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ON THE CORRECTNESS OF LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

M. ASHORDIA

ABSTRACT. The sufficient conditions are established for the correctness of the linear boundary value problem

$$dx(t) = dA(t) \cdot x(t) + df(t), \quad l(x) = c_0,$$

where $A : [a,b] \to \mathbb{R}^{n\times n}$ and $f : [a,b] \to \mathbb{R}^n$ are matrix- and vector-functions of bounded variation, $c_0 \in \mathbb{R}^n$ and l is a linear continuous operator from the space of n-dimentional vector-functions of bounded variation into $\mathbb{R}^n$.

რეზიუმე. დადგენილია

$$dx(t) = dA(t) \cdot x(t) + df(t), \quad l(x) = c_0,$$

სახის წრფივი სასაზღვრო ამოცანის კორექტულობისთვის საკმარისი პირობები, სადაც $A : [a,b] \to \mathbb{R}^{n\times n}$ და $f : [a,b] \to \mathbb{R}^n$ სასრული ვარიაციის მატრიცული და ვექტორული ფუნქციებია, $c_0 \in \mathbb{R}^n$, ხოლო l წრფივი უწყვეტი ოპერატორია, რომელიც მოქმედებს სასრული ვარიაციის მქონე n-განზომილებიან ვექტორულ ფუნქციათა სივრციდან $\mathbb{R}^n$-ში.

Let matrix- and vector-functions, $A : [a,b] \to \mathbb{R}^{n\times n}$ and $f : [a,b] \to \mathbb{R}^n$, respectively, be of bounded variation, $c_0 \in \mathbb{R}^n$, and let $l : BV_n(a,b) \to \mathbb{R}^n$ be a linear continuous operator such that the boundary value problem

$$dx(t) = dA(t) \cdot x(t) + df(t), \tag{1}$$

$$l(x) = c_0 \tag{2}$$

has the unique solution x_0.

Consider the sequences of matrix- and vector-functions of bounded variation $A_k : [a,b] \to \mathbb{R}^{n\times n}$ $(k = 1,2,\dots)$ and $f_k : [a,b] \to \mathbb{R}^n$ $(k = 1,2,\dots)$, respectively, the sequence of constant vectors $c_k \in \mathbb{R}^n$

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($k = 1, 2, \dots$) and the sequence of linear continuous operators $l_k : BV_n(a, b) \to \mathbb{R}^n$ ($k = 1, 2, \dots$).

In this paper the sufficient conditions are given for the problem

$$dx(t) = dA_k(t) \cdot x(t) + df_k(t), \tag{3}$$

$$l_k(x) = c_k \tag{4}$$

to have the unique solution x_k for any sufficiently large k and

$$\lim_{k \to +\infty} x_k(t) = x_0(t) \quad \text{uniformly on } [a, b]. \tag{5}$$

An analogous question is studied in [2–4] for the boundary value problem for a system of ordinary differential equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential and difference equations from the common standpoint. Moreover, the convergence conditions for difference schemes corresponding to boundary value problems for systems of ordinary differential equations can be deduced from the correctness results for aprropriate boundary value problems for systems of generalized ordinary differential equations [1, 5, 6].

The following notations and definitions will be used throughout the paper:

$\mathbb{R} =]-\infty, +\infty[$;

$\mathbb{R}^n$ is a space of real column n-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is a space of real $n \times n$-matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|X\| = \max_{j=1,\dots,n} \sum_{i=1}^n |x_{ij}|;$$

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are the matrix inverse to X and the determinant of X, respectively; E is the identity $n \times n$-matrix;

$\bigvee_a^b x$ and $\bigvee_a^b X$ are the sums of total variations of components of vector- and matrix-functions, $x : [a, b] \to \mathbb{R}^n$ and $X : [a, b] \to \mathbb{R}^{n\times n}$, respectively;

$BV_n(a, b)$ is a space of all vector-functions of bounded variation $x : [a, b] \to \mathbb{R}^n$ (i.e., such that $\bigvee_a^b x < +\infty$) with the norm

$$\|x\|_{\sup} = \sup\{\|x(t)\| : t \in [a,b]\}^1;$$

$x(t-)$ and $x(t+)$ $(x(a-) = x(a),\ x(b+) = x(b))$ are the left and the right limit of the vector-function $x : [a,b] \to \mathbb{R}^n$ at the point t;

$d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$;

$BV_{n\times n}(a,b)$ is a set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times n}$, i.e., such that $\bigvee_a^b X < +\infty$;

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;

If $X = (x_{ij})_{i,j=1}^n \in BV_{n\times n}(a,b)$, then $V(X) : [a,b] \to \mathbb{R}^{n\times n}$ is defined by

$$V(X)(a) = 0, \quad V(X)(t) = \Big(\bigvee_a^t x_{ij}\Big)_{i,j=1}^n \quad (a < t \le b);$$

If $\alpha \in BV_1(a,b)$, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,d\alpha(\tau) = x(t)d_1\alpha(t) + x(s)d_2\alpha(s) + \int_{]s,t[} x(\tau)\,d\alpha(\tau),$$

where $\int_{]s,t[} x(\tau)\,d\alpha(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ (if $s = t$, then $\int_s^t x(\tau)\,d\alpha(\tau) = 0$);

If $A = (a_{ij})_{i,j=1}^n \in BV_{n\times n}(a,b)$, $X = (x_{ij})_{i,j=1}^n : [a,b] \to \mathbb{R}^{n\times n}$, $x = (x_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ and $a \le s \le t \le b$, then

$$\int_s^t dA(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,da_{ik}(\tau)\Big)_{i,j=1}^n,$$

$$\int_s^t dA(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n \int_s^t x_k(\tau)\,da_{ik}(\tau)\Big)_{i=1}^n;$$

$\|l\|$ is the norm of the linear continuous operator $l : BV_n(a,b) \to \mathbb{R}^n$;

If $X \in BV_{n\times n}(a,b)$ is the matrix-function with columns $x_1,\dots,x_n$, then $l(X)$ is the matrix with columns $l(x_1),\dots,l(x_n)$.

A function $x \in BV_n(a,b)$ is said to be a solution of problem (1), (2) if it satisfies condition (2) and

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for } a \le s \le t \le b.$$

Alongside with (1) and (3), we shall consider the corresponding homogeneous systems

$$dx(t) = dA(t)\cdot x(t) \tag{1_0}$$

---

[1] $BV_n(a,b)$ is not the Banach space with respect to this norm.

and

$$dx(t) = dA_k(t) \cdot x(t), \qquad (3_0)$$

respectively.

A matrix-function $Y \in BV_{n\times n}(a,b)$ is said to be a fundamental matrix of the homogeneous system (1_0) if

$$Y(t) = Y(s) + \int_s^t dA(\tau) \cdot Y(\tau) \quad \text{for } a \le s \le t \le b$$

and

$$\det\Big(Y(t)\Big) \ne 0 \quad \text{for } t \in [a,b].$$

Theorem 1. *Let the conditions*

$$\det\Big(E + (-1)^j d_j A(t)\Big) \ne 0 \quad \textit{for } t \in [a,b] \ (j = 1,2), \qquad (6)$$

$$\lim_{k\to+\infty} l_k(y) = l(y) \quad \textit{for } y \in BV_n(a,b), \qquad (7)$$

$$\lim_{k\to+\infty} c_k = c_0, \qquad (8)$$

$$\lim_{k\to+\infty} \sup \|l_k\| < +\infty, \qquad (9)$$

$$\lim_{k\to+\infty} \sup \bigvee_a^b A_k < +\infty \qquad (10)$$

be satisfied and let the conditions

$$\lim_{k\to+\infty} \Big[A_k(t) - A_k(a)\Big] = A(t) - A(a), \qquad (11)$$

$$\lim_{k\to+\infty} \Big[f_k(t) - f_k(a)\Big] = f(t) - f(a) \qquad (12)$$

be fulfilled uniformly on $[a,b]$. Then for any sufficiently large k problem (3), (4) *has the unique solution x_k and* (5) *is valid.*

To prove the theorem we shall use the following lemmas.

Lemma 1. *Let $\alpha_k, \beta_k \in BV_1(a,b)$ $(k = 0,1\dots)$,*

$$\lim_{k\to+\infty} \|\beta_k - \beta_0\|_{\sup} = 0, \qquad (13)$$

$$r = \sup\Big\{ \bigvee_a^b \alpha_k : k = 0,1,\dots \Big\} < +\infty \qquad (14)$$

and the condition

$$\lim_{k\to+\infty} \Big[\alpha_k(t) - \alpha_k(a)\Big] = \alpha_0(t) - \alpha_0(a) \qquad (15)$$

be fulfilled uniformly on $[a, b]$. *Then*

$$\lim_{k\to+\infty} \int_a^t \beta_k(\tau)\, d\alpha_k(\tau) = \int_a^t \beta_0(\tau)\, d\alpha_0(\tau)$$

uniformly on $[a, b]$.

Proof. Let ε be an arbitrary positive number. We denote

$$\mathcal{D}_j(a, b, \varepsilon; g) = \Big\{ t \in [a, b] : d_j g(t) \ge \varepsilon \Big\} \quad (j = 1, 2)$$

where

$$g(t) = V(\beta_0)(t) \quad \text{for } \ t \in [a, b].$$

By Lemma 1.1.1 from [5] there exists a finite subdivision $\{\alpha_0, \tau_1, \alpha_1, \dots, \tau_m, \alpha_m\}$ of $[a, b]$ such that

a) $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$, $\alpha_0 \le \tau_1 \le \alpha_1 \le \cdots \le \tau_m \le \alpha_m$;

b) If $\tau_i \notin \mathcal{D}_1(a, b, \varepsilon; g)$, then $g(\tau_i) - g(\alpha_{i-1}) < \varepsilon$;
If $\tau_i \in \mathcal{D}_1(a, b, \varepsilon; g)$, then $\alpha_{i-1} < \tau_i$ and $g(\tau_i-) - g(\alpha_{i-1}) < \varepsilon$;

c) If $\tau_i \notin \mathcal{D}_2(a, b, \varepsilon; g)$, then $g(\alpha_i) - g(\tau_i) < \varepsilon$;
If $\tau_i \in \mathcal{D}_2(a, b, \varepsilon; g)$, then $\tau_i < \alpha_i$ and $g(\alpha_i) - g(\tau_i+) < \varepsilon$.

We set

$$\eta(t) = \begin{cases} \beta_0(t) & \text{for } t \in \{\alpha_0, \tau_1, \alpha_1, \dots, \tau_m, \alpha_m\}; \\ \beta_0(\tau_i-) & \text{for } t \in]\alpha_{i-1}, \tau_i[,\ \tau_i \in \mathcal{D}_1(a, b, \varepsilon; g); \\ \beta_0(\tau_i) & \text{for } t \in]\alpha_{i-1}, \tau_i[,\ \tau_i \notin \mathcal{D}_1(a, b, \varepsilon; g) \text{ or} \\ & \text{for } t \in]\tau_i, \alpha_i[,\ \tau_i \notin \mathcal{D}_2(a, b, \varepsilon; g); \\ \beta_0(\tau_i+) & \text{for } t \in]\tau_i, \alpha_i[,\ \tau_i \in \mathcal{D}_2(a, b, \varepsilon; g); \\ & \qquad (i = 1, \dots, m). \end{cases}$$

It can be easily shown that $\eta \in BV_1(a, b)$ and

$$|\beta_0(t) - \eta(t)| < 2\varepsilon \quad \text{for } \ t \in [a, b]. \tag{16}$$

For every natural k and $t \in [a, b]$ we assume

$$\gamma_k(t) = \int_a^t \beta_k(\tau)\, d\alpha_k(t) - \int_a^t \beta_0(\tau)\, d\alpha_0(\tau)$$

and

$$\delta_k(t) = \int_a^t \eta(t)\, d[\alpha_k(\tau) - \alpha_0(\tau)].$$

It follows from (15) that

$$\lim_{k\to+\infty} \|\delta_k\|_{\sup} = 0. \tag{17}$$

On the other hand, by (14) and (16) we have

$$\|\gamma_k\|_{\sup} \leq 4r\varepsilon + r\|\beta_k - \beta_0\|_{\sup} + \|\delta\|_{\sup} \quad (k = 1, 2, \dots).$$

Hence in view of (13) and (17)

$$\lim_{k\to+\infty} \|\gamma_k\|_{\sup} = 0$$

since ε is arbitrary. ■

Lemma 2. *Let condition* (6) *be fulfilled and*

$$\lim_{k\to+\infty} Y_k(t) = Y(t) \quad \textit{uniformly on } [a,b], \tag{18}$$

where Y *and* Y_k *are the fundamental matrices of the homogeneous systems* (1_0) *and* (3_0), *respectively. Then*

$$\inf\Big\{\Big|\det(Y(t))\Big| : t \in [a,b]\Big\} > 0, \tag{19}$$

$$\inf\Big\{\Big|\det(Y^{-1}(t))\Big| : t \in [a,b]\Big\} > 0 \tag{20}$$

and

$$\lim_{k\to+\infty} Y_k^{-1}(t) = Y^{-1}(t) \quad \textit{uniformly on } [a,b]. \tag{21}$$

Proof. It is known ([6], Theorem III.2.10) that

$$d_j Y(t) = d_j A(t) \cdot Y(t) \quad \text{for } t \in [a,b] \ (j = 1, 2).$$

Therefore (6) implies

$$\det\Big(Y(t-) \cdot Y(t+)\Big) = \Big[\det(Y(t))^2\Big] \cdot \prod_{j=1}^{2} \det\Big(E + (-1)^j d_j A(t)\Big) \neq 0 \quad \text{for } t \in [a,b]. \tag{22}$$

Let us show that (19) is valid. Assume the contrary. Then it can be easily shown that there exists a point $t_0 \in [a,b]$ such that

$$\det\Big(Y(t_0-) \cdot Y(t_0+)\Big) = 0.$$

But this equality contradicts (22). Inequality (19) is proved.

The proof of inequality (20) is analogous.

In view of (18) and (19) there exists a positive number q such that

$$\inf\Big\{\Big|\det(Y_k(t))\Big| : t \in [a,b]\Big\} > q > 0$$

for any sufficiently large k. From this and (18) we obtain (21). ■

Proof of the Theorem. Let us show that

$$\det\big(E + (-1)^j d_j A_k(t)\big) \neq 0 \quad \text{for } t \in [a,b] \ (j = 1,2) \tag{23}$$

for any sufficiently large k.

By (11)

$$\lim_{k\to+\infty} d_j A_k(t) = d_j A(t) \quad (j = 1,2) \tag{24}$$

uniformly on $[a,b]$. Since $\overset{b}{\underset{a}{\vee}} A < +\infty$, the series $\sum_{t\in[a,b]} \|d_j A(t)\|$ $(j = 1,2)$ converge. Thus for any $j \in \{1,2\}$ the inequality

$$\|d_j A(t)\| \geq \frac{1}{2}$$

may hold only for some finite number of points $t_{j1}, \dots, t_{jm_j}$ in $[a,b]$. Therefore

$$\|d_j A(t)\| < \frac{1}{2} \quad \text{for } t \in [a,b], \ t \neq t_{ji} \ (i = 1,\dots,m_j). \tag{25}$$

It follows from (6), (24) and (25) that for any sufficiently large k and for $j \in \{1,2\}$

$$\det\big(E + (-1)^j d_j A_k(t_{ji})\big) \neq 0 \quad (i = 1,\dots,m_j) \tag{26}$$

and

$$\|d_j A_k(t)\| < \frac{1}{2} \quad \text{for } t \in [a,b], \ t \neq t_{ji} \ (i = 1,\dots,m_j). \tag{27}$$

The latter inequality implies that the matrices $E + (-1)^j d_j A_k(t)$ $(j = 1,2)$ are invertible for $t \in [a,b]$, $t \neq t_{ji}$ $(i = 1,\dots,m_j)$ too. Therefore (23) is proved.

Besides, by (26) and (27) there exists a positive number r_0 such that for any sufficiently large k

$$\Big\| \big[E + (-1)^j d_j A_k(t)\big]^{-1} \Big\| \leq r_0 \quad \text{for } t \in [a,b] \ (j = 1,2). \tag{28}$$

Let k be a sufficiently large natural number. In view of (6) and (23) there exist ([6], Theorem III.2.10) fundamental matrices Y and Y_k of systems (1_0) and (3_0), respectively, satisfying

$$Y(a) = Y_k(a) = E.$$

Moreover,

$$Y_k^{-1} \in BV_{n\times n}(a,b).$$

Let us prove (18). We set

$$Z_k(t) = Y_k(t) - Y(t) \quad \text{for } t \in [a,b]$$

and

$$B_k(t) = A_k(t-) \quad \text{for } t \in [a,b].$$

Obviously, for every $t \in [a,b]$

$$d_1\big[B_k(t) - A_k(t)\big] = -d_2\big[B_k(t) - A_k(t)\big] = -d_1 A_k(t)$$

and

$$\int_a^t d\big[B_k(\tau) - A_k(\tau)\big] \cdot Z_k(\tau) = -d_1 A_k(t) \cdot Z_k(t).$$

Consequently,

$$Z_k(t) \equiv \big[E - d_1 A_k(t)\big]^{-1}\bigg[\int_a^t d\big[A_k(\tau) - A(\tau)\big] \cdot Y(\tau) + \int_a^t dB_k(\tau) \cdot Z_k(\tau)\bigg].$$

From this and (28) we get

$$\|Z_k(t)\| \le r_0\bigg(\varepsilon_k + \int_a^t d\|V(B_k)(\tau)\| \cdot \|Z_k(\tau)\|\bigg) \quad \text{for } t \in [a,b],$$

where

$$\varepsilon_k = \sup\bigg\{\bigg\|\int_a^t d\big[A_k(\tau) - A(\tau)\big] \cdot Y(\tau)\bigg\| : t \in [a,b]\bigg\}.$$

Hence, according to the Gronwall inequality ([6], Theorem I.4.30),

$$\|Z_k(t)\| \le r_0 \varepsilon_k \exp\Big(r_0 \overset{b}{\underset{a}{\vee}} B_k\Big) \le r_0 \varepsilon_k \exp\Big(r_0 \overset{b}{\underset{a}{\vee}} A_k\Big) \quad \text{for } t \in [a,b].$$

By (10), (11) and Lemma 1 this inequality implies (18).

It is known ([6], Theorem III.2.13) that if x_k is the solution of (3), then

$$x_k(t) \equiv Y_k(t) x_k(a) + f_k(t) - f_k(a) - Y_k(t) \int_a^t dY_k^{-1}(\tau) \cdot \big[f_k(\tau) - f_k(a)\big].$$

Thus problem (3), (4) has the unique solution if and only if

$$\det\big(l_k(Y_k)\big) \ne 0. \tag{29}$$

Since problem (1), (2) has the unique solution x_0, we have

$$\det\big(l(Y)\big) \ne 0. \tag{30}$$

Besides, by (7), (9) and (18)

$$\lim_{k\to+\infty} l_k(Y_k) = l(Y).$$

Therefore, in view of (30), there exists a natural number k_0 such that condition (29) is fulfilled for every $k \geq k_0$. Thus problem (3), (4) has the unique solution x_k for $k \geq k_0$ and

$$x_k(t) \equiv Y_k(t)\Big[l_k(Y_k)\Big]^{-1}\Big[c_k - l_k(F_k(f_k))\Big] + F_k(f_k)(t), \qquad (31)$$

where

$$F_k(f_k)(t) = f_k(t) - f_k(a) - Y_k(t)\int_a^t dY_k^{-1}(\tau)\cdot\Big[f_k(\tau) - f_k(a)\Big].$$

According to Lemma 2 condition (21) is fulfilled and

$$\rho = \sup\Big\{\|Y_k^{-1}(t)\| + \|Y_k(t)\| : t\in[a,b],\ k\geq k_0\Big\} < +\infty. \qquad (32)$$

The equality

$$Y_k^{-1}(t) - Y_k^{-1}(s) = Y_k^{-1}(s)\int_t^s dA_k(\tau)\cdot Y_k(\tau)\ Y_k^{-1}(t)$$

implies

$$\|Y_k^{-1}(t) - Y_k^{-1}(s)\| \leq \rho^3 \bigvee_s^t A_k \quad \text{for } a\leq s\leq t\leq b \ \ (k\geq k_0).$$

This inequality, together with (10) and (32) yields

$$\lim_{k\to+\infty} \sup \bigvee_a^b Y_k^{-1} < +\infty.$$

By this, (12) and (21) it follows from Lemma 1 that

$$\lim_{k\to+\infty}\int_a^t dY_k^{-1}(\tau)\cdot\Big[f_k(\tau)-f_k(a)\Big] =$$

$$= \int_a^t dY^{-1}(\tau)\cdot\Big[f(\tau)-f(a)\Big] \qquad (33)$$

uniformly on $[a,b]$.

Using (7)-(9), (12), (18), (29), (30) and (33), from (31) we get

$$\lim_{k\to+\infty} x_k(t) = z(t) \quad \text{uniformly on } [a,b],$$

where

$$z(t) = Y(t)\big[l(Y)\big]^{-1}\big[c_0 - l\big(F(f)\big)\big] + F(f)(t),$$

$$F(f)(t) = f(t) - f(a) - Y(t)\int_a^t dY^{-1}(\tau)\cdot\big[f(\tau) - f(a)\big].$$

It is easy to verify that the vector-function $z : [a,b] \to \mathbb{R}^n$ is the solution of problem (1), (2). Therefore

$$x_0(t) = z(t) \quad \text{for} \quad t \in [a,b]. \blacksquare$$

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OPTIMAL TRANSMISSION OF GAUSSIAN SIGNALS THROUGH A FEEDBACK CHANNEL

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Abstract. Using the methodology and results of the theory of filtering of conditionally Gaussian processes, the optimal schemes of transmission of Gaussian signals through the noisy feedback channel are constructed under the new power conditions.

რეზიუმე. აგებულია გაუსის სიგნალების უკუკავშირიანი ხმაურიანი არხით გადაცემის ოპტიმალური სქემები ახალ ენერგეტიკულ პირობებში პირობითად გაუსის პროცესების ფილტრაციის თეორიის მეთოდოლოგიისა და შედეგების გამოყენებით.

In the present paper, in contrast to our previous work [1], the problem of transmission of Gaussian signals through the noisy feedback channel under new power conditions (see conditions (15),(38) and (40)) is investigated. The obtained results, in our opinion, imply significant simplification and more clearness.

In §1 the optimal (in the sense of mean square criterion) linear schemes of transmission in the case of the discrete time are constructed. In §2 the optimal linear schemes of transmission in the case of continuous time are investigated and it is proved that these schemes are also optimal in the general class of transmission schemes.

§ 1. Optimal Transmission in a Discrete Case

1. Let the signal $\theta = (\theta_t, \mathcal{F}_t)$, $t \in S = \{0, \Delta, 2\Delta, \dots, T\}$, $\Delta > 0$, be a discrete Gaussian process given on the basic probability space $(\Omega, \mathcal{F}, P)$ with the nondecreasing family of σ-algebras $(\mathcal{F}_t)$, $t \in S$, $\mathcal{F}_t \subseteq \mathcal{F}$, $t < s$, satisfying the following finite difference equation

$$\Delta\theta_t = a(t)\theta_t\Delta + b(t)\Delta v_t, \tag{1}$$

where $v = (v_t, \mathcal{F}_t)$, $t \in S$, is a Gaussian random sequence (G.R.S.) $N(0,t)$ with independent increments independent of θ_0 which is a Gaussian $N(m,\gamma)$,$\gamma > 0$, random variable; $a(t)$ and $b(t)$ are the known

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functions on S, for every t $|a(t)| \leq k$, $|b(t)| \leq k$ where k is some constant.

Suppose that θ is transmitted according to the scheme

$$\Delta\xi_t = [A_0(t,\xi) + A_1(t,\xi)\theta_t]\Delta + \Delta w_t, \quad \xi_0 = 0, \tag{2}$$

where $w = (w_t, \mathcal{F}_t)$, $t \in S$ is a G.R.S. $N(0,t)$ with independent increments which is independent of θ_0 and v. Nonanticipating with respect to ξ functionals $A_0(t,\xi)$ and $A_1(t,\xi)$ define the coding.

The transmission performed according to the scheme (2) is a transmission of a Gaussian message θ through a noiseless feedback channel which is an analogue of the additive "White noise" channel in the discrete time case. No instantaneous feedback is required here (which is essential in the continuous time), but we assume that the quantization step Δ is equal to the time of signal return.

Suppose that the coding functionals A_0 and A_1 satisfy the condition

$$E[A_0(t,\xi) + A_1(t,\xi)\theta_t]^2 \leq p, \tag{3}$$

where p is a constant characterizing the energetic potential of the transmitter.

Consider the decoding $\hat{\theta} = \hat{\theta}_t(\xi)$ satisfying for every t the condition

$$E\hat{\theta}_t^2 < \infty. \tag{4}$$

Such kind of $[(A_0, A_1), \hat{\theta}]$ form a class of admissible codings and decodings.

The problem is to find the codings (A_0^*, A_1^*) and the decodings $\hat{\theta}^*$ optimal in the sense of the criterion

$$\delta(t) = \inf_{A_0, A_1, \hat{\theta}} E[\theta_t - \hat{\theta}_t(\xi)]^2 \tag{5}$$

where inf is taken in the class of all admissible $[(A_0, A_1), \hat{\theta}]$.

