I$ ) coefficients. Along with systems (1), we will consider the systems

$$\dot{y} = (A(t) + Q(t))y, \quad y \in \mathbb{R}^n, \quad t \in I, \quad (2)$$

likewise with piecewise continuous and bounded on  $I$  coefficients.

Systems (1) and (2) are asymptotically equivalent (Lyapunov-equivalent, reducible) if there exists a linear transformation  $x = L(t)y$ , transferring one of the systems into another, where the matrix  $L(t)$  is the Lyapunov one, i.e., satisfying the condition

$$\sup_{t \in I} \{ \|L(t)\| + \|L^{-1}(t)\| + \|\dot{L}(t)\| \} < +\infty.$$

One of the tests of asymptotical equivalence of systems (1) and (2) is reflected [1] in the following assertion.

**Theorem 1.** *If  $\left\| \int_t^{+\infty} Q(u) du \right\| \leq Ce^{-\sigma t}$ ,  $t \in I$ ,  $\sigma > 2a$ , where  $C$  is some constant, then the systems (1) and (2) are asymptotically equivalent.*

The following statement [1] establishes that the estimate  $\sigma > 2a$  is unimprovable in a whole set of linear systems (1) with piecewise continuous matrices of coefficients.

**Theorem 2.** *For any number  $a > 0$  there exist system (1) with piecewise continuous matrix of coefficients with the norm  $\|A(t)\| \leq a$  for  $t \in I$  and the piecewise continuous matrix  $Q(t)$ , satisfying the condition  $\left\| \int_t^{+\infty} Q(u) du \right\| \leq Ce^{-2at}$ ,  $t \in I$ , such that systems (1) and (2) are not asymptotically equivalent.*

The following assertion establishes [2] the integral test of asymptotic equivalence of systems (1) and (2).

**Theorem 3.** *If the matrix of perturbations  $Q(t)$  of system (2) satisfies the condition  $\overline{\lim}_{t \rightarrow +\infty} \int_t^{+\infty} \|X_A(t, \tau)Q(\tau)X_A(\tau, t)\| d\tau < 1$ , where  $X_A(t, \tau)$  is the Cauchy matrix of system (1), then system (2) is equivalent to system (1).*

## 2 Coefficients and Exponents of Reducibility of Linear Systems

Let the perturbation  $Q(t)$  satisfy the condition

$$\|Q(t)\| \leq C(Q)e^{-\sigma t}, \quad \sigma \geq 0, \quad t \geq 0, \quad (3)$$

or the more general condition

$$\lambda[Q] \equiv \overline{\lim}_{t \rightarrow +\infty} t^{-1} \ln \|Q(t)\| \leq -\sigma \leq 0. \quad (4)$$

These perturbations for  $\sigma = 0$  in both cases (3) and (4) we assume additionally to be vanishing at infinity:  $Q(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . To every system (1) we put into correspondence the sets  $R(A)$  and  $R_\lambda(A)$  of those values of the parameter  $\sigma$  in (3) and (4) for which perturbed system (2) for any perturbation  $Q(t)$  satisfying (3) or, respectively, (4), is asymptotically equivalent to non-perturbed system (1).

**Definition.** An exact lower bound  $r(A)$  of the set  $R(A)$  (an exact lower bound  $\rho(A)$  of the set  $R_\lambda(A)$ ) will be called a coefficient of reducibility (an exponent of reducibility) of system (1).

**Theorem 4 ([2]).** *The coefficient of reducibility  $r(A)$  and the exponent of reducibility  $\rho(A)$  of every system (1) with piecewise continuous bounded coefficients coincide.*

This fact allows one to define a new asymptotic invariant of linear systems, i.e., the coefficient of reducibility of the system  $r_A$ , as a general value of its coefficient and exponent of reducibility. However, despite the fact that the above-mentioned values coincide for every system (1), the behavior of the coefficient of reducibility  $r_A$  is distinct with respect to perturbations (3) and (4). The following theorem [2] establishes this difference.

**Theorem 5.** *For any number  $a > 0$ , there exists system (1) with the coefficient of reducibility  $r_A = 2a$  such that system (2) with any piecewise continuous perturbation  $Q$  satisfying condition (3) with  $Q$  is reducible to the initial system (1) and non-reducible to that system for some perturbation  $Q$  satisfying the condition (4) with  $\sigma = r_A$ .*

Thus it follows from the above results (see also [3, 4]) that both the sets  $R(A)$  and  $R_\lambda(A)$  are the intervals, and  $(2a, +\infty) \subset R_\lambda(A) \subset R(A)$ . In addition, despite the fact that the values  $r(A)$  and  $\rho(A)$  coincide for any system (1), their properties are distinct: there exist systems (1) in which  $R(A) = R_\lambda(A) = (r_A, +\infty)$  and, at the same time, there exist systems (1) for which  $R(A) = [r_A, +\infty)$  and  $R_\lambda(A) = (r_A, +\infty)$ . Moreover [5], unlike the coefficient  $r(A)$  which can or cannot belong to the set  $R(A)$ , the exponent of reducibility  $\rho(A)$  of system (1) never belongs to the set  $R_\lambda(A)$ .

Establishment of the above-mentioned properties of the coefficient of reducibility of the linear differential system allows one to investigate certain parametric properties of the so-called sets of non-reducibility  $N_r(a, \sigma)$  and  $N_\rho(a, \sigma)$   $\sigma \in (0, 2a]$  of all those systems (1) for every of which there exists a non-reducible to it system (2) with the matrix  $Q(t)$  satisfying, respectively, either condition (3), or the more general condition (4).

These sets are non-empty for  $\sigma \in (0, 2a]$  and empty for  $\sigma > 2a$ . Moreover, these sets get strictly narrow as parameter  $\sigma \in (0, 2a]$  increases and for  $\sigma \in (0, 2a]$  they do not coincide with each other.

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# The Initial-Characteristic Problems for Wave Equations with Nonlinear Damping Term

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For one-dimensional wave equations with nonlinear damping term [1]–[4]

$$u_{tt} - u_{xx} + h(u_t) = f(x, t), \quad (1)$$

in the half-plane  $\Omega := \{(x, t) : x \in \mathbb{R}, t > 0\}$  let us consider the Initial-Cauchy problem with the following conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}, \quad (2)$$

where  $f, h, \varphi, \psi$  are given, and  $u$  unknown real functions.

Conditions, imposed on nonlinear function  $h$ , are obtained guaranteeing the existence of a global classical solution of (1), (2). The violation of those conditions may cause the blow-up of the solution.

**Theorem 1.** *Let the conditions*

$$f \in C^1(\overline{\Omega}), \quad \varphi \in C^2(\mathbb{R}), \quad \psi \in C^1(\mathbb{R})$$

*be fulfilled and*

$$h \in C^2(\mathbb{R}), \quad h'(s) \geq -M, \quad s \in \mathbb{R}, \quad M := \text{const} > 0. \quad (3)$$

*Then there exists a unique global classical solution  $u \in C^2(\overline{\Omega})$  of the problem (1), (2).*

Violation of the conditions (3) may, generally speaking, cause an absence [5] or nonuniqueness of the classical solution of the problem (1), (2).

*Remark 1.* Let  $h(s) = -|s|^\alpha$ ,  $s \in \mathbb{R}$ ,  $0 < \alpha < 1$ . Then the problem (1), (2) together  $u \equiv 0$  has the solution

$$u = \begin{cases} 0, & 0 \leq t \leq c, \\ (1 - \alpha)^{\frac{2-\alpha}{1-\alpha}} (2 - \alpha)^{-1} (t - c)^{\frac{2-\alpha}{1-\alpha}}, & t \geq c, \end{cases}$$

where  $c \geq 0$  arbitrary real constant.

Let  $h(s) = -|s|^\alpha s$ ,  $s \in \mathbb{R}$ ,  $\alpha > 1$  and the function  $\psi \geq 0$  has the compact supports, for example, the segment  $[x_1, x_2] \subset \mathbb{R}$ ;  $T_\infty := (\alpha c_1^\alpha c_2)^{-1} > 0$ ,  $c_2 := (x_2 - x_1)^{-\alpha}$  and

$$c_1 := \frac{1}{2} \int_{x_1}^{x_2} \psi(x) dx > 0. \quad (4)$$

**Theorem 2.** *If*

$$\varphi''(x) \geq 0, \quad \psi(x) \geq 0, \quad x \in \mathbb{R}, \quad f(x, t) \geq 0, \quad (x, t) \in \overline{\Omega}, \quad (5)$$

*then for  $t > T_\infty$  the problem (1), (2) has no classical solution.*

*Remark 2.* Naturally arise a question. What is happening, when some of the conditions (4), (5) are violated.

Let the condition (4) be violated i.e. the function  $\psi \geq 0$  has not compact support

$$\varphi(x) = \alpha x + \beta, \quad \alpha, \beta := \text{const}, \quad x \in \mathbb{R}, \quad \psi \equiv 0, \quad f \equiv 0.$$

Then the problem (1), (2) has the global classical solution  $u = \alpha x + \beta$ .

Let the first condition of (5) be violated:

$$\varphi = -x^2, \quad x \in \mathbb{R}, \quad \psi \equiv 1, \quad f \equiv 1.$$

Then the problem (1), (2) has the global classical solution  $u = -x^2 + t$ .

Let now the second condition of (5) be violated:

$$\varphi \equiv 0, \quad \psi \equiv -1, \quad f \equiv 1.$$

Then the problem (1), (2) has the global classical solution  $u = -t$ .

Let at last the third condition of (5) be violated:

$$\varphi \equiv 1, \quad \psi \equiv 1, \quad f \equiv -1.$$

Then the problem (1), (2) has the global classical solution  $u = 1 + t$ .

Similarly can be considered the characteristic (Goursat) and initial characteristic (First Darboux and Chachy–Goursat) problems and obtained corresponding results.

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# The Multidimensional Darboux Problem for a Class of Nonlinear Wave Equations

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Consider the nonlinear wave equation of the type

$$Lu := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + f(u) = F, \quad n > 1, \quad (1)$$

where  $f$  and  $F$  are given real functions,  $f$  is a nonlinear function, and  $u$  is an unknown real function.

Denote by  $D : t > |x|$ ,  $x_n > 0$ , the half of the light cone of the future which is bounded by the part  $S^0 : D \cap \{x_n = 0\}$  of the hyperplane  $x_n = 0$  and by the half  $S : t = |x|, x_n \geq 0$ , of the characteristic conoid  $C : t = |x|$  of equation (1). Assume  $D_T := \{(x, t) \in D : t < T\}$ ,  $S_T^0 := \{(x, t) \in S^0 : t \leq T\}$ ,  $S_T := \{(x, t) \in S : t \leq T\}$ ,  $T > 0$ . When  $T = \infty$ , it is obvious that  $D_\infty = D$ ,  $S_\infty^0 = S^0$  and  $S_\infty = S$ .

For equation (1) we consider the multidimensional version of the Darboux problem: find in the domain  $D_T$  a solution  $u(x, t)$  of that equation with the boundary conditions

$$u|_{S_T^0} = 0, \quad u|_{S_T} = 0. \quad (2)$$

Below we consider the following conditions imposed on the function  $f$ :

$$f \in C(\mathbb{R}), \quad |f(u)| \leq M_1 + M_2|u|^\alpha, \quad \alpha = \text{const} \geq 0, \quad (3)$$

$$\int_0^u f(s)ds \geq -M_3 - M_4u^2, \quad (4)$$

where  $M_i = \text{const} \geq 0$ ,  $i = 1, 2, 3, 4$ .

Note that in case  $\alpha \leq 1$  the inequality (3) results in the equality (4).

Let  $\overset{0}{W} \frac{1}{2}(D_T, S_T^0 \cup S_T) := \{u \in W_2^1(D_T) : u|_{S_T^0 \cup S_T} = 0\}$ , where  $W_2^k(D_T)$  is the well-known Sobolev space consisting of the functions  $u \in L_2(D_T)$  whose all generalized derivatives up to the  $k$ -th order, inclusive, also belong to the space  $L_2(D_T)$ , while the equality  $u|_{S_T^0 \cup S_T} = 0$  is understood in the sense of the trace theory.

**Definition 1.** Let  $F \in L_2(D_T)$ . A function  $u \in \overset{0}{W} \frac{1}{2}(D_T, S_T^0 \cup S_T)$  is said to be a strong generalized solution of the problem (1), (2) of the class  $W_2^1$  in the domain  $D_T$  if there exists a sequence of functions  $u_m \in \overset{0}{C}^2(\overline{D}_T, S_T^0 \cup S_T) := \{u \in C^2(\overline{D}_T) : u|_{S_T^0 \cup S_T} = 0\}$  such that  $u_m \rightarrow u$  in the space  $\overset{0}{W} \frac{1}{2}(D_T, S_T^0 \cup S_T)$  and  $Lu_m \rightarrow F$  in the space  $L_2(D_T)$ .

**Theorem 1.** Let  $F \in L_{2,loc}(D_\infty)$  and  $F \in L_2(D_T)$  for any  $T > 0$ . Let  $0 \leq \alpha < \frac{n+1}{n-1}$  and the function  $f$  satisfy the inequality (3). Moreover, in case  $\alpha > 1$ , let the function  $f$  satisfy also the condition (4). Then the problem (1), (2) is globally solvable in the class  $W_2^1$ , i.e. for any  $T > 0$  this problem has at least one strong generalized solution of the class  $W_2^1$  in the domain  $D_T$  in the sense of Definition 1.

**Theorem 2.** Let  $F \in L_{2,loc}(D_\infty)$  and  $F \in L_2(D_T)$  for any  $T > 0$ . Let  $1 < \alpha < \frac{n+1}{n-1}$ . For the function  $f$  let the condition (3) be fulfilled but the condition (4) may be violated. Then the problem

(1), (2) is locally solvable in the class  $W_2^1$ , i.e. there exists a number  $T_0 = T_0(F) > 0$  such that for  $T \leq T_0$  this problem has at least one strong generalized solution of the class  $W_2^1$  in the domain  $D_T$  in the sense of Definition 1.

Note that in case  $f(u) = -|u|^\alpha$ ,  $1 < \alpha < \frac{n+1}{n-1}$ , the condition (4) is violated.

**Theorem 3.** Let  $f(u) = -|u|^\alpha$ ,  $1 < \alpha < \frac{n+1}{n-1}$ . If  $F \in L_{2,loc}(D_\infty)$ ,  $F \in L_2(D_T)$  for any  $T > 0$ , and  $F \geq 0$ ,  $F(x, t) \geq ct^{-k}$  for  $t \geq 1$ , where  $c = \text{const} > 0$ ,  $0 < k = \text{const} \leq n + 1$ , then there exists a positive number  $T_1 = T_1(F)$  such that, for  $T > T_1$ , the problem (1), (2) cannot have a strong generalized solution of the class  $W_2^1$  in the domain  $D_T$  in the sense of Definition 1.

# On Two-Point Boundary Value Problems for Higher Order Quasi-Halflinear Differential Equations with Strong Singularities

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In the open interval  $]a, b[$  we consider the differential equation

$$u^{(2m)} = p(t) \left( \prod_{i=1}^m |u^{(i-1)}|^{\alpha_i} \right) \operatorname{sgn} u + q(t, u) \quad (1)$$

with the Dirichlet or the focal boundary conditions

$$\lim_{t \rightarrow a} u^{(i-1)}(t) = 0, \quad \lim_{t \rightarrow b} u^{(i-1)}(t) = 0 \quad (i = 1, \dots, m); \quad (2)$$

$$\lim_{t \rightarrow a} u^{(i-1)}(t) = 0, \quad \lim_{t \rightarrow b} u^{(i-1)}(t) = 0 \quad (i = 1, \dots, m). \quad (3)$$

Here  $m$  is a natural number,  $p : ]a, b[ \rightarrow \mathbb{R}$  and  $q : ]a, b[ \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions and  $\alpha_i$  ( $i = 1, \dots, m$ ) are nonnegative numbers such that

$$\alpha_1 > 0, \quad \sum_{i=1}^m \alpha_i = 1.$$

We say that the equation (1) has a strong singularity at the point  $a$  (at the point  $b$ ) if

$$\int_a^t (s-a)^{\alpha-1} |p(s)| ds = +\infty \quad \left( \int_t^b (b-s)^{\alpha-1} |p(s)| ds = +\infty \right) \quad \text{for } a < t < b,$$

where

$$\alpha = 2m - \sum_{i=1}^m i\alpha_i.$$

The obtained by us sufficient conditions of solvability of the problem (1), (2) (of the problem (1), (3)) cover the case, where the equation (1) has strong singularities at the points  $a$  and  $b$  (has a strong singularity at the point  $a$ ).

By  $C^{2m,m}(]a, b[)$  we denote the space of  $2m$ -times continuously differentiable functions  $u : ]a, b[ \rightarrow \mathbb{R}$  such that  $\int_a^b |u^{(m)}(s)|^2 ds < +\infty$ . Put

$$\gamma_1 = \frac{1}{(2m-1)!!} \prod_{i=1}^m \left( \frac{2^{2m-i+1}}{(2m-2i+1)!!} \right)^{\alpha_i}, \quad \gamma_2 = \frac{1}{(m-1)!\sqrt{2m-1}} \prod_{i=1}^m \left( (m-i)!\sqrt{2m-2i+1} \right)^{-\alpha_i},$$

$$\varphi_1(t) = \left( (t-a)^{-2m} + (b-t)^{-2m} \right)^{\frac{1}{2}} \prod_{i=1}^m \left( (t-a)^{2i-2m-2} + (b-t)^{2i-2m-2} \right)^{\frac{\alpha_i}{2}},$$

$$\varphi_2(t) = \left( (t-a)^{1-2m} + (b-t)^{1-2m} \right)^{\frac{1}{2}} \prod_{i=1}^m \left( (t-a)^{2i-2m-1} + (b-t)^{2i-2m-1} \right)^{\frac{\alpha_i}{2}},$$

$$q^*(t, y) = \max \{ |q(t, x)| : |x| \leq y \}, \quad q_*(t, y) = \inf \{ |q(t, x)| : |x| \geq y \}.$$

Along with (1), we consider the differential equation

$$u^{(n)} = \lambda p(t) \left( \prod_{i=1}^m |u^{(i-1)}|^{\alpha_i} \right) \operatorname{sgn} u, \quad (4)$$

depending on the parameter  $\lambda \in [0, 1]$ .