Theorem 1. *During the transmission a discrete Gaussian process described by the finite difference equation* (1) *according to the transmission scheme* (2) *under conditions* (3),(4) *and the optimal coding functionals A_0^* and A_1^*, have the form*

$$A_1^*(t) = \sqrt{\frac{p}{\gamma}}(1 + p\Delta)^{\frac{t}{2\Delta}}, \quad A_0^*(t,\xi^*) = -A_1^*(t)m_t^*, \tag{6}$$

where the optimal decoding $\hat{\theta}_t^* = m_t^* = E[\theta_t|\mathcal{F}_t^{\xi^*}]$ *and the transmitted signal are defined by the relations*

$$\Delta m_t^* = a(t)m_t^*\Delta + \sqrt{p\gamma_t^*}\frac{(1+a(t)\Delta)}{1+p\Delta}\Delta\xi_t^*, \quad m_0^* = m, \tag{7}$$

$$\Delta\xi_t^* = \sqrt{\frac{p}{\gamma_t^*}}(\theta_t - m_t^*)\Delta + \Delta w_t, \quad \xi_0^* = 0. \tag{8}$$

The minimal error of message reproduction has the following form

$$\delta(t) = \gamma_t^* = \gamma\Big(\prod_{k=0}^{t-\Delta}(1+a(k)\Delta)^2\Big)(1+p\Delta)^{-\frac{t}{\Delta}} +$$
$$+\sum_{k=0}^{t-\Delta}(b^2(k)\Delta)\Big(\prod_{s=k+\Delta}^{t-\Delta}(1+a(s)\Delta)^2\Big)(1+p\Delta)^{-\frac{t-k-2\Delta}{\Delta}}.$$

Proof. For the given A_0 and A_1 it is known that $\hat{\theta}_t = m_t = E(\theta_t|\mathcal{F}_t^{\xi})$. Hence

$$\delta(t) = \inf_{(A_0,A_1)} E(\theta_t - m_t)^2 = \inf_{(A_0,A_1)} E\gamma_t.$$

In order to find m_t and $\gamma_t = E[(\theta_t - m_t)^2|\mathcal{F}_t^{\xi}]$ we shall use filtering equations for the conditionally Gaussian random sequence (see [1] or [2]).

The rest of the proof is similar to that of the theorem on the optimal scheme for the transmission of Gaussian processes through a noiseless feedback channel in continuous time (see [2]). ■

2. It is natural to investigate the case of Gaussian signal transmission when white noise is imposed on the back signal, i.e. the message is transmitted according to the scheme

$$\Delta\xi_t = A(t,\theta,\tilde{\xi})\Delta + \sigma(t)\Delta w_t, \quad \xi_0 = 0, \tag{9}$$

where the back signal $\tilde{\xi}_t$ has the form, say, $\tilde{\xi}_t = \xi_t + \eta_t$ or $\tilde{\xi}_t = \Pi(t,\xi) + \eta_t$, where $\Pi(t,\xi)$ is some nonanticipating functional and η_t is the noise in the back channel.

We shall specify the problem under consideration.

Let the signal $\theta = (\theta_t, \mathcal{F}_t)$, $t \in S$, be a discrete Gaussian process described by the equation (1).

Assume that θ is transmitted according to the scheme

$$\Delta\xi_t = [A_0(t)\tilde{\xi}_t + A_1(t)\theta_t]\Delta + \sigma(t)\Delta w_t, \quad \xi_0 = 0, \tag{10}$$

where the functions A_0 and A_1 define the coding.

The back signal has the form

$$\tilde{\xi}_t = \Pi(t,\xi) + \eta_t. \tag{11}$$

Here $\Pi(t,\xi)$ is a transformator of the back message and η_t is the noise in the back channel governed by the finite difference equation

$$\Delta\eta_t = c(t)\eta_t\Delta + d(t)\Delta\bar{w}_t, \tag{12}$$

where $\bar{w} = (\bar{w}_t, \mathcal{F}_t)$, $t \in S$, is a G.R.S. $N(0,t)$ with independent increments, for every t $|c(t)| \le L$ and $|d(t)| \le L$, where L is some constant.

Denote

$$m_t^{(1)} = m_t = E[\theta_t|\mathcal{F}_t^\xi], \quad m_t^{(2)} = E[\eta_t|\mathcal{F}_t^\xi],$$

$$\gamma_t^{(1)} = \gamma_t = E[(\theta_t - m_t)^2|\mathcal{F}_t^\xi], \quad \gamma_t^{(2)} = E[(\eta_t - m_t^{(2)})^2|\mathcal{F}_t^\xi],$$

$$\gamma_t^{(1,2)} = \gamma_t^{(2,1)} = E[(\theta_t - m_t)(\eta_t - m_t^{(2)})|\mathcal{F}_t^\xi],$$

$$\tilde{\gamma}_t = \gamma_t^{(2)} - \frac{\gamma_t^{2(1,2)}}{\gamma_t}, \quad \hat{\gamma}_t = \frac{\gamma_t^{(1,2)}}{\gamma_t}.$$

The problem is to find the codings (A_0^*, A_1^*), decodings $\hat{\theta}^* = (\hat{\theta}_t^*(\xi^*))$, $t \in S$, and the transformator Π^* optimal in the sense of the square criterion

$$\delta(t) = \inf_{A_0, A_1, \Pi, \hat{\theta}} E[\theta_t - \hat{\theta}_t(\xi)]^2, \tag{13}$$

where inf is taken in the class of admissible $A_0, A_1, \Pi, \hat{\theta}$ for which the following power condition[1]

$$E[A_0(t)\tilde{\xi}_t + A_1(t)\theta_t]^2 \le p(t), \tag{14}$$

$$A_0^2(t)\tilde{\gamma}_t \ge q(t) \tag{15}$$

holds where $p(t)$ and $q(t)$ are summable functions on S characterizing the changes of the energetic transmitter potential, and $q(t) \le p(t)$, $t \in S$. Let

$$E\hat{\theta}_t^2 < \infty. \tag{16}$$

Theorem 2. *When Gaussian random sequence $\theta = (\theta_t, \mathcal{F}_t)$, $t \in S$, governed by the finite difference equation* (1) *is transmitted according to the scheme of transmission* (10)–(12) *through a noisy feedback channel under conditions* (14)–(16), *then the optimal in the sense of*

[1]The fact that condition (15) is the power one indeed will be shown at the end of the section.

square criterion (13) *coding functions* A_0^*, A_1^*, *decoding functional* $\hat{\theta}^*$ *and transformator of the back message* Π^* *have the form*

$$A_0^*(t) = \Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}},$$

$$A_1^*(t) = -\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\hat{\gamma}_t^* + \Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac{1}{2}},$$

$$\Pi^*(t,\xi) = -m_t^{*(2)} - \frac{A_1^*(t)}{A_0^*(t)}m_t^*,$$

$$\hat{\theta}_t^*(\xi^*) = m_t^*, \tag{17}$$

$$\Delta m_t^* = a(t)m_t^*\Delta + (1+a(t)\Delta)[\gamma_t^*(p(t)-q(t))]^{\frac{1}{2}} \times$$
$$\times(\sigma^2(t)+p(t)\Delta)^{-1}\Delta\xi_t^*, \quad m_0^* = m.$$

The optimal transmission has the following form

$$\Delta\xi_t^* = \Big\{\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}(\eta_t - m_t^{*(2)}) + \Big[-\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\hat{\gamma}_t^* + \Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac{1}{2}}\Big] \times$$
$$\times(\theta_t - m_t^*)\Big\}\Delta + \sigma(t)\Delta w_t, \quad \xi_0^* = 0, \tag{18}$$

where $m_t^{*(2)}$ *satisfies the finite difference equation*

$$\Delta m_t^{*(2)} = c(t)m_t^{*(2)}\Delta + (1+c(t)\Delta) \times$$
$$\times\{\hat{\gamma}_t^*[\gamma_t^*(p(t)-q(t))]^{\frac{1}{2}} + [q(t)\tilde{\gamma}_t^*]^{\frac{1}{2}}\} \times$$
$$\times(\sigma^2(t)+p(t)\Delta)^{-1}\Delta\xi_t^*, \quad m_0^{*(2)} = m^{(2)} = E\eta_0^{(2)}$$

and $\tilde{\gamma}_t^* = \gamma_t^{*(2)} - \frac{(\gamma_t^{*(1,2)})^2}{\gamma_t^*}$ *and* $\hat{\gamma}_t^* = \frac{\gamma_t^{*(1,2)}}{\gamma_t^*}$ *are defined by the filtering equations* (19)–(23) *given below, where* A_0^* *and* A_1^* *are substituted from* (17).

The minimal message reproduction error $\delta(t)$ *has the form*

$$\delta(t) = \gamma_t^* = \gamma\prod_{k=0}^{t-\Delta}(1+a(k)\Delta)^2(\sigma^2(k)+q(k)\Delta)(\sigma^2(k)+p(k)\Delta)^{-1} +$$
$$+\sum_{k=0}^{t-\Delta} b^2(k)\Big(\prod_{m=k+\Delta}^{t-\Delta}(1+a(m)\Delta)^2(\sigma^2(m)+q(m)\Delta)(\sigma^2(m)+p(m)\Delta)^{-1}\Big).$$

Corollary. *Let* $a(t) = b(t) \equiv 0$, *i.e. according to the scheme* (10)–(12), *a Gaussian* $N(m,\gamma)$, $\gamma > 0$, *random variable* θ *is transmitted.*

Then the reproduction error $\delta(t)$ is

$$\delta(t)=\gamma\prod_{k=0}^{t-\Delta}\Big(1+\frac{q(k)}{\sigma^2(k)}\Delta\Big)\Big(1+\frac{p(k)}{\sigma^2(k)}\Delta\Big)^{-1}.$$

In this case $\tilde{\gamma}_t^*$ and $\hat{\gamma}_t^*$ satisfy the following finite difference equations

$$\frac{\Delta\tilde{\gamma}_t^*}{\Delta}=d^2(t)+\tilde{\gamma}_t^*(c(t)\sigma^2(t)-q(t))(\sigma^2(t)+q(t)\Delta)^{-1},\quad \tilde{\gamma}_0^*=0,$$

$$\frac{\Delta\hat{\gamma}_t^*}{\Delta}=\Big\{\hat{\gamma}_t^*[c(t)\sigma^2(t)-q(t)\gamma_t^*+q(t)(1+c(t)\Delta)]-$$
$$-(1+c(t)\Delta)\Big(\frac{q(t)}{\gamma_t^*}\Big)^{\frac12}\Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac12}\tilde{\gamma}_t^*\Big\}\times$$
$$\times[\sigma^2(t)+q(t)\Delta]^{-1},\quad \hat{\gamma}_0^*=0.$$

Proof of the theorem. It can be easily seen that

$$\delta(t)=\inf_{(A_0,A_1,\Pi)}E\gamma_t=\inf_{(A_0,A_1,\Pi)}\gamma_t.$$

Rewrite (10) in the following form

$$\Delta\xi_t=[A_0(t)\Pi(t,\xi)+A_0(t)\eta_t+A_1(t)\theta_t]\Delta+\sigma(t)\Delta w_t,\quad \xi_0=0.$$

Then one can see that the equation of optimal filtering of a partially observable conditionally Gaussian process $(\tilde{\theta}_t,\xi_t)$, $t\in S$, with an unobservable component $\tilde{\theta}_t=(\theta_t,\eta_t)$ (see [1] or [2]) leads to the following closed system of finite difference equations:

$$\Delta m_t=a(t)m_t\Delta+(1+a(t)\Delta)(A_1(t)\gamma_t+A_0(t)\gamma_t^{(1,2)})\times$$
$$\times[\sigma^2(t)+(A_1^2(t)\gamma_t+2A_1(t)A_0(t)\gamma_t^{(1,2)}+A_0^2(t)\gamma_t^{(2)})\Delta]^{-1}\times$$
$$\times[\Delta\xi_t-(A_0(t)(\Pi(t,\xi)+m_t^{(2)})+A_1(t)m_t)\Delta],\quad m_0=m,\qquad(19)$$

$$\Delta m_t^{(2)}=c(t)m_t^{(2)}\Delta+(1+c(t)\Delta)(A_1(t)\gamma_t^{(1,2)}+A_0(t)\gamma_t^{(2)})\times$$
$$[\sigma^2(t)+(A_1^2(t)\gamma_t+2A_1(t)A_0(t)\gamma_t^{(1,2)}+A_0^2(t)\gamma_t^{(2)})\Delta]^{-1}\times$$
$$\times[\Delta\xi_t-(A_0(t)(\Pi(t,\xi)+m_t^{(2)})+A_1(t)m_t)\Delta],\quad m^{(2)}=m,\qquad(20)$$

$$\frac{\Delta\gamma_t^{(1)}}{\Delta}=b^2(t)+a^2(t)\gamma_t^{(1)}\Delta+2a(t)\gamma_t^{(1)}-$$
$$-(1+a(t)\Delta)^2(A_1(t)\gamma_t^{(1)}+A_0(t)\gamma_t^{(1,2)})^2[\sigma^2(t)+$$
$$+(A_1^2(t)\gamma_t^{(1)}+2A_1(t)A_0(t)\gamma_t^{(1,2)}+A_0^2(t)\gamma_t^{(2)})\Delta]^{-1},$$
$$\gamma_0^{(1)}=\gamma,\qquad(21)$$

$$\frac{\Delta\gamma_t^{(2)}}{\Delta} = d^2(t) + c^2(t)\gamma_t^{(2)}\Delta + 2c(t)\gamma_t^{(2)} -$$
$$-(1 + c(t)\Delta)(A_1(t)\gamma_t^{(1,2)} + A_0(t)\gamma_t^{(2)})^2[\sigma^2(t) +$$
$$+(A_1^2(t)\gamma_t^{(1)} + 2A_1(t)A_0(t)\gamma_t^{(1,2)} + A_0^2(t)\gamma_t^{(2)})\Delta]^{-1},$$
$$\gamma_0^{(2)} = \gamma_2, \tag{22}$$

$$\frac{\gamma_t^{(1,2)}}{\Delta} = a(t)\gamma_t^{(1,2)} + c(t)\gamma_t^{(1,2)} -$$
$$-(1 + a(t)\Delta)(1 + c(t)\Delta)(A_1(t)\gamma_t^{(1)} + A_0(t)\gamma_t^{(1,2)}) \times$$
$$\times(A_1(t)\gamma_t^{(1,2)} + A_0(t)\gamma_t^{(2)})[\sigma^2(t) + A_1^2(t)\gamma_t^{(1)} + 2A_1(t)A_0(t)\gamma_t^{(1,2)} +$$
$$+A_0^2(t)\gamma_t^{(2)})\Delta]^{-1}. \tag{23}$$

Equation (21) can be reduced to the form

$$\gamma_t = \Big(\prod_{k=0}^{t-\Delta}(1 + a(k)\Delta)^2(\sigma^2(k) + A_0^2(k)\tilde{\gamma}_k\Delta)[\sigma^2(k) +$$
$$+(A_1^2(k)\gamma_k + 2A_1(k)A_0(k)\gamma_k^{(1,2)} + A_0^2(k)\gamma_k^{(2)})\Delta]^{-1}\Big) \times$$
$$\times\Big[\sum_{l=0}^{t-\Delta} b^2(l)\Big(\prod_{m=0}^{l}(1 + a(m)\Delta)^2(\sigma^2(m) + A_0^2(m)\tilde{\gamma}_m\Delta) \times$$
$$\times[\sigma^2(m) + (A_1^2(m)\gamma_m + 2A_1(m)A_0(m)\gamma_m^{(1,2)} +$$
$$+A_0^2(m)\gamma_m^{(2)})\Delta]^{-1}\Big)^{-1} + \gamma\Big]. \tag{24}$$

Using inequality (15) and the resulting from (14) inequality

$$A_1^2(t)\gamma_t + 2A_1(t)A_0(t)\gamma_t^{(1,2)} + A_0^2(t)\gamma_t^{(2)} \le p \tag{25}$$

we obtain from (24)

$$E(\theta_t - m_t)^2 = \gamma_t \ge \psi(t)$$

where

$$\psi(t) = \Big(\prod_{k=0}^{t-\Delta}(1 + a(k)\Delta)^2(\sigma^2(k) + q(k)\Delta)(\sigma^2(k) + p(k)\Delta)^{-1}\Big) \times$$
$$\times\Big[\sum_{l=0}^{t-\Delta} b^2(l)\Big(\prod_{m=0}^{l}(1 + a(m)\Delta)^2(\sigma^2(m) + q(m)\Delta) \times$$
$$\times(\sigma^2(m) + p(m)\Delta)^{-1}\Big)^{-1} + \gamma\Big]$$

and since $\psi(t)$ is a known function, we have

$$\delta(t) \ge \psi(t) \tag{26}$$

for all $t \in S$.

The equality in (26) is obtained when

$$A_0(t)\tilde{\gamma}_t = q(t) \tag{27}$$

and

$$A_1^2(t)\gamma_t + 2A_1(t)A_0(t)\gamma_t^{(1,2)} + A_0^2(t)\gamma_t^{(2)} = p(t). \tag{28}$$

Since

$$p(t) \geq E(A_0(t)\tilde{\xi}_t + A_1(t)\theta_t)^2 = E[A_0(t)\Pi(t,\xi) +$$
$$+A_0(t)m_t^{(2)} + A_1(t)m_t]^2 + A_1^2(t)\gamma_t + 2A_1(t)A_0(t)\gamma_t^{(1,2)} + A_0^2(t)\gamma_t(2),$$

(28) implies (P-a.s.)

$$A_0(t)\Pi(t,\xi) + A_0(t)m_t^{(2)} + A_1(t)m_t = 0. \tag{29}$$

Consequently, (27),(28) and (29) are the relations from which optimal codings (A_0^*, A_1^*) and the transformator Π^* are obtained. This completes the proof of the theorem. ■

Remark. As it can be seen from relation (17), the optimal transformator of the back message Π_0^* is constructed in such a way that the back message is multiplied by the value of some deterministic function of time at the moment t and $m_t^{(2)}$ with a negative sign, i.e. the noise η_t is compensated by the best (in the sense of square criterion) estimate $m_t^{(2)} = E[\eta_t|\mathcal{F}_t^\xi]$ (see optimal transmission scheme (18)).

To conclude this section we can show that condition (15) is indeed a power-type one.

It can be easily seen that

$$E(A_0(t)\tilde{\xi}_t + A_1(t)\theta_t)^2 = E[A_0(t)\Pi(t,\xi) + A_0(t)m_t^{(2)} +$$
$$+A_1(t)m_t]^2 + A_0^2(t)\gamma_t^{(2)} + 2A_0(t)A_1(t)\gamma_t^{(1,2)} + A_1^2(t)\gamma_t =$$
$$= E[A_0(t)\Pi(t,\xi) + A_0(t)m_t^{(2)} + A_1(t)m_t]^2 + A_0^2(t)\tilde{\gamma}_t +$$
$$+(A_0^2(t)\gamma_t^{(1,2)} + A_1(t)\gamma_t)^2\gamma_t^{-1} \tag{30}$$

and since

$$A_0^2(t)\tilde{\gamma}_t \geq q(t)$$

(30) implies

$$E[A_0(t)\tilde{\xi}_t + A_1(t)\theta_t]^2 \geq q(t). \tag{31}$$

Consequently (15) implies power condition (31).

§ 2. Optimal transmission in a continuous case

Consider the following problem. A Gaussian message θ is transmitted through an additive white noise instantaneous feedback channel described by the following stochastic differential equation

$$d\xi_t = A(t,\theta,\tilde{\xi})dt + \sigma(t)dw_t \tag{32}$$

where $w = (w_t, \mathcal{F}_t)$ is a Wiener process. In contrast to the traditional schemes (see, e.g. [2]–[6]), the feedback here is not assumed to be noiseless.

The functional A in (32) defines the coding, and the back signal $\tilde{\xi}$ has the following form

$$\tilde{\xi}_t = \Pi(t,\xi) + \eta_t \tag{33}$$

where Π is the transformator of the back signal, η is a noise in the back channel.

In this section optimal transmission schemes are constructed under certain power restrictions in a linear case when

$$A(t,\theta,\tilde{\xi}) = A_0(t)\tilde{\xi}_t + A_1(t)\theta_t$$

and it is proved that these particular linear schemes are also optimal in the general class given by (32),(33).

1. Let the transmitted message $\theta = (\theta_t, \mathcal{F}_t)$, $t \in [0,T]$, be a Gaussian process described by the stochastic differential equation

$$d\theta_t = a(t)\theta_t dt + b(t)dv_t, \tag{34}$$

where $v = (v_t, \mathcal{F}_t)$ is a Wiener process independent of the Gaussian $N(m,\gamma)$, $\gamma > 0$, random variable θ_0, $|a(t)| \le k$, $|b(t)| \le k$, k is some constant.

Suppose that θ is transmitted according to the following linear scheme

$$d\xi_t = [A_0(t)\tilde{\xi}_t + A_1(t)\theta_t]dt + \sigma(t)dw_t, \quad \xi_0 = 0, \tag{35}$$

where $w = (w_t, \mathcal{F}_t)$ is a Wiener process independent of v; $A_0(t)$ and $A_1(t)$ are the coding functions, $\sigma(t) > 0$. The back signal $\tilde{\xi}$ has the form (33). The noise in the back channel η admits the stochastic differential

$$d\eta_t = \bar{a}(t)\eta_t dt + \bar{b}(t)d\bar{w}_t \tag{36}$$

where $\bar{w} = (\bar{w}_t, \mathcal{F}_t)$ is a Wiener process independent of w and v and of the Gaussian $N(m_2,\gamma_2)$, $\gamma_2 > 0$, random variable η_0, $|\bar{a}(t)| \le k$, $|\bar{b}(t)| \le k$.

A class of admissible codings, transformator and decodings is formed by such $[(A_0, A_1), \Pi, \hat{\theta}]$ for which the following conditions are satisfied:

1) stochastic differential equation (35) has a unique strong solution, $\sup_{t\in[0,T]} |A_i(t)| < \infty,\ i = 0, 1$;

2) $E[A_0(t)\tilde{\xi}_t + A_1(t)\theta_t]^2 \le p(t),$ (37)

$A_0^2(t)\tilde{\gamma}_t \ge q(t),$ (38)

where $p(t)$ and $q(t)$ are some functions integrable on $[0,T]$ and for every t

$$q(t) \le p(t).$$

Let

$$\Delta(t) = \inf E[\theta_t - \hat{\theta}_t(\xi)]^2, \tag{39}$$

where inf is taken in a class of admissible $[(A_0, A_1), \Pi, \hat{\theta}]$.

Theorem 3. *When a Gaussian random process θ_t governed by a stochastic differential equation* (34) *is transmitted through a noisy feedback channel* (35),(33),(36) *under conditions* 1) *and* 2) *optimal in the sense of the square criterion* (39), *the coding functions A_0^*, A_1^* and the transformator of the back message Π^* have the following form*[2]

$$A_0^*(t) = \Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}},$$

$$A_1^*(t) = -\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\hat{\gamma}_t + \Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac{1}{2}},$$

$$\Pi^*(t, \xi^*) = -m_t^{*(2)} + \frac{A_0^*(t)}{A_1^*(t)} m_t^*,$$

where γ_t^* is equal to the minimal message reproduction error

$$\Delta(t) = \gamma_t^* = \gamma \exp\{2\int_0^t a(s)ds - \int_0^t \frac{(p(s)-q(s))}{\sigma^2(s)}ds\} +$$

$$+ \int_0^t b(s)\exp\{2\int_s^t a(u)ds - \int_s^t \frac{(p(u)-q(u))}{\sigma^2(u)}du\}ds,$$

and $\tilde{\gamma}_t^*$ and $\hat{\gamma}_t^*$ are defined by the following equations

$$\frac{d\tilde{\gamma}_t^*}{dt} = \bar{b}^2(t) + 2\tilde{\gamma}_t^*\Big(\bar{a}(t) - \frac{q(t)}{\sigma^2(t)}\Big) + \hat{\gamma}_t^*(b^2(t) + 2a(t)\gamma_t^*),$$

$$\tilde{\gamma}_0^* = \gamma_2,$$

$$\frac{d\hat{\gamma}_t^*}{dt} = \hat{\gamma}_t^*\Big[\bar{a}(t) - \frac{q(t)}{\sigma^2(t)} - b^2(t) - 2a(t)\gamma_t^*\Big] - A_1^*(t)A_0^*(t)\frac{\tilde{\gamma}_t}{\sigma^2(t)},$$

$$\hat{\gamma}_0^* = 0.$$

[2]The notation of §1 is used.

The optimal decoding m_t^* satisfies the following stochastic differential equation

$$dm_t^* = a(t)m_t^*dt + [\gamma_t^*(p(t) - q(t))]^{\frac{1}{2}}\sigma^{-2}(t)d\xi_t^*, \quad m_t^* = m.$$

The optimal transmission

$$d\xi_t^* = \Big\{\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\tilde{\xi}_t^* + \Big[-\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\hat{\gamma}_t + \Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac{1}{2}}\Big]\theta_t\Big\} + \\ +\sigma(t)dw_t$$

or

$$d\xi_t^* = \Big\{\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}(\eta_t - m_t^{*(2)}) + \Big[-\Big(\frac{q(t)}{\tilde{\gamma}_t^*}\Big)^{\frac{1}{2}}\hat{\gamma}_t^* + \\ +\Big(\frac{p(t)-q(t)}{\gamma_t^*}\Big)^{\frac{1}{2}}\Big](\theta_t - m_t^*)\Big\}dt + \sigma(t)dw_t, \quad \xi_0^* = 0,$$

where $m_t^{*(2)}$ is defined by the equation

$$dm_t^{*(2)} = \bar{a}(t)m_t^{*(2)}dt + \{\hat{\gamma}_t^*[\gamma_t^*(p(t) - q(t))]^{\frac{1}{2}} + \\ +[q(t)\tilde{\gamma}_t^*]^{\frac{1}{2}}\sigma^{-2}(t)d\xi_t^*, \quad m_0^{*(2)} = m_2.$$

The proof of the theorem is similar to that of an analogous theorem (Theorem 2, §1) for the discrete case, and the equations of nonlinear filtering of conditionally Gaussian type processes (see [2]) are used.

Corollary. *When a Gaussian $N(m,\gamma)$, $\gamma > 0$, random variable θ is transmitted through the channel* (35),(33),(36), *the minimal message reproduction error is*

$$\Delta(t) = \gamma \exp\Big[-\int_0^t \frac{(p(s) - q(s))ds}{\sigma^2(s)}\Big].$$

Now we shall consider the simplest case of a Gaussian $N(m,\gamma)$ random variable θ transmission through a noisy feedback channel (32),(33) with $A(t,\theta,\tilde{\xi}) = \tilde{\xi}_t + A(t)\theta$. Let $\sigma(t) \equiv 1$.

For simplicity we assume that $\eta_t = \bar{b}w(t)$, where $\bar{b}$ is some constant. Then the optimization problem is simplified and instead of obtaining optimal A_0, A_1, Π, $\hat{\theta}$ as in Part 1 of Section 1 we must find optimal A, Π, $\hat{\theta}$.

The necessity of condition (38) is eliminated but it should be required that

$$p(t) \geq \tilde{\gamma}_t, \quad t \in [0,T].$$

Optimal A^*, Π^* and $\hat{\theta}^*$ will have the following form

$$A^*(t) = -\hat{\gamma}_t^* + \Big(\frac{p(t) - \tilde{\gamma}_t}{\gamma_t^*}\Big)^{\frac{1}{2}},$$

$$\Pi^*(t, \xi^*) = -m_t^{*(2)} + \Big[\Big[\hat{\gamma}_t^* - \Big(\frac{p - \tilde{\gamma}_t}{\gamma_t^*}\Big)^{\frac{1}{2}}\Big] m_t^*,$$

$$\hat{\theta}_t^* = \hat{\theta}_t^*(\xi^*) = m_t^*,$$

where m_t^* admits the representation

$$dm_t^* = [\gamma_t^*(p(t) - \tilde{\gamma}_t)]^{\frac{1}{2}} d\xi_t^*, \quad m_0^* = m$$

and $\tilde{\gamma}_t$ and $\hat{\gamma}_t$ are found from the relations

$$\frac{d\tilde{\gamma}_t}{dt} = \bar{b}^2 - \tilde{\gamma}_t^2, \quad \tilde{\gamma}_0 = 0,$$

$$\hat{\gamma}_t = -\int_0^t \Big(\frac{p(s) - \tilde{\gamma}_s}{\gamma_s}\Big)^{\frac{1}{2}} \tilde{\gamma}_s ds,$$

while $m_t^{*(2)}$ is found from the stochastic differential equation

$$dm_t^{*(2)} = \{\hat{\gamma}_t^*[\gamma_t^*(p(t) - \tilde{\gamma}_t)]^{\frac{1}{2}} + \tilde{\gamma}_t\} d\xi_t^*, \quad m_0^{*(2)} = 0.$$

The minimal message reproduction error is

$$\Delta(t) = \gamma \exp\Big[-\int_0^t (p(s) - \tilde{\gamma}_s) ds\Big].$$

In the case $\bar{b} = 1$ we have

$$\tilde{\gamma}_t = \frac{e^{2t} - 1}{e^{2t} + 1}$$

and

$$\Delta(t) = \gamma \exp\Big[-\int_0^t p(s) ds\Big] \operatorname{ch} t,$$

where $\operatorname{ch} t$ is a hyperbolic cosine.

In the case $\bar{b} = 0$, i.e. when the noise η in the back channel is absent, we have

$$\Delta(t) = \gamma \exp\Big(-\int_0^t p(s) ds\Big),$$

which coincides with the transmission through a feedback noiseless channel (see [2]) and our optimal Π and A coincide with the optimal codings A_0 and A_1 by using the notation of [2], i.e. in this case the transformator Π can be placed in the coding device by virtue of a noiseless feedback.

2. Consider the transmission of the Gaussian process described by stochastic differential equation (34) through the channel (32),(33). Let

$$\tilde{A}(t,\theta_t,\xi)=E[A(t,\theta_t,\tilde{\xi})|\mathcal{F}_t^{\theta_t,\xi}],\quad \mathcal{F}^{\theta_t,\xi}=\sigma\{\theta_t,\xi_s,s\le t\}$$

and

$$\bar{A}(t,\xi)=E[A(t,\theta_t,\tilde{\xi})|\mathcal{F}_t^{\xi}].$$

Assume that the following conditions are satisfied:

1) Equation (1) has a unique strong solution,

2) $EA^2(t,\theta_t,\tilde{\xi})\le p(t)$,

$$E[A(t,\theta_t,\tilde{\xi})-\tilde{A}(t,\theta_t,\xi)]^2\ge q(t) \tag{40}$$

where $p(t)$ and $q(t)$ are some functions integrable on $[0,T]$ and for every t

$$p(t)\ge q(t).$$

Let $I_T(\theta,\xi)$ be mutual information of signals θ and ξ and let $I_T(\tilde{\theta},\xi)$ be mutual information of $\tilde{\theta}=(\theta_t,\eta_t)$ and $\xi, t\in[0,T]$.

Lemma 1. *Mutual informations $I_T(\theta,\xi)$ and $I_T(\tilde{\theta},\xi)$ have the following forms (see [7])*

$$I_T(\theta,\xi)=\frac{1}{2}\int_0^T E[\tilde{A}^2(t,\theta_t,\xi)-\bar{A}^2(t,\xi)]\sigma^{-2}(t)dt, \tag{41}$$

$$I_T(\tilde{\theta},\xi)-I_T(\theta,\xi)=$$

$$+\frac{1}{2}\int_0^T E[A(t,\theta_t,\tilde{\xi})-\tilde{A}(t,\theta_t,\xi)]^2\sigma^{-2}(t)dt. \tag{42}$$

Corollary. *Under conditions* 1) *and* 2) *we have*

$$I_T(\tilde{\theta},\xi)\le\frac{1}{2}\int_0^T\frac{p(t)}{\sigma^2(t)}dt,$$

$$I_T(\tilde{\theta},\xi)-I_T(\theta,\xi)\ge\frac{1}{2}\int_0^T\frac{q(t)}{\sigma^2(t)}dt.$$

Let

$$I_t=\sup I_t(\theta,\xi)$$

where sup is taken in the class of all admissible transmission schemes (32),(33), i.e. schemes for which conditions 1) and 2) are satisfied. Then from the corollary of the lemma we have

$$I_t\le\frac{1}{2}\int_0^t\frac{p(s)-q(s)}{\sigma^2(s)}ds.$$

For the linear case $A(t,\theta_t,\tilde{\xi}) = A_0(t)\tilde{\xi}_t + A_1(t)\theta_t$, since $\bar{A}(t,\xi) = 0$ we have

$$I_t(\theta,\xi^*) = \frac{1}{2}\int_0^t E\tilde{A}^2(s,\theta_s,\xi^*)ds =$$

$$= \frac{1}{2}\int_0^t \frac{p(s)-q(s)}{\sigma^2(s)}ds.$$

Hence the following theorem is true.

Theorem 4. *Optimal codings A_0^*, A_1^*, the decoding m_t^* and the transformator Π^* constructed in Theorem 3 are also optimal in the sense of maximum of mutual information.*

3. Finally we shall prove the following

Theorem 5. *When a Gaussian $N(m,\gamma)$, $\gamma > 0$, random variable θ is transmitted according to the transmission scheme (32),(33) under conditions 1) and 2), the minimal reproduction error*

$$\delta(t) = \inf_{(A,\Pi,\hat{\theta})} E(\theta - \hat{\theta}_t(\xi))^2$$

has the following form

$$\delta(t) = \Delta(t) = \gamma \exp\left\{-\int_0^t (p(s)-q(s))\sigma^{-2}ds\right\},$$

where $\Delta(t)$ is the minimal message reproduction error for the optimal linear transformation constructed in Theorem 3, i.e. among all admissible schemes the transmission constructed in Theorem 3 is optimal in the sense of the square criterion.

Proof. Since $\delta(t) \le \Delta(t)$, the theorem will be proved if we show that

$$\delta(t) \ge \gamma \exp\left\{-\int_0^t (p(s)-q(s))\sigma^{-2}ds\right\}. \tag{43}$$

Let $\hat{\theta} = \hat{\theta}_t(\xi)$ be some decoding. Then by Lemma 16.8 from [2] we have

$$E(\theta - \hat{\theta}_t(\xi))^2 \ge \gamma e^{-2I(\theta,\hat{\theta}_t(\xi))}.$$

But $I(\theta,\hat{\theta}_t(\xi)) \le I_t(\theta,\xi)$, and according to Theorem 4

$$I_t(\theta,\xi) \le I_t(\theta,\xi^*) = \frac{1}{2}\int_0^t (p(s)-q(s))\sigma^{-2}(s)ds.$$

Hence inequality (43) takes place and Theorem 5 is true. ■

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TWO-WEIGHTED L_p-INEQUALITIES FOR SINGULAR INTEGRAL OPERATORS ON HEISENBERG GROUPS

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ABSTRACT. Some sufficient conditions are found for a pair of weight functions, providing the validity of two-weighted inequalities for singular integrals defined on Heisenberg groups.

რეზიუმე. ნაშრომში ნაპოვნია ზოგიერთი საკმარისი პირობა წონათა წყვილისათვის, რომლებიც უზრუნველყოფენ ორწონიანი უტოლობების მართებულობას ჰაიზენბერგის ჯგუფებზე განსაზღვრული სინგულარული ინტეგრალებისათვის.

Estimates for singular integrals of the Calderon–Zygmund type in various spaces (including weighted spaces and the anisotropic case) have attracted a great deal of attention on the part of researchers. In this paper we will deal with singular integral operators T on the Heisenberg group H^n which have an essentially different character as compared with operators of the Calderon–Zygmund type. We have obtained the two-weighted L_p-inequality with monotone weights for singular integral operators T on H^n. Applications are given.

Let H^n be the Heisenberg group (see [1], [2]) realized as a set of points $x = (x_0, x_1, \dots, x_{2n}) = (x_0, x') \in \mathbb{R}^{2n+1}$ with the multiplication

$$xy = \Big(x_0 + y_0 + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i), \quad x' + y'\Big).$$

The corresponding Lie algebra is generated by the left-invariant vector fields

$$X_0 = \frac{\partial}{\partial x_0}, \quad X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} x_{n+i} \frac{\partial}{\partial x_0},$$
$$X_{n+i} = \frac{\partial}{\partial x_{n+i}} - \frac{1}{2} x_i \frac{\partial}{\partial x_0}, \quad i = 1, \dots, n,$$

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which satisfy the commutation relation

$$[X_i, X_{n+i}] = \frac{1}{4}X_0,$$
$$[X_0, X_i] = [X_0, X_{n+i}] = [X_i, X_j] = [X_{n+i}, X_{n+j}] = [X_i, X_{n+j}] = 0,$$
$$i, j = 1, \dots, n \quad i \neq j.$$

The dilation $\delta_t : \delta_t x = (t^2 x_0, tx')$, $t > 0$, is defined on H^n. The Haar measure on this group coincides with the Lebesgue measure $dx = dx_0 dx_1 \cdots dx_{2n}$. The identity element in H^n is $e = 0 \in \mathbb{R}^{2n+1}$, while the element x^{-1} inverse to x is $(-x)$.

The function f defined in H^n is said to be H-homogeneous of degree m, on H^n, if

$$f(\delta_t x) = t^m f(x), \quad t > 0.$$

We also define the norm on H^n

$$|x|_H = \Big[x_0^2 + \Big(\sum_{i=1}^{2n} x_i^2\Big)^2\Big]^{1/4}$$

which is H-homogeneous of degree one. This also yields the distance function, namely, the distance

$$d(x, y) = d(y^{-1}x, e) = |y^{-1}x|_H,$$
$$|y^{-1}x|_H = \Big[\Big(x_0 - y_0 - \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)\Big)^2 +$$
$$+ \Big(\sum_{i=1}^{2n}(x_i - y_i)^2\Big)^2\Big]^{1/4}.$$

d is left-invariant in the sense that $d(x, y)$ remains unchanged when x and y are both left-translated by some fixed vector in H^n. Furthermore, d satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z), \quad x, y, z \in H^n.$$

For $r > 0$ and $x \in H^n$ let

$$B(x, r) = \{y \in H^n;\ |y^{-1}x|_H < r\}$$
$$(S(x, r) = \{y \in H^n;\ |y^{-1}x|_H = r\})$$

be the H-ball (H-sphere) with centre x and radius r.

The number $Q = 2n + 2$ is called the homogeneous dimension of H^n. Clearly,

$$d(\delta_t x) = t^Q dx.$$

Given functions $f(x)$ and $g(x)$ defined in H^n, the Heisenberg convolution (H-convolution) is obtained by

$$(f*g)(x)=\int_{H^n} f(y)g(y^{-1}x)dy=\int_{H^n} f(xy^{-1})g(y)dy,$$

where dy is the Haar measure on H^n.

The kernel $K(x)$ admitting the estimate

$$|K(x)|\le C|x|_H^{\alpha-Q}$$

is summable in the neighbourhood of e for $\alpha>0$ and in that case $K*g$ is defined for the function g with bounded support. If however the kernel $K(x)$ has a singularity of order Q at zero, i.e. $|K(x)|\sim |x|_H^{-Q}$ near e, then there arises a singular integral on H^n.

Let $\omega(x)$ be a positive measurable function on H^n. Denote by $L_p(H^n,\omega)$ a set of measurable functions $f(x)$, $x\in H^n$, with the finite norm

$$\|f\|_{L_P(H^n,\omega)}=\Big(\int_{H^n}|f(x)|^p\omega(x)dx\Big)^{1/p},\quad 1\le p<\infty.$$

We say that a locally integrable function $\omega: H^n\to(0,\infty)$ satisfies Muckenhoupt's condition $A_p=A_p(H^n)$ (briefly, $\omega\in A_p$), $1<p<\infty$, if there is a constant $C=C(\omega,p)$ such that for any H-ball $B\subset H^n$

$$\Big(|B|^{-1}\int_B\omega(x)dx\Big)\Big(|B|^{-1}\int_B\omega^{1-p'}(x)dx\Big)\le C,\quad \frac{1}{p}+\frac{1}{p'}=1,$$

where the second factor on the left is replaced by $\operatorname{ess\,sup}\{\omega^{-1}(x):x\in B\}$ if $p=1$.

Let $K(x)$ be a singular kernel defined on $H^n\backslash\{e\}$ and satisfying the conditions: $K(x)$ is an H-homogeneous function of degree $-Q$, i.e. $K(\delta_t x)=t^{-Q}K(x)$ for any $t>0$ and $\int_{S_H}K(x)d\sigma(x)=0$, where $d\sigma(x)$ is a measure element on $S_H=S(e,1)$.

Denote by $\omega_K(\delta)$ the modulus of continuity of the kernel on S_H:

$$\omega_K(\delta)=\sup\{|K(x)-K(y)|: x,y\in S_H,\quad |y^{-1}x|_H\le\delta\}.$$

It is assumed that

$$\int_0^1\omega_K(t)\frac{dt}{t}<\infty.$$

We consider the singular integral operator T:

$$Tf(x)=\int_{H^n}K(xy^{-1})f(y)dy=:\lim_{\varepsilon\to0+}\int_{|xy^{-1}|_H>\varepsilon}K(xy^{-1})f(y)dy.$$

As known, T acts boundedly in $L_p(H^n)$, $1 < p < \infty$ (see [3], [4]). For singular integrals with Cauchy–Szegö kernels the weighted estimates were established in the norms of $L_p(H^n, \omega)$ with weights ω satisfying the condition A_p [5]. These results extend to the more general kernels considered above [4].

Theorem 1 [4]. *Let $1 < p < \infty$ and $\omega \in A_p$, then T is bounded in $L_p(H^n, \omega)$.*

In the sequel we will use

Theorem 2. *Let $1 \le p \le q < \infty$ and $U(t)$, $V(t)$ be positive functions on $(0, \infty)$.*

1) *The inequality*

$$\left(\int_0^\infty U(t)\left|\int_0^t \varphi(\tau)d\tau\right|^q dt\right)^{1/q} \le$$
$$\le K_1\left(\int_0^\infty |\varphi(t)|^p v(t)dt\right)^{1/p}$$

with the constant K_1 not depending on φ holds iff the condition

$$\sup_{t>0}\left(\int_t^\infty U(\tau)d\tau\right)^{p/q}\left(\int_0^t V(\tau)^{1-p'}d\tau\right)^{p-1} < \infty$$

is fulfilled;

2) *The inequality*

$$\left(\int_0^\infty U(t)\left|\int_t^\infty \varphi(\tau)d\tau\right|^q dt\right)^{1/q} \le K_2\left(\int_0^\infty |\varphi(t)|^p V(t)dt\right)^{1/p}$$

with the constant K_2 not depending on φ holds iff the condition

$$\sup_{t>0}\left(\int_0^t U(\tau)d\tau\right)^{p/q}\left(\int_t^\infty V(\tau)^{1-p'}d\tau\right)^{p-1} < \infty$$

is fulfilled.

Note that Theorem 2 was proved by G.Talenti, G.Tomaselli, B.Muckenhoupt [7] for $1 \le p = q < \infty$, and by J.S.Bradley [8], V.M.Kokilashvili [9], V.G.Maz'ya [10] for $p < q$.

We say that the weight pair (ω, ω_1) belongs to the class $\widetilde{A}_{pq}(\gamma)$, $\gamma > 0$, if either of the following conditions is fulfilled:

a) $\omega(t)$ and $\omega_1(t)$ are increasing functions on $(0, \infty)$ and

$$\sup_{t>0}\left(\int_t^\infty \omega(\tau)\tau^{-1-\gamma q/p'}d\tau\right)^{p/q}\left(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{\gamma-1}d\tau\right)^{p-1} < \infty;$$

b) $\omega(t)$ and $\omega_1(t)$ are decreasing functions on $(0,\infty)$ and

$$\sup_{t>0}\left(\int_0^{t/2}\omega_1(\tau)\tau^{\gamma-1}d\tau\right)^{p/q}\left(\int_t^{\infty}\omega(\tau)^{1-p'}\tau^{-1-\gamma p'/q}d\tau\right)^{p-1}<\infty.$$

Theorem 3. *Let $1<p<\infty$ and the weight pair $(\omega,\omega_1)\in\tilde{A}_p(Q)\equiv\tilde{A}_{pp}(Q)$. Then for $f\in L_p(H^n,\omega(|x|_H))$ there exists $Tf(x)$ for almost all $x\in H^n$ and*

$$\int\limits_{H^n}|Tf(x)|^p\omega_1(|x|_H)dx\le C\int_{H^n}|f(x)|^p\omega(|x|)_H)dx, \tag{1}$$

where the constant C does not depend on f.

Corollary. *If $\omega(t)$, $t>0$ is increasing (decreasing) and the function $\omega(t)t^{-\beta}$ is decreasing (increasing) for some $\beta\in(0,Q(p-1))$ ($\beta\in(-Q,0)$), then T is bounded on $L_p(H^n,\omega(|x|_H))$.*

Proof. Let $f\in L_p(H^n,\omega(|x|_H))$ and ω, ω_1 be positive increasing functions on $(0,\infty)$. We will prove that $Tf(x)$ exists for almost all $x\in H^n$. We take any fixed $\tau>0$ and represent the function f in the norm of the sum f_1+f_2, where

$$f_1(x)=\begin{cases}f(x), & \text{if } |x|_H>\tau/2\\ 0, & \text{if } |x|_H\le\tau/2\end{cases},\qquad f_2(x)=f(x)-f_1(x).$$

Let $\omega(t)$ be a positive increasing function on $(0,\infty)$ and $f\in L_p(H^n,\omega(|x|_H))$. Then $f_1\in L_p(H^n)$ and therefore $Tf_1(x)$ exists for almost all $x\in H^n$. Now we will show that Tf_2 converges absolutely for all $x:|x|_H\ge\tau$. Note that $C(K)=\sup_{x\in S_H}|K(x)|<\infty$. Hence

$$\begin{aligned}|Tf_2(x)|&\le C(K)\int\limits_{|y|_H\le\tau/2}\frac{|f(y)|}{|xy_{-1}|_H^Q}dy\le\\ &\le\left(\frac{2}{\tau}\right)^{\frac{Q}{p}}\int\limits_{|y|_H\le\tau/2}\frac{|f(y)|\omega(|y|_H)^{\frac{1}{p}}}{\omega(|y|_H)^{\frac{1}{p}}}dy,\end{aligned} \tag{2}$$

since $|xy^{-1}|_H\ge|x|_H-|y|_H\ge\tau/2$. Thus, by the Hölder inequality we can estimate (2) as

$$|Tf_2(x)|\le C\tau^{-Q/p}\|f\|_{L_p(H^n,\omega(|x|_H))}\left(\int_0^{\tau/2}\omega(t)^{1-p'}t^{Q-1}dt\right)^{1/p'}.$$

Therefore $Tf_2(x)$ converges absolutely for all $x : |x|_H \geq \tau$ and thus $Tf(x)$ exists for almost all $x \in H^n$. Assume $\bar{\omega}_1(t)$ to be an arbitrary continuous increasing function on $(0,\infty)$ such that $\bar{\omega}_1(t) \leq \omega_1(t), \bar{\omega}_1(0) = \omega_1(0+)$ and $\bar{\omega}_1(t) = \int_0^t \varphi(\tau)d\tau + \bar{\omega}_1(0)$, $t \in (0,\infty)$ (it is obvious that such $\bar{\omega}_1(t)$ exists; for example, $\bar{\omega}_1(t) = \int_0^t \omega_1'(\tau)d\tau + \omega_1(t)$).

We observe that the condition a) implies

$$\exists C_1 > 0, \ \forall t > 0, \quad \omega_1(t) \leq C_1\omega(t/2). \tag{3}$$

Indeed, from

$$\exists C_2 > 0, \ \forall t > 0,$$

$$\left(\int_t^\infty \varphi(\tau)\tau^{-Q(p-1)}d\tau\right)\left(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{Q-1}d\tau\right)^{p-1} \leq C_2 \tag{4}$$

we obtain (3), since

$$\int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau \geq C\omega_1(t)t^{-Q(p-1)},$$

$$\left(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{Q-1}d\tau\right)^{p-1} \leq C\omega(t/2)^{-1}t^{Q(p-1)}$$

and, besides,

$$\frac{1}{Q(p-1)}\int_t^\infty \varphi(\tau)\tau^{-Q(p-1)}d\tau = \int_t^\infty \varphi(\tau)d\tau \int_\tau^\infty \lambda^{-1-Q(p-1)}d\lambda =$$

$$= \int_t^\infty \lambda^{-1-Q(p-1)}d\lambda \int_t^\lambda \varphi(\tau)d\tau \leq \int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau.$$

We have

$$\|Tf\|_{L_{p,\bar{\omega}_1}(H^n)} \leq \left(\int_{H^n} |Tf(x)|^p dx \int_0^{|x|_H} \varphi(t)dt\right)^{1/p} +$$

$$+\left(\bar{\omega}_1(0)\int_{H^n} |Tf(x)|^p dx\right)^{1/p} = A_1 + A_2.$$

If $\omega(0+) > 0$, then $L_p(H^n, \omega(|x|_H)) \subset L_p(H^n)$, and if $\omega(0+) = 0$, then $\bar{\omega}(t) \leq \omega_1(t) \leq C\omega(t/2)$ implies $\bar{\omega}_1(0) = 0$. Therefore in the case $\omega(0+) = 0$ we have $A_2 = 0$.

If $\omega(0) > 0$, then $f \in_{L_p(\mathbb{R}^n)}$ and we have

$$A_2 \leq C\left(\bar{\omega}_1(0)\int_{H^n} |f(x)|^p dx\right)^{1/p} \leq C\left(\int_{H^n} |f(x)|^p \omega_1(|x|_H)dx\right)^{1/p} \leq$$

$$\leq C\|f\|_{L_p(H^n, \omega(|x|_H))}.$$

Now we can write

$$A_1 \le \Big(\int_0^\infty \varphi(t)dt \int\limits_{|x|_H > t} |Tf(x)^p dx \Big)^{1/p} \le A_{11} + A_{12}.$$

where

$$A_{11}^p = \int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t} \Big| \int\limits_{|y|_H>t/2} K(x, y^{-1}) f(y) dy \Big|^p dx,$$

$$A_{12}^p = \int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t} \Big| \int\limits_{|y|_H<t/2} K(x, y^{-1}) f(y) dy \Big|^p dx.$$

The relation

$$\int\limits_{|y|_H>t/2} |f(y)|^p dy \le \frac{1}{\omega(t/2)} \int\limits_{|y|_H>t/2} |f(y)|^p \omega(|y|_H) dy$$

implies $f \in L_p(\{y \in H^n : |y|_H > t\})$ for any $t > 0$.