**Theorem 1.** *Let*

$$(-1)^m p(t) \leq \ell \varphi_1(t) + p_0(t) \varphi_2(t) \text{ for } a < t < b, \quad (5)$$

where  $\ell$  and  $p_0 : ]a, b[ \rightarrow [0, +\infty[$  are, respectively, a nonnegative number and a continuous function such that

$$\gamma_1 \ell < 1, \quad \int_a^b p_0(t) dt < +\infty.$$

If, moreover,

$$\lim_{\rho \rightarrow +\infty} \int_a^b [(t-a)(b-t)]^{m-\frac{1}{2}} \frac{q(t, [(t-a)(b-t)]^{m-\frac{1}{2}} \rho)}{\rho} dt = 0 \quad (6)$$

and for an arbitrary  $\lambda \in [0, 1]$  the problem (4), (2) has only the trivial solution in the space  $C^{2m,n}(]a, b[)$ , then the problem (1), (2) has at least one solution in the same space.

**Theorem 2.** *Let the conditions (5) and (6) hold, where  $\ell$  and  $p_0 : ]a, b[ \rightarrow [0, +\infty[$  are, respectively, a nonnegative number and a continuous function such that*

$$\gamma_1 \ell + \gamma_2 \int_a^b p_0(t) dt < 1. \quad (7)$$

Then the problem (1), (2) in the space  $C^{2m,m}(]a, n[)$  has at least one solution.

**Theorem 3.** *Let along with (5) and (6) the conditions*

$$(-1)^m p(t) \geq 0, \quad (-1)^m q(t, x) \geq 0 \text{ for } a < t < b, \quad x \in \mathbb{R},$$

$$\lim_{\rho \rightarrow 0} \int_a^b (t-a)^m (b-t)^m \frac{q_*(t, (t-a)^m (b-t)^m \rho)}{\rho} dt = +\infty$$

hold, where  $\ell$  and  $p_0 : ]a, b[ \rightarrow [0, +\infty[$  are, respectively, a nonnegative number and a continuous function satisfying the inequality (7). Then the problem (1), (2) in the space  $C^{2m,m}(]a, b[)$  along with the trivial solution has a positive and a negative on  $]a, b[$  solutions.

Analogous results have been established for the problem (1), (3).

*Remark.* In Theorem 1 (in Theorems 2 and 3) the condition  $\gamma_1 \ell < 1$  (the condition (7)) is unimprovable and it cannot be replaced by the condition

$$\gamma_1 \ell \leq 1 \quad \left( \gamma_1 \ell + \gamma_2 \int_a^b p_0(t) dt \leq 1 \right).$$

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# On Conditional Well-Posedness of Nonlocal Problems for Fourth Order Singular Linear Hyperbolic Equations

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In the rectangle  $\Omega = [0, a] \times [0, b]$  consider the linear hyperbolic equation

$$u^{(2,2)} = \sum_{i=1}^2 \sum_{k=1}^2 h_{ik}(x, y) u^{(i-1, k-1)} + h(x, y) \quad (1)$$

with the nonlocal boundary conditions

$$\int_0^a u(s, y) d\alpha_i(s) = 0 \text{ for } 0 \leq y \leq b, \quad \int_0^b u(x, t) d\beta_k(t) = 0 \text{ for } 0 \leq x \leq a \quad (i, k = 1, 2). \quad (2)$$

Here

$$u^{(i,k)}(x, y) = \frac{\partial^{i+k} u(x, y)}{\partial x^i \partial y^k} \quad (i, k = 0, 1, 2),$$

$h_{ik} : \Omega \rightarrow \mathbb{R}$  ( $i, k = 1, 2$ ) are measurable functions,  $h \in L(\Omega)$ , and  $\alpha_i : [0, a] \rightarrow \mathbb{R}$  and  $\beta_i : [0, b] \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) are functions of bounded variation. Moreover,

$$\alpha_i(0) = 0, \quad \beta_i(0) = 0, \quad \Delta_i(0) = 1 \quad (i = 1, 2), \quad (3)$$

where

$$\Delta_1(x) = \alpha_2(a) \int_x^a \alpha_1(s) ds - \alpha_1(a) \int_x^a \alpha_2(s) ds, \quad \Delta_2(y) = \beta_2(b) \int_y^b \beta_1(t) dt - \beta_1(b) \int_y^b \beta_2(t) dt.$$

We employ the concepts of well-posedness and conditional well-posedness for problem (1), (2) that were introduced in [1].

Along with the equation (1) consider the corresponding homogeneous equation

$$u^{(2,2)} = \sum_{i=1}^2 \sum_{k=1}^2 h_{ik}(x, y) u^{(i-1, k-1)}, \quad (1_0)$$

and introduce the functions:

$$\begin{aligned} \chi(s, t) &= \begin{cases} 1 & \text{for } s \geq t, \\ 0 & \text{for } s < t, \end{cases} \\ g_1(x, s) &= \int_0^a \alpha_1(\tau) d\tau \int_s^a \alpha_2(\tau) d\tau - \int_s^a \alpha_1(\tau) d\tau \int_0^a \alpha_2(\tau) d\tau \\ &\quad + (s - a)\Delta_1(0) + (a - x)\Delta_1(s) + \chi(x, s)(x - s) \text{ for } 0 \leq x, s \leq a, \\ g_2(y, t) &= \int_0^b \beta_1(\tau) d\tau \int_t^b \beta_2(\tau) d\tau - \int_t^b \beta_1(\tau) d\tau \int_0^b \beta_2(\tau) d\tau \\ &\quad + (t - b)\Delta_2(0) + (b - y)\Delta_2(t) + \chi(y, t)(y - t) \text{ for } 0 \leq y, t \leq b, \end{aligned}$$

$$\varphi_{11}(x) = \max \{|g_1(x, s)| : 0 \leq s \leq a\}, \quad \varphi_{12}(x) = \sup \{|\Delta_1(s) - \chi(x, s)| : 0 \leq s \leq a, s \neq x\},$$

$$\varphi_{21}(y) = \max \{|g_2(y, t)| : 0 \leq t \leq b\}, \quad \varphi_{22}(y) = \sup \{|\Delta_2(t) - \chi(y, t)| : 0 \leq t \leq b, t \neq y\}.$$

**Theorem 1.** *If along with (3) the condition*

$$\int_0^b \int_0^a \varphi_{1i}(x) \varphi_{2k}(y) |h_{ik}(x, y)| dx dy < +\infty \quad (i, k = 1, 2)$$

*holds, then problem (1), (2) is conditionally well-posed if and only if the corresponding homogeneous problem (1<sub>0</sub>), (2) has only the trivial solution.*

**Theorem 2.** *If along with (3) the inequality*

$$\sum_{i=1}^2 \sum_{k=1}^2 \int_0^b \int_0^a \varphi_i(x) \psi_k(y) |h_{ik}(x, y)| dx dy < 1 \quad (4)$$

*holds, then problem (1), (2) is conditionally well-posed. Moreover, if  $h_{ik} \in L(\Omega)$  ( $i, k = 1, 2$ ), then problem (1), (2) is well-posed.*

**Theorem 3.** *If conditions (3) and (4) hold, and*

$$\int_0^b \int_0^a |h_{11}(x, y)| dx dy = +\infty,$$

*then problem (1), (2) is conditionally well-posed but not well-posed.*

## References

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# Relaxational Oscillations and Diffusive Chaos in Belousov's Reaction

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The surprising phenomenon of chemistry, the reaction discovered by B. P. Belousov in 1951 (frequently called also as Belousov–Zhabotinsky's reaction), is an instructive episode in the history of home natural science deserving separate narration. The mathematical model of the reaction earns the attention and it should be included in the compulsory course of ordinary differential equations, since its discussion may, on the one hand, be anticipated by a rather simply organizable visual experiment and, on the other hand, it shows an exclusive value and might of mathematization for penetration into the essence of natural phenomena.

We will consider modification of the mathematical model of Belousov's reaction, namely, a system of three ordinary differential equations

$$\dot{x} = r_1[1 + a(1 - z) - x]x, \quad \dot{y} = r_2[x - y]y, \quad \dot{z} = r_3[\alpha x + (1 - \alpha)y - z]z, \quad (1)$$

where  $x, y, z$  are the analogues of concentration densities of chemical substances; the parameters  $r_1, r_2, r_3$  and  $a$  are positive ones, and the parameter  $\alpha \in (0, 1)$ . The most natural from the chemical point of view is the assumption that the parameter  $a$  is “very large” and the rest parameters are of order 1.

Belousov's reaction has, for the first time, shown experimentally the possibility for the chemical reaction to run periodically, and moreover, the stages running “very fast” in the course of the reaction alternate with those running “rather slowly”. Such periodical processes are called relaxational oscillations. Therefore of interest is the study of relaxation regime in system (1) in which we pass from the parameter  $a$  to the small parameter  $\varepsilon = 1/a$ .

We fix an arbitrary compact set  $\Omega_0$  of the semi-strip  $\{(u_0, v_0) : u_0 > 0, 0 < v_0 < 1\}$  and denote by

$$L_\varepsilon(u_0, v_0) = \left( x(t, u_0, v_0, \varepsilon), y(t, u_0, v_0, \varepsilon), z(t, u_0, v_0, \varepsilon) \right) : t \geq 0, (u_0, v_0) \in \Omega_0 \quad (2)$$

the trajectory of the system

$$\varepsilon \dot{x} = r_1[1 - z + \varepsilon(1 - x)]x, \quad \dot{y} = r_2[x - y]y, \quad \dot{z} = r_3[\alpha x + (1 - \alpha)y - z]z, \quad (3)$$

emanating for  $t = 0$  from the point  $(x, y, z) = (1, u_0, v_0)$ . We introduce into consideration the second positive root  $t = T(u_0, v_0, \varepsilon)$  (if it exists) of the equation  $x(t, u_0, v_0, \varepsilon) = 1$  and on the intersecting plane  $\{(x, y, z) : x = 1\}$  we define the Poincaré successor operator  $\Pi_\varepsilon(u_0, v_0)$ :

$$\Pi_\varepsilon(u_0, v_0) = \left( y(t, u_0, v_0, \varepsilon), z(t, u_0, v_0, \varepsilon) \right) \Big|_{t=T(u_0, v_0, \varepsilon)}. \quad (4)$$

**Theorem 1.** *On the set  $\Omega_0$ , in the metric of the space  $C^1(\Omega_0; \mathbb{R}^2)$  there exists the limiting operator*

$$\lim_{\varepsilon \rightarrow 0} \Pi_\varepsilon(u_0, v_0) = \Pi_0(u_0, v_0) \quad (5)$$

*describing constructively.*

**Theorem 2.** *Under the condition*

$$\alpha < \frac{r_2}{r_1 + r_2 + r_3} \quad (6)$$

*in the limiting mapping of  $\Pi_0$  in the semi-strip  $\{(u_0, v_0) : u_0 > 0, 0 < v_0 < 1\}$  there exists at least one stable fixed point, and for all sufficiently small  $\varepsilon > 0$  the initial operator  $\Pi_\varepsilon$  has a stable fixed point  $(u_\varepsilon, v_\varepsilon)$  with asymptotically close to  $(u_0, v_0)$  components. In system (3), this point is associated with the stable relaxational cycle  $L_\varepsilon$ .*

**Theorem 3.** *The time of motion of the phase point of system (3) along the “rapid segments” of the trajectory  $L_\varepsilon$  is of order  $\varepsilon \ln(1/\varepsilon)$ , while the time of motion along its “slow segments” admits as  $\varepsilon \rightarrow 0$  a finite positive limit.*

The distributed model, associated with system (1), i.e., the parabolic boundary value problem

$$\begin{aligned} \frac{\partial x}{\partial t} &= dD_1^0 \frac{\partial^2 x}{\partial s^2} + r_1[1 + a(1 - z) - x]x, & \frac{\partial x}{\partial s} \Big|_{s=0} &= \frac{\partial x}{\partial s} \Big|_{s=1} = 0, \\ \frac{\partial y}{\partial t} &= dD_2^0 \frac{\partial^2 y}{\partial s^2} + r_2[x - y]y, & \frac{\partial y}{\partial s} \Big|_{s=0} &= \frac{\partial y}{\partial s} \Big|_{s=1} = 0, \\ \frac{\partial z}{\partial t} &= dD_3^0 \frac{\partial^2 z}{\partial s^2} + r_3[\alpha x + (1 - \alpha)y - z]z, & \frac{\partial z}{\partial s} \Big|_{s=0} &= \frac{\partial z}{\partial s} \Big|_{s=1} = 0 \end{aligned} \quad (7)$$

on the segment  $0 \leq s \leq 1$ , is also considered. Here  $d, D_j, j = 1, 2, 3$  are positive parameters. Of interest is the investigation of attractors arising in its phase space  $(x, y, z) \in C([0, 1]; \mathbb{R}^3)$  as  $d$  decreases.

For the distributed model (7), by means of numerical experiments we have managed to establish a phenomenon of diffusion chaos, an unrestricted growth of dimensions of chaotic attractors as  $d \rightarrow 0$ , i.e., for proportional decrease of coefficients of diffusion. Two types of chaotic autooscillations, the relaxation chaos and that of the type of self-organization, are discovered.

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# On Three Problems of the Qualitative Theory of Differential Equations

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**I.** The asymptotics as  $t \rightarrow +\infty$  of solutions of monotone type for a real nonlinear first order differential equation (ODE-1)

$$\sum_{k=1}^s p_k(t)y^{\alpha_k}(y')^{\beta_k} + \sum_{k=s+1}^n p_k(t)y^{\alpha_k}(y')^{\beta_k} = 0 \quad (1)$$

is considered.

To find formal asymptotic representations for solutions of monotone type, it is assumed that there exists at least one such a solution  $y(t)$  of equation (1), and for that solution asymptotically basic are the summands appearing in the sum  $\sum_{k=1}^s$ . Under that assumption, we have obtained for  $y(t)$  possible formal asymptotic representations (exact, or requiring more precise determination). An asymptotic character of the obtained formal asymptotic representations is investigated. (L. L. Kol'tsova and A. V. Kostin, The results of the work are submitted for publication in *Mem. Differential Equations Math. Phys.* (Tbilisi).

**II.** We investigate a classical problem dealing with asymptotic stability (AS) of the real linear homogeneous differential equation ODE- $n$ ,  $n'geq2$  ( $n$  is order of ODE),

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = 0, \quad t \in I = [t_0, +\infty[ \quad (2)$$

under the condition that the roots  $\lambda_i(t)$  ( $i = \overline{1, n}$ ) of the corresponding characteristic equation are such that  $\lambda_i(t) \in C^1(I)$ ,  $\text{Re}\lambda_i(t) < 0$ ,  $\int_{t_0}^{+\infty} \text{Re}\lambda_i(t)dt = -\infty$ ,  $\exists \text{Re}\lambda_i(+\infty)$ ,  $-\infty \leq \text{Re}\lambda_i(+\infty) \leq 0$  ( $i = \overline{1, n}$ ).

The case  $n = 2$  is considered rather thoroughly. We have managed to prove the property of asymptotic stability in the case of real  $\lambda_i(t)$  ( $i = 1, 2$ ) in the following subcases:

- (1)  $\lambda_1(+\infty) \in \mathbb{R}_- = ]-\infty, 0[$ ,  $\lambda_2(+\infty) = 0$ ,  $\lambda_1'(t) = o(\lambda_2(t))$ ;
- (2)  $\lambda_i(+\infty) = 0$  ( $i = 1, 2$ ),  $\lambda_1'(t) = o(\lambda_1(t)\lambda_2(t))$ ;
- (3)  $\lambda_1(+\infty) = 0$ ,  $\lambda_2(+\infty) = -\infty$ ,  $\lambda_1'(t) = o(\lambda_1^2(t))$ ;
- (4)  $\lambda_i(+\infty) = -\infty$  ( $i = 1, 2$ ),  $\lambda_1'(t) = o(\lambda_1^2(t))$ ,  $\lambda_1'(t) = o(\lambda_1(t)\lambda_2(t))$ .

The subcase  $\lambda_i(+\infty) \in \mathbb{R}_-$  ( $i = 1, 2$ ) is known.

The case of complex-conjugate roots  $\lambda_i(t)$  ( $i = 1, 2$ ) is considered analogously. (T. Yu. Kosetskaya, A. V. Kostin).

**III.** We investigate the problem on the existence of a particular solution  $y(t)$  from some class of real functions

$$K \left\{ f(t) : f(t) \in C^2(\mathbb{R}), \sup_{\mathbb{R}} |f(t)| < +\infty \right\}$$

in ODE-2,

$$y'' + ay' + by = f(t) + \mu F(t, y, y'), \quad (3)$$

where  $(t, y, y') \in G\{t \in \mathbb{R}, |y - u(t)| \leq h_1, |y' - u'(t)| \leq h_2\}$ ,  $h_i < +\infty$  ( $i = 1, 2$ ),  $a, b \in \mathbb{R}$ ,  $f(t) \in K$ , the roots  $\lambda_i$  ( $i = 1, 2$ ) of the equation  $\lambda^2 + a\lambda + b = 0$  are such that  $\operatorname{Re} \lambda_i \neq 0$  ( $i = 1, 2$ ), and the condition

$$\int_{A_i}^t f(\tau) \exp \operatorname{Re} \lambda_i(t - \tau) d\tau \in K, \quad A_i = \begin{cases} +\infty & (\operatorname{Re} \lambda_i > 0) \\ -\infty & (\operatorname{Re} \lambda_i < 0) \end{cases}, \quad f(t) \in K$$

is fulfilled,  $u(t) \in K$  is a unique solution of the class  $K$  of the equation

$$y'' + ay' + by = f(t),$$

$F, F'_y, F'_{y'} \in C(G)$ ,  $\sup_G (|F| + |F'_y| + |F'_{y'}|) < +\infty$  ( $F(t, f(t), f'(t)) \in K$ , if  $f(t) \in K$ ).