Hence, on account of (3), we have

$$A_{11} \le C\Big(\int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t/2} |f(x)|^p dx \Big)^{1/p} =$$

$$= C\Big(\int\limits_{H^n} |f(x)|^p dx \int_0^{2|x|_H} \varphi(t)dt \Big)^{1/p} \le$$

$$\le C\Big(\int\limits_{H^n} |f(x)|^p \omega_1(2|x|_H) dx \Big)^{1/p} \le$$

$$\le C \|f\|_{L_{p,\omega(|x|_H)}}(H^n).$$

Obviously, if $|x|_H > t$, $|y|_H < t/2$, then $\frac{1}{2}|x|_H \le |y^{-1}x|_H \le \frac{3}{2}|x|_H$. Therefore

$$\int\limits_{|x|_H>t} \Big| \int\limits_{|y|_H<t/2} K(xy^{-1}) f(y) dy \Big|^p dx \le$$

$$\le C(K) \int\limits_{|x|_H>t} \Big(\int\limits_{|y|_H<t/2} |xy^{-1}|_H^{-Q} |f(y)| dy \Big)^p dx \le$$

$$\le 2^{Qp} C(K) \int\limits_{|x|_H>t} |x|_H^{-Qp} dx \Big(\int\limits_{|y|_H<t/2} |f(y)| dy \Big)^p.$$

Taking the H-polar coordinates $x = \delta_\varrho \bar{x}$, $\varrho = |x|_H$, $\bar{x} \in S_H$ we can write

$$\int\limits_{|x|_H>t} |x|_H^{-Qp} dx = \int\limits_{S_H} d\sigma(\bar{x}) \int_0^\infty \varrho^{Q-1-Qp} d\varrho = Ct^{Q-Qp}.$$

For $\alpha > Q(1+\frac{1}{p'})$, by virtue of the Hölder inequality, we have

$$\int\limits_{|y|_H<t/2} |f(y)|dy = \alpha \int\limits_{S_H} d\sigma(\bar{y}) \int_0^{t/2} \varrho^{Q-\alpha-1} |f(\delta_\varrho \bar{y})| d\varrho \int_0^\varrho s^{\alpha-1} ds =$$

$$= \alpha \int_0^{t/2} s^{\alpha-1} ds \int\limits_{s<|y|_H<t/2} |f(y)|\, |y|_H^{-\alpha} dy \leq$$

$$\leq \int_0^{t/2} s^{\alpha-1} ds \Big(\int\limits_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} \times$$

$$\times \Big(\int\limits_{s<|y|_H<t/2} |y|_H^{(Q-\alpha)p'} dy \Big)^{1/p'} \leq$$

$$\leq C \int_0^{t/2} s^{Q+\frac{Q}{p'}} \Big(\int\limits_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds.$$

Consequently

$$A_{12} \leq C \Big\{ \int_0^\infty \varphi(2t) t^{-Q(p-1)} \times$$

$$\times \Big[\int_0^t s^{Q(1+\frac{1}{p'})} \Big(\int\limits_{|y|_H \geq s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds \Big]^p dt \Big\}^{1/p}.$$

By (4) and Theorem 2

$$A_{12} \leq C \Big[\int_0^\infty s^{Qp(1+\frac{1}{p'})} \Big(\int\limits_{|y|_H>s} |f(y)|^p \times$$

$$\times |y|^{-Qp} dy \Big) \omega(s) s^{-(Q-1)(p-1)} ds \Big]^{1/p} =$$

$$= C \Big(\int_0^\infty s^{-1+Qp} \omega(s) ds \int\limits_{|y|_H>s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} =$$

$$= C \Big(\int\limits_{H^n} |f(y)|^p |y|_H^{-Qp} \int_0^{|y|_H} \omega(s) s^{-1+Qp} ds \Big)^{1/p} \leq$$

$$\leq C\Big(\int\limits_{H^n} |f(y)|^p \omega(|y|_H)dy\Big)^{1/p}.$$

Hence we obtain (1) for $\omega_1(t) = \bar{\omega}_1(t)$. Now, by the Fatou theorem, the inequality (1) is fulfilled. ■

Theorem 3 was earlier announced in [11].

A similar reasoning can be used to prove the analogue of Theorem 3 for the operator $T_\alpha : f \to T_\alpha f$ where

$$T_\alpha f(x) = \int\limits_{H^n} |xy^{-1}|_H^{\alpha-Q} f(y)dy, \quad 0 < \alpha < Q.$$

Namely, we have

Theorem 4. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ *and the weights* (ω, ω_1) *be monotone positive functions on* $(0, \infty)$. *Then the inequality*

$$\Big(\int\limits_{H^n} |T_\alpha f(x)|^q \omega_1(|x|_H)dx\Big)^{1/q} \leq$$
$$\leq C\Big(\int\limits_{H^n} |f(x)|^p \omega(|x|_H)dx\Big)^{1/p}$$

holds if and only if $(\omega, \omega_1) \in \widetilde{A}_{p,q}(Q)$.

Remark. In the case of a homogeneous group the analogue of Theorem 4 is also valid (see [12]).

For monotone weights one can find the weighted L_p-estimates for a Calderon–Zygmund operator in [13] and [14], and for the anisotropic case in [15].

As known [16], if $f \in C_0^\infty(H^n)$, then the function

$$g(x) = C_n \int\limits_{H^n} |xy^1|_H^{-2n} f(y)dy$$

is a solution of the equation $L_0 g = f$, where $L_0 = -\sum_{i=1}^{2n} X_j^2$. In particular, our results lead to

Theorem 5. *Let* $1 < p < \infty$, $(\omega, \omega_1) \in \widetilde{A}(Q)$, $f \in L_p(H^n, \omega(|x|_H))$ *and* $L_0(g) = f$. *Then*

$$\|X_0 g\|_{L_p(H^n, \omega_1(|x|_H))} \leq c\|f\|_{L_p(H^n, \omega(|x|_H))},$$
$$\|X_i X_j g\|_{L_p(H^n, \omega_1(|x|)_H)} \leq C\|f\|_{L_p(H^n, \omega(|x|_H))},$$
$$i, j = 1, 2, \dots, 2n.$$

Theorem 6. *Let* $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$, $(\omega, \omega_1) \in \tilde{A}_{pq}(Q)$, $f \in L_p(H^n, \omega(|x|_H))$ *and* $L_0 g = f$. *Then*

$$\|X_i g\|_{L_q(H^n, \omega_1(|x|_H))} \leq C \|f\|_{L_p(H^n, \omega(|x|_H))}, \quad i = 1, 2, \dots, 2n.$$

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SINGULAR INTEGRAL OPERATORS ON MANIFOLDS WITH A BOUNDARY

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ABSTRACT. The paper deals with the questions of singular integral operators being bounded, completely continuous and Noetherian on manifolds with a boundary in weighted Hölder spaces.

რეზიუმე. ნაშრომში შესწავლილია საზღვრიან მრავალსახეობებზე გავრცელებული სინგულარული ინტეგრალური ოპერატორების შემოსაზღვრულობის, სავსებით უწყვეტობისა და ნეტერისეულობის საკითხები ჰელდერის წონიან სივრცეებში.

We shall investigate the matrix singular operator

$$A(u)(x) = a(x)u(x) + \int_D f\Big(x, \frac{x-y}{|x-y|}\Big)|x-y|^{-m}u(y)dy,$$
$$x \in D, \quad D \subset \mathbb{R}^m,$$

in weighted Hölder spaces and develop the results obtained in [1] for one-dimensional singular operators and in [2–8] for multidimensional singular operators in Lebesgue spaces.

The paper consists of two sections. In Section I we shall prove the theorems of integral operators being bounded and completely continuous in Hölder spaces with weight. Section II will contain the proof of the theorem of factorization of matrix-functions and present the theorem of singular operators being Noetherian in weighted spaces.

1. Let $\mathbb{R}^m$ $(m \geq 2)$ be an m-dimensional Euclidean space, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$ be points of the space $\mathbb{R}^m$,

$$|x| = \Big(\sum_{i=1}^m x_i^2\Big)^{\frac{1}{2}}, \quad \Gamma = \{x : x \in \mathbb{R}^m, \ x_m = 0\},$$
$$\mathbb{R}^m_+ = \{x : x \in \mathbb{R}^m, \ x_m > 0\}, \quad x' = (x_1, \dots, x_{m-1}),$$
$$B(x, a) = \{y : y \in \mathbb{R}^m, \ |y - x| < a\},$$

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$$S(x,a) = \{y : \ y \in \mathbb{R}^m, \ |y - x| = a\}.$$

Definition 1. A function u defined on $\mathbb{R}^m \backslash \Gamma$ belongs to the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)$ $(0 < \nu, \ \alpha < 1, \ \beta \geq 0, \ \alpha + \beta < m)$ if

$$\text{(i)} \quad \forall x \in \mathbb{R}^m \backslash \Gamma \quad |u(x)| \leq c|x_m|^{-\alpha}(1 + |x|)^{-\beta},$$

$$\text{(ii)} \quad \forall x \in \mathbb{R}^m \backslash \Gamma, \quad \forall y \in B(x, \frac{1}{2}|x_m|)$$

$$|u(x) - u(y)| \leq c|x_m|^{-(\alpha+\nu)}(1 + |x|)^{-\beta}|x - y|^\nu.$$

The norm in the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)$ is defined by the equality

$$\|u\| = \sup_{x \in \mathbb{R}^m \backslash \Gamma} |x|^\alpha (1 + |x|)^\beta |u(x)| +$$

$$+ \sup_{\substack{x \in \mathbb{R}^m \backslash \Gamma \\ y \in B(x, \frac{1}{2}|x_m|)}} |x_m|^{\alpha+\nu}(1 + |x|)^\beta \frac{|u(x) - u(y)|}{|x - y|^\nu}.$$

The space $H^\nu_{\alpha,\beta}(\mathbb{R}^m_+)$ is defined similarly.

Note that if $y \in B(x, \frac{1}{2}|x_m|)$, then

$$|y| \leq \frac{3}{2}|x|, \quad |y| \geq \frac{1}{2}|x|; \quad |y_m| \leq \frac{3}{2}|x_m|, \quad |y_m| \geq \frac{1}{2}x_m|. \tag{1}$$

Thus for $y \in B(x, \frac{1}{2}x_m)$ we have $|x| \sim |y|$, $|x_m| \sim |y_m|$.

Let $x, y \in \mathbb{R}^m \backslash \Gamma$ and $|x - y| \geq \frac{1}{2}\min(|x_m|, |y_m|)$. Then the condition (i) implies

$$|u(x) - u(y)| \leq c(\min(|x_m|, |y_m|))^{-\alpha}(\min(1 + |x|, 1 + |y|))^{-\beta} \leq$$
$$\leq c|x - y|^\nu (\min(|x_m|, |y_m|)^{-\alpha-\nu}(\min(1 + |x|, 1 + |y|)^{-\beta};$$

and therefore the condition (ii) can be replaced by the condition

$$\forall x, y \in \mathbb{R}^m \backslash \Gamma$$

$$|u(x) - u(y)| \leq c|x - y|^\nu(\min(|x_m|, |y_m|)^{-\alpha-\nu}(\min(1 + |x|, 1 + |y|)^{-\beta}.$$

One can easily prove that the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m \backslash \Gamma)[H^\nu_{\alpha,\beta}(\mathbb{R}^m_+)]$ is the Banach one.

Consider the singular integral operator

$$v(x) = A(u)(x) = \int_{\mathbb{R}^m} K(x, x - y)u(y)dy,$$

where

$$K(x,z) = f\Big(x, \frac{z}{|z|}\Big)|z|^{-m}.$$

Theorem 1. *Let the characteristic f defined on $(\mathbb{R}^m\backslash\Gamma)\times(S(0,1)\backslash\Gamma)$ satisfy the conditions*

(a) $\forall x \in \mathbb{R}^m\backslash\Gamma \quad \int_{S(0,1)} f(x,z)d_zS = 0;$

(b) $\forall x \in \mathbb{R}^m\backslash\Gamma, \ \forall z \in S(0,1)\backslash\Gamma \quad |f(x,z)| \le c|z_m|^{-\sigma} \quad (0 \le \sigma \le \alpha);$

(c) $\forall x, y \in \mathbb{R}^m\backslash\Gamma, \ \forall z, \theta, \omega \in S(0,1)\backslash\Gamma$

$$|f(x,z) - f(y,z)| \le c|x-y|^{\nu}(\min(|x_m|,|y_m|))^{-\nu}|z_m|^{-\sigma}$$
$$|f(x,\theta) - f(x,\omega)| \le c|\theta-\omega|^{\nu_1}(\min(|\theta_m|,|\omega_m|))^{-\nu_1-\sigma}$$
$$\nu_1 > \nu, \quad \nu_1 + \sigma < 1.$$

Then the operator A is bounded in the space $H_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$.

Proof. In the first place note that the second inequality of the condition (c) yields the inequality

$$\forall x, y, z \in \mathbb{R}^m\backslash\Gamma$$
$$\left|f\left(x,\frac{y}{|y|}\right) - f\left(x,\frac{z}{|z|}\right)\right| \le c|y-z|^{\nu_1}\left(\frac{|y|^{\sigma}}{|y_m|^{\nu_1+\sigma}} + \frac{|z|^{\sigma}}{|z_m|^{\nu_1+\sigma}}\right). \quad (2)$$

Set

$$D_1 = B(x,\tfrac{1}{2}|x_m|), \quad D_2 = B(x,\tfrac{1}{2}(1+|x|))\backslash D_1,$$
$$D_3 = B(0,\tfrac{1}{2}(1+|x|))\backslash(D_1\cup D_2), \quad D_4 = \mathbb{R}^m\backslash(D_1\cup D_2\cup D_3), \quad (3)$$
$$D = \{y: \ (y',y_m)\in\mathbb{R}^m, \ |y_m| \le 2(|y'|+1)\}.$$

We have

$$v(x) = \int_{D_1} K(x,x-y)[u(y)-u(x)]dy + \sum_{i=2}^{4}\int_{D_i} K(x,x-y)u(y)dy \equiv$$
$$\equiv \sum_{i=1}^{4} I_i(x).$$

By virtue of the condition (ii) and the inequalities (1)

$$|I_1(x)| \le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}\int_{D_1}\left|f\left(\frac{x-y}{|x-y|}\right)\right||x-y|^{\nu-m}dy\|u\| \le$$
$$\le c|x_m|^{-\alpha}(1+|x|)^{-\beta}\int_{S(0,1)}|f(z)|d_zS\|u\|. \quad (4)$$

If $y \notin D_1$, then

$$|x-y| \ge \frac{1}{6}(|x'-y'| + |x_m| + |y_m|)$$

and if $y \in D_2$, then $1+|y| \geq 1+|x|-|x-y| \geq \frac{1+|x|}{2}$. Therefore due to the conditions (i) and (b)

$$|I_2(x)| \leq c(1+|x|)^{-\beta} \int_{D_2} \frac{|y_m|^{-\alpha}|y_m - x_m|^{-\sigma}}{(|x'-y'|+|x_m|+|y_m|)^{m-\sigma}} dy \|u\|.$$

After performing the spherical transformation of $y'-x'$, we obtain

$$|I_2(x)| \leq c(1+|x|)^{-\beta} \int_0^\infty r^{m-2} dr \int_{-\infty}^\infty \frac{|y_m|^{-\alpha}|x_m - y_m|^{-\sigma}}{(r+|x_m|+|y_m|)^{m-\sigma}} dy_m \|u\|.$$

The transformation of $r=|x_m|\widetilde{r}$, $y_m=|x_m|\widetilde{y}_m$ leads to

$$|I_2(x)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta} \int_{-\infty}^\infty |y_m|^{-\alpha}|y_m - \operatorname{sign} x_m|^{-\sigma} dy_m \times$$
$$\times \int_0^\infty (r+|y_m|+1)^{\sigma-2} dr \|u\| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta} \times$$
$$\times \int_{-\infty}^\infty |y_m|^{-\alpha}|y_m - \operatorname{sign} x_m|^{-\sigma}(1+|y_m|)^{\sigma-1} dy_m \|u\|. \tag{5}$$

The term $I_4(x)$ is evaluated in the same manner, since in that case, too,

$$1+|y| \geq \frac{1}{2}(|x|+1).$$

Represent $I_3(x)$ in the form

$$I_3(x) = \int_{D_3 \cap B(0,\frac{1}{2}|x_m|)} K(x,x-y)u(y)dy +$$
$$+ \int_{D_3 \setminus B(0,\frac{1}{2}|x_m|)} K(x,x-y)u(y)dy \equiv J_1(x)+J_2(x).$$

If $y \in D_3$, then $|x-y| \geq \frac{1}{2}(1+|x|) \geq |y|$; if $y \notin B(0,\frac{1}{2}|x_m|)$, then $|y| \geq \frac{|x_m|}{2}$ and hence $|y| \geq \frac{1}{4}(|y'|+|x_m|+|y_m|)$. Therefore, in evaluating $J_2(x)$, we shall have

$$|J_2(x)| \leq c(1+|x|)^{-\beta} \times$$
$$\times \int_{D_3 \setminus B(0,\frac{1}{2}|x_m|)} |y_m|^{-\alpha}|x_m - y_m|^{-\sigma}(|y'|+|x_m|+|y_m|)^{\sigma-m} dy \|u\|.$$

After performing the spherical transformation of y', we obtain, as in the case of evaluating $I_2(x)$,

$$|J_2(x)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{6}$$

Write $J_1(x)$ in the form

$$J_1(x) = \int_{D_3 \cap B(0,\frac{1}{2}|x_m|)\cap D} K(x, x-y)u(y)dy +$$

$$+ \int_{D_3 \cap B(0,\frac{1}{2}|x_m|)\cap(\mathbb{R}^m \setminus D)} K(x, x-y)u(y)dy \equiv J_1'(x) + J_1''(x).$$

If $y \in B(0, \frac{1}{2}|x_m|)$, then $|x_m - y_m| \geq |x_m| - |y_m| \geq \frac{1}{2}|x_m|$. We have

$$|J_1'(x)| \leq c(1+|x|)^{-m+\sigma}|x_m|^{-\sigma} \int_{|y'|\leq\frac{1}{2}(1+|x|)} (1+|y'|)^{-\beta}dy' \times$$

$$\times \int_{|y_m|\leq 2(|y'|+1)} |y_m|^{-\alpha}dy_m\|u\| \leq$$

$$\leq c|x_m|^{-\sigma}(1+|x|)^{\sigma-m} \int_{|y'|\leq\frac{1}{2}(1+|x|)} (1+|y'|)^{1-\beta-\alpha}dy'\|u\| \leq$$

$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}|x_m|^{\alpha-\sigma}(1+|x|)^{\beta+\sigma-m}\left(c+(1+|x|)^{m-(\beta+\alpha)}\right)\|u\| \leq$$

$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|, \tag{7}$$

since $\sigma \leq \alpha$, $\alpha + \beta < m$.

If $y \in \mathbb{R}^m \setminus D$, then $1 + |y| < 1 + |y'| + |y_m| < \frac{3}{2}|y_m|$. Therefore

$$|J_1''(x)| \leq c|x_m|^{-\sigma}(1+|x|)^{\sigma-m} \int_{|y|\leq\frac{1}{2}(1+|x|)} (1+|y|)^{-\beta-\alpha}dy\|u\| \leq$$

$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{8}$$

From the estimates (4)–(8) we obtain

$$|v(x)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\|u\|. \tag{9}$$

Let us evaluate the difference $v(x) - v(z)$. It is assumed that $|x-z| \leq \frac{1}{8}|x_m|$. Then $|z| \sim |x|$, $|z_m| \sim |x_m|$.

We introduce the set

$$\widetilde{D}_1 = B(x, 2|x-z|), \quad \widetilde{D}_2 = B(z, 3|x-z|), \quad \widetilde{D}_3 = B(z, \tfrac{1}{2}|x_m| - |x-z|).$$

Clearly,

$$\widetilde{D}_1 \subset \widetilde{D}_2 \subset \widetilde{D}_3 \subset B(x, \tfrac{1}{2}|x_m|) = D_1.$$

We have the representation

$$v(x)-v(z)=\int_{\mathbb{R}^m}[K(x,x-y)-K(z,z-y)]u(y)dy=$$
$$=\int_{\mathbb{R}^m}[K(x,x-y)-K(x,z-y)]u(y)dy+$$
$$+\int_{\mathbb{R}^m}[K(z,x-y)-K(z,z-y)]u(y)dy\equiv$$
$$\equiv I_1(x,z)+I_2(x,z).$$

By virtue of the first inequality of the condition (c) the term $I_1(x,z)$ is evaluated exactly in the same manner as $v(x)$ and we obtain the estimate

$$|I_1(x,y)|\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{10}$$

Rewrite the term $I_2(x,z)$ in the form

$$I_2(x,z)=\int_{D_1}[K(z,x-y)-K(z,z-y)]u(y)dy+$$
$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy=$$
$$=\int_{D_1}K(z,x-y)[u(y)-u(x)]dy-\int_{D_1\setminus\widetilde{D}_3}[K(z,z-y)u(y)dy-$$
$$-\int_{\widetilde{D}_3\setminus\widetilde{D}_2}K(z,z-y)[u(y)-u(x)]dy-\int_{\widetilde{D}_2}K(z,z-y)[u(y)-u(z)]dy+$$
$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy=$$
$$=\left(\int_{\widetilde{D}_2}+\int_{D_1\setminus\widetilde{D}_3}\right)K(z,x-y)[u(y)-u(x)]dy-$$
$$-\int_{\widetilde{D}_2}K(z,z-y)[u(y)-u(z)]dy-$$
$$-\int_{\widetilde{D}_3\setminus\widetilde{D}_2}[K(z,x-y)-K(z,z-y)][u(y)-u(x)]dy-$$
$$-\int_{D_1\setminus\widetilde{D}_3}K(z,z-y)u(y)dy+$$
$$+\int_{\mathbb{R}^m\setminus D_1}[K(z,x-y)-K(z,z-y)]u(y)dy\equiv\sum_{i=1}^{5}J_i(x,z).$$

In evaluating $J_1(x,z)$, note that

$$\widetilde{D}_2\subset B(x,4|x-z|),\quad B(x,\tfrac{1}{2}|x_m|-2|x-z|)\subset\widetilde{D}_3.$$

Therefore by virtue of the condition (b) and the inequalities (1) we obtain

$$|J_1(x,z)| \leq \int_{B(x,4(x-z))} |K(z,x-y)||u(y)-u(x)|dy +$$
$$+ \int_{D_1\setminus B(x,\frac{1}{2}|x_m|)-2|x-z|)} |K(z,x-y)||u(y)-u(x)|dy \leq$$
$$\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|, \tag{11}$$

since

$$\left(\tfrac{1}{2}|x_m|\right)^{\nu} - \left(\tfrac{1}{2}|x_m| - 2|z-z|\right)^{\nu} \leq c|x-z|^{\nu}.$$

Similarly, if $y \in \widetilde{D}_2$, then

$$|y-z| \leq 3|x-z| \leq \frac{3|x_m|}{8} \leq \frac{3}{7}|z_m|.$$

Therefore

$$|J_2(x,z)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{12}$$

It is clear that $B(x,\frac{1}{2}|x_m|) \subset B(z,\frac{1}{2}|x_m|+|x-z|)$ and hence

$$|J_4(x,z)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\int_{D_1\setminus\widetilde{D}_3} |K(z,z-y)|dy\|u\| \leq$$
$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\ln\frac{|x_m|+2|x-z|}{|x_m|-2|x-z|}\|u\| \leq$$
$$\leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}\frac{|x-z|}{|x_m|-2|x-z|}\|u\| \leq$$
$$\leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{13}$$

Note that if $y \notin \widetilde{D}_2$, then

$$|x-y| > 2|x-z|, \quad |z-y| > 3|x-z|,$$
$$|x-y| < |x-z| + |z-y| < \frac{4}{3}|z-y|, \quad |z-y| < \frac{3}{2}|x-y|.$$

Taking these inequalities into account, the inequality (2) readily implies that for $y \notin \widetilde{D}_2$

$$|K(z,x-y) - K(z,z-y)| \leq$$
$$\leq c\frac{|x-z|^{\nu_1}}{|x-y|^{m+\nu_1}}\left(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}} + \frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\right), \tag{14}$$

using which we obtain

$$|J_3(x,z)| \le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1} \times$$
$$\times \int_{\widetilde{D}_3\setminus\widetilde{D}_2} \frac{1}{|x-y|^{m+\nu_1-\nu}}\left(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}} + \frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\right)dy\|u\| \le$$
$$\le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1} \times$$
$$\times\left(\int_{D_1\setminus\widetilde{D}_1} \frac{1}{|x-y|^{m+\nu_1-\nu}}\left(\frac{|x-y|}{|x_m-y_m|}\right)^{\nu_1+\sigma} dy +\right.$$
$$\left. + \int_{\widetilde{D}_3\setminus\widetilde{D}_2} \frac{1}{|z-y|^{m+\nu_1-\nu}}\left(\frac{|z-y|}{|z_m-y_m|}^{\nu_1+\sigma}\right)dy\right).$$

Passing to the spherical coordinates and keeping in mind that $\frac{x_m-y_m}{|x-y|}$, $\frac{z_m-y_m}{|z-y|}$ do not depend on the radius, we have

$$|J_3(x,z)| \le$$
$$\le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu_1}\left(|x_m|^{\nu-\nu_1} + |x-z|^{\nu-\nu_1}\right)\|u\| \le$$
$$\le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{15}$$

In deriving the estimate, we took into account that $\nu < \nu_1$, $\nu_1+\sigma < 1$. By virtue of the inequality (14)

$$|J_5(x,z)| \le c|z-x|^{\nu_1}\left(\int_{\mathbb{R}^m\setminus D_1} |x-y|^{\sigma-m}|x_m-y_m|^{-\sigma-\nu_1}|u(y)|dy +\right.$$
$$\left. + \int_{\mathbb{R}^m\setminus B(z,\frac{1}{3}|z_m|)} |z-y|^{\sigma-m}|z_m-y_m|^{-\sigma-\nu_1}|u(y)|dy\right).$$

We evaluate the obtained integral expression by the same technique as was used to evaluate the integral expression

$$\int_{\mathbb{R}^m\setminus D_1} |K(x,x-y)||u(y)|dy$$

(see the estimates (5)–(8)) and finally obtain

$$|J_5(x,z)| \le c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{16}$$

The estimates (10)–(13), (15), (16) show that

$$|v(x)-v(z)| \le c|x_m^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|,$$

which, with the equality (9) taken into account, proves the theorem. ■

Corollary 1. *In the conditions of Theorem* 1 *the operator*

$$\int_{\mathbb{R}^m_+} K(x, x-y)u(y)dy$$

is bounded, when acting from the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m_+)$ *into the space* $H^\nu_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$.

Definition 2. Let M be a closed set in $\mathbb{R}^m$. The set M is called an $(m-1)$-dimensional manifold without a boundary of the class $C^{1,\delta}$ $(0 \le \delta \le 1)$, if for each $x \in M$ there exist a positive number r_x and a neighbourhood $Q(x)$ of the point x in $\mathbb{R}^m$, which is mapped by means of the orthogonal transform T_x onto the cylinder

$$\Omega_0 = \{\xi : \xi \in \mathbb{R}^m,\ |\xi'| < r_x,\ |\xi_m| < r_x\}$$

and if the following conditions are fulfilled:
$T_x(x) = 0$, the set $T_x(M \cap Q(x))$ is given by the equation $\xi_m = \varphi_x(\xi')$, $|\xi'| < r_x$; $\varphi_x \in C^{1,\delta}$ in the domain $|\xi'| < r_x$ and $\partial_{\xi_i}\varphi_x(0) = 0$, $i = 1, \dots, m-1$.

Clearly, $Q(x)$ is the cylinder to be denoted by $C(x, r_x)$.

In what follows the manifold M will be assumed compact.

We introduce the notation

$$d(x) \equiv d(x, M) = \inf_{y \in M} |x-y|, \quad M(\tau) = \{x \in \mathbb{R}^m, d(x) < \tau\}.$$

Note some properties of the function $d(x)$:

$$\begin{gathered} d(x) \le c(1+|x|), \quad |d(x)-d(y)| \le c|x-y|; \\ \forall x \in \mathbb{R}^m\backslash M,\ \forall y \in B(x, \tfrac{1}{2}d(x)) \\ d(y) \le \frac{3}{2}d(x) \le 3d(y), \quad 1+|x| \sim 1+|y|; \end{gathered} \tag{17}$$

$$\begin{gathered} \forall x \in M,\ \forall y \in C(x, \tfrac{1}{3}r_x) \\ d(y, M) = d(y, M \cap C(x, r_x)) \ \text{ and if } \ y = T_x^{-1}(\eta), \ \text{ then} \\ d(y) \le |\eta_m - \varphi_x(\eta')| \le 2(1+a_x)d(y), \end{gathered} \tag{18}$$

where a_x is the Lipshitz constant of the function φ_x.

Definition 3. A function u defined on $\mathbb{R}^m\backslash M$ belongs to the space $H^\nu_{\alpha,\beta}(\mathbb{R}^m\backslash M)$ $(0 < \nu, \alpha < 1,\ \beta \ge 0,\ \alpha+\beta < m)$, if:

(i) $\forall x \in \mathbb{R}^m\backslash M \quad |u(x)| \le c d^{-\alpha}(x)(1+|x|)^{-\beta}$,

(ii) $\forall x \in \mathbb{R}^m\backslash M, \quad \forall y \in B(x, \tfrac{1}{2}d(x))$

$$|u(x)-u(y)| \le c d^{-(\alpha+\nu)}(x)(1+|x|)^{-\beta}|x-y|^\nu.$$

The norm in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash M)$ is defined by the equality

$$\|u\| = \sup_{x\in\mathbb{R}^m\backslash M} d^{\alpha}(x)(1+|x|)^{\beta}|u(x)| +$$
$$+ \sup_{\substack{x\in\mathbb{R}^m\backslash M \\ y\in B(x,\frac{1}{2}d(x))}} d^{\nu+\alpha}(x)(1+|x|)^{\beta}\frac{|u(x)-u(y)|}{|x-y|^{\nu}}.$$

The space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash M)$ is the Banach one.

Theorem 2. *Let $M\in C^{1,\delta}$, the characteristic f of the singular operator A be defined on $(\mathbb{R}^m\backslash M)\times S(0,1)$ and satisfy the conditions:*

(a) $\forall x\in\mathbb{R}^m\backslash M$, $\forall z\in S(0,1)$

$$|f(x,z)|\le c, \quad \int_{S(0,1)} f(x,z)d_zS = 0;$$

(b) $\forall x,y\in\mathbb{R}^m\backslash M$, $\forall\theta,\omega\in S(0,1)$

$$|f(x,\theta)-f(y,\theta)|\le c|x-y|^{\nu}\Big(\min(d(x),d(y))\Big)^{-\nu},$$
$$|f(x,\theta)-f(x,\omega)|\le c|\theta-\omega|^{\nu_1}, \quad \nu<\nu_1.$$

Then the operator A is bounded in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash M)$.

Proof. Let $M\subset B(0,r_0)$ and e_1 be an infinitely differentiable function such that $e_1(x)=1$ for $|x|\le r_0+1$, $e_1(x)=0$ for $|x|\ge r_0+2$. Setting $e_2=1-e_1$, we have

$$v(x) = \int_{\mathbb{R}^m} K(x,x-y)e_1(y)u(y)dy +$$
$$+\int_{\mathbb{R}^m} K(x,x-y)e_2(y)u(y)dy \equiv v_1(x)+v_2(x).$$

Let us evaluate the integral

$$v_1(x) = \int_{\mathbb{R}^m} K(x,x-y)u_1(y)dy \quad (u_1=e_1u).$$

Choose a constant r^* $(0<r^*<1)$ such that the system $\{C(\overset{i}{x},\frac{1}{4}r^*)\}_{i=1}^{l}$ ($\overset{i}{x}$, $i=1,\dots,e$, are points of the manifold M) cover the manifold M and $C(\overset{i}{x},4r^*)$, $i=1,\dots,e$, be again the coordinate neighbourhoods.

We introduce the sets

$$D_1 = B(x,\tfrac{1}{2}d(x)), \quad D_2 = (B(x,\tfrac{1}{4}r^*)\backslash D_1)\cap B(0,r_0+2)$$
$$D_3 = B(0,r_0+2)\backslash(D_1\cup D_2), \quad D_4=\{y:\ y\in\mathbb{R}^m,\ d(y)<\tfrac{1}{2}r^*\}$$

Now

$$v_1(x) = \int_{D_1} K(x, x-y)[u_1(y) - u_1(x)]dy +$$
$$+ \sum_{i=2}^{3} \int_{D_i} K(x, x-y) u_1(y) dy \equiv \sum_{i=1}^{3} I_i(x).$$

By virtue of the inequality (17) and the condition (ii) we obtain

$$|I_1(x)| \leq c d^{-\alpha}(x)(1+|x|)^{-\beta}\|u_1\| \leq c d^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \quad (19)$$

Next,

$$|I_2(x)| \leq \int_{D_2 \cap D_4} |K(x, x-y)||u_1(y)|dy +$$
$$+ \int_{D_2 \setminus D_4} |K(x, x-y)||u_1(y)|dy \equiv I_2'(x) + I_2''(x).$$

If $y \in D_2$, then

$$1 + |x| \sim 1 + |y| \sim c.$$

Moreover, $d(y) \leq \frac{1}{2} r^*$ for $y \in D_2 \cap D_4$ and therefore there exists i ($i = 1, \ldots, e$) such that $x, y \in C(\overset{i}{x}, \frac{3+\sqrt{2}}{2} r^*)$. Let $y = T_{\underset{x}{i}}^{-1}(\eta)$, $x = T_{\underset{x}{i}}^{-1}(\xi)$. By virtue of the inequality (18)

$$|\eta_m - \varphi_{\underset{x}{i}}(\eta')| \leq 2(1 + a_{\underset{x}{i}})d(y), \quad |\xi_m - \varphi_{\underset{x}{i}}(\xi')| \leq 2(1 + a_{\underset{x}{i}})d(x).$$

Taking into account that $|x - y| > \frac{1}{2} d(x)$, we therefore obtain

$$|x - y| = |\xi - \eta| \geq \frac{1}{4}(|\xi' - \eta'| + |\xi_m - \eta_m| + d(x)) \geq$$
$$\geq \frac{1}{16}(1 + a_{\underset{x}{i}})^{-1}\big(4(1 + a_{\underset{x}{i}})|\xi' - \eta'| + |\xi_m - \eta_m| + 4(1 + a_{\underset{x}{i}})d(x)\big) \geq$$
$$\geq \frac{1}{16}(1 + a_{\underset{x}{i}})^{-1}\big(|\xi' - \eta'| + |\eta_m - \varphi_{\underset{x}{i}}(\eta')| + d(x)\big), \quad (20)$$

since

$$|\xi_m - \eta_m| \geq |\eta_m - \varphi_{\underset{x}{i}}(\eta')| - |\varphi_{\underset{x}{i}}(\eta') - \varphi_{\underset{x}{i}}(\xi')| - |\xi_m - \varphi_{\underset{x}{i}}(\xi')|.$$

Thus

$$|I_2'(x)| \leq c \int\limits_{|\xi' - \eta'| \leq 4r^*} d\eta' \int_{-2r^*}^{2r^*} |\eta_m - \varphi_{\underset{x}{i}}(\eta')|^{-\alpha} \times$$
$$\times \big(|\xi' - \eta| + |\eta_m - \varphi_{\underset{x}{i}}(\eta')| + d(x)\big)^{-m} d\eta_m \|u\|.$$

Using the transform $\eta_m - \varphi_{\frac{i}{x}}(\eta') = \tilde{\eta}_m$, we obtain

$$|I_2'(x)| \leq c\int_0^{4r^*} r^{m-2}dr \int_{-4r^*}^{4r^*} |\eta_m|^{-\alpha}\Big(r + |\eta_m| + d(x)\Big)^{-m} d\eta_m \|u\|,$$

which, upon applying the transform $r = d(x)\tilde{r}$, $\eta_m = d(x)\tilde{\eta}_m$ gives

$$|I_2'(x)| \leq cd^{-\alpha}(x)\|u\| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{21}$$

If $y \in D_2 \backslash D_4$, then $d(y) \geq \frac{1}{2}r^*$, $\frac{1}{2}d(x) \leq |x-y| < \frac{1}{4}r^*$ by virtue of which

$$d(x) \geq d(y) - |x-y| \geq \tfrac{1}{4}r^*, \quad |x-y| \geq \tfrac{1}{8}r^*.$$

Therefore

$$|I_2''(x)| \leq c\|u\| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{22}$$

Finally, if $y \in D_3$, then $1+|x| \leq 1+|y|+|x-y| \leq c|x-y|$. Therefore

$$|I_3(x)| \leq c(1+|x|)^{-m}\|u\| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{23}$$

The inequalities (19), (21)–(23) show that

$$|v_1(x)| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{24}$$

In evaluating the difference $v_1(x) - v_1(z)$, it will be assumed that $|x-z| < \frac{1}{8}d(x)$. Then $1+|z| \sim 1+|x|$, $d(x) \sim d(y)$.

We introduce the set

$$\widetilde{D}_1 = B(x, 2|x-y|), \quad \widetilde{D}_2 = B(z, 3|x-z|), \quad \widetilde{D}_3 = B(z, \tfrac{1}{2}d(x) - |x-z|).$$

Proceeding as in the case of proving Theorem 1, we obtain

$$|v_1(x) - v_1(z)| \leq cd^{-(\alpha+\nu)}(x)(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \tag{25}$$

To evaluate the integral

$$v_2(x) = \int_{\mathbb{R}^m} K(x, x-y)u_2(y)dy \quad (u_2 = e_2u)$$

note that the function u_2 is defined on $\mathbb{R}^m$ and satisfies the conditions of Definition 3, if the function $d(x)$ is replaced by the function $1+|x|$. Therefore, after introducing the sets

$$D_1 = B(x, \tfrac{1}{2}(1+|x|)), \quad D_2 = B(0, 2|x|+1)\backslash D_1, \quad D_3 = \mathbb{R}^m \backslash (D_1 \cup D_2),$$

we readily obtain the estimate

$$|v_2(x)| \leq cd^{-\alpha}(x)(1+|x|)^{-\beta}\|u\|. \tag{26}$$

Now, considering the sets

$$\widetilde{D}_1 = B(x, 2|x-z|), \quad \widetilde{D}_2 = B(z, 3|x-z|),$$
$$\widetilde{D}_3 = B(z, \tfrac{1}{2}(1+|x|) - |x-z|).$$

it is easy to show that

$$|v_2(x) - v_2(z)| \leq cd^{-(\alpha+\nu)}(x)(1+|x|)^{-\beta}|x-z|^{\nu}\|u\|. \qquad (27)$$

The estimates (24)–(27) prove the theorem. ■

The result close to the one presented here is obtained in [9] (see also [10]).

Definition 4. A function u defined on $\mathbb{R}^m$ belongs to the space $H_\lambda^\nu(\mathbb{R}^m)$ $(\nu, \lambda > 0)$, if

$$|u(x)| \leq c(1+|x|)^{-\beta}, \quad |u(x) - u(y)| \leq c|x-y|^{\nu}\rho_{xy}^{-\nu-\lambda},$$

where

$$\rho_{xy} = \min(1+|x|, 1+|y|).$$

Theorem 3. *Let the characteristic f of the singular operator A satisfy the conditions of Theorem 1, assuming that $\sigma < \alpha$ and the first inequality of the condition* (c) *is fulfilled in the strong form*

$$|f(x,z) - f(y,z)| \leq c|x-z|^{\nu_1}(\min(|x_m|, |y_m|))^{\nu_1}|z_m|^{-\sigma}.$$

It is also assumed that $a \in C(\dot{\mathbb{R}}^m)$ $(\dot{\mathbb{R}}^m = \mathbb{R}^m \cup \infty)$ and $(a - a(\infty)) \in H_\delta^{\nu_1}(\mathbb{R}^m)$. Then the integral operator

$$v(x) = B(u)(x) = \int_{\mathbb{R}^m} [a(x) - a(y)]K(x, x-y)u(y)dy$$

is completely continuous in the space $H_{\alpha,\beta}^\nu(\mathbb{R}^m \backslash \Gamma)$.

Proof. From the proof of Theorem 1 it follows that B is the bounded operator from the space $H_{\alpha,\beta}^\nu(\mathbb{R}^m \backslash \Gamma)$ into the space $H_{\alpha-\gamma,\beta+2\gamma}^{\nu+\gamma}(\mathbb{R}^m \backslash \Gamma)$, where γ is an arbitrary positive number satisfying the condition

$$\gamma < \min\{\lambda, \nu_1 - \nu, \alpha - \sigma, \tfrac{1}{2}(m - \beta - \alpha)\}.$$

Indeed, it is clear that

$$|v(x)| \leq \int_{D_1 \cup D_2 \cup D_4} |a(x) - a(y)||K(x, x-y)||u(y)|dy +$$
$$+ \int_{D_3} (|a(x) - a(\infty)| + |a(y) - a(\infty)|)|K(x, x-y)||u(y)|dy.$$

Taking into account that $(a - a(\infty)) \in H_\gamma^\gamma(\mathbb{R}^m)$ and repeating the proof of Theorem 1, we obtain

$$|v(x)| \leq c|x_m|^{\gamma-\alpha}(1+|x|)^{-\beta-2\gamma}\|u\|. \tag{28}$$

Let us now assume that $|x-z| \leq \frac{1}{8}|x_m|$ and evaluate the difference $v(x) - v(z)$. We have

$$\begin{aligned}|v(x)v(y)| \leq \int_{\widetilde{D}_2} |a(x)-a(y)||K(x,x-y)||u(y)|dy + \\ + \int_{\widetilde{D}_2} |a(z)-a(y)||K(z,z-y)||u(y)|dy + \\ + \int_{\mathbb{R}^m\setminus\widetilde{D}_1} |a(x)-a(z)||K(x,x-y)||u(y)|dy + \\ + \int_{\mathbb{R}^m\setminus\widetilde{D}_2} |a(z)-a(y)||K(x,x-y)-K(z,x-y)||u(y)|dy \equiv \\ \equiv \sum_{i=1}^4 I_i(x,z).\end{aligned}$$

Hence

$$\begin{aligned}|I_i(x,z)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta}(1+|x|)^{-\nu_1-\lambda}|x-z|^{\nu_1}\|u\| \leq \\ \leq c|x_m|^{-\alpha-\gamma}(1+|x|)^{-\beta-2\gamma}\|u\| \quad (i=1,2).\end{aligned} \tag{29}$$

Write the term I_3 in the form

$$\begin{aligned}I_3(x,z) = |a(x)-a(z)|\Big(\int_{D_1\setminus\widetilde{D}_1} |K(x,x-y)||u(y)|dy + \\ + \int_{\mathbb{R}^m\setminus\widetilde{D}_1} |K(x,x-y)||u(y)|dy\Big).\end{aligned}$$

This representation gives

$$\begin{aligned}|I_3(x,z)| \leq \\ \leq c|x-z|^{\nu_1}(1+|x|)^{-\nu_1-\lambda}|x_m|^{-\alpha}(1+|x|)^{-\beta}\Big(\ln\frac{|x_m|}{|x-z|}+c_1\Big)\|u\| \leq \\ \leq c|x_m|^{-\alpha-\nu}(1+|x|)^{-\beta-2\gamma}|x-z|^{\nu+\gamma}\|u\|.\end{aligned} \tag{30}$$

To evaluate the integral term I_4 note that for $y \notin \widetilde{D}_1$ we have

$$\begin{aligned}|K(x,x-y)-K(z,z-y)| \leq c\frac{|x-z|^{\nu_1}}{|x-y|^m}|x_m|^{-\nu_1}\frac{|x-y|^\sigma}{|x_m-y_m|^\sigma} + \\ + c\frac{|x-z|^{\nu_1}}{|x-y|^{m+\nu_1}}\Big(\frac{|x-y|^{\nu_1+\sigma}}{|x_m-y_m|^{\nu_1+\sigma}} + \frac{|z-y|^{\nu_1+\sigma}}{|z_m-y_m|^{\nu_1+\sigma}}\Big).\end{aligned}$$

Using this estimate in the same manner as in proving Theorem 1, we obtain

$$|I_4(x,z)| \leq c|x_m|^{-\alpha-\nu}(1+|x|)^{-\beta-2\gamma}|x-z|^{\nu+\gamma}\|u\|. \qquad (31)$$

The estimates (28)–(31) show that the operator B is bounded from the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$ into the space $H^{\nu+\gamma}_{\alpha-\gamma,\beta+2\gamma}(\mathbb{R}^m\backslash\Gamma)$. The validity of the theorem now follows from the complete continuity of the operator of the embedding of the space $H^{\nu+\gamma}_{\alpha-\gamma,\beta+2\gamma}(\mathbb{R}^m\backslash\Gamma)$ into the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash\Gamma)$. ∎

In a similar manner we prove

Theorem 4. *Let $m \in C^{1,\delta}$, the characteristic f of the operator A satisfy the conditions of Theorem 2, the first inequality of the condition* (b) *being replaced by a stronger inequality*

$$|f(x,\theta)-f(y,\theta)| \leq c|x-y|^{\nu_1}\Big(\min(d(x),d(y))\Big)^{-\nu_1},$$

and the function a satisfy the conditions of Theorem 3. Then the operator B is completely continuous in the space $H^{\nu}_{\alpha,\beta}(\mathbb{R}^m\backslash M)$.

2. We shall consider the matrix-function $A(\xi) = \|A_{ij}(\xi)\|_{n\times n}$. Let

$$A(\lambda\xi) = A(\xi) \quad (\lambda > 0), \quad A_{ij} \in C^{\infty}(\mathbb{R}^m\backslash 0), \quad \det A(\xi) \neq 0 \quad (\xi \neq 0).$$

We set

$$A_0 = A^{-1}(0,\dots,0,-1)A(0,\dots,0,+1).$$

It is assumed that λ_j $(j=1,\dots,s)$ is the eigenvalue of the matrix A_0 and r_j is its multiplicity $(\sum_{j=1}^{s} r_j = n)$.

We introduce the matrices $B_r(\alpha) \equiv \|B_{\nu k}(\alpha)\|_{r\times r}$ where

$$B_{\nu k}(\alpha) = \begin{cases} 0, & \nu < k; \\ 1, & \nu = k; \\ \frac{\alpha^{\nu-k}}{(\nu-k)!}, & \nu > k, \end{cases}$$

$$B(r_i;\alpha) = \operatorname{diag}\left[B_{r_{i1}}(\alpha),\dots,B_{r_{ip_i}}(\alpha)\right]$$

$$(r_{i1}+\dots+r_{ip_i} = r_i).$$

By the Jordan theorem the matrix A_0 is representable in the form

$$A_0 = gBg^{-1},$$

where $\det g \neq 0$, B is the modified Jordan form of the matrix A_0,

$$B = \operatorname{diag}[\lambda_1 B(r_1;1),\dots,\lambda_s B(r_s;1)].$$

We introduce the notation

$$\delta'_j = \frac{1}{2\pi i}\ln\lambda_j, \quad \delta_j = \delta'_k \quad \text{for} \quad \sum_{\nu=1}^{k-1} r_\nu < j \le \sum_{\nu=1}^{k} r_\nu, \quad j = 1,\dots,n;$$

$$\alpha_\pm(\xi) = \frac{1}{2\pi i}\ln\frac{\xi_m \pm |\xi'|}{|\xi'|} \quad (\xi' = (\xi_1,\dots,\xi_{m-1}));$$

by $\ln z$ we denote a logarithm branch which is real on the positive semi-axis, i.e., $-\pi < \arg z \le \pi$,

$$\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|}\right)^{\delta} \equiv \operatorname{diag}\left[\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|}\right)^{\delta_1},\dots,\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|}\right)^{\delta_n}\right],$$

$$B_\pm(\xi) \equiv \operatorname{diag}\left[B(r_1;\alpha_\pm(\xi)),\dots,B(r_s;\alpha_\pm(\xi))\right].$$

Theorem 5. *Let the matrix A be strongly elliptic. Then A admits the factorization*

$$A(\xi) = cgA_-(\xi',\xi_m)D(\xi)A_+(\xi',\xi_m)g^{-1},$$

where

$$c = A(0,\dots,+1), \quad D(\xi) = B_-(\xi)(\xi_m - i|\xi'|)^{\delta}(\xi_m + i|\xi'|)^{-\delta}B_+^{-1}(\xi),$$
$$A_\pm(\lambda\xi) = A_\pm(\xi) \quad (\lambda > 0), \quad \det A_\pm(\xi) \ne 0.$$

For $|\xi'| \ne 0$ the matrices A_+, A_+^{-1} (accordinly, A_-, A_-^{-1}) admit analytic continuations with respect to ξ_m into the upper (lower) complex half-plane and these continuations are bounded.

Moreover, for any natural number k the matrix $A_\pm$ admits the expansion

$$A_\pm(\xi',\xi_m) = I + \\ + \sum_{p=1}^{k}\sum_{q=0}^{(p+1)(2n-1)} c^{pq}\left(\frac{\xi'}{|\xi'|}\ln^q\frac{\xi_m \pm |\xi'|}{|\xi'|}\left(\frac{\xi_m \pm |\xi'|}{|\xi'|}\right)^{-p} + \overset{k}{A}(\xi), \quad (32)$$

where

$$c^{pq} \in C^\infty(\mathbb{R}^{m-1}\backslash 0), \quad \overset{k}{A}(\lambda\xi) = \overset{k}{A}(\xi) \quad (\lambda > 0) \quad \overset{k}{A} \in C^k(\mathbb{R}^m\backslash 0).$$

The similar expansions hold for the inverse matrices $A_\pm^{-1}$ too.

Let us outline a scheme for proving the theorem.

We set

$$A_*(\xi) = \left(\frac{\xi_m - i|\xi'|}{|\xi'|}\right)^{-\delta} B_-^{-1}(\xi) g^{-1} A^{-1}(0,\dots,0,+1) \times$$
$$\times A(\xi) g B_+(\xi) \left(\frac{\xi_m + i|\xi'|}{|\xi'|}\right)^{\delta},$$
$$Z_+ = \{z = x_1 + ix_2, x_2 > 0\}, \quad Z_- = \{z = x_1 + ix_2,\ x_2 < 0\}.$$

Consider the homogeneous Hilbert problem: Find an analytic in the domain $Z_+ \cup Z_-$ matrix-function $\Phi(\xi', z)$, which is left and right continuously extendable on $\mathbb{R}$, by the boundary condition

$$\Phi^-(\xi', t) = A_*(\xi', t)\Phi^+(\xi', t),$$
$$\lim_{Z_+ \ni z \to \infty} \Phi(\xi', z) = I, \quad \lim_{Z_- \ni z \to \infty} \Phi(\xi', z) = I.$$

The solution is to be sought in the form

$$\Phi(\xi', z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi(\xi', z)}{t - z} dt + I.$$

To define the matrix φ we obtain the system of singular integral equations

$$\left(A_*(\xi', t_0) + I\right)\varphi(\xi', t_0) + \frac{1}{\pi i}\left(A_*(\xi', t_0) - I\right) \int_{\mathbb{R}} \frac{\varphi(\xi', t)}{t - t_0} dt =$$
$$= 2\left(I - A_*(\xi', t_0)\right). \qquad (33)$$

One can prove that the system (33) is unconditionally and uniquely solvable and obtain, after a rather sophisticated reasoning, the desired result.

Remark 1. The fact that partial indices of the strongly elliptic matrix A are equal to zero is proved in [5]. An expansion of the form (32) when A is a scalar function is also obtained therein.

Let D be a finite or infinite domain in $\mathbb{R}^m$ bounded by the compact manifold without a boundary M from the class C^{1,ν_1}.