Using the Perron transformation, we have obtained the estimate for a small parameter  $\mu$  which guarantees the solvability of the problem. In the capacity of the class  $K$  one can consider a class of almost-periodic, slowly varying and another functions. (A. V. Kostin, T. V. Kondratenko)

# Investigation of Stability of Stochastic Differential Equations by Using Lyapunov Functions of Constant Signs

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The method of Lyapunov functions is one of the most effective one for the investigation of stability of differential systems, in particular, of stochastic differential systems. The main purpose of the report is the theorem on the stability of stochastic differential equations by using Lyapunov functions of constant signs.

Consider the stochastic differential equation

$$dx(t) = f(x(t))dt + g(x(t))dW(t), \quad (1)$$

with the Borel-measurable functions  $f : R^d \rightarrow R^d$ ,  $g : R^d \rightarrow R^{d \times d}$ .

**Definition 1.** If there exists a process  $x(t)$  given on some probability space  $(\Omega, \mathcal{F}, P)$  with a flow of  $\sigma$ -algebras  $\mathcal{F}_t$ , satisfying the following conditions:

- (1) there exists a  $(\mathcal{F}_t)$ -moment of the stop  $e$  such that the process  $x(t)1_{[0,e)}(t)$  is  $(\mathcal{F}_t)$ -coordinated, has continuous trajectories for  $t < e$  a.s. and  $\limsup_{t \uparrow e} \|x(t)\| = \infty$  if  $e < \infty$ ;
- (2) there exists the  $(\mathcal{F}_t)$ -Brownian motion  $W(t)$ ,  $W(0) = 0$  a.s.;
- (3) the processes  $f(x(t))$  and  $g(x(t))$  belong, respectively, to the spaces  $L_1^{\text{loc}}$  and  $L_2^{\text{loc}}$ , where  $L_i^{\text{loc}}$  is a set of all measurable  $(\mathcal{F}_t)$ -coordinated processes  $\psi$  such that for every moment of the stop  $\sigma$ ,  $0 \leq \sigma < e$  the condition  $\int_0^\sigma \|\psi(s, \omega)\|^i ds < \infty$  a.s.  $i \in \{1, 2\}$  is fulfilled;
- (4) with probability  $I$  for all  $t \in [0, e)$ , the equality

$$x(t) = x(0) + \int_0^t f(x(\tau)) d\tau + \int_0^t g(x(\tau)) dW(\tau)$$

holds, and the set  $(x(t), \Omega, \mathcal{F}, P, \mathcal{F}_t, W(t), e)$  (or briefly,  $x(t)$ ) is called a weak solution of equation (1).

We choose rows of the matrix  $g$  with numbers  $\beta_1, \dots, \beta_l$ ,  $\beta_1 < \dots < \beta_l$ , and let  $\beta_{l+1} < \dots < \beta_d$  be numbers of the rest rows. We construct the matrix

$$\sigma_{\beta_1, \dots, \beta_l}(x_1, \dots, x_d) = \begin{pmatrix} g_{\beta_1} g_{\beta_1}^\top & \dots & g_{\beta_1} g_{\beta_l}^\top \\ \dots & \dots & \dots \\ g_{\beta_l} g_{\beta_1}^\top & \dots & g_{\beta_l} g_{\beta_l}^\top \end{pmatrix},$$

where  $g_{\beta_j}$  is the row with the number  $\beta_j$  of the matrix  $g$ , and also we construct the sets  $H_1, H_2$ ;  $H_1(\beta_1, \dots, \beta_l) = \{(x_{\beta_1}, \dots, x_{\beta_l}) \mid \text{for any open neighborhood } U_{(x_{\beta_1}, \dots, x_{\beta_l})} \text{ of the point } (x_{\beta_1}, \dots, x_{\beta_l}) \text{ there exists a number } a > 0 \text{ such that the integral}$

$$\int_{U_{(x_{\beta_1}, \dots, x_{\beta_l})}} \sup_{(x_{\beta_{l+1}}, \dots, x_{\beta_d}) \in D_2(0, a)} (\det \sigma_{\beta_1, \dots, \beta_l}(x_1, \dots, x_d))^{-1} dx_{\beta_1} \dots dx_{\beta_l}$$

is either indeterminate, or is equal to  $\infty$ }, where  $D_2(0, a) = \{(x_{\beta_{l+1}}, \dots, x_{\beta_d}) \mid (x_{\beta_{l+1}}^2 + \dots + x_{\beta_d}^2)^{1/2} \leq a\}$ ;  $H_2(\beta_1, \dots, \beta_l) = \{(x_{\beta_1}, \dots, x_{\beta_l}) \in H_1^c(\beta_1, \dots, \beta_l) \mid \text{for any open neighborhood } U_{(x_{\beta_1}, \dots, x_{\beta_l})} \text{ of the point } (x_{\beta_1}, \dots, x_{\beta_l}) \text{ there exists a number } a > 0 \text{ such that the function}$

$$\sup_{(x_{\beta_{l+1}}, \dots, x_{\beta_d}) \in D_2(0, a)} (\det \sigma_{\beta_1, \dots, \beta_l}(x_1, \dots, x_d))^{-1} : U \rightarrow [0, \infty]$$

is not Borel-measurable} (under the complement  $H_1^c$  of the set  $H_1$  is understood the complement in the space of variables  $(x_{\beta_1}, \dots, x_{\beta_l})$ , and under the open neighborhood is understood the neighborhood which is open in the space of the same variables  $(x_{\beta_1}, \dots, x_{\beta_l})$ ).

Let  $\widehat{H}(\beta_1, \dots, \beta_l) = H_1(\beta_1, \dots, \beta_l) \cup H_2(\beta_1, \dots, \beta_l)$ .

We will say that the real function  $h(x) = h(x_1, \dots, x_d)$  satisfies **Condition C**) if there exist indices  $\beta_1, \dots, \beta_l$ ,  $\beta_1 < \dots < \beta_l \leq d$  such that:

- 1) the function  $h$  for every fixed  $(x_{\beta_1}, \dots, x_{\beta_l})$  is continuous with respect to the rest components  $(x_{\beta_{l+1}}, \dots, x_{\beta_d})$  of the vector  $x$ ;
- 2) in the space of variables  $(x_{\beta_1}, \dots, x_{\beta_l})$  there is a closed set  $H$  with the properties:
  - (2a)  $H \supset \widehat{H}(\beta_1, \dots, \beta_l)$ ;
  - (2b) the set  $\{(x_1, \dots, x_d) \mid (x_{\beta_1}, \dots, x_{\beta_l}) \in H\}$  belongs to the set of points of continuity of the mapping  $h$ ;
  - (2c) the function  $\sigma_{\beta_1, \dots, \beta_l}(x_1, \dots, x_d)$  for every fixed  $(x_{\beta_1}, \dots, x_{\beta_l}) \in H^c$  is continuous with respect to the variables  $(x_{\beta_{l+1}}, \dots, x_{\beta_d})$ .

Let the functions  $f(x)$  and  $g(x)$  be Borel-measurable and locally bounded, the components of the functions  $f(x)$ ,  $\sigma(x) = g(x)g^\top(x)$  satisfy Condition C). Then for any given probability  $\nu$  on  $(R^d, \beta(R^d))$  equation (1) has a weak solution with an initial distribution  $\nu$  [1].

**Definition 2.** A zero solution is said to be  $\varpi$ -stable, if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for a weak solution  $x(t)$  of equation (1), for which  $\|x(0)\| \leq \delta$  a.s., we have  $E(\|x(t)\|^\varpi) \leq \varepsilon \forall t \geq 0$  ( $E$  is a mathematical expectation).

**Definition 3.** A zero solution is said to be asymptotically  $\varpi$ -stable, if it is  $\varpi$ -stable and there exists  $M > 0$  such that, for any weak solution  $x(t)$  for which  $\|x(0)\| \leq M$  a.s., the relation  $\lim_{t \rightarrow +\infty} E(\|x(t)\|^\varpi) = 0$  is fulfilled.

**Condition L).** There exists a twice continuously differentiable function  $V : R^d \rightarrow R_+$  such that  $\forall x \in R^d$ ,

$$BV(x) = \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 V(x)}{\partial x^2} g(x)g^\top(x) \right) \leq 0.$$

Assume  $M_V = \{x \in R^d \mid BV(x) = 0\}$ . We say that a weak solution  $(x(t), W(t), \Omega, \mathcal{F}, P, \mathcal{F}_t)$  belongs to the set  $M_V$  if

$$\frac{\partial V(x(t))}{\partial x} f(x(t)) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 V(x(t))}{\partial x^2} g(x(t))g^\top(x(t)) \right) = 0$$

for  $(\mu \times P)$ - almost all  $(t, \omega) \in R_+ \times \Omega$ .

**Condition A).** There exist the constants  $r > 1$ ,  $\sigma > 0$ ,  $M > 0$  such that for any weak solution  $x(t)$  of the system (1), satisfying the condition  $\|x(0)\| \leq \sigma$ , the inequality  $E(\|x(t)\|^r) \leq M \forall t \geq 0$  is fulfilled.

**Condition B).** The system (1) has no nonzero weak solutions  $x(t)$  such that  $x(0) = 0$  a.s.

**Definition 4.** The process  $x(t)$ ,  $t \in ] - \infty, 0]$  given on some probability space  $(\Omega, \mathcal{F}, P)$  is said to be a weak solution of equation (1) on the interval  $] - \infty, 0]$ , if:

- 1) for every  $t_0 \in ] - \infty, 0[$ , there exists the extension  $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$  of the space  $(\Omega, \mathcal{F}, P)$  and on that extension there exists the flow  $(\widetilde{\mathcal{F}}_t)$ ,  $t \in [t_0, 0]$ , such that on  $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$  with  $(\widetilde{\mathcal{F}}_t)$  one can define the  $(\widetilde{\mathcal{F}}_t)$ -Brownian motion  $W(t)$ ,  $W(t_0) = 0$  a.s.;
- 2) the processes  $f(x(t))$  and  $g(x(t))$  belong, respectively, to the spaces  $L_1$  and  $L_2$ , where  $L_i$  is the set of all measurable  $(\widetilde{\mathcal{F}}_t)$ -coordinated processes  $\psi(t)$ ,  $t \in [t_0, 0]$  such that  $\int_{t_0}^0 \|\psi(s, \omega)\|^i ds < \infty$  a.s.,  $i \in \{1, 2\}$ ;
- 3) with probability 1 for all  $t \in [t_0, 0]$  the equality  $x(t) = x(t_0) + \int_{t_0}^t f(x(\tau)) d\tau + \int_{t_0}^t g(x(\tau)) dW(\tau)$  holds.

The other necessary definitions and notation can be found in [1].

**Theorem.** *Let the functions  $f(x)$  and  $g(x)$  be Borel-measurable and locally bounded, the components of the functions  $f(x)$ ,  $\sigma(x) = g(x)g^\top(x)$  satisfy Condition C), the system (1) satisfy Conditions A), B) and L), and let  $0 < s < r$ . If there exists a constant  $a > 0$  such that the system (1) has no nonzero weak solutions  $x(t)$  on the interval  $] - \infty, 0]$  possessing the properties:  $x(t) \in m_V = \{x \in R^d \mid V(x) = \} \forall t \in ] - \infty, 0]$  a.s.;  $E(\|x(t)\|^s) \leq a \forall t \in ] - \infty, 0]$ , then a zero solution of the system (1) is  $s$ -stable. If, moreover, there exists a constant  $b > 0$  such that the system has no nonzero weak solutions  $x(t)$ ,  $t \in [0, \infty[$ , satisfying the conditions  $x(t) \in M_V$ ,  $E(\|x(t)\|^s) \leq b \forall t \in [0, \infty[$ , then a zero solution is asymptotically  $s$ -stable.*

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# On Hybrid Stabilization

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## 1 Introduction

Hybrid systems are those combining both discrete and continuous dynamics. Many examples of hybrid systems can be found in manufacturing systems, intelligent vehicle highway systems, various chemical plants. Hybrid systems also arise when there is a necessity of combining logical decision with the generation of continuous control laws.

An important question is how to stabilize a continuous control plant through an interaction with a discrete time controller. Such a "hybrid" feedback may help when the ordinary feedback fails to stabilize the system.

An example of a linear system which cannot be stabilized by the ordinary output feedback is **the harmonic oscillator**:

$$\frac{d\xi}{dt} = \eta, \quad \frac{d\eta}{dt} = -\xi + u, \quad u = u(y), \quad y = \xi. \quad (1.1)$$

The only measured quantity (output), which is allowed to control, is the position variable  $\xi$ . Although this last system is both controllable and observable, it cannot be stabilized by (even discontinuous) output feedbacks.

It was however shown by Z. Artstein (1995), there exists a special *hybrid feedback control*, under which system (1.1) becomes asymptotically stable. Z. Artstein conjectured also that hybrid controls can stabilize general linear systems of ordinary differential equations.

## 2 The Main Result

We give here the following affirmative answer to Artstein's conjecture on the existence of a hybrid stabilizer.

**Theorem 2.1.** *Under assumptions of controllability of  $(A, B)$  and observability of  $(A, C)$  the system*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ u &= u(y), \quad y = Cx \end{aligned}$$

*is stabilizable by a hybrid feedback control designed with the help of a discrete automaton which has at most countable number of locations.*

*Proof* is based on the classical stabilization technique as well as on some recent results in the **theory of functional-differential equations** in an essential way.

### 3 Applications to Theorem 2.1

We give here two examples.

#### 1. Predator-Prey Interactions

An example of a situation, where hybrid feedback controls may be of use, is given by a population model with an arbitrary number of species, some of them being observable and the others not. Although such a model is nonlinear, but linearization about the equilibrium state provides a linear system with a control depending on *a part* of variables while the rest variables may be not observable at all, so that these variables cannot be used for setting up a control function. A stabilization of the unstable equilibrium state may become then problematical if we use ordinary feedback, only. What does help is *hybrid feedback controls*.

#### 2. A Contribution to the Theory of Love Affairs

Here is another example which illustrates the power of stabilization by hybrid feedback controls. The mathematical model for the dynamics of love affairs is given by a  $2 \times 2$ -system of linear equations (Strogatz, 1994). We consider the following particular case, which is called *the star-crossed romance between Romeo and Juliet*.

$$\begin{aligned} \dot{R} &= aJ, & \dot{J} &= -bR, \\ R(t) &= \text{Romeo's love/hate for Juliet at time } t, \\ J(t) &= \text{Juliet's love/hate for Romeo at time } t \end{aligned} \tag{3.1}$$

(love gives positive sign to variables, while hate makes variables negative).

From the system (3.1) we obtain the following tragic picture.

*The more Romeo loves Juliet, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her; he warms up when she loves him, and grows cold when she hates him. . . The sad outcome of their affair is, of course, a never-ending cycle of love and hate, because solutions of the system (3.1) are ellipses with a center at (0, 0).*

We observe that ordinary feedback controls do not help Romeo and Juliet in this unpleasant situation. If, for instance, we insert a feedback control like  $u = \alpha R$  in the first equation suddenly turns Romeo either into an “eager beaver” ( $\alpha > 0$ ), or into a “cautious lover” ( $\alpha < 0$ ) which seems to be quite unrealistic as soon as one particular Juliet is concerned. The same applies to Juliet. The only possibility is therefore to try making influence on the constants  $a$  and  $b$  in the system (3.1). But any feedback control like  $u = \alpha J$  or/and  $v = \beta R$  will never change the sad and tragic picture of never-ending ellipses in the phase-plane, because the corresponding coefficient matrix will always have imaginary eigenvalues!

We propose a *hybrid feedback scenario*, which according to Theorem 2.1 does make solutions of the system (3.1) asymptotically stable. This scenario can be described explicitly and does lead to a kind of “happy end”. Naturally, the relationship between Romeo and Juliet will fizzle out to mutual indifference and the disaster will be prevented.

Unfortunately, it is impossible to use a similar procedure to help Romeo and Juliet becoming eventually daring to each other, because hybrid feedback controls can stabilize solutions, but they cannot exclude oscillations.

# On Oscillation and Nonoscillation of Two-Dimensional Linear Differential Systems

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Consider the system

$$\begin{aligned} u' &= q(t)v, \\ v' &= -p(t)u, \end{aligned} \tag{1}$$

where  $p, q: [0, +\infty[ \rightarrow \mathbb{R}$  are locally Lebesgue integrable functions such that

$$q(t) \geq 0 \text{ for a. e. } t \geq 0, \quad \int_0^{+\infty} q(s) \, ds < +\infty,$$

and  $q \not\equiv 0$  in any neighbourhood of  $+\infty$ . Under a *solution* of system (1) we understand a vector-function  $(u, v): [0, +\infty[ \rightarrow \mathbb{R}^2$  with locally absolutely continuous components satisfying equalities (1) almost everywhere in  $[0, +\infty[$ . A solution  $(u, v)$  of system (1) is said to be *nontrivial* if  $u \not\equiv 0$  in any neighbourhood of  $+\infty$ . A nontrivial solution  $(u, v)$  of system (1) is called *oscillatory* if the function  $u$  has a sequence of zeros tending to infinity.

**Definition 1.** System (1) is said to be *oscillatory* if every nontrivial solution of this system is oscillatory, and *nonoscillatory* otherwise.

For any  $\lambda > 1$ , we put

$$c(t; \lambda) := (\lambda - 1)f^{\lambda-1}(t) \int_0^t \frac{q(s)}{f^\lambda(s)} \left( \int_0^s f^\lambda(\xi)p(\xi) \, d\xi \right) ds \text{ for } t \geq 0,$$

where

$$f(t) := \int_t^{+\infty} q(s) \, ds \text{ for } t \geq 0.$$

The following theorem is an analogue of the well-known Hartman–Wintner theorem.

**Theorem 1.** *Let there exist  $\lambda > 1$  such that either*

$$\lim_{t \rightarrow +\infty} c(t; \lambda) = +\infty$$

*or*

$$-\infty < \liminf_{t \rightarrow +\infty} c(t; \lambda) < \limsup_{t \rightarrow +\infty} c(t; \lambda).$$

*Then system (1) is oscillatory.*

If we take this theorem into account it is obvious that, for given  $\lambda > 1$ , the following two cases remain uncovered. The first case, where

$$\text{there exists a finite limit } \lim_{t \rightarrow +\infty} c(t; \lambda), \tag{2}$$



and the second case, where

$$\liminf_{t \rightarrow +\infty} c(t; \lambda) = -\infty.$$

Below, we establish new oscillation and nonoscillation criteria assuming that (2) holds for some  $\lambda > 1$ . Having such  $\lambda$ , we denote

$$Q(t; \lambda) := \frac{1}{f^{\lambda-1}(t)} \left( c_0(\lambda) - \int_0^t f^\lambda(s) p(s) ds \right) \text{ for } t \geq 0,$$

where

$$c_0(\lambda) = \lim_{t \rightarrow +\infty} c(t; \lambda). \quad (3)$$

Moreover, for any  $\mu < 1$ , we put

$$H(t; \mu) := f^{1-\mu}(t) \int_0^t f^\mu(s) p(s) ds \text{ for } t \geq 0.$$

Finally, let

$$\begin{aligned} Q_*(\lambda) &= \liminf_{t \rightarrow +\infty} Q(t; \lambda), & Q^*(\lambda) &= \limsup_{t \rightarrow +\infty} Q(t; \lambda), \\ H_*(\mu) &= \liminf_{t \rightarrow +\infty} H(t; \mu), & H^*(\mu) &= \limsup_{t \rightarrow +\infty} H(t; \mu). \end{aligned}$$

Now we formulate our main results.

**Theorem 2.** *Let there exist  $\lambda > 1$  such that condition (2) holds and*

$$\limsup_{t \rightarrow +\infty} \frac{-1}{f^{\lambda-1}(t) \ln f(t)} (c_0(\lambda) - c(t; \lambda)) > \frac{1}{4},$$

where the number  $c_0(\lambda)$  is defined by formula (3). Then system (1) is oscillatory.

**Corollary 1.** *Let there exist  $\lambda > 1$  and  $\mu < 1$  such that condition (2) holds and*

$$\liminf_{t \rightarrow +\infty} (Q(t; \lambda) + H(t; \mu)) > \frac{1}{4(\lambda - 1)} + \frac{1}{4(1 - \mu)}. \quad (4)$$

Then system (1) is oscillatory.

**Corollary 2.** *Let there exist  $\lambda > 1$  such that condition (2) holds and either*

$$Q_*(\lambda) > \frac{1}{4(\lambda - 1)}$$

or

$$H_*(\mu) > \frac{1}{4(1 - \mu)} \quad (5)$$

for some  $\mu < 1$ . Then system (1) is nonoscillatory.

*Remark 1.* It might seem that if assumption (5) is satisfied in the previous corollary then assumption (2) is redundant. However, one can show that, under assumption (5), the function  $c(\cdot; \lambda)$  possesses a limit for every  $\lambda > 1$  and  $\lim_{t \rightarrow +\infty} c(t; \lambda) > -\infty$ . If this limit is equal to  $+\infty$ , then system (1) is oscillatory according to Theorem 1. Therefore, assumption (2) in the previous corollary is necessary in a certain sense also in the case where inequality (5) is supposed to be satisfied.

The next theorem deals with the upper limit of the sum on the left-hand side of inequality (4) and thus it complements Corollary 1 in a certain sense.

**Theorem 3.** *Let there exist  $\lambda > 1$  and  $\mu < 1$  such that condition (2) holds and*

$$\limsup_{t \rightarrow +\infty} (Q(t; \lambda) + H(t; \mu)) > \frac{\lambda^2}{4(\lambda - 1)} + \frac{\mu^2}{4(1 - \mu)}.$$

*Then system (1) is oscillatory.*

The following statement complements Corollary 2.

**Theorem 4.** *Let there exist  $\lambda > 1$  and  $\mu < 1$  such that condition (2) holds and either*

$$\frac{\lambda(2 - \lambda)}{4(\lambda - 1)} \leq Q_*(\lambda) \leq \frac{1}{4(\lambda - 1)}, \quad H^*(\mu) > \frac{\mu^2}{4(1 - \mu)} + \frac{1 + \sqrt{1 - 4(\lambda - 1)Q_*(\lambda)}}{2}$$

*or*

$$\frac{\mu(2 - \mu)}{4(1 - \mu)} \leq H_*(\mu) \leq \frac{1}{4(1 - \mu)}, \quad Q^*(\lambda) > \frac{\lambda^2}{4(\lambda - 1)} - \frac{1 - \sqrt{1 - 4(1 - \mu)H_*(\mu)}}{2}.$$

*Then system (1) is oscillatory.*

At last, we present two results dealing with nonoscillation of system (1).

**Theorem 5.** *Let there exist  $\lambda > 1$  such that condition (2) holds and either*

$$-\frac{(2\lambda - 3)(2\lambda - 1)}{4(\lambda - 1)} < Q_*(\lambda), \quad Q^*(\lambda) < \frac{1}{4(\lambda - 1)}$$

*or*

$$-\frac{(3 - 2\mu)(1 - 2\mu)}{4(1 - \mu)} < H_*(\mu), \quad H^*(\mu) < \frac{1}{4(1 - \mu)} \quad (6)$$

*for some  $\mu < 1$ . Then system (1) is nonoscillatory.*

**Theorem 6.** *Let there exist  $\lambda > 1$  such that condition (2) holds and either*

$$-\infty < Q_*(\lambda) \leq -\frac{(2\lambda - 3)(2\lambda - 1)}{4(\lambda - 1)}, \quad Q^*(\lambda) < Q_*(\lambda) + 1 - \lambda + \sqrt{1 - 4(\lambda - 1)Q_*(\lambda)}$$

*or*

$$-\infty < H_*(\mu) \leq -\frac{(3 - 2\mu)(1 - 2\mu)}{4(1 - \mu)}, \quad H^*(\mu) < H_*(\mu) + \mu - 1 + \sqrt{1 - 4(1 - \mu)H_*(\mu)} \quad (7)$$

*for some  $\mu < 1$ . Then system (1) is nonoscillatory.*

*Remark 2.* It might seem that if assumptions (6) and (7) are satisfied in the previous theorems, then assumption (2) is redundant. However, one can show that, under assumption (6) as well as under (7), the function  $c(\cdot; \lambda)$  possesses a finite limit for every  $\lambda > 1$ . Therefore, assumption (2) in Theorems 5 and 6 is necessary also in the case where inequalities (6) and (7) are supposed to be satisfied.

# An Infinite Dimensional Generalization of Weak Exponents for Solutions of Equations in Total Derivatives

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Let  $E$  and  $F$  be real Banach spaces,  $K \subset E$  be a closed convex cone with a bounded base. Consider a linear completely integrable equation in total derivatives

$$y'h = A(x)hy, \quad y \in F, \quad h \in E, \quad x \in E, \quad (1)$$

with a bounded continuous coefficient  $A : E \rightarrow L(E, L(F, F))$  (here and in the sequel, we use notation and notions from [1]). Let  $\mathcal{E}(y)$  be a set of all linear continuous functionals  $\mu \in E^*$  such that the inequality  $\limsup_{x \rightarrow \infty, x \in K} \|x\|^{-1}(\ln y(x) + \mu x) \leq 0$  holds.

In [2], the interrelation is established between characteristic functionals and (weak) characteristic exponents of solutions of equation (1) for a finite-dimensional  $E$  in the form  $\mathcal{E}(y) = \mathcal{E}(\exp \psi[y])$ , where  $\psi[y](x) := \overline{\lim}_{t \rightarrow +\infty} t^{-1} \ln y(tx)$  is a modified exponent of a solution  $y$ . This result is valid only for a finite-dimensional  $E$ . Therefore it is necessary to generalize the above notions in order to obtain some analog of the statement in [2] for the settings of infinite-dimensional  $E$ .

For this, we introduce new exponent  $\psi[y](\varphi)$  by means of the formula

$$\psi[y](\varphi) := \overline{\lim}_{t \rightarrow +\infty} t^{-1} \ln \|y(\varphi(t))\|,$$

as the functional on the space  $\Phi$  of continuous functions  $\varphi : [0, +\infty[ \rightarrow K$  such that  $\|\varphi(t)\| \rightarrow \infty$ , as  $t \rightarrow +\infty$  and  $\sup \|\varphi(t)\|/t < +\infty$ .

**Theorem.** *The inclusion  $\lambda \in \mathcal{E}(y)$  holds if and only if for any  $\varphi \in \Phi$  the inequality  $\psi[y](\varphi) + \lambda(\varphi) \leq 0$ , where  $\lambda(\varphi) := \underline{\lim}_{t \rightarrow \infty} t^{-1} \lambda\varphi(t)$ , holds.*

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# A Certain Class of Discontinuous Dynamical Systems in the Plane

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In the monographs [1]–[3], a theory of impulsive differential equations is built. Mainly, the mathematical models of evolutionary processes that undergo impulsive perturbations at fixed moments of time or at the moments, when the moving point meets the given hypersurfaces in the extended phase space are considered. However, in the monographs [1]–[3], the importance of studying of systems with impulsive perturbations that occur at the moments, when the phase point meets the given sets in the phase space is emphasized. In this report, we investigate a linear differential systems in the plane that are subjected to impulsive perturbations on the given line, i.e. systems of the form

$$\frac{dx}{dt} = Ax, \quad \langle a, x \rangle \neq 0; \quad \Delta x|_{\langle a, x \rangle=0} = Bx,$$

where  $x \in R^2$ ,  $A$  and  $B$  are constant matrices,  $a$  is a constant vector.

The motion of the phase point is defined by the differential system  $\dot{x} = Ax$ , when this point is outside of the line  $\langle a, x \rangle = 0$  and immediately transfers to a point  $x^+ = (E + B)x(t^*)$  at the time when phase point meets with the line  $\langle a, x \rangle = 0$ .

We have indicated the necessary and sufficient conditions for the existence of one-impulsive and two-impulsive discontinuous cycles of this system, as well as conditions for asymptotic stability of the zero equilibrium position.

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# Generalized Linear Differential Equations in a Banach Space: Continuous Dependence on a Parameter

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In what follows,  $X$  is a Banach space and  $L(X)$  is the Banach space of bounded linear operators on  $X$ . By  $\|\cdot\|_X$  we denote the norm in a Banach space  $X$ . Further,  $BV([a, b], X)$  is the set of  $X$  valued functions of bounded variation on  $[a, b]$  and  $G([a, b], X)$  is the set of  $X$  valued functions having on  $[a, b]$  all one-sided limits (i.e.  $X$  valued functions regulated on  $[a, b]$ ). A couple  $P=(D, \xi)$ , where  $D=\{\alpha_0, \alpha_1, \dots, \alpha_m\}$  and  $\xi=(\xi_1, \dots, \xi_m) \in [a, b]^m$ , is said to be a partition of  $[a, b]$  if  $a=\alpha_0 < \alpha_1 < \dots < \alpha_m=b$  and  $\alpha_{j-1} \leq \xi_j \leq \alpha_j$  for  $j=1, 2, \dots, m$ . For such a partition  $P$  and functions  $F: [a, b] \rightarrow L(X)$  and  $g: [a, b] \rightarrow X$  we define

$$S(dF, g, P) = \sum_{j=1}^m [F(\alpha_j) - F(\alpha_{j-1})]g(\xi_j) \quad \text{and} \quad S(F, dg, P) = \sum_{j=1}^m F(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})].$$

For a gauge  $\delta: [a, b] \rightarrow (0, \infty)$ , the partition  $P$  is called  $\delta$ -fine if

$$[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for all } j \in \mathbb{N}.$$

The integrals are the abstract *Kurzweil-Stieltjes integrals* (*KS-integrals*) defined as follows:

**Definition.** For  $F: [a, b] \rightarrow L(X)$ ,  $g: [a, b] \rightarrow X$  and  $I \in X$  we say that  $\int_a^b d[F]g = I$  if for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that

$$\|S(dF, g, P) - I\|_X < \varepsilon \quad \text{for all } \delta\text{-fine partitions } P \text{ of } [a, b].$$

Similarly we define the KS-integral  $\int_a^b Fd[g]$  using sums of the form  $S(F, dg, P)$ .

It is known that the integrals  $\int_a^b d[F]g$ ,  $\int_a^b Fd[g]$  exist if  $F \in G([a, b], L(X))$ ,  $g \in G([a, b], X)$  and at least one of the functions  $F$ ,  $g$  has a bounded variation on  $[a, b]$  (cf. [2]). Further basic properties of the abstract KS-integral, like e.g. the substitution theorem, the integration-by-parts theorem or the convergence theorems, have been described in [6] and [2].

Let  $A, A_k \in BV([a, b], L(X))$ ,  $\tilde{x}, \tilde{x}_k \in X$  and  $f, f_k \in G([a, b], X)$  be given for  $k \in \mathbb{N}$ . Consider the generalized linear differential equations

$$x(t) = \tilde{x} + \int_a^t d[A(s)]x(s) + f(t) - f(a), \quad t \in [a, b], \quad (1)$$

and

$$x_k(t) = \tilde{x}_k + \int_a^t d[A_k(s)]x_k(s) + f_k(t) - f_k(a), \quad t \in [a, b], \quad k \in \mathbb{N}. \quad (1_k)$$

The following assumptions are crucial for the existence of solutions to (1) and (1<sub>k</sub>)

$$[I - \Delta^- A(t)]^{-1} \in L(X) \quad \text{for all } t \in (a, b), \quad (2)$$

and

$$[I - \Delta^- A_k(t)]^{-1} \in L(X) \quad \text{for all } t \in (a, b), \quad k \in \mathbb{N}. \quad (2_k)$$

For the basic properties of generalized linear differential equations in a Banach space, see [7].

Our first result extends that by M. Ashordia [1] valid for the case  $X = \mathbb{R}^n$ .

**Theorem 1.** *Let  $A, A_k$  satisfy (1) and (1<sub>k</sub>), and let*

$$A_k \rightrightarrows A \quad \text{on } [a, b], \quad (3)$$

$$\alpha^* := \sup_{k \in \mathbb{N}} (\text{var}_a^b A_k) < \infty, \quad (4)$$

$$f_k \rightrightarrows f \quad \text{on } [a, b], \quad (5)$$

$$\tilde{x}_k \rightarrow \tilde{x} \quad \text{in } X. \quad (6)$$

*Then (1) has a unique solution  $x$  on  $[a, b]$ . Furthermore, for each  $k \in \mathbb{N}$  large enough there is a unique solution  $x_k$  on  $[a, b]$  to (1<sub>k</sub>) and  $x_k \rightrightarrows x$ .*

The next result extends that by Z. Opial [5] to homogeneous generalized linear differential equations in a general Banach space  $X$ .

**Theorem 2.** *Let  $f(t) \equiv f(a), f_k(t) \equiv f_k(a)$  on  $[a, b]$  for  $k \in \mathbb{N}$  and let  $A, A_k$  satisfy (1), (1<sub>k</sub>). Let  $\tilde{x}, \tilde{x}_k \in X$  satisfy (6) and let*

$$\lim_{k \rightarrow \infty} \left( \sup_{t \in [a, b]} \|A_k(t) - A(t)\|_{L(X)} \right) (1 + \text{var}_a^b A_k) = 0. \quad (6)$$

*Then the conclusions of Theorem 1 are true.*

For the proofs of Theorems 1 and 2, see [3]. The case when (3) (and hence also (6)) is not satisfied is treated in [4].

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# Two-Point Boundary Value Problems for Strongly Singular Higher-Order Linear Differential Equations with Deviating Arguments

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Consider the boundary value problem

$$u^{(n)}(t) = \sum_{j=1}^m p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b \quad (1)$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \quad (j = 1, \dots, m), \quad u^{(i-1)}(b) = 0 \quad (j = 1, \dots, n - m), \quad (2)$$

where  $n \geq 2$ ,  $m$  is the integer part of  $n/2$ ,  $p_j, q \in L_{loc}([a, b])$  ( $j = 1, \dots, m$ ), and  $\tau_j : ]a, b[ \rightarrow ]a, b[$  are measurable functions. By  $u^{(j-1)}(a)$  (by  $u^{(j-1)}(b)$ ) we denote the right (the left) limit of the function  $u^{(j-1)}$  at the point  $a$  (at the point  $b$ ).

The Agarwal–Kiguradze type theorems [1] are obtained by us, which contains unimprovable in a certain sense conditions guaranteeing the unique solvability of problem (1), (2). The results below cover the strongly singular case, where  $\sum_{j=1}^m \int_a^b (t-a)^{n-j}(b-t)^{n-j}|p_j(t)| dt = +\infty$ .

We use the following notations.

$L_{\alpha, \beta}([a, b])$  ( $L_{\alpha, \beta}^2([a, b])$ ) is the space of integrable (square integrable) with the weight  $(t-a)^\alpha(b-t)^\beta$  functions  $y : ]a, b[ \rightarrow R$ , with the norm  $\|y\|_{L_{\alpha, \beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta |y(s)| ds$  ( $\|y\|_{L_{\alpha, \beta}^2} = (\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s) ds)^{1/2}$ );

$\tilde{L}_{\alpha, \beta}^2([a, b])$  is the space of functions  $y \in L_{loc}([a, b])$  such that  $\tilde{y} \in L_{\alpha, \beta}^2([a, b])$ , where  $\tilde{y}(t) = \int_c^t y(s) ds$ ,  $c = (a+b)/2$ . The norm in  $\tilde{L}_{\alpha, \beta}^2([a, b])$  is defined by the equality

$$\begin{aligned} \|y\|_{\tilde{L}_{\alpha, \beta}^2} = & \max \left\{ \left[ \int_a^t (s-a)^\alpha \left( \int_s^t y(\xi) d\xi \right)^2 ds \right]^{1/2} : a \leq t \leq c \right\} + \\ & + \max \left\{ \left[ \int_t^b (b-s)^\beta \left( \int_t^s y(\xi) d\xi \right)^2 ds \right]^{1/2} : c \leq t \leq b \right\}. \end{aligned} \quad (3)$$

$\tilde{C}^{n-1, m}([a, b])$  is the space of  $(n-1)$ -times continuously differentiable functions  $y : ]a, b[ \rightarrow R$  such that  $\int_a^b |u^{(m)}(s)|^2 ds < +\infty$  and  $y^{n-1}$  is absolutely continuous on every closed interval from  $]a, b[$ .

If  $n = 2m$ , we assume that  $p_j \in L_{loc}([a, b])$  ( $j = 1, \dots, m$ ), and if  $n = 2m + 1$ , we assume that along with  $p_j \in L_{loc}([a, b])$  ( $j = 1, \dots, m$ ), the condition  $\limsup_{t \rightarrow b} |(b-t)^{2m-1} \int_c^t p_1(s) ds| < +\infty$  holds.