Consider the matrix singular operator

$$A(u)(x) = a(x)u(x) + \int_D f\left(x, \frac{x - y}{|x - y|}\right)|x - y|^{-m} u(y) dy, \qquad (34)$$
$$a(x) = \|a_{ij}(x)\|_{n\times n}, \quad f(x, z) = \|f_{ij}(x, z)\|_{n\times n},$$
$$u = (u_1, \dots, u_n)$$

in the spaces $[H^{\nu}_{\alpha,\beta}(D)]^n$ $(0 < \alpha, \nu < 1, \nu < \nu_1, \beta > 0, \alpha+\beta < m)$ and $[L_p(D,(1+|x|)^{\gamma})]^n$ $(p > 1, -\frac{m}{p} < \gamma < \frac{m}{p'}, p' = \frac{p}{p-1})$, $u \in L_p(D,(1+|x|^{\gamma}) \leftrightarrow \int_D |u(x)^p(1+|x|)^{p\gamma}dx < \infty$.

Taking into account the character of the linear bounded operator acting in the spaces with two norms (see [11]), the proved theorems enable us to prove

Theorem 6. *Let* $a \in H^{\nu_1}_{\lambda}(\bar{D})$, $f(x,\cdot) \in C^{\infty}(\mathbb{R}^m \backslash 0)$, $\int_{S(0,1)} f(x,z)d_zS = 0$; $\partial^p_z f(\cdot,z) \in H^{\nu_1}_{\lambda}(\bar{D})$, $|p| = 0,1,\dots$, *if the domain* D *is bounded;* $\lim_{x\to\infty} f(x,z) \equiv f(\infty,z)$ *and* $\partial^p_z(f(\cdot,z)-f(\infty,z)) \in H^{\nu_1}_{\lambda}(\bar{D})$, *if the domain is unbounded. The determinant of the symbol matrix* $\Phi(A)(x,\xi)$ *of the integral operator* (34) *is different from zero and either of the following two conditions is fulfilled:*
(i) $\forall x \in M$ *the matrix* $\Phi(A)(x,\xi)$ *is strongly elliptic and Hermitian;*
(ii) $\forall x \in M$ *the matrix* $\Phi(A)(x,\xi)$ *is strongly elliptic and odd with respect to the variable* ξ.

Then the operator A *is the Noether operator both in the space* $[L_p(D,(1+|x|)^{\gamma})]^n$ *and in the space* $[H^{\nu}_{\alpha,\beta}(D)]^n$. *Any solution of the equation*

$$A(u)(x) = g(x), \quad g \in [L_p(D,(1+|x|)^{\gamma})]^n \cap [H^{\nu}_{\alpha,\beta}(D)]^n \tag{35}$$

from the space $[L_p(D,(1+|x|)^{\gamma})]^n$ *belongs to the space* $[L_p(D,(1+|x|)^{\gamma})]^n \cap [H^{\nu}_{\alpha,\beta}(D)]^n$. *For the equation* (35) *to be solvable it is necessary and sufficient that* $(g,v) = 0$ *where* v *is an arbitrary solution of the formally conjugate equation* $A'(v) = 0$ *from the space* $[L_{p'}(D,(1+|x|)^{-\gamma})]^n$ *into* $[H^{\nu}_{\alpha,\beta}(D)]^n$.

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ON THE BOUNDEDNESS OF CAUCHY SINGULAR OPERATOR FROM THE SPACE L_p TO L_q, $p > q \geq 1$

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ABSTRACT. It is proved that for a Cauchy type singular operator, given by equality (1), to be bounded from the Lebesgue space $L_p(\Gamma)$ to $L_q(\Gamma)$, as $\Gamma = \cup_{n=1}^{\infty}\Gamma_n$, $\Gamma_n = \{z : |z| = r_n\}$, it is necessary and sufficient that either condition (4) or (5) be fulfilled.

ᲠᲔᲖᲘᲣᲛᲔ. შესწავლილია (1) ტოლობით განსაზღვრული კოშის სინგულარული S_Γ ოპერატორის ლებეგის სივრცეებში შემოსაზღვრულობის საკითხი იმ შემთხვევაში, როცა $\Gamma = \cup_{n=1}^{\infty}\Gamma_n$, $\Gamma_n = \{z : |z| = r_n\}$. დამტკიცებულია, რომ ამ ოპერატორის შემოსაზღვრულობისათვის $L_p(\Gamma)$ სივრციდან $L_q(\Gamma)$ სივრცეში, როდესაც $p > q \geq 1$, აუცილებელია და საკმარისი (4) ან (5) პირობათაგან ერთ-ერთის შესრულება.

1. Let Γ be a plane rectifiable Jordan curve, $L_p(\Gamma)$, $p \geq 1$, a class of functions summable to the p-th degree on Γ, and S_Γ a Cauchy singular operator

$$S_\Gamma(f)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau - t}, \quad f \in L_p(\Gamma), \quad t \in \Gamma. \tag{1}$$

Numerous studies have been devoted to problems of the existence of $S_\Gamma(f)(t)$ and boundedness of the operator $S_\Gamma : f \to S_\Gamma(f)$ in the space $L_p(\Gamma)$ (see, e.g., [1–3]). The final solution of these problems is given in [4,5]. It was proved by G.David that for the operator S_Γ to be bounded in $L_p(\Gamma)$, it is necessary and sufficient that the condition

$$l(t,r) \leq Cr, \tag{2}$$

be fulfilled, where $l(t,r)$ is a length of the part of Γ is contained in the circle with center at $t \in \Gamma$ and radius r, and C is a constant.[1]

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[1]Following [7,8], the necessity of condition (2) is shown also in [6]. In the same work its sufficiency is proved for special classes of curves.

The present paper is devoted to the problem of boundedness of the operator S_Γ from $L_p(\Gamma)$ to $L_q(\Gamma)$, $p > q \geq 1$ (see also [6–9]).

2. Throughout the rest of this paper under $\{r_n\}_{n=1}^{\infty}$ is meant a strictly decreasing sequence of positive numbers satisfying the condition $\sum_{k=1}^{n} r_k < \infty$ and under Γ the family of concentric circumferences on a complex plane $\Gamma_n = \{z : |z| = r_n\}$, $n = 1, 2, \ldots$.

It has been shown in [10,11] that for the operator S_Γ to be bounded in $L_p(\Gamma)$, $p > 1$, it is necessary and sufficient that the conditions

$$\sum_{k=n}^{\infty} r_k \leq C r_n, \quad n = 1, 2, \ldots, \tag{3}$$

be fulfilled, where C is an absolute constant.

We shall prove

Theorem. *Let $p > q \geq 1$ and $\sigma = pq/(p-q)$. Then the following statements are equivalent:*

(A) *operator S_Γ is bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$;*

(B) $$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} r_k}{r_n} \right)^{\sigma} r_n < \infty; \tag{4}$$

(C) $$\sum_{n=1}^{\infty} n^{\sigma} r_n < \infty. \tag{5}$$

Remark. A family of concentric circumferences with finite a sum of their lengths, as a set of integration, principally, "simulates" rectifiable curves with isolated singularities. Analogy of conditions (2) and (3) also indicates this fact. Taking into account the above-said, we assume that the following statement (an analogue of the theorem from Subsection 2) is valid: for the operator S_Γ to be bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$, where Γ is an arbitrary rectifiable curve, $p > q \geq 1$, it is necessary and sufficient that the condition

$$\int_{\Gamma} [\chi(t)]^{pq/(p-q)} |dt| < \infty$$

be fulfilled, where

$$\chi(t) = \sup_{r} \frac{l(t, r)}{r}, \quad t \in \Gamma.$$

3. In proving this theorem, use will often be made of the well-known Abel equality (see, e.g., [12], p.307)

$$\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} u_k \Big) v_n = \sum_{n=1}^{\infty} u_n \Big(\sum_{k=n}^{\infty} v_k \Big), \tag{6}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences of positive numbers and $\sum_{k=1}^{\infty} v_k < \infty$, as well as of its particular case

$$\sum_{n=1}^{\infty} n v_n = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} v_k. \tag{7}$$

We shall also need

Lemma. *Let $p > 0$. If f is a function analytic in the circle $|z| < 1$, then for $r < R < 1$,*

$$\int_{|z|=r} |f(z)|^p |dz| \le \frac{r}{R} \int_{|z|=R} |f(z)|^p |dz|. \tag{8}$$

If f is a function analytic in the domain $|z| > 1$ and $f(\infty) = 0$ then for $1 < R < r$

$$\int_{|z|=r} |f(z)|^p |dz| \le \left(\frac{R}{r}\right)^{p-1} \int_{|z|=R} |f(z)|^p |dz|. \tag{9}$$

If, in addition, f belongs to the Hardy class H_p in the domains $|z| < 1$ or $|z| > 1$, i.e. $\sup_\rho \int_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta < \infty$ (in particular, if f is represented by a Cauchy type integral), then we can take $R = 1$ in inequalities (8) and (9).

Proof. Since $|dz| = |d\rho e^{i\vartheta}| = \rho d\vartheta$, inequality (8) follows from the fact that the mean value $\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta$ of $|f(\rho e^{i\vartheta})|^p$ is a nondecreasing function of ρ (see, e.g., [13], p.9).

Under the conditions of the lemma, if $|z| > 1$, then the function $g(\zeta) = \frac{1}{\zeta} f(\frac{1}{\zeta})$ is analytic in the circle $|\zeta| < 1$. Using inequality (8) for g, we get

$$\int_{|\zeta|=\frac{1}{r}} \left|f\left(\frac{1}{\zeta}\right)\right|^p |d\zeta| \le \left(\frac{R}{r}\right)^{p+1} \int_{|\zeta|=\frac{1}{R}} \left|f\left(\frac{1}{\zeta}\right)\right|^p |d\zeta|.$$

Applying the transformation of $\zeta = \frac{1}{z}$, the latter inequality reduces to (9).

If $f \in H_p$, then by the Riesz theorem

$$\lim_{\rho \to 1} \int_0^{2\pi} |f(\rho e^{i\vartheta})|^p d\vartheta = \int_0^{2\pi} |f(e^{i\vartheta})|^p d\vartheta$$

(see, e.g., [13], p.21), which enables us to suppose that $R = 1$. ■

4. Let us prove the equivalence of conditions (B) and (C). This follows from equality (7) for $\sigma = 1$ and therefore we shall assume that $\sigma > 1$.

(C) follows from (B). We use Abel–Dini's theorem (see, e.g., [12], p. 292): if a series with positive terms $\sum_{n=1}^{\infty} a_n$ diverges and S_n means its n-th partial sum, then the series $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ also diverges, while the series $\sum_{n=1}^{\infty} \frac{a_n}{S_n^{1+\varepsilon}}$ $(\varepsilon > 0)$ converges. Assume that the series $\sum_{n=1}^{\infty} n^{\sigma} r_n$ diverges. Then, setting $a_n = n^{\sigma} r_n$ and $\omega_n = 1\Big/\sum_{k=1}^{n} k^{\sigma} r_k$, we shall see by this theorem that the series $\sum_{n=1}^{\infty} \omega_n n^{\sigma} r_n$ diverges while the series $\sum_{n=1}^{\infty} \omega_n^{\sigma'} r_n n^{\sigma}$ converges, where $\sigma' = \frac{\sigma}{\sigma-1} > 1$.

Using equality (6) and the Hölder inequality, we obtain

$$\sum_{n=1}^{\infty} \omega_n n^{\sigma} r_n \le 2\sum_{n=1}^{\infty} \omega_n \Big(\sum_{k=1}^{n} k^{\sigma-1}\Big) r_n \le 2\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \omega_k k^{\sigma-1}\Big) r_n =$$
$$= 2\sum_{n=1}^{\infty} \omega_n n^{\sigma-1} \bigg(\frac{\sum_{k=n}^{\infty} r_k}{r_n}\bigg) r_n \le$$
$$\le 2\bigg[\sum_{n=1}^{\infty} \bigg(\frac{\sum_{k=n}^{\infty} r_k}{r_n}\bigg)^{\sigma} r_n\bigg]^{1/\sigma} \Big(\sum_{n=1}^{\infty} \omega_n^{\sigma'} n^{\sigma} r_n\Big)^{1/\sigma'} < \infty.$$

The obtained contradiction shows that (C) follows from (B).

Let us now show that (B) follows from (C). If $m \le n$, then

$$A_m = \frac{\sum_{k=m}^{\infty} r_k}{r_m} = \frac{r_m + r_{m+1} + \cdots + r_{n-1}}{r_m} + $$
$$+ \frac{\sum_{k=n}^{\infty} r_k}{r_m} \le (n-m) + A_n. \tag{10}$$

Let $1 \le s \le \sigma$. Using equality (6) and inequality (10), we get

$$\sum_{n=1}^{\infty} n^{s-1} A_n^{\sigma-s+1} r_n = \sum_{n=1}^{\infty} n^{s-1} A_n^{\sigma-s} \sum_{k=n}^{\infty} r_k =$$
$$= \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} k^{s-1} A_k^{\sigma-s}\Big) r_n \le \sum_{n=1}^{\infty} \Big(k^{s-1}\big[A_n + (n-k)\big]^{\sigma-s} r_n\Big) \le$$
$$\le 2^{\sigma} \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} k^{s-1}\big[A_n^{\sigma-s} + (n-k)^{\sigma-s}\big]\Big) r_n \le$$
$$\le 2^{\sigma} \sum_{n=1}^{\infty} A_n^{\sigma-s} \Big(\sum_{k=1}^{n} k^{s-1}\Big) r_n + 2^{\sigma} \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} k^{s-1}(n-k)^{\sigma-s}\Big) r_n \le$$
$$\le 2^{\sigma} \sum_{n=1}^{\infty} A_n^{\sigma-s} n^s r_n + 2^{\sigma} \sum_{n=1}^{\infty} n^{\sigma} r_n. \tag{11}$$

Let $[\sigma]$ be the integer part of σ and $\alpha = \sigma - [\sigma]$. Using inequality (11) successively $[\sigma]$ times for $s = 1, 2, \ldots, [\sigma]$, we arrive at the inequality

$$\sum_{n=1}^{\infty} A_n^{\sigma} r_n \leq C_1 \sum_{n=1}^{\infty} A_n^{\alpha} n^{[\sigma]} r_n + C_2, \tag{12}$$

where the constants C_1 and C_2 depend on σ only.

If σ is an integer, then $\alpha = 0$, and consequently the proof is completed. Let $\alpha > 0$. Then making use of the Hölder inequality and equality (7), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} A_n^{\alpha} n^{[\sigma]} r_n &= \sum_{n=1}^{\infty} A_n^{\alpha} n^{\alpha(\sigma-1)} n^{\sigma(1-\alpha)} r_n \leq \\
&\leq \Big(\sum_{n=1}^{\infty} A_n n^{\sigma-1} r_n\Big)^{\alpha} \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} = \\
&= \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} n^{\sigma-1} \sum_{k=n}^{\infty} r_k\Big)^{\alpha} \leq \\
&\leq \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} k^{\sigma-1} r_k\Big)^{\alpha} = \\
&= \Big(\sum_{n=1}^{\infty} n^{\sigma} r_n\Big)^{1-\alpha} \Big(\sum_{n=1}^{\infty} n(n^{\sigma-1} r_n)\Big)^{\alpha} = \sum_{n=1}^{\infty} n^{\sigma} r_n < \infty,
\end{aligned}$$

which completes the proof.

5. Let us show that (A) follows from (B) or (C). Consider first the case when $q = 1$ and show that if $p > 1$ and $\sigma = p' = p/(p-1)$, then S_Γ is bounded from $S_p(\Gamma)$ to $L_1(\Gamma)$.

Let ϕ_n^i and ϕ_n^l be the functions determined respectively in $\operatorname{Int}\Gamma_n$ and $\operatorname{Ext}\Gamma_n$ by the Cauchy type integral

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{\varphi_n(t)\,dt}{t - z}, \quad \varphi_n \in L_p(\Gamma_n), \quad p \geq 1, \quad z \notin \Gamma_n. \tag{13}$$

Using the Sokhotsky–Plemelj formula

$$\phi_n^i(t) - \phi_n^e(t) = \varphi_n(t), \quad \phi_n^i(t) + \phi_n^e(t) = S_\Gamma(\varphi_n)(t)$$

and the Cauchy formula

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{\varphi_n(t)\,dt}{t - z} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\phi_n^i(t) - \phi_n^e(t)}{t - z}\,dt = \begin{cases} \phi_n^i(z), & z \in \operatorname{Int}\Gamma_n, \\ \phi_n^e(z), & z \in \operatorname{Ext}\Gamma_n, \end{cases}$$

we obtain by direct calculations

$$S_\Gamma(\varphi)(t) = 2\sum_{k=1}^{n-1} \phi_k^i(t) + [\phi_n^i(t) + \phi_n^e(t)] + 2\sum_{k=n+1}^{\infty} \phi_k^e(t) \tag{14}$$

for $t \in \Gamma_n$.

Let us evaluate the integrals of the sums

$$S_1(t) = 2\sum_{k=1}^{n-1} \phi_k^i(t) + \phi_n^i(t), \quad S_2(t) = \phi_n^e(t) + 2\sum_{k=n+1}^{\infty} \phi_k^e(t).$$

Using the lemma from Subsection 3 and the Hölder inequality, we can write

$$\int_{\Gamma_n} |S_1(t)|\,ds \le 2\sum_{k=1}^{n} \int_{\Gamma_n} |\phi_k^i(t)|\,ds \le 2\sum_{k=1}^{n} \frac{r_n}{r_k} \int_{\Gamma_k} |\phi_k^i(t)|\,ds \le$$
$$\le 2(2\pi)^{1/p'} \sum_{k=1}^{n} \frac{r_k}{r_k^{1/p}} \Big(\int_{\Gamma_k} |\phi_k^i(t)|^p\,ds \Big)^{1/p},$$

where ϕ_k^i is a limiting function of the Cauchy type integral (13) on Γ_k, $k = 1, 2, \dots, n$.

Next, changing the order of summation and using the Riesz's inequality for the Cauchy singular operator in the case of the circle as well as the Hölder inequality, we get

$$\int_{\Gamma} |S_1(t)|\,ds = \sum_{n=1}^{\infty} \int_{\Gamma_n} |S_1(t)|\,ds \le$$
$$\le 2(2\pi)^{1/p'} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{r_n}{r_k^{1/p}} \Big(\int_{\Gamma_k} |\phi_k^i(t)|^p ds \Big)^{1/p} =$$
$$= 2(2\pi)^{1/p'} \sum_{n=1}^{\infty} \frac{\sum_{k=n}^{\infty} r_k}{r_n^{1/p}} \Big(\int_{\Gamma_n} |\phi_n^i(t)|^p ds \Big)^{1/p} \le$$
$$\le 2(2\pi)^{1/p'} C_p \sum_{n=1}^{\infty} \frac{\sum_{k=n}^{\infty} r_k}{r_n^{1/p}} \Big(\int_{\Gamma_n} |\varphi_n^i(t)|^p ds \Big)^{1/p} \le$$
$$\le 2(2\pi)^{1/p'} C_p \left[\sum_{n=1}^{\infty} \Big(\frac{\sum_{k=n}^{\infty} r_k}{r_n} \Big)^{p'} r_n \right]^{1/p'} \Big(\int_{\Gamma} |\varphi(t)|^p\,ds \Big)^{1/p}, \tag{15}$$

where C_p is the constant from the Riesz inequality (which depends on p only).

The integral of $S_2(t)$ can be evaluated analogously. Using inequality (9), as well as the Hölder and Riesz inequalities, we obtain

$$\int_{\Gamma_n}|S_2(t)|dt\le 2\sum_{k=n}^{\infty}\int_{\Gamma_n}|\phi_k^e(t)|ds\le 2(2\pi)^{1/p'}\sum_{k=n}^{\infty}\Big(\int_{\Gamma_n}|\phi_k^e(t)|^p ds\Big)^{1/p} r_n^{1/p'}\le$$

$$\le 2(2\pi)^{1/p'}\sum_{k=n}^{\infty}\Big(\frac{r_k}{r_n}\Big)^{\frac{p-1}{p}}\Big(\int_{\Gamma_k}|\phi_k^e(t)|^p ds\Big)^{1/p} r_n^{1/p'}\le$$

$$\le 2(2\pi)^{1/p'}C_p\sum_{k=n}^{\infty}r_k^{1/p'}\Big(\int_{\Gamma_k}|\varphi_k(t)|^p ds\Big)^{1/p}.$$

Next, changing the order of summation and using the Hölder inequality, we can write

$$\int_{\Gamma}|S_2(t)|\,dt=\sum_{n=1}^{\infty}\int_{\Gamma_n}|S_2(t)|\,ds\le 2(2\pi)^{1/p'}C_p\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}r_k^{1/p'}\times$$

$$\times\Big(\int_{\Gamma_k}|\varphi_k(t)|^p ds\Big)^{1/p}=2(2\pi)^{1/p'}C_p\sum_{n=1}^{\infty}n r_n^{1/p'}\Big(\int_{\Gamma_n}|\varphi_n(t)|^p ds\Big)^{1/p}\le$$

$$\le 2(2\pi)^{1/p'}C_p\Big(\sum_{n=1}^{\infty}n^{p'}r_n\Big)^{1/p'}\Big[\sum_{n=1}^{\infty}\Big(\int_{\Gamma_n}|\varphi_n(t)|^p ds\Big)^{\frac{1}{p}\cdot p}\Big]^{\frac{1}{p}}=$$

$$=2(2\pi)^{1/p'}C_p\Big(\sum_{n=1}^{\infty}n^{p'}r_n\Big)^{1/p'}\Big(\int_{\Gamma}|\varphi(t)|^p ds\Big)^{1/p}. \qquad (16)$$

It follows from (14),(15) and (16) that if conditions (B) and (C) are fulfilled for $\sigma = p'$, then the operator S_Γ is bounded from $L_p(\Gamma)$ to $L_1(\Gamma)$.

Let us now consider the general case. Let conditions (B) and (C) be fulfilled for $p > q \ge 1$ and $\sigma = pq/(p-q)$. Then, by virtue of the above arguments, S_Γ is continuous from $L_{\sigma'}(\Gamma)$, $\sigma' = \frac{\sigma}{\sigma-1}$, to $L_1(\Gamma)$. But then S_Γ is also continuous from $L_\infty(\Gamma)$ to $L_\sigma(\Gamma)$ ($L_\infty(\Gamma)$ is a class of functions essentially bounded on Γ). This statement can be proved by the well-known method using the Riesz equality

$$\int_{\Gamma}\varphi S_\Gamma\psi\,dt=-\int_{\Gamma}\psi S_\Gamma\varphi\,dt,\quad \varphi\in L_{\sigma'}(\Gamma),\ \ \psi\in L_\infty(\Gamma),$$

whose validity in our case can be immediately verified.

Further, since S_Γ is bounded from $L_{\sigma'}(\Gamma)$ and $L_\infty(\Gamma)$ to $L_1(\Gamma)$ and $L_\sigma(\Gamma)$, respectively, according to Riesz–Torin's theorem on interpolation of linear operators (see, e.g., [14], p.144), it follows that S_Γ is bounded from $L_\alpha(\Gamma)$, $\sigma' \le \alpha \le \infty$, to $L_{\alpha\sigma/(\alpha+\sigma)}(\Gamma)$. Letting $\alpha = p$, we get that S_Γ is bounded from $L_p(\Gamma)$ to $L_q(\Gamma)$.

6. Let us now show that (C) and consequently (B) follow from (A). Let for a pair p and q, $p > q \ge 1$, $\sigma = pq/(p-q)$, the series $\sum_{n=1}^\infty n^\sigma r_n$ diverge. Then, according to the above-mentioned Abel–Dini's theorem, if $\omega_n = (\sum_{k=1}^n k^\sigma r_k)^{-1/q}$, then

$$\sum_{n=1}^\infty \omega_n^p n^\sigma r_n = \sum_{n=1}^\infty \frac{n^\sigma r_n}{S_n^{p/q}} < \infty, \quad S_n = \sum_{k=1}^n k^\sigma r_k,$$

$$\sum_{n=1}^\infty \omega_n^q n^\sigma r_n = \sum_{n=1}^\infty \frac{n^\sigma r_n}{S_n} = \infty.$$

Consider, on Γ, the function $\varphi(t) = \omega_n n^{\sigma/p}$ for $t \in \Gamma_n$, $n = 1, 2, \dots$. Then

$$\int_\Gamma |\varphi(t)|^p\,|dt| = \sum_{n=1}^\infty \int_{\Gamma_n} |\varphi(t)|^p\,ds = 2\pi \sum_{n=1}^\infty \omega_n^p n^\sigma r_n < \infty. \tag{17}$$

Next, by equality (14) we have

$$S_\Gamma(\varphi)(t) = 2\sum_{k=1}^{n-1} \omega_k k^{\sigma/p} + \omega_n n^{\sigma/p} > \sum_{k=1}^n \omega_k k^{\sigma/p}$$

for $t \in \Gamma_n$. Consequently,

$$\int_\Gamma |S_\Gamma(\varphi)(t)|^q\,|dt| = \sum_{n=1}^\infty \int_{\Gamma_n} |S_\Gamma(\varphi)(t)|^q\,|dt| > 2\pi \sum_{n=1}^\infty \Big(\sum_{k=1}^n \omega_k k^{\sigma/p}\Big)^q r_n >$$

$$> 2\pi \sum_{n=1}^\infty \omega_n^q \Big(\sum_{k=1}^n k^{\sigma/p}\Big)^q r_n \ge 2\pi \sum_{n=1}^\infty \omega_n^q n^{(\frac{\sigma}{p}+1)q} r_n =$$

$$= 2\pi \sum_{n=1}^\infty \omega_n^q n^\sigma r_n = \infty. \tag{18}$$

It follows from (17) and (18) that if condition (C) is not fulfilled for $p > q \ge 1$, then there exists a function $\varphi \in L_p(\Gamma)$ for which $S_\Gamma(\varphi) \notin L_q(\Gamma)$. Consequently, for condition (A) to be fulfilled, it is necessary that condition (C) (and hence (B)) be fulfilled.