By  $h_j$  and  $f_j$  ( $j = 1, \dots, m$ ) we denote the functions and operator, respectively, defined by the equalities

$$h_1(t, s) = \left| \int_s^t (\xi - a)^{n-2m} [(-1)^{n-m} p_1(\xi)]_+ d\xi \right|, \quad h_j(t, s) = \left| \int_s^t (\xi - a)^{n-2m} p_j(\xi) d\xi \right| \quad (j=2, \dots, m),$$

$$f_j(c)(t, s) = \left| \int_s^t (\xi - a)^{n-2m} |p_j(\xi)| \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} d\xi_1 \right|^{1/2} d\xi \right| \quad (j = 1, \dots, m).$$

Along with (1), we consider the corresponding homogeneous equation

$$u^{(n)}(t) = \sum_{j=1}^m p_j(t) u^{(j-1)}(\tau_j(t)). \quad (1_0)$$

**Theorem 1.** *Let there exist  $a_0 \in ]a, b[$ ,  $b_0 \in ]a_0, b[$  and positive numbers  $l_{kj}$  and  $\gamma_{kj}$  ( $k = 0, 1$ ,  $j = 1, \dots, m$ ) such that*

$$(t - a)^{2m-j} h_j(t, s) \leq l_{0j} \text{ for } a \leq t \leq s < a_0, \quad \limsup_{t \rightarrow a} (t - a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a)(t, s) < +\infty,$$

$$(b - t)^{2m-j} h_j(t, s) \leq l_{1j} \text{ for } b_0 \leq s \leq t < b, \quad \limsup_{t \rightarrow b} (b - t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b)(t, s) < +\infty,$$

$$\sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{kj} < 1 \quad (k = 0, 1).$$

If, moreover, problem (1<sub>0</sub>), (2) has only the trivial solution in the space  $\tilde{C}^{n-1,m}(]a, b[)$ , then problem (1), (2) is uniquely solvable in this space for every  $q \in \tilde{L}_{2n-2m-2, 2m-2}(]a, b[)$ .

**Theorem 2.** *Let there exist numbers  $t^* \in ]a, b[$ ,  $\ell_{kj} > 0$ ,  $\bar{l}_{kj} \geq 0$ , and  $\gamma_{kj} > 0$  ( $k = 0, 1$ ,  $j = 1, \dots, m$ ) such that along with inequalities*

$$\sum_{j=1}^m \left( \frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1} (t^* - a)^{\gamma_{0j}} \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!! \sqrt{2\gamma_{0j}}} \right) < \frac{1}{2},$$

$$\sum_{j=1}^m \left( \frac{(2m-j)2^{2m-j+1} l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1} (b - t^*)^{\gamma_{1j}} \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!! \sqrt{2\gamma_{1j}}} \right) < \frac{1}{2},$$

the conditions

$$(t - a)^{2m-j} h_j(t, s) \leq l_{0j}, \quad (t - a)^{m-\gamma_{0j}-1/2} f_j(a)(t, s) \leq \bar{l}_{0j} \text{ for } a < t \leq s \leq t^*,$$

$$(b - t)^{2m-j} h_j(t, s) \leq l_{1j}, \quad (b - t)^{m-\gamma_{1j}-1/2} f_j(b)(t, s) \leq \bar{l}_{1j} \text{ for } t^* \leq s \leq t < b$$

hold. Then for every  $q \in \tilde{L}_{2n-2m-2, 2m-2}^2(]a, b[)$ , (1), (2) is uniquely solvable in the space  $\tilde{C}^{n-1,m}(]a, b[)$ .

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# Asymptotics of Solutions of a Linear Homogeneous System of Differential Equations in the Case of Asymptotically Equivalent, as $t \rightarrow +\infty$ , Roots of a Characteristic Equation

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The investigation of linear homogeneous systems (LHS) was carried out by the author by combining the method of generalized shearing transformations [1] and the method of  $L$ -diagonal systems [2]. The cases which are particular for the well-known methods are considered.

We study the 2-dimensional linear homogeneous system

$$\varepsilon(t)Y' = (D^{-1}\Lambda D + \alpha(t)B + Q(t))Y, \quad (1)$$

where  $t \in [t_0, +\infty[ = I$ ,  $Y = (y_1, y_2)^T$ , the scalar functions  $\varepsilon(t)$  and  $\alpha(t)$ , the constant matrices  $D$ ,  $\Lambda$ ,  $B$  and the matrix  $Q(t)$  are, in a general case, complex, and the following conditions are fulfilled:

1)  $0 \neq \varepsilon(t) \in C^1(I)$ ,  $\int_{t_0}^{+\infty} |\varepsilon^{-1}(t)| dt = +\infty$ ,

2)  $0 \neq \alpha(t) \in C^1(I)$ ,  $\alpha(+\infty) = 0$ ,

3)  $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ ,  $\det D \neq 0$ ,

$$\Lambda = \begin{pmatrix} \lambda & e \\ 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0, \quad e \in \{0, 1\},$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad B \neq O \quad (O \text{ is a zero matrix}),$$

$$Q(t) = \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix}, \quad Q(t) \in C(I), \quad Q(t) = o(|\alpha(t)|),$$

4)  $\exists$  a finite or an infinite limit  $\lim_{t \rightarrow +\infty} \varepsilon(t) \frac{w'(t)}{w(t)} = a$ , where  $w = w(t) = \alpha^{\frac{1}{2}}(t)$  (for a specific choice of root).

Denote

$$A = D^{-1}BD = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$P(t) = D^{-1}Q(t)D = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}, \quad P(t) = o(|\alpha(t)|).$$

**Theorem 1.** *Let  $e = 1$ ,  $|a| < +\infty$ , the conditions (1)–(4) be fulfilled, and*

$$\frac{a^2}{4} + a_{21} \neq 0, \quad |\alpha'| + |w'| + \left| \left( \varepsilon \frac{w'}{w^2} \right)' \right| + \|P'\| + \left\| \left( \frac{P}{w} \right)' \right\| + \left\| \left( \frac{P}{\alpha} \right)' \right\| \in L_1(I),$$

$\operatorname{Re} \left( \frac{w}{\varepsilon} (\tilde{\mu}_1 - \tilde{\mu}_2) \right)$  do not change its sign in  $I$ , where  $\tilde{\mu}_i(t)$  ( $i = 1, 2$ ) are the roots of the equation

$$\tilde{\mu}^2 - \tilde{\mu} \left( wa_{12} - \varepsilon \frac{w'}{w^2} + \frac{p_{22}}{w} + wa_{11} + \frac{p_{11}}{w} \right) - \left( a_{21} + \frac{p_{21}}{w} \right) (1 + \alpha a_{12} + p_{12}) = 0.$$

Then LHS (1) has two linearly independent solutions

$$Y_i(t) = e^{\int_0^t \left( \frac{\lambda}{\varepsilon(\tau)} + \frac{w'(\tau)}{\varepsilon(\tau)} \tilde{\mu}_i(\tau) \right) d\tau} \cdot \begin{pmatrix} d_{1i} + o(1) \\ d_{2i} + o(1) \end{pmatrix} \quad (i = 1, 2). \quad (2)$$

If all  $d_{ik} \neq 0$  ( $i, k = 1, 2$ ), then asymptotics (2) are exact.

The cases, where among  $d_{ik}$  ( $i, k = 1, 2$ ) there are those equal to zero, and also the case  $e = 1$ ,  $a = \infty$ , are considered.

Generalization of the obtained results to the case of LHS (1) of dimension  $n > 2$  ( $\Lambda$  is Jordan's block) is also considered.

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# On Oscillations of Solutions to Second-Order Linear Delay Differential Equations

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On the half-line  $\mathbb{R}_+ = [0, +\infty[$ , we consider the second-order linear delay differential equation

$$u''(t) + p(t)u(\tau(t)) = 0, \quad (1)$$

where  $p: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a locally Lebesgue integrable function and  $\tau: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a measurable function such that

$$\tau(t) \leq t \text{ for a. e. } t \geq 0, \quad \lim_{t \rightarrow +\infty} \text{ess inf}\{\tau(s) : s \geq t\} = +\infty.$$

Solutions to equation (1) can be defined in various ways. Since we are interested in properties of solutions in the neighbourhood of  $+\infty$ , we introduce the following commonly used definition.

**Definition 1.** Let  $t_0 \in \mathbb{R}_+$  and  $a_0 = \text{ess inf}\{\tau(t) : t \geq t_0\}$ . A continuous function  $u: [a_0, +\infty[ \rightarrow \mathbb{R}$  is said to be a *solution to equation (1) on the interval*  $[t_0, +\infty[$  if it is absolutely continuous together with its first derivative on every compact interval contained in  $[t_0, +\infty[$  and satisfies equality (1) almost everywhere in  $[t_0, +\infty[$ . A solution  $u$  to equation (1) on the interval  $[t_0, +\infty[$  is called *proper* if the inequality  $\sup\{|u(s)| : s \geq t\} > 0$  holds for  $t \geq t_0$ .

**Definition 2.** A proper solution  $u$  to equation (1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

It is known that if the integral  $\int_0^{+\infty} \tau(s)p(s) ds$  is convergent, then equation (1) has proper non-oscillatory solutions. Therefore, we will assume throughout the paper that

$$\int_0^{+\infty} \tau(s)p(s) ds = +\infty. \quad (2)$$

Let us put

$$G_* = \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t s\tau(s)p(s) ds, \quad G^* = \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t s\tau(s)p(s) ds.$$

**Proposition 1.** *Let condition (2) hold and  $G^* > 1$ . Then every proper solution to equation (1) is oscillatory.*

In view of Proposition 1, it is natural to suppose in the sequel that

$$G_* \leq 1. \quad (3)$$

A Wintner type oscillation criterion is presented in the next theorem.

**Theorem 1.** *Let conditions (2) and (3) be fulfilled and let there exist  $\lambda < 1$  and  $\varepsilon \in [0, 1[$  such that*

$$\int_0^{+\infty} s^\lambda \left( \frac{\tau(s)}{s} \right)^{1-\varepsilon G_*} p(s) ds = +\infty. \quad (4)$$

*Then every proper solution to equation (1) is oscillatory.*

**Theorem 2.** *Let conditions (2) and (3) hold and let there exist  $\varepsilon \in [0, 1[$  such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{\ln t} \int_0^t s \left( \frac{\tau(s)}{s} \right)^{1-\varepsilon G_*} p(s) ds > \frac{1}{4}. \quad (5)$$

*Then every proper solution to equation (1) is oscillatory.*

In view of Theorem 1, we can assume in the sequel that

$$\int_0^{+\infty} s^\lambda \left( \frac{\tau(s)}{s} \right)^{1-\varepsilon G_*} p(s) ds < +\infty \text{ for all } \lambda < 1, \varepsilon \in [0, 1[.$$

It allows one to define, for any  $\lambda < 1$  and  $\varepsilon \in [0, 1[$ , the function

$$Q(t; \lambda, \varepsilon) := t^{1-\lambda} \int_t^{+\infty} s^\lambda \left( \frac{\tau(s)}{s} \right)^{1-\varepsilon G_*} p(s) ds \text{ for } t > 0.$$

Moreover, for any  $\mu > 1$  and  $\varepsilon \in [0, 1[$ , we put

$$H(t; \mu, \varepsilon) := \frac{1}{t^{\mu-1}} \int_0^t s^\mu \left( \frac{\tau(s)}{s} \right)^{1-\varepsilon G_*} p(s) ds \text{ for } t > 0.$$

By using the lower and upper limits

$$\begin{aligned} Q_*(\lambda, \varepsilon) &= \liminf_{t \rightarrow +\infty} Q(t; \lambda, \varepsilon), & Q^*(\lambda, \varepsilon) &= \limsup_{t \rightarrow +\infty} Q(t; \lambda, \varepsilon), \\ H_*(\mu, \varepsilon) &= \liminf_{t \rightarrow +\infty} H(t; \mu, \varepsilon), & H^*(\mu, \varepsilon) &= \limsup_{t \rightarrow +\infty} H(t; \mu, \varepsilon), \end{aligned}$$

we establish new Hille and Nehari type oscillation criteria, which coincide with the well-known results in the case of ordinary differential equations.

**Theorem 3.** *Let conditions (2) and (3) be fulfilled and let there exist  $\lambda < 1$ ,  $\mu > 1$ , and  $\varepsilon \in [0, 1[$  such that*

$$\limsup_{t \rightarrow +\infty} (Q(t; \lambda, \varepsilon) + H(t; \mu, \varepsilon)) > \frac{\lambda^2}{4(1-\lambda)} + \frac{\mu^2}{4(\mu-1)}. \quad (6)$$

*Then every proper solution to equation (1) is oscillatory.*

As a corollary of Theorem 3 (with  $\mu = 2$  and  $\lambda = 0$ , respectively) we obtain

**Corollary 1.** *Let conditions (2) and (3) be fulfilled and let there exist  $\varepsilon \in [0, 1[$  such that*

$$\text{either } Q^*(\lambda, \varepsilon) > \frac{(2-\lambda)^2}{4(1-\lambda)} \text{ for some } \lambda < 1, \text{ or } H^*(\mu, \varepsilon) > \frac{\mu^2}{4(\mu-1)} \text{ for some } \mu > 1.$$

*Then every proper solution to equation (1) is oscillatory.*

The next theorem deals with the lower limit of the sum on the left-hand side of inequality (6) and thus it complements Theorem 3 in a certain sense.

**Theorem 4.** *Let conditions (2) and (3) be fulfilled and let there exist  $\lambda < 1$ ,  $\mu > 1$ , and  $\varepsilon \in [0, 1[$  such that*

$$\liminf_{t \rightarrow +\infty} (Q(t; \lambda, \varepsilon) + H(t; \mu, \varepsilon)) > \frac{1}{4(1-\lambda)} + \frac{1}{4(\mu-1)}. \quad (7)$$

*Then every proper solution to equation (1) is oscillatory.*

Theorem 4 yields

**Corollary 2.** *Let conditions (2) and (3) be fulfilled and let there exist  $\varepsilon \in [0, 1[$  such that*

$$\text{either } Q_*(\lambda, \varepsilon) > \frac{1}{4(1-\lambda)} \text{ for some } \lambda < 1, \text{ or } H_*(\mu, \varepsilon) > \frac{1}{4(\mu-1)} \text{ for some } \mu > 1.$$

*Then every proper solution to equation (1) is oscillatory.*

Now we give a statement complementing Corollary 2.

**Theorem 5.** *Let conditions (2) and (3) be fulfilled and let there exist  $\lambda < 1$ ,  $\mu > 1$ , and  $\varepsilon \in [0, 1[$  such that either*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \leq Q_*(\lambda, \varepsilon) \leq \frac{1}{4(1-\lambda)}, \quad (8)$$

$$H^*(\mu, \varepsilon) > \frac{\mu^2}{4(\mu-1)} - \frac{1}{2} \left( 1 - \sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} \right) \quad (9)$$

or

$$\frac{\mu(2-\mu)}{4(\mu-1)} \leq H_*(\mu, \varepsilon) \leq \frac{1}{4(\mu-1)}, \quad (10)$$

$$Q^*(\lambda, \varepsilon) > \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2} \left( 1 + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)} \right). \quad (11)$$

*Then every proper solution to equation (1) is oscillatory.*

If both conditions (8) and (10) are satisfied then oscillation criteria (9) and (11) can be slightly refined as is presented in the last statement.

**Theorem 6.** *Let conditions (2) and (3) be fulfilled and let there exist  $\lambda < 1$ ,  $\mu > 1$ , and  $\varepsilon \in [0, 1[$  such that inequalities (8) and (10) are satisfied. If*

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} (Q(t; \lambda, \varepsilon) + H(t; \mu, \varepsilon)) > \\ & > Q_*(\lambda, \varepsilon) + H_*(\mu, \varepsilon) + \frac{1}{2} \left( \sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)} \right), \end{aligned}$$

*then every proper solution to equation (1) is oscillatory.*

*Remark 1.* If we assume, in addition, that there are numbers  $\alpha > 0$  and  $t_0 \geq 0$  such that  $\frac{\tau(t)}{t} \geq \alpha$  for a. e.  $t \geq t_0$ , then we can put  $\varepsilon = 1$  in all above presented statements.

# Solvability and Well-Posedness of Two-Point Weighted Singular Boundary Value Problems

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In an open interval  $]a, b[$ , we consider the second order nonlinear differential equation

$$u'' = f(t, u) \tag{1}$$

with two-point weighted boundary conditions of one of the following two types:

$$\limsup_{t \rightarrow a} \frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \limsup_{t \rightarrow b} \frac{|u(t)|}{(b-t)^\beta} < +\infty \tag{2}$$

and

$$\limsup_{t \rightarrow a} \frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \lim_{t \rightarrow b} u'(t) = 0. \tag{3}$$

Here  $f : ]a, b[ \times R \rightarrow R$  is a continuous function,  $\alpha \in ]0, 1[$ , and  $\beta \in ]0, 1[$ .

Following R. P. Agarwal and I. Kiguradze [1,2] we say that Eq. (1) with respect to the time variable has a strong singularity at the point  $a$  (at the point  $b$ ) if for any  $t_0 \in ]a, b[$  and  $x > 0$  the condition  $\int_a^{t_0} (t-a)[|f(t, x)| - f(t, x) \operatorname{sgn} x] dt = +\infty$  ( $\int_{t_0}^b (b-t)[|f(t, x)| - f(t, x) \operatorname{sgn} x] = +\infty$ ) is satisfied.

Theorems obtained by us contain unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of the problem (1), (2) (of the problem (1), (3)), at that these theorems, unlike the previous well-known results, cover the case, where Eq. (1) with respect to the time variable has strong singularities at the points  $a$  and  $b$  (has a strong singularity at the point  $a$ ).

Before passing to the formulation of the main results, we introduce some notation.

$$f^*(t, x) = \max \{|f(t, y)| : 0 \leq y \leq x\} \quad \text{for } a < t < b, \quad x \geq 0.$$

By  $G_0$  and  $G_1$  we denote the Green functions of the problems  $u'' = 0$ ;  $u(a) = u(b) = 0$  and  $u'' = 0$ ;  $u(a) = u'(b) = 0$ , respectively, i.e.,

$$G_0(t, s) = \begin{cases} \frac{(s-a)(t-b)}{b-a} & \text{for } a \leq s \leq t \leq b, \\ \frac{(t-a)(s-b)}{b-a} & \text{for } a \leq t < s \leq b, \end{cases} \quad \text{and} \quad G_1(t, s) = \begin{cases} a-s & \text{for } a \leq s \leq t \leq b, \\ a-t & \text{for } a \leq t < s \leq b. \end{cases}$$

For any continuous function  $h : ]a, b[ \rightarrow R$ , we assume

$$\nu_{\alpha, \beta}(h) = \sup \left\{ (t-a)^{-\alpha} (b-t)^{-\beta} \int_a^b |G_0(t, s)h(s)| ds : a < t < b \right\},$$

$$\nu_\alpha(h) = \sup \left\{ (t-a)^{-\alpha} \int_a^b |G_1(t, s)h(s)| ds : a < t < b \right\}.$$

**Definition.** The problem (1), (2) (the problem (1), (3)) is said to be **well-posed** if for any continuous function  $h : ]a, b[ \rightarrow \mathbb{R}$ , satisfying the condition  $\nu_{\alpha, \beta}(h) < +\infty$  ( $\nu_{\alpha}(h) < +\infty$ ), the perturbed differential equation

$$v'' = f(t, v) + h(t) \quad (4)$$

has a unique solution, satisfying the boundary conditions (2) (the boundary conditions (3)), and there exists a positive constant  $r$ , independent of the function  $h$ , such that in the interval  $]a, b[$  the inequality  $|u(t) - v(t)| \leq r\nu_{\alpha, \beta}(h)(t-a)^{\alpha}(b-t)^{\beta}$  ( $|u(t) - v(t)| \leq r\nu_{\alpha}(h)(t-a)^{\alpha}$ ) is satisfied, where  $u$  and  $v$  are the solutions of (1), (2) and (4), (2) (of (1), (3) and (4), (3)), respectively.

The following statements are valid.

**Theorem 1.** *Let there exist a constant  $\ell \in [0, 1[$  and a continuous function  $q : ]a, b[ \rightarrow [0, +\infty[$  such that*

$$f(t, x) \operatorname{sgn} x \geq -\ell \left( \frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2} \right) |x| - q(t) \quad \text{for } a < t < b, \quad x \in \mathbb{R},$$

and  $\nu_{\alpha, \beta}(q) < +\infty$ . Then the problem (1), (2) has at least one solution.

**Theorem 2.** *Let there exist a constant  $\ell \in [0, 1[$  such that*

$$f(t, x) - f(t, y) \geq -\ell \left( \frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2} \right) (x - y) \quad \text{for } a < t < b, \quad x > y,$$

and  $\nu_{\alpha, \beta}(f(\cdot, 0)) < +\infty$ . Then the problem (1), (2) is well-posed.

**Theorem 3.** *Let there exist a constant  $\ell < \alpha(1-\alpha)$  and a continuous function  $q : ]a, b[ \rightarrow [0, +\infty[$  such that*

$$f(t, x) \operatorname{sgn} x \geq -\frac{\ell}{(t-a)^2} |x| - q(t) \quad \text{for } a < t < b, \quad x \in \mathbb{R}$$

and  $\nu_{\alpha}(q) < +\infty$ . If, moreover, the condition

$$\int_t^b f^*(s, x) ds < +\infty \quad \text{for } a < t < b, \quad x > 0 \quad (5)$$

holds, then the problem (1), (3) has at least one solution.

**Theorem 4.** *Let there exist a constant  $\ell < \alpha(1-\alpha)$  such that*

$$f(t, x) - f(t, y) \geq -\frac{\ell}{(t-a)^2} (x - y) \quad \text{for } a < t < b, \quad x > y.$$

If, moreover,  $\nu_{\alpha}(f(\cdot, 0)) < +\infty$  and the condition (5) holds, then the problem (1), (3) is well-posed.

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# On Invariant Sets of Impulsive Multi-Frequency Systems

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We consider a system of differential equations, defined in the direct product of an  $m$ -dimensional torus  $\mathcal{T}_m$  and an  $n$ -dimensional Euclidean space  $E^n$  that undergo impulsive perturbations at the moments when the phase point  $\varphi$  meets a given set in the phase space

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} &= B(\varphi)x + g(\varphi), \end{aligned} \tag{1}$$

where  $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$ ,  $x = (x_1, \dots, x_n)^T \in E^n$ ,  $a(\varphi)$  is a continuous  $2\pi$ -periodic with respect to each of the components  $\varphi_v$ ,  $v = 1, \dots, m$  vector function that satisfies a Lipschitz condition with respect to  $\varphi$ . Functions  $A(\varphi), B(\varphi)$  are continuous  $2\pi$ -periodic with respect to each of the components  $\varphi_v$ ,  $v = 1, \dots, m$  square matrices;  $f(\varphi), g(\varphi)$  are continuous (piecewise continuous with first kind discontinuities in the set  $\Gamma$ )  $2\pi$ -periodic with respect to each of the components  $\varphi_v$ ,  $v = 1, \dots, m$  vector functions.

We assume that the set  $\Gamma$  is a subset of the torus  $\mathcal{T}_m$ , which is a manifold of dimension  $m - 1$  defined by the equation  $\Phi(\varphi) = 0$  for some continuous scalar  $2\pi$ -periodic with respect to each of the components  $\varphi_v$ ,  $v = 1, \dots, m$  function.

Denote by  $t_i(\varphi)$ ,  $i \in \mathbb{Z}$  the solutions of the equation  $\Phi(\varphi_t(\varphi)) = 0$  that are the moments of impulsive action in system (1). Let the function  $\Phi(\varphi)$  be such that the solutions  $t = t_i(\varphi)$  exist since otherwise system (1) would not be an impulsive system.

We call a point  $\varphi^*$  an  $\omega$ -limit point of the trajectory  $\varphi_t(\varphi)$  if there exists a sequence  $\{t_n\}_{n \in \mathbb{N}}$  in  $\mathbb{R}$  so that

$$\lim_{n \rightarrow +\infty} t_n = +\infty, \quad \lim_{n \rightarrow +\infty} \varphi_{t_n}(\varphi) = \varphi^*.$$

The set of all  $\omega$ -limit points for a given trajectory  $\varphi_t(\varphi)$  is called  $\omega$ -limit set of the trajectory  $\varphi_t(\varphi)$  and denoted by  $\Omega_\varphi$ . Denote

$$\Omega = \bigcup_{\varphi \in \mathcal{T}_m} \Omega_\varphi,$$

and assume that the matrices  $A(\varphi)$  and  $B(\varphi)$  are constant in the domain  $\Omega$ :

$$A(\varphi)|_{\varphi \in \Omega} = \tilde{A}, \quad B(\varphi)|_{\varphi \in \Omega} = \tilde{B}.$$

We will obtain sufficient conditions for the existence and asymptotic stability of an invariant set of the system (1) in terms of the eigenvalues of the matrices  $\tilde{A}$  and  $\tilde{B}$ . Denote

$$\gamma = \max_{j=1, \dots, n} \operatorname{Re} \lambda_j(\tilde{A}), \quad \alpha^2 = \max_{j=1, \dots, n} \lambda_j((E + \tilde{B})^T (E + \tilde{B})).$$

**Theorem 1.** *Let the moments of impulsive perturbations  $\{t_i(\varphi)\}$  be such that uniformly with respect to  $t \in \mathbb{R}$  there exists a finite limit*

$$\lim_{\tilde{T} \rightarrow \infty} \frac{i(t, t + \tilde{T})}{\tilde{T}} = p. \tag{2}$$

If the following inequality holds

$$\gamma + p \ln \alpha < 0, \quad (3)$$

then system (1) has an asymptotically stable invariant set.

Such approach may be extended to the nonlinear system of the form

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A_0(\varphi)x + A_1(\varphi, x)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} &= B_0(\varphi)x + B_1(\varphi, x)x + g(\varphi). \end{aligned} \quad (4)$$

**Theorem 2.** Let the matrices  $A_0(\varphi)$  and  $B_0(\varphi)$  be constant in the domain  $\Omega$ , uniformly with respect to  $t \in \mathbb{R}$  there exist a finite limit (2) and the inequality (3) hold. Then there exist sufficiently small constants  $a_1$  and  $b_1$  and sufficiently small Lipschitz constants  $L_A$  and  $L_B$  such that for any continuous  $2\pi$ -periodic with respect to each of the components  $\varphi_v$ ,  $v = 1, \dots, m$  functions  $A_1(\varphi, x)$  and  $B_1(\varphi, x)$  such that  $\max_{\varphi \in \mathcal{T}_m, x \in \bar{J}_h} \|A_1(\varphi, x)\| \leq a_1$ ,  $\max_{\varphi \in \mathcal{T}_m, x \in \bar{J}_h} \|B_1(\varphi, x)\| \leq b_1$  and for any  $x', x'' \in \bar{J}_h$ ,

$$\|A_1(\varphi, x') - A_1(\varphi, x'')\| \leq L_A \|x' - x''\|, \quad \|B_1(\varphi, x') - B_1(\varphi, x'')\| \leq L_B \|x' - x''\|,$$

system (4) has an asymptotically stable invariant set.

In summary, we have obtained sufficient conditions for the existence and asymptotic stability of invariant sets of a linear impulsive system of differential equations defined in  $\mathcal{T}_m \times E^n$  that has specific properties in the  $\omega$ -limit set  $\Omega$  of the trajectories  $\varphi_t(\varphi)$ . We have proved that it is sufficient to impose some restrictions on system (1) only in the domain  $\Omega$  to guarantee the existence and asymptotic stability of the invariant set.

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# On the Well-Possedness of the Weighted Cauchy Problem for Nonlinear Singular Differential Equations of Higher Orders with Deviating Arguments

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In a finite interval  $]a, b[$  we consider the differential equation

$$u^{(n)}(t) = f\left(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\right) \quad (1)$$

with the weighted initial conditions

$$\limsup_{t \rightarrow a} \left( \frac{|u^{(i-1)}(t)|}{\rho^{(i-1)}(t)} \right) < +\infty \quad (i = 1, \dots, n), \quad (2)$$

where  $f : ]a, b[ \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a function, satisfying the local Carathéodory conditions,  $\tau_i : ]a, b[ \rightarrow ]a, b[$  ( $i = 1, \dots, n$ ) are measurable functions, and  $\rho : [a, b] \rightarrow [0, +\infty[$  is the  $(n - 1)$ -times continuously differentiable function such that

$$\rho^{(i-1)}(a) = 0, \quad \rho^{(i-1)}(t) > 0 \quad \text{for } a < t \leq b \quad (i = 1, \dots, n).$$

Let  $q : ]a, b[ \rightarrow ]0, +\infty[$  be some nonincreasing function. The problems (1), (2) is said to be **well-posed with the weight**  $q$ , if for any integrable with the weight  $q$  function  $h : ]a, b[ \rightarrow \mathbb{R}$ , satisfying the condition

$$\nu_{q,n}(h; \rho) \stackrel{\text{def}}{=} \sup \left\{ \left( \int_a^t q(s)h(s) ds \right) / q(t)\rho^{(n-1)}(t) : a < t \leq b \right\} < +\infty,$$

the differential equation

$$u^{(n)}(t) = f\left(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\right) + h(t)$$

under the initial conditions (2) has a unique solution  $u_h$  and there exists the positive not depending on  $r$  function  $h$ , such that

$$|u_h^{(i-1)}(t) - u_0^{(i-1)}(t)| \leq r\nu_{q,n}(h; \rho)\rho^{(i-1)}(t) \quad \text{for } a \leq t \leq b \quad (i = 1, \dots, n),$$

where  $u_0$  is a solution of the problem (1), (2).

The problem (1), (2) is said to be **well-posed** if it is well-posed with the weight  $q(t) \equiv 1$ .

**Theorem 1.** *Let in the domain  $]a, b[ \times \mathbb{R}^n$  the condition*

$$|f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)| \leq \sum_{i=1}^n h_i(t)|x_i - y_i|, \quad (3)$$

be fulfilled, where  $h_i \in L_{loc}([a, b])$  ( $i = 1, \dots, n$ ). Let, moreover,

$$\sup \left\{ \left( \int_a^t |f(s, 0, \dots, 0)| ds \right) / \rho^{(n-1)}(t) : a < t \leq b \right\} < +\infty$$

and there exist a number  $\gamma \in ]0, 1[$  such that

$$\sum_{i=1}^n \int_a^t \rho^{(i-1)}(\tau_i(s)) h_i(s) ds \leq \gamma \rho^{(n-1)}(t) \text{ for } a \leq t \leq b.$$

Then the problem (1), (2) is well-posed.

**Theorem 2.** Let  $n \geq 2$ , the function  $f_0(t) \stackrel{\text{def}}{=} f(t, 0, \dots, 0)$  be integrable with the weight  $q$ , and  $\nu_{n,q}(f_0; \rho) < +\infty$ , where  $q : ]a, b[ \rightarrow ]0, +\infty[$  is some nonincreasing function, satisfying the equality

$$\lim_{t \rightarrow a} (q(t) \rho^{(n-1)}(t)) = 0.$$

Let, moreover, in the domain  $]a, b[ \times \mathbb{R}^n$  the condition (3) be fulfilled, where  $h_i \in L_{loc}([a, b])$  ( $i = 1, \dots, n$ ), and there exist the numbers  $m \in \{1, \dots, n-1\}$  and  $\gamma \in ]0, 1[$  such that

$$\begin{aligned} \exp \left( \sum_{i=1}^m \int_s^t \frac{(x-a)^{n-i}}{(n-i)!} h_i(x) dx \right) &\leq \frac{q(s)}{q(t)} \text{ for } a < s \leq t \leq b, \\ \int_a^t q(s) \left[ \sum_{i=1}^m |\rho^{(i-1)}(\tau_i(s)) - \rho^{(i-1)}(s)| h_i(s) + \sum_{i=m+1}^n \rho^{(i-1)}(\tau_i(s)) h_i(s) \right] ds &\leq \\ &\leq \gamma q(t) \rho^{(n-1)}(t) \text{ for } a < t \leq b. \end{aligned}$$

Then the problem (1), (2) is well-posed with the weight  $q$ .

The above-formulated theorems cover the case in which the equation (1) is strongly singular, i.e., the case, where

$$\int_a^b (t-a)^\mu f^*(t, x) dt = +\infty \text{ for } \mu \geq 0, x > 0,$$

where  $f^*(t, x) = \max \left\{ |f(t, x_1, \dots, x_n)| : \sum_{i=1}^n |x_i| \leq x \right\}$ . It should be also noted that the condition  $\gamma \in ]0, 1[$  in these theorems is unimprovable and it cannot be replaced by the condition  $\gamma = 1$ .

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# Three Types of Solutions of Some Singular Nonlinear Differential Equation

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The differential equation

$$(p(t)u')' = p(t)f(u) \tag{1}$$

is investigated on the positive half line under the assumptions (2)–(6):

$$f \in Lip_{loc}(\mathbb{R}), \quad \exists L \in (0, \infty) : f(L) = 0, \tag{2}$$

$$\exists L_0 \in [-\infty, 0) : \quad xf(x) < 0, \quad x \in (L_0, 0) \cup (0, L), \tag{3}$$

$$\exists \bar{B} \in (L_0, 0) : F(\bar{B}) = F(L), \quad \text{where } F(x) = - \int_0^x f(z) \, dz, \quad x \in \mathbb{R}, \tag{4}$$

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0, \tag{5}$$

$$p'(t) > 0, \quad t \in (0, \infty), \quad \lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} = 0. \tag{6}$$

Due to  $p(0) = 0$ , equation (1) has a singularity at  $t = 0$ .

The following results are proved.

**1.** For each  $B < 0$  equation (1) has a unique solution  $u_B \in C^1[0, \infty) \cap C^2(0, \infty)$  which satisfies the initial conditions

$$u(0) = B, \quad u'(0) = 0. \tag{7}$$

**2.** A solution  $u_B$  of problem (1), (7) satisfying  $\sup\{u(t) : t \in [0, \infty)\} < L$  is called a *damped* solution. If  $\mathcal{M}_d$  is the set of all  $B < 0$  such that  $u_B$  is a damped solution, then  $\mathcal{M}_d$  is nonempty and open in  $(-\infty, 0)$ .

**3.** A solution  $u_B$  of problem (1), (7) satisfying  $\sup\{u(t) : t \in [0, \infty)\} > L$  is called an *escape* solution. If  $\mathcal{M}_e$  is the set of all  $B < 0$  such that  $u_B$  is an escape solution, then  $\mathcal{M}_e$  is open in  $(-\infty, 0)$ . In addition,  $\mathcal{M}_e$  is nonempty provided one of the following additional assumptions (A1), (A2), (A3), or (A4) is valid:

- (A1):  $L_0 \in (-\infty, 0)$ ,  $f(L_0) = 0$ .
- (A2):  $f(x) > 0$  for  $x \in (-\infty, 0)$  and

$$0 \leq \limsup_{x \rightarrow -\infty} \frac{f(x)}{|x|} < \infty.$$

- (A3):  $f(x) > 0$  for  $x \in (-\infty, 0)$  and there exists  $k \geq 2$  such that

$$\lim_{t \rightarrow 0^+} \frac{p'(t)}{t^{k-2}} \in (0, \infty).$$

Further, there exists  $r \in (1, \frac{k+2}{k-2})$  such that  $f$  fulfils

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{|x|^r} \in (0, \infty).$$

- (A4):

$$\int_0^1 \frac{ds}{p(s)} < \infty.$$

4. If  $\mathcal{M}_e$  is nonempty, then problem (1), (7) has a solution  $u$  such that

$$\sup \{u(t) : t \in [0, \infty)\} = L.$$

Such solution is called *homoclinic*. It is increasing and  $\lim_{t \rightarrow \infty} u(t) = L$ .

5. Some other additional conditions for  $p$  and  $f$  which give asymptotic formulas for damped and homoclinic solutions are discussed.