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ON SOME GENERALIZATIONS OF THE VANDERMONDE MATRIX AND THEIR RELATIONS WITH THE EULER BETA-FUNCTION

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ABSTRACT. Multiple Vandermonde matrix which, besides the powers of variables, also contains their derivatives is introduced and an explicit expression of its determinant is obtained. For the case of arbitrary real powers, when the variables are positive, it is proved that such generalized multiple Vandermonde matrix is positive definite for appropriate enumerations of rows and columns. As an application of these results, some relations are obtained which in the one-dimensional case give the well-known formula for the Euler beta-function.

რეზიუმე. შემოღებულია ვანდერმონდის ჯერადი მატრიცა, როგორც მატრიცა, რომელიც ცვლადების ხარისხებთან ერთად შეიცავს მათ ყველა წარმოებულსაც და მიღებულია მისი დეტერმინანტის ცხადი გამოსახულება. იმ შემთხვევაში, როცა ცვლადები დადებითია, ხოლო ხარისხის მაჩვენებლები ნამდვილი რიცხვებია, დამტკიცებულია, რომ სტრიქონებისა და სვეტების სათანადო ნუმერაციისას ვანდერმონდის განზოგადებული ჯერადი მატრიცა დადებითადაა განსაზღვრული. მიღებული შედეგების გამოყენებით ნაპოვნია თანაფარდობები, რომელთა კერძო შემთხვევაა ეილერის ფორმულა ბეტა-ფუნქციისათვის.

It is known from the classical problem of multiple interpolation (cf. [1], pp. 104-106; [2], pp. 13-16) that there exists the unique polynomial P_{N-1} of degree $N-1$ which satisfies the conditions

$$P_{N-1}^{(k-1)}(x_j) = y_j^{(k-1)}, \quad k = \overline{1, r_j}, \quad j = \overline{1, n}, \tag{1}$$

where x_j, $y_j^{(k-1)}$ are given elements of an arbitrary field F; $x_j \neq x_l$ if $j \neq l$, $r_j \geq 1$ and $N = \sum\limits_{j=1}^{n} r_j$. Hence the determinant of the $N \times N$ matrix $V(x_1, r_1; \dots; x_n, r_n) \in M_N(F)$ is nonzero. Here by $M_l(F)$ with

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natural l we denote the set of all $l \times l$ matrices with elements from F and by $V(x_1, r_1; \ldots; x_n, r_n)$ the $N \times N$ block-matrix

$$V(x_1, r_1; \ldots; x_n, r_n) = \left[w_1(x_1)|w_2(x_2)|\cdots|w_n(x_n)\right], \tag{2}$$

where w_j, $j = \overline{1, n}$, stand for the rectangular matrices

$$w_j(x) = \left[(x^{i-1})^{(k-1)}\right]_{\substack{1 \le i \le N \\ 1 \le k \le r_j}}. \tag{3}$$

There are other cases when one has to deal with matrices of the form (2). For instance, the determinant of such a matrix can be effectively used when proving the proposition on functional independence of elements from the complete set of invariants of operators (no matter whether they are Hermitian or not) in an N-dimensional unitary space if these operators have r_j-dimensional eigenspaces, $r_j > 1$ (this prorosition will be published in the forthcoming paper of the author). The case $r_j = 1$ has been investigated in [3].

For $r_1 = \cdots = r_n = 1$ we have $N = n$ and (2) becomes the Vandermonde matrix $V = V(x_1, \ldots, x_n) \in M_n(F)$

$$V(x_1, 1; \ldots; x_n, 1) = \left[x_j^{i-1}\right]_1^n$$

with the determinant

$$\det \left[x_j^{i-1}\right]_1^n = \prod_{1 \le k < j \le n} (x_j - x_k). \tag{4}$$

1. In what follows $\mathbb{N} = \{1, 2, \ldots\}$, F stands for the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$ and we shall call (2) the $(r_1, \ldots, r_n)$-*tiple Vandermonde matrix*. For $r_1 = \cdots = r_n = r$ this matrix is said to be the r-*tiple Vandermonde matrix* and we denote it by $V(x_1, \ldots, x_n; r)$.

Theorem 1. *For arbitrary* $r_1, r_2, \ldots, r_n \in \mathbb{N}$ *the* $(r_1, r_2, \ldots, r_n)$-*tiple Vandermonde matrix* (2) *with* $x_1, \ldots, x_n \in F$ *satisfies the identity*

$$\det V(x_1, r_1; \ldots; x_n, r_n) = \left(\prod_{j=1}^{n} \prod_{k=0}^{r_j - 1} (k!)\right) \prod_{1 \le i < k \le n} (x_k - x_i)^{r_k r_i}. \tag{5}$$

In particular,

$$\det V(x_1, \ldots, x_n; r) = \left(\prod_{k=0}^{r-1} (k!)\right)^n \prod_{1 \le i < k \le n} (x_k - x_i)^{r^2}. \tag{5$_1$}$$

Before proving Theorem 1 we need a certain preparation.

Lemma 1. *Let n, $N \in \mathbb{N}$, $1 \le n \le N$ and ϕ_{jk}, $f_i : F \to F$ be such that*

$$\phi_{jk} \in C^{k-1}(F,F), \quad k = \overline{1,r_j}, \quad j = \overline{1,n},$$

$$f_i \in C^{r-1}(F,F), \quad r = \max\{r_j | j = \overline{1,n}\}, \quad i = \overline{1,N}, \quad N = \sum_{j=1}^{n} r_j,$$

and let $U \in M_N(F)$ be the square $N \times N$ block-matrix

$$U = \Big[u_1(x_1)| \cdots |u_n(x_n)\Big], \tag{6}$$

where

$$u_j(x) = \Big[(f_i(x)\phi_{jk}(x))^{(k-1)}\Big]_{\substack{1\le i\le N \\ 1\le k\le r_j}}, x \in F. \tag{7}$$

Then we have

$$\det U = \det \Big[v_1(x_1)| \cdots |v_n(x_n)\Big] \prod_{j=1}^{n} \prod_{k=1}^{r_j} (\phi_{jk}(x_j)), \tag{8}$$

with

$$v_j(x) = \Big[(f_i(x))^{(k-1)}\Big]_{\substack{1\le i\le N \\ 1\le k\le r_j}}.$$

Proof of Lemma 1. Let us denote by $[k]_j$ the column having number k in the matrix u_j, $k = \overline{1,r_j}$, $j = \overline{1,n}$. It is clear that (8) is trivial if $\phi_{j1}(x_j) = 0$ for some $j, 1 \le j \le n$. Now if $\phi_{j1}(x_j) \ne 0$ for each $j = \overline{1,n}$, then it is sufficient to show that by elementary transformations of the columns of (7) it is possible to reduce them to the form

$$[k]_j \to [\widetilde{k}]_j = \Big[(f_i(x))^{(k-1)}\phi_{jk}(x)\Big]_{1\le i\le N}. \tag{9}$$

This will be proved by induction. The column $[1]_j$ in (7) already has the desired form (9)

$$[\widetilde{1}]_j = [1]_j = [f_i(x)\phi_{j1}(x)]_{1\le i\le N}, j = \overline{1,n}.$$

Let us assume that the columns $\{[k]_j | k = \overline{1,m}\}$, $1 \le m \le r_j - 1$, also have the desired form (9) for each j, $j = \overline{1,n}$. If now $\phi_{jk}(x_j) = 0$ for some k, $1 \le k \le m$, then (8) is proved. Next, if

$$\phi_{jk}(x_j) \ne 0, \tag{10}$$

for each j, k, $j = \overline{1,n}$, $k = \overline{1,m}$, then we add the sum

$$\sum_{q=1}^{m} C_m^{q-1}\Big(-\phi_{j,m+1}^{(m-q+1)}(x)/\phi_{jq}(x)\Big)[q]_j = \sum_{q=0}^{m-1} C_m^{q}\Big(-\phi_{j,m+1}^{(m-q)}(x)\Big) f_i^{(q)}(x)$$

(which is correctly defined according to (10)) to the column $[m+1]_j$ written by virtue of the Leibniz formula as

$$[m+1]_j = \left[(f_i(x)\phi_{j,m+1}(x))^{(m)}\right]_{1\le i\le N} = \left[\sum_{q=0}^{m} C_m^q f_i^{(q)}(x)\phi_{j,m+1}^{(m-q)}(x)\right]_{1\le i\le N}.$$

Finally we obtain

$$[m+1]_j \to \widetilde{[m+1]}_j = \left[f_i^{(m)}(x)\phi_{jm+1}(x)\right]_{1\le i\le N}. \quad \blacksquare$$

Proof of Theorem 1. Denote by $\{i\}$ the row with the number i, $i=\overline{1,N}$, in the matrix (2). By adding the sum $\sum_{q=1}^{r_1} c_q\{q+i-1\}$ with the coefficients $c_q \in F$ to the row $\{r_1+i\}$, $i=\overline{1,N-r_1}$, we get

$$\det V(x_1,r_1;\dots;x_n,r_n) = \det\begin{bmatrix} \widetilde{w}_1(x_1) & | & & | & \widetilde{w}_n(x_n) \\ -- & | & \cdots & | & -- \\ \widetilde{u}_1(x_1) & | & & | & \widetilde{u}_n(x_n) \end{bmatrix}, \tag{11}$$

where the matrix $\widetilde{w}_j(x)$, $j=\overline{1,n}$, is obtained from the matrix (3) after eliminating the last $N-r_1$ rows, and the matrix $\widetilde{u}_j(x)$, $j=\overline{1,n}$, has the form

$$\widetilde{u}_j(x) = \left[(x^{i-1}Q(x))^{(k-1)}\right]_{\substack{1\le i\le N-r_1 \\ 1\le k\le r}}. \tag{12}$$

Here

$$Q(x) = x^{r_1} + \sum_{q=1}^{r_1} c_q x^{q-1}$$

is a polynomial of degree r_1.

Note that the matrix $\widetilde{w}_1(x) = V(x,r_1)$ is the $r_1\times r_1$ Wronski matrix of r_1 functions $\{f_i(x) = x^{i-1} | i=\overline{1,r_1}\}$ and therefore

$$\det V(x,r_1) = \det\left[\widetilde{w}_1(x)\right] = \prod_{k=0}^{r_1-1} k!. \tag{13}$$

Combining the coefficients $c_q\in F$, $q=\overline{1,N-n}$, in such a way that Q has a r_1-tiple root at $x=x_1$, we obtain

$$Q(x) = (x-x_1)^{r_1}, \tag{14}$$

$$Q^{(k)}(x_1) = 0, \quad k=\overline{0,r_1-1}, \tag{15}$$

and now the matrix $\tilde{u}_1(x_1)$ vanishes by virtue of the Leibniz formula and (12). Therefore

$$\det V(x_1,r_1;\dots;x_n,r_n) = \det \begin{bmatrix} \tilde{w}_1(x_1) & | & \tilde{w}_2(x_2) & | & & | & \tilde{w}_n(x_n) \\ -- & | & -- & | & \cdots & | & -- \\ 0 & | & \tilde{u}_2(x_2) & | & & | & \tilde{u}_n(x_n) \end{bmatrix},$$

which, taking into account (13) and the Laplace theorem on determinant expansion, gives

$$\det V(x_1,r_1;\dots;x_n,r_n) = \Big(\prod_{k=0}^{r_1-1} k!\Big) \det \tilde{U}, \tag{16}$$

where the matrix $\tilde{U} \in M_{N-r_1}(F)$,

$$\tilde{U} = \Big[\tilde{u}_2(x_2)|\cdots|\tilde{u}_n(x_n)\Big],$$

satisfies the conditions of Lemma 1. Hence according to (8) and (3)

$$\det \tilde{U} = \det \Big[w_2(x_2)|\cdots|w_n(x_n)\Big] \prod_{j=2}^{n} (Q(x_j))^{r_j}$$

and from (16) together with (2) and (14) we obtain a recurrence relation

$$\det V(x_1,r_1;\dots;x_n,r_n) =$$
$$\Big(\prod_{k=0}^{r_1-1} k!\Big) \prod_{j=2}^{n} (x_j - x_1)^{r_1 r_j} \det V(x_2,r_2;\dots;x_n,r_n). \tag{17}$$

The induction by n gives (5) from (17) and (13). ∎

2. Let the reals α_i, x_j, $i,j = \overline{1,n}$, be given. We introduce the notation

$$(\alpha)_n = \{\alpha_i | \alpha_i \neq \alpha_j \text{ if } i \neq j,\ i = \overline{1,n}\}.$$

It is known that the generalized Vandermonde matrix

$$\Big[x_j^{\alpha_i}\Big]_1^n = V^{(G)}(x_1,\dots,x_n;(\alpha)_n) \equiv V^{(G)}$$
$$(0 < x_1 < \cdots < x_n;\ \alpha_1 < \alpha_2 < \cdots < \alpha_n)$$

is completely positive (cf.[4], p.372), i.e. all its minors are positive.

Let now $1 \le n \le N$ and $r_1, r_2, \dots, r_n \in \mathbb{N}$. We shall call the matrix

$$V^{(G)}(x_1,r_1;\dots;x_n,r_n;(\alpha)_N) = \Big[w_1^{(G)}(x_1)|\cdots|w_n^{(G)}(x_n)\Big] \tag{18}$$

the $(r_1, r_2, \dots, r_n)$-tiple generalized Vandermonde matrix. Here $w_j^{(G)}$, $j = \overline{1,n}$, is the rectangular matrix

$$w_j^{(G)}(x) = \left[(x^{\alpha_i})^{(k-1)}\right]_{\substack{1\le i\le N,\\ 1\le k\le r_j}} \tag{19}$$

$$\left(r_j \ge 1, \quad j = \overline{1,n}, \quad \sum_{j=1}^{n} r_j = N\right).$$

Theorem 2. *the $(r_1, r_2, \dots, r_n)$-tiple generalized Vandermonde matrix*

$$V^{(G)}(x_1, r_1; \dots; x_n, r_n; (\alpha)_N)$$
$$(0 < x_1 < \dots < x_n; \alpha_1 < \alpha_2 < \dots < \alpha_n)$$

is positively definite.

Lemma 2. *For arbitrary simultaneously nonzero reals $c_i \in \mathbb{R}$, $i = \overline{1,N}$, $\alpha_i \ne \alpha_k$ if $i \ne k$, the function $f: \mathbb{R} \to \mathbb{R}$*

$$f(x) = \sum_{i=1}^{N} c_i x^{\alpha_i}$$

has at most $N-1$ positive zeros counted according to their multiplicities:

$$f(x) = \prod_{j=1}^{n} (x - x_j)^{r_j} \phi(x; (\alpha)_N),$$

$$\sum_{j=1}^{n} r_j \le N-1; \quad \phi(x;(\alpha)_N) \ne 0, \quad x>0; \quad x_j>0, \quad j = \overline{1,n}, \quad 1 \le n \le N.$$

Proof of Lemma 2. If f has only single zeros, i.e., $r_1 = \dots = r_N = 1$, then $n = N$, and the assertion of Lemma 2 follows from the inequality $\det V^{(G)} \ne 0$ ([4], p.372). Assume that the assertion of Lemma 2 is true for $1 \le r_j \le r$, $j = \overline{1,n}$, and prove it for N power-summands also in the case $r_{j'} = r+1$ for certain j', $1 \le j' \le n$.

Let the opposite be true: say, there exist reals c_i, $i = \overline{1,N}$, at least one of which is nonzero, such that

$$f(x) = \sum_{i=1}^{N} c_i x^{\alpha_i} = \prod_{j=1}^{n} (x - x_j)^{r_j} \phi(x; (\alpha)_N),$$

$$\sum_{j=1}^{n} r_j \ge N; \quad 1 \le r_j \le r+1, \quad j = \overline{1,n}, \quad 1 \le n \le N.$$

Let m, $1 \leq m \leq n$, be the number of multiple zeros of f and zeros x_j, $j = \overline{1,n}$, be arranged according to the decreasing of their multiplisity. Then f satisfies the conditions

$$f(x_j) = 0, \quad j = \overline{1,n}, \tag{20}$$

$$f'(x_j) = \cdots = f^{(r_j-1)}(x_j) = 0, \quad 2 \leq r_j \leq r+1, \quad j = \overline{1,m}; \tag{21}$$

here

$$n - m + \sum_{j=1}^{m} r_j \geq N. \tag{22}$$

The Rolle theorem, together with (20), implies that the function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$

$$\widetilde{f}(x) = (f(x)x^{-\alpha_1})' = \sum_{i=1}^{N-1} \widetilde{c}_i x^{\widetilde{\alpha}_i} \tag{23}$$

vanishes at the points ξ_j, $0 < x_j < \xi_j < x_{j+1}$, $j = \overline{1,n-1}$, i.e.

$$\widetilde{f}(\xi_j) = 0, \quad j = \overline{1,n-1}.$$

Besides, from (21) it follows that

$$\widetilde{f}(x_j) = \widetilde{f}'(x_j) = \cdots = \widetilde{f}^{(r_j-2)}(x_j) = 0, \quad j = \overline{1,m},$$

i.e., $\widetilde{f}$ has the form

$$\widetilde{f}(x) = \prod_{k=1}^{n-1}(x - \xi_k) \prod_{j=1}^{m}(x - x_j)^{r_j-1} \widetilde{\phi}(x; (\widetilde{\alpha})_N) =$$

$$= P_{\widetilde{N}}(x) \widetilde{\phi}(x; (\widetilde{\alpha})_n), \tag{24}$$

all roots of the polynomial $P_{\widetilde{N}}$ being positive and having the multiplicity $\leq r$. From (24) and (22) we find

$$\widetilde{N} = n - 1 + \sum_{j=1}^{m}(r_j - 1) = n - 1 + \sum_{j=1}^{m} r_j - m \geq N - 1,$$

but this contradicts our assumption, since the function (23) is the sum of $N-1$ power-summands. ■

Proof of Theorem 2. For $\alpha_i = i - 1$, $i = \overline{1,N}$, we have

$$V^{(G)}(x_1, r_1; \ldots; x_n, r_n; (\alpha)_N) = V(x_1, r_1; \ldots; x_n, r_n)$$

and, according to (5),

$$\det V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N) > 0$$
$$(0 < x_1 < \dots < x_n; \quad \alpha_i = i-1, \quad i = \overline{1, N}).$$

It is possible to pass to arbitrary values $\alpha_1 < \dots < \alpha_N$ starting from the integers $\alpha_i = i-1$, $i = \overline{1, N}$, and changing them continuously, but preserving the inequalities among them. In doing so, the determinant $\det V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N)$ does not vanish according to Lemma 2 and therefore for all $0 < x_1 < \dots < x_n$; $\alpha_1 < \alpha_2 < \dots < \alpha_N$ we have

$$\det V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N) > 0.$$

Since each principal minor of the matrix $V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N)$ can be considered as a determinant of a certain $(r_1, r_2, \dots , r_n)$-tiple generalized Vandermonde matrix, all such minors are positive. ■

Corollary 1. *It follows from* $\det V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N) \neq 0$ *that for arbitrary reals* α_i, $\alpha_i \neq \alpha_k$ *if* $i \neq k$, $i, k = \overline{1, N}$, *there exists the unique collection of reals* c_i, $i = \overline{1, N-1}$, *such that*

$$\sum_{i=1}^{N-1} c_i x^{\alpha_i} + x^{\alpha_n} = P_{N-1}(x)\phi(x; (\alpha)_N),$$

where P_{N-1} *is any given polynomial of degree* $N-1$, *having only positive roots and* $\phi(x; (\alpha)_N) \neq 0$ *for* $x > 0$.

Note that for $\alpha_i \in \mathbb{N}$ the above proposition is nothing but the Viete theorem, since it enables us to determine the coefficients at power-summands provided we know zeros of the sum.

3. Let n, N be integers, $1 \leq n \leq N$, and $0 < \alpha_1 < \alpha_2 < \dots < \alpha_N$, $x_j > 0$, $j = \overline{1, n}$. Transform the determinant $\det V^{(G)}(x_1, r_1; \dots ; x_n, r_n; (\alpha)_N)$, taking the factor α_i, $i = \overline{1, N}$, out of the i-th row and taking into account that

$$\alpha_i^{-1}(x^{\alpha_i})^{(k-1)} = \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i + 2 - k)} x^{\alpha_i - k + 1} = (x^{\alpha_i - 1})^{(k-2)}, \quad k = 1, 2, \dots .$$

Here we put

$$(x^\alpha)^{(-q)} = \underbrace{\int_0^x dx \cdots \int_0^x}_{q-\text{tiple}} dx x^\alpha = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} t^\alpha \, dt,$$

$$\alpha \geq 0, \quad q = 1, 2, \dots,$$

$$(x^\alpha)^{(-1)} = \int_0^x t^\alpha dt, \quad \alpha \geq 0, \quad x > 0, \tag{25}$$

and $\Gamma(z) = (z-1)!$ is the gamma-function. Taking into account the corollary of Theorem 2, after proper transformations we get

$$\det V^{(G)}(x_1, r_1; \dots; x_n, r_n; (\alpha)_N) =$$

$$= (-1)^{\sigma_1} \prod_{i=1}^{N} \alpha_i \det V^{(G)}(x_1, r_1 - 1; \dots; x_n, r_n - 1; (\alpha - 1)_{N-n}) \det A, \tag{26}$$

where

$$\sigma_1 = \sum_{j=1}^{n} j(r_j - 1),$$

$$(\alpha - 1)_N = \{\alpha_i - 1 | \alpha_i \neq \alpha_j \text{ if } i \neq j, \, i, j = \overline{1, N}\},$$

and $A \in M_n(F)$ is a square matrix of the form

$$A = [a_{ij}] = \left[\int_{x_{j-1}}^{x_j} \phi_i(t; (\alpha - 1)_{\widetilde{N}}) \prod_{p=1}^{n} (t - x_p)^{r_p - 1} dt \right]. \tag{27}$$

Here we use the notation

$$\widetilde{N} = N - n + 1 \tag{28}$$

and $\phi_i(t; (\alpha - 1)_{\widetilde{N}})$ is defined by the formula

$$\phi_i(t; (\alpha - 1)_{\widetilde{N}}) \prod_{p=1}^{n} (t - x_p)^{r_p - 1} = \sum_{q=0}^{N-n} c_{i+q} t^{\alpha_{i+q} - 1}, \quad i = \overline{1, n}, \tag{29}$$

the coefficients c_{i+q} being defined according to the corollary of Theorem 2.