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# Slowly Growing Solutions of Singular Linear Functional Differential Systems

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We consider the system of linear functional differential equations

$$x'_i(t) = \sum_{k=1}^n (l_{ik}x_k)(t) + q_i(t), \quad t \in [a, b], \quad i = 1, 2, \dots, n, \quad (1)$$

subjected to the initial conditions

$$x_i(a) = \lambda_i, \quad i = 1, 2, \dots, n, \quad (2)$$

where  $-\infty < a < b < \infty$ , the functions  $q_i$ ,  $i = 1, 2, \dots, n$ , are locally integrable,  $l_{ik} : C([a, b], \mathbb{R}) \rightarrow L_{1; \text{loc}}([a, b], \mathbb{R})$ ,  $i, k = 1, 2, \dots, n$ , are linear mappings and  $L_{1; \text{loc}}([a, b], \mathbb{R})$  is the set of functions  $u : (a, b) \rightarrow \mathbb{R}$  such that  $u|_{[a+\varepsilon, b]} \in L_1([a+\varepsilon, b], \mathbb{R})$  for any  $\varepsilon \in (0, b-a)$ .

Our aim is to find conditions sufficient for the existence and uniqueness of a slowly growing solution of the initial value problem (1), (2). The "slow growth" of a solution  $x = (x_i)_{i=1}^n : [a, b) \rightarrow \mathbb{R}^n$  is understood in the sense that its components satisfy the conditions

$$\sup_{t \in [a, b)} h_i(t) |x_i(t)| < +\infty, \quad i = 1, 2, \dots, n, \quad (3)$$

where  $h_i : [a, b) \rightarrow [0, +\infty)$ ,  $i = 1, 2, \dots, n$ , are certain given continuous functions possessing the properties

$$\lim_{t \rightarrow b^-} h_i(t) = 0, \quad i = 1, 2, \dots, n. \quad (4)$$

Solutions of system (1) are sought for in the class of functions that are only locally absolutely continuous and, in particular, may be unbounded in a neighbourhood of the point  $b$ .

**Definition.** By a *solution* of the functional differential system (1), we mean a locally absolutely continuous vector function  $x = (x_i)_{i=1}^n : [a, b) \rightarrow \mathbb{R}^n$  with components possessing the properties  $h_i x'_i \in L_1([a, b), \mathbb{R})$ ,  $i = 1, 2, \dots, n$ , and satisfying equalities (1) almost everywhere on the interval  $[a, b)$ . We say that a solution  $x = (x_i)_{i=1}^n : [a, b) \rightarrow \mathbb{R}^n$  of system (1) is *slowly growing* if it has property (3).

The theorem formulated below concerns the case where the right-hand side terms of equations (1) are determined by linear operators which are positive with respect to the pointwise partial orderings of the linear manifolds  $C([a, b), \mathbb{R})$  and  $L_{1; \text{loc}}([a, b), \mathbb{R})$ .

**Definition.** An operator  $l : C([a, b), \mathbb{R}) \rightarrow L_{1; \text{loc}}([a, b), \mathbb{R})$  is said to be *positive* if  $(lu)(t) \geq 0$  for a. e.  $t \in [a, b)$  whenever  $u$  is non-negative on  $[a, b)$ .

**Theorem ([1]).** *Let us assume that the linear mappings  $l_{ik} : C([a, b), \mathbb{R}) \rightarrow L_{1; \text{loc}}([a, b), \mathbb{R})$ ,  $i, k = 1, 2, \dots, n$ , are positive, the relations*

$$\sum_{k=1}^n h_k l_{ik} \left( \frac{1}{h_k} \right) \in L_1([a, b), \mathbb{R}), \quad i = 1, 2, \dots, n, \quad (5)$$

hold, and there exists a certain  $\delta \in [0, 1)$  such that the inequality

$$\sum_{k=1}^n l_{ik} \left( \sum_{j=1}^n \int_a^t l_{kj} \left( \frac{1}{h_j} \right) (s) ds \right) (t) \leq \delta \sum_{k=1}^n l_{ik} \left( \frac{1}{h_k} \right) (t) \quad (6)$$

is satisfied for a. e.  $t \in [a, b)$  and every  $i = 1, 2, \dots, n$ .

Then the initial value problem (1), (2) has a unique slowly growing solution for arbitrary locally integrable functions  $q_i : [a, b) \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, n$ , possessing the property

$$\{h_i q_i \mid i = 1, 2, \dots, n\} \subset L_1([a, b), \mathbb{R}). \quad (7)$$

Furthermore, if  $q_i$  and  $\lambda_i$ ,  $i = 1, 2, \dots, n$ , satisfy the condition

$$-\sum_{k=1}^n \lambda_k (l_{ik} 1)(t) \leq q_i(t), \quad t \in [a, b), \quad i = 1, 2, \dots, n, \quad (8)$$

then the unique solution of problem (1), (2), (3) has non-negative components.

The symbol  $l_{ik} 1$  in (8) stands for the result of application of the operator  $l_{ik}$  to the function equal identically to 1.

By virtue of the positivity of the mappings  $l_{ik}$ ,  $i, k = 1, 2, \dots, n$ , conditions (8) are satisfied, in particular, if  $\{\lambda_i \mid i = 1, 2, \dots, n\} \subset [0, +\infty)$  and the functions  $q_i$ ,  $i = 1, 2, \dots, n$ , are non-negative almost everywhere on  $[a, b)$ .

*Remark.* The condition  $\delta < 1$  on the constant  $\delta$  appearing in assumption (6) of Theorem is sharp and cannot be weakened.

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# On the Existence of Special Class Solutions of a Quasi-Linear Differential System in One Particular Case

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The system of differential equations

$$\frac{dz_j}{dt} = (j-1)b(t, \varepsilon)z_1 + (2-j)\varepsilon\alpha(t, \varepsilon)z_2 + h_j(t, \varepsilon, \theta) + \mu Z_j(t, \varepsilon, \theta, z_1, z_2), \quad j = 1, 2 \quad (1)$$

is considered, where  $\alpha \in S_{m-1}$ ,  $b \in S_m$ ,  $h_1, h_2 \in B_m$  (the definitions of classes  $S_m$  and  $B_m$  are given in [1]),  $\inf_G |b| > 0$ ,  $z_1, z_2 \in \bar{D}$ , the functions  $Z_1$  and  $Z_2$  belong to the class  $B_m$  with respect to  $t, \varepsilon, \theta$  and have in  $\bar{D}$  continuous partial derivatives with respect to  $x_1, x_2$  up to some order  $2q+3$ , inclusive, and if  $z_1, z_2 \in B_m$ , then these partial derivatives are also of the class  $B_m$ .  $\mu \in \mathbf{R}^+$ .

Denote  $\forall f \in B_m$ :

$$\Gamma_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t, \varepsilon, \theta) \exp(-in\theta) d\theta, \quad n \in \mathbf{Z},$$

and introduce the functions

$$\begin{aligned} \xi_{10}(t, \varepsilon, \theta) &= \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\Gamma_n(h_1(t, \varepsilon, \theta))}{in\varphi(t, \varepsilon)} \exp(in\theta) - \frac{\Gamma_0(h_2(t, \varepsilon, \theta))}{b(t, \varepsilon)}, \\ \xi_{20}(t, \varepsilon, \theta) &= \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left( \frac{-b(t, \varepsilon)\Gamma_n(h_1(t, \varepsilon, \theta))}{n^2\varphi^2(t, \varepsilon)} + \frac{\Gamma_n(h_2(t, \varepsilon, \theta))}{in\varphi(t, \varepsilon)} \right) \exp(in\theta) + N_0(t, \varepsilon), \end{aligned}$$

where the function  $N_0(t, \varepsilon)$  is defined from the equation

$$Q(t, \varepsilon, N_0) = \Gamma_0(Z_1(t, \varepsilon, \theta, \xi_{10}, \xi_{20})) = 0. \quad (2)$$

Denote, further,  $(Z)_0 = Z(t, \varepsilon, \theta, \xi_{10}, \xi_{20})$ ,

$$\begin{aligned} \eta_{11}(t, \varepsilon, \theta) &= \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\Gamma_n((Z_1)_0)}{in\varphi(t, \varepsilon)} \exp(in\theta) - \frac{\Gamma_0((Z_2)_0)}{b(t, \varepsilon)}, \\ \eta_{21}(t, \varepsilon, \theta) &= \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left( \frac{-b(t, \varepsilon)\Gamma_n((Z_1)_0)}{n^2\varphi^2(t, \varepsilon)} + \frac{\Gamma_n((Z_2)_0)}{in\varphi(t, \varepsilon)} \right) \exp(in\theta), \\ Q_1(t, \varepsilon, N_0) &= \Gamma_0 \left( \left( \frac{\partial Z_1}{\partial \xi_1} \right)_0 \eta_{11}(t, \varepsilon, \theta) + \left( \frac{\partial Z_1}{\partial \xi_2} \right)_0 \eta_{21}(t, \varepsilon, \theta) \right). \end{aligned}$$

**Lemma.** *Let the system (1) satisfy the following conditions:*

- (1)  $\Gamma_0(h_1(t, \varepsilon, \theta)) \equiv 0 \quad \forall t, \varepsilon \in G$ ;
- (2) *the equality (2) is fulfilled  $\forall t, \varepsilon \in G$  and  $\forall N_0$ ;*

(3) the equation  $Q_1(t, \varepsilon, N_0) = 0$  has a root  $N_0(t, \varepsilon)$  satisfying the condition

$$\inf_G \left| \frac{dQ_1(t, \varepsilon, N_0)}{dN_0} \right| > 0.$$

Then for sufficiently small values of the parameter  $\mu$  there exists transformation of the type

$$z_j = \tilde{\xi}_j(t, \varepsilon, \theta, \mu) + \sum_{k=1}^2 \psi_{jk}(t, \varepsilon, \mu) \tilde{z}_k, \quad j = 1, 2, \quad (3)$$

where  $\tilde{\xi}_j, \psi_{jk} \in B_m$  ( $j, k = 1, 2$ ), reducing the system (2) to the form

$$\begin{aligned} \frac{d\tilde{z}_j}{dt} &= (j-1)b(t, \varepsilon)\tilde{z}_j + \sum_{k=1}^2 \left( \sum_{l=1}^q a_{jkl}(t, \varepsilon)\mu^l \right) \tilde{z}_k + \varepsilon c_j(t, \varepsilon, \theta, \mu) + \mu^{2q}(t, \varepsilon, \theta, \mu) + \\ &+ \varepsilon \sum_{k=1}^2 r_{jk}(t, \varepsilon, \theta, \mu) \tilde{z}_k + \mu^{q+1} \sum_{k=1}^2 w_{jk}(t, \varepsilon, \theta, \mu) \tilde{z}_k + \mu \tilde{Z}_j(t, \varepsilon, \theta, \tilde{z}_1, \tilde{z}_2, \mu), \quad j = 1, 2, \end{aligned} \quad (4)$$

where  $a_{jkl} \in S_m$ ,  $c_j, d_j, r_{jk}, w_{jk} \in B_{m-1}$ ,  $\tilde{Z}_j$  contain summands, not lower than of the 2nd order with respect to  $\tilde{z}_1, \tilde{z}_2$ .

Introduce the matrices  $A_l(t, \varepsilon) = (a_{jkl}(t, \varepsilon))_{j,k=1,2}$ , ( $l = \overline{1, q}$ ) ( $a_{jkl}$  are defined in Lemma 1),

$$J(t, \varepsilon) = \begin{pmatrix} 0 & 0 \\ b(t, \varepsilon) & 0 \end{pmatrix}, \quad A^*(t, \varepsilon, \mu) = J(t, \varepsilon) + \sum_{l=1}^q A_l(t, \varepsilon)\mu^l.$$

**Theorem.** *Let:*

- (1) eigenvalues  $\lambda_j^*(t, \varepsilon, \mu)$  ( $j = 1, 2$ ) of the matrix  $A^*(t, \varepsilon, \mu)$  be such that  $\inf_G |\operatorname{Re} \lambda_j^*(t, \varepsilon, \mu)| \geq \gamma_0 \mu^{q_0}$  ( $\gamma_0 > 0$ ,  $0 < q_0 \leq q$ );
- (2) for the matrix  $A^*(t, \varepsilon, \mu)$ , there exist a matrix  $U(t, \varepsilon, \mu)$  such that
  - (a)  $\inf_G |\det U(t, \varepsilon, \mu)| > 0$ ;
  - (b)  $U^{-1} A^* U = \Lambda(t, \varepsilon, \mu)$  is a diagonal matrix;
- (3) the conditions of the lemma be fulfilled.

Then for sufficiently small values  $\mu, \varepsilon/\mu^{2q_0-1}$ , the system (1) has a particular solution of the class  $B_{m-1}$ .

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# Asymptotic Representation for Solutions of Systems of Differential Equations with Regularly and Fast Varying Nonlinearities

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We consider the system of differential equations

$$\begin{cases} y'_i = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) & (i = \overline{1, n-1}), \\ y'_n = \alpha_n p_n(t) \varphi_1(y_1), \end{cases} \quad (1)$$

where  $\alpha_i \in \{-1, 1\}$  ( $i = \overline{1, n}$ ),  $p_i : [a, \omega[ \rightarrow ]0, +\infty[$  ( $i = \overline{1, n}$ ) are continuous functions,  $-\infty < a < \omega \leq +\infty$ ,  $\varphi_i : \Delta(Y_i^0) \rightarrow ]0; +\infty[$  ( $i = \overline{1, n}$ ) ( $\Delta(Y_i^0)$  is a one-sided neighborhood of  $Y_i^0$ ,  $Y_i^0$  equals either 0, or  $\pm\infty$ ) are twice continuously differentiable functions that satisfy the conditions  $\varphi'_i(z) \neq 0$  when  $z \in \Delta(Y_i^0)$ ,  $\lim_{\substack{z \rightarrow Y_i^0 \\ z \in \Delta(Y_i^0)}} \varphi_i(z) = \Phi_i^0 \in \{0, +\infty\}$ ,  $\lim_{\substack{z \rightarrow Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{\varphi''_i(z) \varphi_i(z)}{[\varphi'_i(z)]^2} = \gamma_i$ , where  $\prod_{i=1}^n (1 - \gamma_i) \neq 1$ .

Such system of differential equations when  $\varphi_i(y_i) = |y_i|^{\sigma_i}$  ( $i = \overline{1, n}$ ) is called the system of differential equations of Emden–Fowler type. While  $t \uparrow \omega$ , the asymptotic representations for its non-oscillating solutions were established in [1], [2] for  $n = 2$ . This work covers situations, when functions  $\varphi_i(y_i)$  ( $i = \overline{1, n}$ ) are close to power functions, when  $\gamma_i \neq 1$ , as well as situations when functions  $\varphi_i(y_i)$  ( $i = \overline{1, n}$ ) have an exponential rate, when  $\gamma_i = 1$ , that means that these functions are fast varying (see [3], [4]).

A solution  $(y_i)_{i=1}^n$  of the system (1), defined on the interval  $[t_0, \omega[ \subset [a, \omega[$ , is called  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solution, if functions  $u_i(t) = \varphi_i(y_i(t))$  satisfy the following conditions:

$$\lim_{t \uparrow \omega} u_i(t) = \Phi_i^0, \quad \lim_{t \uparrow \omega} \frac{u_i(t) u'_{i+1}(t)}{u'_i(t) u_{i+1}(t)} = \Lambda_i \quad (i = \overline{1, n}).$$

Note that the second condition in the definition of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solution implies:

$$\prod_{i=1}^n \Lambda_i = 1. \quad (2)$$

For the system (1) in case, when  $\Lambda_i \neq 0$  ( $i = \overline{1, n}$ ), the necessary and sufficient conditions for the existence of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solutions are established, as well as the asymptotic representation for these solutions when  $t \uparrow \omega$ .

In order to formulate the theorem, we introduce several auxiliary notations:

$$\mathfrak{J} = \left\{ i \in \{1, \dots, n\} : 1 - \Lambda_i - \gamma_i \neq 0 \right\}, \quad \bar{\mathfrak{J}} = \{1, \dots, n\} \setminus \mathfrak{J}.$$

Note that the set  $\mathfrak{J} \neq \emptyset$ , because (2) and  $\prod_{i=1}^n (1 - \gamma_i) \neq 1$ . Further, let  $l = \min \mathfrak{J}$ .

$$I_i(t) = \begin{cases} \int_t^t p_i(\tau) d\tau & \text{for } i \in \mathfrak{J}, \\ \int_{A_i}^{A_i} I_l(\tau) p_i(\tau) d\tau & \text{for } i \in \bar{\mathfrak{J}}, \end{cases}$$

$$\beta_i = \begin{cases} 1 - \Lambda_i - \gamma_i, & \text{if } i \in \mathfrak{J}, \\ \frac{\beta_l}{\Lambda_l \cdots \Lambda_{i-1}}, & \text{if } i \in \{l+1, \dots, n\} \setminus \mathfrak{J}, \\ \frac{\beta_l}{\Lambda_l \cdots \Lambda_n \Lambda_1 \cdots \Lambda_{i-1}}, & \text{if } i \in \{1, \dots, l-1\} \setminus \mathfrak{J}, \end{cases}$$

where limits of integration  $A_i \in \{\omega, a\}$  are chosen in such a way that corresponding integral  $I_i$  aims either to zero, or to  $\infty$  when  $t \uparrow \omega$ .

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \quad (i = 1, \dots, n).$$

**Theorem.** Let  $\Lambda_i \in \mathbb{R} \setminus \{0\}$  ( $i = \overline{1, n}$ ) and  $l = \min \mathfrak{J}$ . Then for the existence of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solutions of (1) it is necessary and, if algebraic equation

$$\prod_{i=1}^n \left( (1 - \gamma_i) \prod_{j=1}^{i-1} \Lambda_j + \nu \right) - \prod_{i=1}^n \prod_{j=1}^{i-1} \Lambda_j = 0 \quad (3)$$

does not have roots with zero real part, it is also sufficient that for each  $i \in \{1, \dots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied  $A_i^* \beta_i > 0$  when  $\Phi_i^0 = +\infty$ ,  $A_i^* \beta_i < 0$  when  $\Phi_i^0 = 0$ ,  $\text{sign} [\alpha_i A_i^* \beta_i] = \text{sign} \varphi'_i(z)$ . Moreover, components of each solution of that type admit following asymptotic representation when  $t \uparrow \omega$

$$\begin{aligned} \frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t)) \varphi_{i+1}(y_{i+1}(t))} &= \alpha_i \beta_i I_i(t) [1 + o(1)], \quad \text{if } i \in \mathfrak{J}, \\ \frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t)) \varphi_{i+1}(y_{i+1}(t))} &= \alpha_i \beta_i \frac{I_i(t)}{I_l(t)} [1 + o(1)], \quad \text{if } i \in \overline{\mathfrak{J}}, \end{aligned}$$

and there exists the whole  $k$ -parametric family of these solutions if there are  $k$  positive roots (including multiple roots) among the solutions of (3), with signs of real parts different from those of the number  $A_l^* \beta_l$ .

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# Limit Properties of Positive Solutions of Fractional Boundary Value Problems

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We discuss the set of the scalar fractional boundary value problems

$${}^cD^{\alpha_n}u(t) + f(t, u(t), u'(t)), \quad {}^cD^{\mu_n}u(t) = 0, \quad n \in \mathbb{N}, \quad (1)$$

$$u'(0) = 0, \quad u(1) + au'(1) = 0, \quad (2)$$

where  $\{\alpha_n\} \subset (1, 2)$ ,  $\{\mu_n\} \subset (0, 1)$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 2$ ,  $\lim_{n \rightarrow \infty} \mu_n = 1$ ,  $a \geq 0$ , and  $f$  satisfies the conditions:

(H<sub>1</sub>)  $f \in C([0, 1] \times \mathcal{D})$ ,  $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_-^2$ ,  $\mathbb{R}_+ = [0, \infty)$ ,  $\mathbb{R}_- = (-\infty, 0]$ ,

(H<sub>2</sub>) the estimate

$$0 \leq \varphi(t) \leq f(t, x, y, z) \leq \omega(x, |y|, |z|),$$

is fulfilled for  $(t, x, y, z) \in [0, 1] \times \mathcal{D}$ , where  $\varphi \in C[0, 1]$ ,  $\varphi(t_0) > 0$  for some  $t_0 \in [0, 1]$ ,  $\omega \in C(\mathbb{R}_+^3)$ ,  $\omega$  is nondecreasing in all its arguments, and

$$\lim_{x \rightarrow \infty} \frac{\omega(x, x, x)}{x} = 0.$$

Here  ${}^cD$  is the Caputo fractional derivative. The Caputo fractional derivative  ${}^cD^\gamma x$  of order  $\gamma > 0$  of a function  $x : [0, 1] \rightarrow \mathbb{R}$  is defined as (see, e.g., [1], [2])

$${}^cD^\gamma x(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1} \left( x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) ds & \text{if } \gamma \notin \mathbb{N}, \\ x^{(\gamma)}(t) & \text{if } \gamma \in \mathbb{N}, \end{cases}$$

where  $n = [\gamma] + 1$  and  $[\gamma]$  means the integral part of  $\gamma$  and where  $\Gamma$  is the Euler gamma function.

We say that a function  $u \in C^1[0, 1]$  is a *positive solution of problem (1), (2)* if  $u > 0$  on  $[0, 1]$ ,  ${}^cD^{\alpha_n}u \in C[0, 1]$ ,  $u$  satisfies the boundary conditions (2) and equality (1) holds for  $t \in [0, 1]$ .

Together with equation (1) we investigate the differential equation

$$u''(t) + f(t, u(t), u'(t), u'(t)) = 0. \quad (3)$$

A function  $u \in C^2[0, 1]$  is called a *positive solution of problem (3), (2)* if  $u > 0$  on  $[0, 1]$ ,  $u$  satisfies (2) and equality (3) is fulfilled for  $t \in [0, 1]$ .

The following result is proved by the Leray–Schauder nonlinear alternative [3].

**Theorem 1.** *Let conditions (H<sub>1</sub>) and (H<sub>2</sub>) hold. Then for each  $n \in \mathbb{N}$  problem (1), (2) has a solution  $u_n$  and there exists a positive constant  $S$  independent of  $n$  such that the estimate*

$$(1 - t^{\alpha_*-1})Q \leq u_n(t) < (1 + a)S, \quad 0 \geq u'_n(t) > -S, \quad 0 \geq {}^cD^{\mu_n}u_n(t) > -\frac{S}{\Delta} \quad (4)$$

is fulfilled for  $t \in [0, 1]$ , where

$$Q = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)\varphi(s) ds, \quad \Delta = \min\{\Gamma(s) : 1 \leq s \leq 2\}, \quad \alpha_* = \inf\{\alpha_n : n \in \mathbb{N}\}.$$

Let  $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$  be the norm in  $C[0, 1]$  and let  $\|x\|_{C^1} = \max\{\|x\|, \|x'\|\}$ . The following theorem states the relation between positive solutions of problem (1), (2) and positive solutions of problem (3), (2).

**Theorem 2.** *Let  $(H_1)$  and  $(H_2)$  hold. Let  $u_n$  be a positive solution of problem (1), (2) satisfying inequality (4). Then there exist a subsequence  $\{u_{k_n}\}$  of  $\{u_n\}$  and a positive solution  $u$  of problem (3), (2) such that*

$$\lim_{n \rightarrow \infty} \|u_{k_n} - u\|_{C^1} = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{D}^{\mu_{k_n}} u_{k_n} - u'\| = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{D}^{\alpha_{k_n}} u_{k_n} - u''\| = 0.$$

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# Optimization of Initial Data for Nonlinear Delay Functional-Differential Equations with the Mixed Initial Condition

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In the present paper, for initial data, the necessary conditions of optimality are obtained. Under initial data we imply the collection of initial moment and constant delays, initial vector and function. Let  $R_x^n$  be an  $n$ -dimensional vector space of points  $x = (x^1, \dots, x^n)^T$ , where  $T$  means transpose;  $P \subset R_p^k$  and  $Z \subset R_z^m$  be open sets and  $O = (P, Z)^T = \{x = (p, z)^T \in R_x^n : p \in P, z \in Z\}$ , with  $k + m = n$ ; next  $t_{01} < t_{02} < t_1, 0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2$  be given numbers, with  $t_1 - t_{02} > \tau_2$ ; the  $n$ -dimensional function  $f(t, x, p, z)$  be continuous on the set  $[t_{01}, t_1] \times O \times P \times Z$  and continuously differentiable with respect to  $x, p, z$ ; let  $P_0 \subset P$  be a compact convex set of initial vectors  $p_0$ ;  $\Phi$  and  $G$  be sets of continuous initial functions  $\varphi(t) \in P_1, t \in [\hat{\tau}, t_{02}]$  and  $g(t) \in Z_1, t \in [\hat{\tau}, t_{02}]$ , respectively, where  $\hat{\tau} = t_{01} - \max\{\tau_2, \sigma_2\}$ ,  $P_1 \subset P, Z_1 \subset Z$  are convex and compact sets. Suppose that functions  $q^i(t_0, \tau, \sigma, p, z, x), i = \overline{0, l}$  are continuously differentiable with respect to all arguments  $t_0 \in [t_{01}, t_{02}], \tau \in [\tau_1, \tau_2], \sigma \in [\sigma_1, \sigma_2], p \in P, z \in Z, x \in O$ .

To each element  $w = (t_0, \tau, \sigma, p_0, \varphi, g) \in W = (t_{01}, t_{02}) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P_0 \times \Phi \times G$ , we assign the delay functional-differential equation

$$\dot{x}(t) = f(t, x(t), p(t - \tau), z(t - \sigma)) \quad (1)$$

with the mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, \quad t \in [\hat{\tau}, t_0), \quad x(t_0) = (p_0, g(t_0))^T. \quad (2)$$

**Definition 1.** Let  $w = (t_0, \tau, \sigma, p_0, \varphi, g) \in W$ . A function  $x(t) = x(t; w) = (p(t; w), z(t; w))^T \in O, t \in [\hat{\tau}, t_1]$ , is called a solution of Eq. (1) with the initial condition (2) if it satisfies condition (2) and is absolutely continuous on the interval  $[t_0, t_1]$  and satisfies Eq. (1) almost everywhere on  $[t_0, t_1]$ .

**Definition 2.** An element  $w \in W$  is said to be admissible if the solution  $x(t) = x(t; w)$  is defined on the interval  $[\hat{\tau}, t_1]$  and satisfies the conditions

$$q^i(t_0, \tau, \sigma, p_0, g(t_0), x(t_1)) = 0, \quad i = \overline{1, l}.$$

We denote the set of admissible elements by  $W_0$ .

**Definition 3.** An element  $w_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0) \in W_0$  is said to be optimal if for any  $w = (t_0, \tau, \sigma, p_0, \varphi, g) \in W_0$  we have

$$q^0(t_{00}, \tau_0, \sigma_0, p_{00}, g_0(t_{00}), x_0(t_1)) \leq q^0(t_0, \tau, \sigma, p_0, g(t_0), x(t_1)), \quad x_0(t) = x(t; w_0).$$

**Theorem.** Let  $w_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0)$  be an optimal element and the following condition hold: the functions  $\varphi_0(t)$ ,  $g_0(t)$  are continuously differentiable. Then there exist a non-zero vector  $\pi = (\pi_0, \dots, \pi_l)$ , where  $\pi_0 \leq 0$  and the solution  $\Psi(t)$  to the equation

$$\begin{cases} \dot{\Psi}(t) = -\Psi(t)f_x[t] - \left( \Psi(t + \tau_0)f_p[t + \tau_0], \Psi(t + \sigma_0)f_z[t + \sigma_0] \right), & t \in [t_{00}, t_1], \\ \Psi(t_1) = \pi Q_{0x}, \quad \Psi(t) = 0, & t > t_1 \end{cases}$$

such that the conditions listed below hold: the conditions for the function  $\Psi(t) = (\psi(t), \chi(t)) = (\psi_1(t), \dots, \psi_k(t), \chi_1(t), \dots, \chi_m(t))$  and vectors  $p_{00}, g_0(t_{00})$

$$\begin{aligned} (\pi Q_{0p_0} + \psi(t_{00}))p_{00} &= \max_{p_0 \in P_1} (\pi Q_{0p_0} + \psi(t_{00}))p_0, & (\pi Q_{0z} + \chi(t_{00}))g_0(t_{00}) &= \max_{g \in Z_1} (\pi Q_{0z} + \chi(t_{00}))g, \\ \text{where } Q_{0p_0} &= Q_{p_0}(t_{00}, \tau_0, \sigma_0, p_{00}, x_0(t_1)), & Q &= (q^0, \dots, q^l)^T; \end{aligned}$$

the integral maximum principles for the optimal initial functions  $\varphi_0(t)$  and  $g_0(t)$

$$\begin{aligned} \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t + \tau_0)f_p[t + \tau_0]\varphi_0(t) dt &= \max_{\varphi(\cdot) \in \Phi} \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t + \tau_0)f_p[t + \tau_0]\varphi(t) dt, \\ \int_{t_{00}-\sigma_0}^{t_{00}} \Psi(t + \sigma_0)f_z[t + \sigma_0]g_0(t) dt &= \max_{g(\cdot) \in G} \int_{t_{00}-\sigma_0}^{t_{00}} \Psi(t + \sigma_0)f_z[t + \sigma_0]g(t) dt; \end{aligned}$$

the condition for the optimal initial moment  $t_{00}$

$$\begin{aligned} \pi Q_{0t_0} + (\pi Q_{0z} + \chi(t_{00}))\dot{g}(t_{00}) &= \\ &= \Psi(t_{00})f[t_{00}] + \Psi(t_{00} + \tau_0) \left\{ f[t_0 + \tau_0; p_{00}] - f[t_0 + \tau_0; \varphi_0(t_{00})] \right\}; \end{aligned}$$

the conditions for the optimal delays  $\tau_0, \sigma_0$

$$\begin{aligned} \pi Q_{0\tau} &= \Psi(t_{00} + \tau_0) \left\{ f[t_0 + \tau_0; p_{00}] - f[t_0 + \tau_0; \varphi_0(t_{00})] \right\} + \int_{t_{00}}^{t_1} \Psi(t)f_{0p}[t]\dot{p}_0(t - \tau_0) dt, \\ \pi Q_{0\sigma} &= \int_{t_{00}}^{t_1} \Psi(t)f_z[t]\dot{z}_0(t - \sigma_0) dt. \end{aligned}$$

Here

$$\begin{aligned} f_x[t] &= f_x(t, x_0(t), p_0(t - \tau_0), z_0(t - \sigma_0)), & f[t] &= f(t, x_0(t), p_0(t - \tau_0), z_0(t - \sigma_0)), \\ f[t; p_0] &= f(t, x_0(t), p_0, z_0(t - \sigma_0)). \end{aligned}$$

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# On Asymptotics of Solutions of Cyclic Systems of Differential Equations

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We consider the system of differential equations

$$\begin{cases} y'_i = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) & (i = \overline{1, n-1}), \\ y'_n = \alpha_n p_n(t) \varphi_1(y_1), \end{cases} \quad (1)$$

where  $\alpha_i \in \{-1, 1\}$  ( $i = \overline{1, n}$ ),  $p_i : [a, \omega[ \rightarrow ]0, +\infty[$  ( $i = \overline{1, n}$ ) are continuous functions,  $-\infty < a < \omega \leq +\infty$ ,  $\varphi_i : \Delta(Y_i^0) \rightarrow ]0; +\infty[$  ( $i = \overline{1, n}$ ) are continuous and regularly varying functions (see [1]) when  $y_i \rightarrow Y_i^0$  of  $\sigma_i$  orders. Constants  $\sigma_i$  satisfy the equality:  $\prod_{i=1}^n \sigma_i \neq 1$ , where  $\Delta(Y_i^0)$  is a one-sided neighborhood of  $Y_i^0$ ,  $Y_i^0$  equals either 0, or  $\pm\infty$ .

If  $\varphi_i(y_i) = |y_i|^{\sigma_i}$  ( $i = \overline{1, n}$ ), system (1) is called an Emden–Fowler system. When  $n = 2$ , the asymptotic behavior of its nonoscillating solutions while  $t \uparrow \omega$  is thoroughly investigated in [3]–[7].

In T. A. Chanturia's paper [2], for systems of differential equations that are close to (1) in a certain sense and include (1), the signs for  $A$  and  $B$ -properties' existence are established.

A solution  $(y_i)_{i=1}^n$  of the system (1), defined on the interval  $[t_0, \omega[ \subset [a, \omega[$ , is called  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solution, if it satisfies following conditions:

$$y_i(t) \in \Delta(Y_i^0) \text{ while } t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0, \quad \lim_{t \uparrow \omega} \frac{y_i(t) y'_{i+1}(t)}{y'_i(t) y_{i+1}(t)} = \Lambda_i \quad (i = \overline{1, n}).$$

It follows from the third condition in the definition of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solution that parameters  $\Lambda_1, \dots, \Lambda_n$  are connected with such a relation:

$$\prod_{i=1}^n \Lambda_i = 1. \quad (2)$$

For the system (1), in case when  $\Lambda_i \neq 0$  ( $i = \overline{1, n}$ ), the necessary and sufficient conditions for the existence of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solutions are established. In order to formulate the main result, we introduce auxiliary notations.

First, we introduce the sets of indices  $\mathfrak{J} = \{i \in \{1, \dots, n\} : 1 - \Lambda_i \sigma_{i+1} \neq 0\}$ ,  $\bar{\mathfrak{J}} = \{1, \dots, n\} \setminus \mathfrak{J}$ . It is obvious that  $\mathfrak{J} \neq \emptyset$ , because of (2) and the fact that  $\prod_{i=1}^n \sigma_i \neq 1$ . Let  $l$  be the minimum element of the set  $\mathfrak{J}$ .

Further, denote:

$$\mu_i = \begin{cases} 1, & \text{as } Y_i^0 = +\infty, \text{ or } Y_i^0 = 0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of } 0, \\ -1, & \text{as } Y_i^0 = -\infty, \text{ or } Y_i^0 = 0 \text{ and } \Delta(Y_i^0) \text{ is left neighborhood of } 0, \end{cases}$$

$$I_i(t) = \begin{cases} \int_{A_i}^t p_i(\tau) d\tau & \text{for } i \in \mathfrak{J}, \\ \int_{A_i}^{A_l} I_l(\tau) p_i(\tau) d\tau & \text{for } i \in \bar{\mathfrak{J}}, \end{cases} \quad \beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{if } i \in \mathfrak{J}, \\ \frac{\beta_l}{\prod_{k=l}^{i-1} \Lambda_k}, & \text{if } i \in \{l+1, \dots, n\} \setminus \mathfrak{J}, \\ \beta_l \prod_{k=i}^{l-1} \Lambda_k, & \text{if } i \in \{1, \dots, l-1\} \setminus \mathfrak{J}, \end{cases}$$

where limits of integration  $A_i \in \{\omega, a\}$  are chosen in such a way that corresponding integral  $I_i$  aims either to zero, or to  $\infty$  as  $t \uparrow \omega$ .

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \quad (i = 1, \dots, n).$$

**Theorem.** Let  $\Lambda_i \in \mathbb{R} \setminus \{0\}$  ( $i = \overline{1, n}$ ) and  $l = \min \mathfrak{J}$ . Then for the existence of  $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ -solutions of (1) it is necessary and, if algebraic equation

$$\prod_{i=1}^n \left( \prod_{j=1}^{i-1} \Lambda_j + \nu \right) - \prod_{i=1}^n \left( \sigma_i \prod_{j=1}^{i-1} \Lambda_j \right) = 0 \quad (3)$$

does not have roots with zero real part, it is also sufficient that for each  $i \in \{1, \dots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied  $A_i^* \beta_i > 0$  if  $Y_i^0 = \pm\infty$ ,  $A_i^* \beta_i < 0$  if  $Y_i^0 = 0$ ,  $\text{sign} [\alpha_i A_i^* \beta_i] = \mu_i$ . Moreover, components of each solution of that type admit following asymptotic representation when  $t \uparrow \omega$ :

$$\begin{aligned} \frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} &= \alpha_i \beta_i I_i(t) [1 + o(1)], \quad \text{when } i \in \mathfrak{J}, \\ \frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} &= \alpha_i \beta_i \frac{I_i(t)}{I_l(t)} [1 + o(1)], \quad \text{when } i \in \overline{\mathfrak{J}}, \end{aligned}$$

and there exists the whole  $k$ -parametric family of these solutions if there are  $k$  positive roots (including multiple roots) among the solutions of (3) with signs of real parts different from those of the number  $A_l^* \beta_l$ .

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