Rearranging the columns of $\det V^{(G)}(x_1, r_1; \dots; x_n, r_n; (\alpha)_N)$ which contain higher derivatives, we can reduce it to the form

$$\det V^{(G)}(x_1, r_1; \dots; x_n, r_n; (\alpha)_N) =$$

$$= (-1)^{\sigma_2} \det V^{(G)}(x_1, r_1 - 1; \dots; x_n, r_n - 1; (\alpha)_{N-n}) \det \widetilde{A}, \tag{30}$$

where $\sigma_2 = \sum_{j=1}^{n}(j-1)(r_j-1)$ and the matrix $\tilde{A} \in M_n(F)$ is written as

$$\tilde{A} = [\tilde{a}_{ij}]_1^n = \left[\left(\tilde{\phi}_i(x;(\alpha)_{\tilde{N}}) \prod_{p=1}^{n}(x-x_p)^{r_p-1}\right)^{(r_j-1)}_{|x=x_j}\right]_1^n =$$

$$= \left[(r_j-1)!\tilde{\phi}_i(x_j;(\alpha)_{\tilde{N}}) \prod_{\substack{p=1\\p\neq j}}^{n}(x_j-x_p)^{r_p-1}\right]_1^n. \tag{31}$$

Here, by definition,

$$\tilde{\phi}_i(x;(\alpha)_{\tilde{N}}) \prod_{p=1}^{n}(x-x_p)^{r_p-1} = \sum_{q=0}^{N-n} c_{i+q}x^{\alpha_{i+q}}, \quad i=\overline{1,n},$$

from which one can find

$$\tilde{\phi}_i(x;(\alpha)_{\tilde{N}}) = x\phi_i(x;(\alpha-1)_{\tilde{N}}), \quad i=\overline{1,n}. \tag{32}$$

Equating the right-hand sides of (26) and (30) and taking into account (31) and (32), we get

$$\frac{\det\left[\int_{x_{j-1}}^{x_j}\phi_i(t;(\alpha-1)_{\tilde{N}})\prod_{\substack{p=1\\p\neq j}}^{n}\left(\frac{t-x_p}{x_j-x_p}\right)^{r_p-1}(x_j-t)^{r_j-1}dt\right]_1^n}{\det[\phi_i(x_j;(\alpha-1)_{\tilde{N}})]_1^n\prod_{p=1}^{n}x_j} =$$

$$= \prod_{j=1}^{n}(r_j-1)!\prod_{i=1}^{N}\alpha_i^{-1}\frac{\det V^{(G)}(x_1,r_1-1;\dots;x_n,r_n-1;(\alpha)_N)}{\det V^{(G)}(x_1,r_1-1;\dots;x_n,r_n-1;(\alpha-1)_N)}.$$

$$(x_p \neq x_j \quad \text{if} \quad p\neq j; \quad x_0=0; \quad i,j=\overline{1,n}).$$

For the right-hand side ratio Lemma 1 obviously gives

$$\frac{\det V^{(G)}(x_1,r_1-1;\dots;x_n,r_n-1;(\alpha)_{N-n})}{\det V^{(G)}(x_1,r_1-1;\dots;x_n,r_n-1;(\alpha-1)_{N-n})} = \prod_{j=1}^{n}x_j^{r_j-1}.$$

Hence we have the formula

$$\frac{\det\left[\int_{x_{j-1}}^{x_j}\phi_i(t;(\alpha-1)_{\tilde{N}})\prod_{\substack{p=1\\p\neq j}}^{n}\left(\frac{t-x_p}{x_j-x_p}\right)^{r_p-1}(x_j-t)^{r_j-1}dt\right]_1^n}{\det[\phi_i(x_j;(\alpha-1)_{\tilde{N}})]_1^n\prod_{p=1}^{n}x_j^{r_j}} =$$

$$= \frac{\prod_{p=1}^{n}(r_p-1)!}{\prod_{i=1}^{N}\alpha_i}$$

$$(x_p\neq x_j \quad \text{if} \quad p\neq j; \quad r_p\in\mathbb{N}, \quad \sum_{p=1}^{n}r_p=N, \quad x_p>0,$$

$$p,i,j=\overline{1,n}; \quad x_0=0),$$

which can be rewritten as

$$\frac{\det\left[\int_{x_{j-1}/x_j}^{1}(1-u)^{r_j-1}\phi_i(ux_j;(\alpha-1)_{\tilde N})\prod_{\substack{p=1\\p\neq j}}^{n}\left(\dfrac{x_ju-x_p}{x_j-x_p}\right)^{r_p-1}du\right]_1^n}{\det[\phi_i(x_j;(\alpha-1)_{\tilde N})]_1^n}=$$

$$=\frac{\prod_{k=1}^{n}\Gamma(r_k)}{\prod_{i=1}^{N}\alpha_i} \tag{33}$$

$$\Big(x_p\neq x_j \ \text{ if } p\neq j;\ r_p\in\mathbb{N},\ \sum_{p=1}^{n} r_p=N,\ x_p>0, p,i,j=\overline{1,n};\ \ x_0=0\Big).$$

In the particular case, when $\alpha_i=r_0-1+i$, $r_0>0$, $i=\overline{1,N}$, we have

$$\phi_i(t;(\alpha-1)_{\tilde N})=\Big(\prod_{p=1}^{n}(t-x_p)^{r_p-1}\Big)^{-1}\sum_{q=0}^{N-n}c_{i+q}t^{r_0-1+i+q-1}=$$

$$=t^{r_0+i-2}\Big(\prod_{p=1}^{n}(t-x_p)^{r_p-1}\Big)^{-1}\sum_{q=0}^{N-n}c_{i+q}t^{q}=t^{r_0+i-2},\quad i=\overline{1,N}.$$

The latter equality is valid because, according to the Viete theorem, there exist reals c_{i+q} such that

$$\sum_{q=0}^{N-n}c_{i+q}t^{q}=\Big(\prod_{p=1}^{n}(t-x_p)^{r_p-1}\Big).$$

Besides,

$$\prod_{i=1}^{N}(r_0-1+i)=\prod_{i=1}^{N}\alpha_i=\frac{\Gamma(r_0+N)}{\Gamma(r_0)}=\frac{\Gamma(r_0+\sum_{p=1}^{n}r_p)}{\Gamma(r_0)}$$

and (33) transforms to the equality

$$\frac{\det\left[x_j^{i-1}\int_{x_j-1/x_j}^{1}u^{r_0+i-2}(1-u)^{r_j-1}\prod_{\substack{p=1\\p\neq j}}^{n}\left(\dfrac{x_j-x_p}{x_j-x_p}\right)^{r_p-1}du\right]_1^n}{\det[x_j^{i-1}]_1^n}=$$

$$=\frac{\prod_{p=0}^{n}\Gamma(r_p)}{\Gamma(\sum_{p=0}^{n}r_p)} \tag{34}$$

$$\big(x_p\neq x_j \ \text{ if } p\neq j;\ \ r_p\in\mathbb{N},\ \ x_p>0,\ \ p,i,j=\overline{1,n};\ \ x_0=0,\ \ r_0>0\big).$$

For the case $n=1$ the formula (34) gives the well-known expression

$$\int_0^1 u^{r_0-1}(1-u)^{r_1-1}du=\frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)}=B(r_0,r_1), \tag{35}$$

for the Euler integral of first kind.

Now, we introduce the notation

$$B_n(\mathbf{r}) = B(r_0, r_1, \ldots, r_n) =$$
$$= \begin{cases} 1, & n = 0, \\ \det\left[x_j^{i-1} \int\limits_{x_{j-1}/x_j}^{1} u^{i-1}(1-u)^{r_j-1} \prod\limits_{\substack{k=0 \\ k\neq j}}^{n} \left(\frac{x_j u - x_k}{x_j - x_k}\right)^{r_k-1} du\right]_1^n \Big/ \det[x_j^{i-1}]_1^n, \\ n \geq 1 \ \ (x_p \neq x_j \text{ if } p \neq j,\ p, j = \overline{0,n},\ x_0 = 0) \end{cases} \tag{36}$$

for arbitrary $\{r_j | r_j > 0,\ j = \overline{0,n}\} \equiv \mathbf{r}$.

Since the Euler formula (35) is valid for arbitrary complex r_0, r_1, $\operatorname{Re} r_0 > 0$, $\operatorname{Re} r_1 > 0$, there arises a problem:

Problem. Is the equality

$$B_n(\mathbf{r}) = B(r_0, r_1, \ldots, r_n) = \frac{\prod_{j=0}^{n} \Gamma(r_j)}{\Gamma(\sum_{j=0}^{n} r_j)} \tag{37}$$

fulfilled for an arbitrary complex r_j, $j = \overline{0,n}$, $n \geq 2$? (Note that this case is not covered neither by (34) nor by (35)).

4. Put $2 \leq n \leq N$ and $1 \leq m \leq \min\{r_j | j = \overline{1,n}\}$. Transform the determinant on the left-hand side of (5), taking the factor $\Gamma(i)/\Gamma(i-m)$ out of each row having the number $i \geq m+1$ and keeping in mind that

$$\left(\frac{\Gamma(i-m)}{\Gamma(i)} x^{i-1}\right)^{(k-1)} = \frac{\Gamma(i-m)}{\Gamma(i-k+1)} x^{i-k} = \left(x^{i-m-1}\right)^{(k-m-1)}$$
$$(i = \overline{m+1, N},\ \ 1 \leq m \leq \min\{r | j = \overline{1,n}\},\ \ k = 1, 2, \ldots).$$

After easy but rather long calculations, using (25) and (5) we get the formula

$$\frac{\det[B^{(1)} | \ldots | B^{(n)}]}{\det V(x_1, \ldots, x_n; m)} = \prod_{k=1}^{m} \left(\frac{\Gamma(k)}{\Gamma(N-k+1)} \prod_{j=0}^{n} \Gamma(r_j - k + 1)\right), \tag{38}$$

where $B^{(j)}$, $j = \overline{1,n}$, is the rectangular matrix of the form

$$B^{(j)} = [b_{ik}^{(j)}]_{\substack{1 \leq i \leq mn \\ 1 \leq k \leq m}} =$$
$$= \left[x_j^{i-k} \int_{x_{j-1}/x_j}^{1} (1-u)^{r_j-m} u^{k+i-2} \prod_{\substack{p=0 \\ p\neq j}}^{n} \left(\frac{x_j u - x_p}{x_j - x_p}\right)^{r_p-m} du\right]_{\substack{1 \leq i \leq mn \\ 1 \leq k \leq m}}$$
$$(x_p \neq x_j \text{ if } p \neq j;\ \ x_0 > 0;\ \ j = \overline{1,n}). \tag{39}$$

In particular, for $m = 1$ (38) reduces to (34).

5. Let us now apply the above results to the case $n = 1$, $N = r$, $x > 0$. It is easy to show that for the determinant of the matrix

$$V^{(G)}(x, r; (\alpha)_r) = \left[\left(x^{\alpha_i}\right)^{(k-1)}\right]_1^r,$$

which is nothing but the Wronski matrix for r functions $\{f_i(x) = x^{\alpha_i} | i = \overline{1, r}; x > 0\}$, one gets

$$\det V^{(G)}(x, r; (\alpha)_r) = x^{\beta} \prod_{1 \le k < j \le r} (\alpha_j - \alpha_k), \tag{40}$$

where

$$\beta = \sum_{i=1}^{N} \alpha_i - r(r-1)/2.$$

Obviously, the formula (40) generalizes (13).

Using (40), one can find values of $\phi(x; (\alpha)_N)$ and coefficients c_i for which

$$\sum_{i=1}^{N-1} c_i x^{\alpha_i} + x^{\alpha_N} = (x - x_1)^{N-1} \phi(x; (\alpha)_N).$$

Namely, the Cramer formulae give

$$c_i = -x_1^{\alpha_N - \alpha_i} L_i(\alpha_N), \quad i = \overline{1, N-1},$$
$$\phi(x; (\alpha)_N) =$$
$$= (x - x_1)^{-(N-1)} x_1^{\alpha_N} \Big((x/x_1)^{\alpha_N} - \sum_{i=1}^{N-1} (x/x_1)^{\alpha_i} L_i(\alpha_N) \Big), \tag{41}$$

where

$$L_i(\alpha) = \prod_{\substack{p=1 \\ p \ne i}}^{N-1} \left(\frac{\alpha - \alpha_p}{\alpha_i - \alpha_p} \right), \quad i = \overline{1, N-1},$$

is the Lagrange elementary interpolation polynomial. (41) gives

$$\phi(x_1; (\alpha - 1)_{\tilde{N}}) =$$
$$= \frac{x_1^{\alpha_N - N}}{(N-1)!} \left(\frac{\Gamma(\alpha_N)}{\Gamma(\alpha_N - N + 1)} - \sum_{i=1}^{N-1} \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i - N + 1)} L_i(\alpha_N) \right), \tag{42}$$

$$\phi(u x_1; (\alpha - 1)_{\tilde{N}}) = \frac{x_1^{\alpha_N - N}}{(u-1)^{(N-1)}} \left(u^{\alpha_N - 1} - \sum_{i=1}^{N-1} u^{\alpha_i - 1} L_i(\alpha_N) \right). \tag{43}$$

Substituting (42) and (43) in (33) for $n = 1$, $N = r_1 \geq 2$, $\widetilde{N} = N$ and performing integration we obtain

$$\frac{\Gamma(\alpha_n)}{\Gamma(\alpha_N - N + 1)} - \sum_{i=1}^{N-1} \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i - N + 1)} L_i(\alpha_N) = = (-1)^N \Big(\sum_{i=1}^{N-1} \alpha_i^{-1} L_i(\alpha_N) - \alpha_N^{-1} \Big) \prod_{k=1}^{N} \alpha_k. \tag{44}$$

Finally, the substitution of (44) in (42) gives

$$\phi(x_1; (\alpha - 1)_{\widetilde{N}}) = = \frac{x_1^{\alpha_N - N}}{(N-1)!} (-1)^N \Big(\sum_{i=1}^{N-1} \alpha_i^{-1} L_i(\alpha_N) - \alpha_N^{-1} \Big) \prod_{k=1}^{N} \alpha_k. \tag{45}$$

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ON THE INITIAL VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT. For a system of functional differential equations of an arbitrary order the conditions are established for the initial value problem to be solvable on an infinite interval and the structure of the set of solutions to this problem is studied.

რეზიუმე. ნებისმიერი რიგის ფუნქციონალურ-დიფერენციალურ განტოლებათა სისტემისათვის დადგენილია პირობები, რომლებიც უზრუნველყოფენ საწყისი ამოცანის ამოხსნადობას უსასრულო შუალედში და გამოკვლეულია ამ ამოცანის ამონახსნთა სიმრავლის სტრუქტურა.

INTRODUCTION

For the p-th order functional differential system

$$x^{(p)}(t) = f(t, x_t, \dots, x_t^{(p-1)}), \quad b \le t < +\infty, \tag{1}$$

we consider the initial value problem

$$x_b^{(k)} = \psi^{(k)} \quad (k = 0, \dots, p-1). \tag{2}$$

The invesigation is based on Kubáček's theorem [2] asserting that under certain conditions the set of all fixed points of the compact map in the Fréchet space is a compact R_δ-set. It is shown that some restrictions on the growth of the right-hand side of the functional differential system imply that the set of all solutions of the initial value problem for that system is a compact R_δ-set in the Fréchet space of C^{p-1}-functions. The result extends a similar theorem for first order functional differential systems proved in [2] and the theorem for second order functional differential systems proved in [3].

In the sequel we shall use the following notations and assumptions:

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Let $h > 0$, $b \in R$, $d \in N$, and let $|\cdot|$ be a norm in R^d. Further, let $H_l = C^l([-h,0], R^d)$ be provided with the norm

$$\|x\|_l = \max\left\{\sum_{k=0}^{l} |x^{(k)}(s)| : -h \le s \le 0\right\}$$

for each $x \in H_l$ and $l = 0, \dots, p-1$. For brevity $\|\cdot\|_{p-1}$ will be denoted by $\|\cdot\|$.

Let $X = C^{p-1}([b,\infty), R^d)$ be equipped with a topology of locally uniform convergence of the functions and of their $p-1$ derivatives on $[b,\infty)$. In the Fréchet space X the topology is given by the metric

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)},$$

where

$$p_m(x) = \sup\left\{\sum_{k=0}^{p-1} |x^{(k)}(t)| : b \le t \le b+m\right\}, \quad x, y \in X, \ m \ge 1.$$

Let $X^* = C^{p-1}([b-h,\infty), R^d)$ be a Fréchet space whose topology is determined by seminorms

$$p_m^*(x) = \sup\left\{\sum_{k=0}^{p-1} |x^{(k)}(t)| : b-h \le t \le b+m\right\}, \quad x \in X^*, \ m \ge 1.$$

For $x \in C([b-h,\infty), R^d)$ we shall denote by $x_t \in H_0$ the function $x_t(s) = x(t+s)$, $s \in [-h,0]$, $t \ge b$. Clearly $(x_t)^{(k)}(s) = (x^{(k)})_t(s)$, $s \in [-h,0]$, $k = 0, \dots, p-1$, and $x \in X^*$, $t \ge b$.

It is assumed throughout the paper that $f \in X([b,\infty) \times H_{p-1} \times \dots \times H_0, R^d)$, $\psi \in H_{p-1}$.

A solution x of (1), (2) is a function $x \in X^*$ such that $x\big|_{[b,\infty)} \in C^p([b,\infty), R^d)$, x satisfies (2) and the functional differential system (1) at each point $t \ge b$.

§ 1. Auxiliary Propositions

Now Kubáček's theorem in [2] will be stated as Lemma 1. In that lemma the compact R_δ-set in the metric space (E,ρ) means a nonempty subset F of E which is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. By [1], p. 92, a metric space G is called an absolute retract when each continuous map $f : K \to G$ has a continuous extension $g : H \to G$ for each metric space H and each closed $K \subset H$. For example, a nonempty convex subset of the Fréchet space is an absolute retract.

Lemma 1. *Let M be a nonempty closed set in the Fréchet space (E,ρ), $T : M \to E$ a compact map (i.e., T is continuous and $T(M)$ is a relatively compac set). Denote by S the map $I - T$ where I is the identity map on E. Let there exist a sequence $\{U_n\}$ of closed convex sets in E fulfilling the conditions*

(i) $0 \in U_n$ *for each* $n \in N$;

(ii) $\lim_{n\to\infty} \operatorname{diam} U_n = 0$,

and a sequence $\{T_n\}$ of maps $T_n : M \to E$ fulfilling the conditions

(iii) $T(x) - T_n(x) \in U_n$ *for each $x \in M$ and each $n \in N$;*

(iv) *the map $S_n = I - T_n$ is a homeomorphism of the set $S_n^{-1}(U_n)$ onto U_n.*

Then the set F of all fixed points of the map T is a compact R_δ-set.

In the special case $E = X$, $\rho = d$ Lemma 1 implies

Lemma 2. *Let (X,d) be the Fréchet space given above; let $\varphi, \varphi_n \in C([b,\infty),[0,\infty))$ and let the following conditions be fulfilled:*

(v) *For each $t \in [b,\infty)$ the sequence $\{\varphi_n(t)\}$ is nonincreasing and* $\lim_{n\to\infty} \varphi_n(t) = 0$. *Let $r_k \in R^d$, $k = 0,\dots,p-1$ and let*

$$M = \Big\{ x \in X : \sum_{k=0}^{p-1} |x^{(k)}(t) - r_k| \le \varphi(t),\ t \ge b,\ x^{(k)}(b) = r_k,\ k = 0,\dots,p-1 \Big\}.$$

It is assumed that $T : M \to X$ is a compact map with the property $(T(x))^{(k)}(b) = r_k$, $k = 0,\dots,p-1$ for each $x \in M$ and there exists a sequence $\{T_n\}$ of compact maps $T_n : M \to X$ such that $(T_n(x)^{(k)}(b) = r_k$, $k = 0,\dots,p-1$ for each $x \in M$ and

(vi) $\sum_{k=0}^{p-1} \Big| \big(T_n(x)\big)^{(k)}(t) - \big(T(x)\big)^{(k)}(t) \Big| \le \varphi_n(t),\ x \in M,\ t \ge b;$

(vii) *for each $n \in N$ there exists a function $\varphi_{*n} \in C([b,\infty),[0,\infty))$ such that*

$$\varphi_{*n} + \varphi_n \le \varphi \quad on\ [b,\infty)$$

and

$$\sum_{k=0}^{p-1} \Big| \big(T_n(x)\big)^{(k)}(t) - r_k \Big| \le \varphi_{*n}(t),\quad x \in M,\quad t \ge b;$$

(viii) *the map $S_n = I - T_n$ is injective on M where I is the identity on X.*

Then the set F of all fixed points of the map T is a compact R_δ.

Proof. The set

$$U_n = \Big\{x \in X : \sum_{k=0}^{p-1} |x^{(k)}(t)| \le \varphi_n(t),\ t \ge b,\ x^{(k)}(b) = 0,\\ k = 0, \dots, p-1\Big\}$$

is convex and closed on X for each $n \in N$. We shall show that the sequence $\{U_n\}$ satisfies all conditions of Kubáček's theorem when $E = X$, $\rho = d$ and thus Lemma 2 will follow from Lemma 1.

Clearly, (i) is fulfilled. As to the condition (ii), we choose an arbitrary $\varepsilon > 0$. Then there is an $m_0 \in N$ such that $\sum_{m=m_l+1}^{\infty}(1/2^m) < \varepsilon/2$. The condition (v) and the Dini theorem imply that the sequence $\{\varphi_n\}$ converges on $[b,\infty)$ locally uniformly to 0. Therefore for $\varepsilon > 0$ and $m_0 \in N$ there is an $n_0 \in N$ such that $g_m(\varphi_n) = \sup\{|\varphi_n(t)| : b \le t \le b+m\} \le \varepsilon/4m_0$ for $n \ge n_0$ and $m = 1, 2, \dots, m_0$. Hence for $n \ge n_0$ and $x, y \in U_n$ we have

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)} \le \sum_{m=1}^{m_0} p_m(x-y) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \le \\ \le \sum_{m=1}^{m_0} 2g_m(\varphi_n) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \le 2m_0 \frac{\varepsilon}{4m_0} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that (ii) is satisfied. The assumption (iii) follows from (vi) and from the definition of $T, T_n, n \in N$.

Now we show the inclusion $U_n \subset S_n(M)$. The condition (viii) then implies that S_n is the bijection of $S_n^{-1}(U_n)$ onto U_n and is fulfilled, since the continuity of $S_n^{-1}\big|_{U_n}$ is the consequence of the compactness of T_n.

Thus we have to prove that for each $y \in U_n$ there is an $x_y \in M$ such that $x_y - T_n(x_y) = y$. This means that the map $P_n(x) = y + T_n(x)$ has a fixed point for each $y \in U_n$. The condition (vii) implies

$$\sum_{k=0}^{p-1} \Big|y^{(k)}(t) + \big(T_n(x)\big)^{(k)}(t) - r_k\Big| \le \sum_{k=0}^{p-1} |y^{(k)}(t)| + \\ + \sum_{k=0}^{p-1} \Big|\big(T_n(x)\big)^{(k)}(t) - r_k\Big| \le \varphi_n(t) + \varphi_{*n}(t) \le \varphi(t),\ \ t \ge b,$$

and $(P_n(t))^{(k)}(b) = r_k$, $k = 0, \dots, p-1$, for each $x \in M$. Therefore $P_n(M) \subset M$. As M is a closed convex bounded set and $P_n(M) \subset M$

is a compact map, by Tikhonov's fixed point theorem, P_n has a fixed point. ∎

Now the function φ in Lemma 2 will be given as a solution of an integral equation. The existence and some properties of the solution to that equation will be discussed in the following two lemmas.

Lemma 3. *Let* $\psi \in H_{p-1}$, $\omega \in C([b,\infty),[0,\infty))$, $g \in C([0,\infty),[0,\infty))$ *be a nondecreasing function. Further, let*

$$\sigma(t) \equiv 0, \quad b \le t < \infty, \quad \text{if } p = 1, \quad \text{and}$$

$$\sigma(t) = \sum_{l=0}^{p-2} \left(\sum_{k=l+1}^{p-1} \frac{|\psi^{(k)}(0)|}{(k-l)!}(t-b)^{k-l}, \quad b \le t < \infty, \quad \text{if } p \ge 2, \right. \tag{3}$$

$$K(t,s) = \sum_{l=0}^{p-1} \frac{(t-s)^{p-1-l}}{(p-1-l)!}, \quad b \le s \le t. \tag{4}$$

Then the following statements hold:

1. *A solution* $\varphi \in C([b,\infty),[0,\infty))$ *of the integral equation*

$$\varphi(t) = \sigma(t) + \int_b^t K(t,s)\omega(s)g(\|\psi\| + \varphi(s))\,ds, \quad b \le t < \infty, \tag{5}$$

exists and satisfies

$$0 \le \varphi(t) \le \lambda(t), \quad b \le t < \infty, \tag{6}$$

if and only if there exists a function $\lambda \in C([b,\infty),[0,\infty))$ *such that*

$$\lambda(t) \ge \sigma(t) + \int_b^t K(t,s)\omega(s)g(\|\psi\| + \lambda(s))\,ds, \quad b \le t < \infty, \tag{7}$$

i.e., if and only if there exists an upper solution λ *of* (5).

2. *A solution* φ *of* (5) *(whenever it exists) is a nondecreasing function in* $[b,\infty)$.

3. *If* (5) *has a solution* φ *and* $\omega_1 \in C([b,\infty),(0,\infty))$ *satisfies* $0 \le \omega_1(t) \le \omega(t)$ *for* $b \le t < \infty$, *then the equation*

$$\varphi(t) = \sigma(t) + \int_b^t K(t,s)\omega_1(s)g(\|\psi\| + \varphi(s))\,ds \tag{8}$$

has a solution φ_1 *such that*

$$0 \le \varphi_1(t) \le \varphi(t), \quad b \le t < \infty.$$

Proof. 1. The necessity is clear. To prove the sufficiency we shall proceed by the method of steps. Hence we prove by mathematical induction that for each $m = 1, 2, \dots$:

(a) There exists a solution $y_m \in C([b, b+m], [0, \infty))$ of (5) satisfying the inequalities $0 \le y_m(t) \le \lambda(t)$, $b \le t \le b+m$, and

(b) $y_{m+1}(t) = y_m(t), \quad b \le t \le b+m.$

Consider the partially ordered Banach space $X_1 = C([b, b+1], R)$ with the sup-norm where $z_1 \le z_2$ if and only if $z_1(t) \le z_2(t)$ for each $t \in [b, b+1]$ and each pair z_1, z_2 from that space. Then, by definition, the interval $\langle z_1, z_2 \rangle = \{y \in X_1 : z_1(t) \le y(t) \le z_2(t),\ b \le t \le b+1\}$. The operator $U_1 : X_1 \to X_1$ defined by

$$U_1(t) = \sigma(t) + \int_b^t K(t,s)\omega(s)g(\|\psi\| + y(s))\,ds, \quad b \le t \le b+1,$$

is completely continuous, nondecreasing and, in view of (7), maps the interval $\langle 0, \lambda\big|_{[b,b+1]}\rangle$ into itself. Hence by Schauder's fixed point theorem U_1 has a fixed point Y_1 satisfying (6) on $[b, b+1]$.

Suppose now that there exists a solution y_m of (5) on $[b, b+m]$. Consider the space $X_{m+1} = C([b, b+m+1], R)$ with the sup-norm and with the natural ordering. Let U_{m+1} be the operator given by the right-hand side of (5) on $[b, b+m+1]$. U_{m+1} is completely continuous, nondecreasing and maps the interval $\langle 0, \lambda\big|_{[b,b+m+1]}\rangle = \{y \in X_{m+1} : 0 \le y(t) \le \lambda(t),\ b \le t \le b+m+1\}$ into itself. Similarly, U_{m+1} maps the closed and convex set $Y_{m+1} = \{x \in X_{m+1} : x(t) = y_m(t),\ b \le t \le b+m\}$ into itself. Hence $U_{m+1}\big(\langle 0, \lambda\big|_{[b,b+m+1]}\rangle \cap Y_{m+1}\big) \subset \langle 0, \lambda\big|_{[b,b+m+1]}\rangle \cap Y_{m+1}$ and there exists a fixed point y_{m+1} of U_{m+1} in $\langle 0, \lambda\big|_{[b,b+m+1]}\rangle \cap Y_{m+1}$. This is the searched function y_{m+1} with the properties (a) and (b). Then the function $\varphi(t) = y_m(t)$ for $b \le t \le b+m$, $m = 1, 2, \dots$, is a solution of (5) in $[b, \infty)$, satisfying (6).

2. The statement follows from (3), (4) and (5).

3. Since each solution φ of (5) is an upper solution of (8), Statement 1 implies Statement 3. ∎

The existence of an upper solution λ of (5) is provided by

Lemma 4. *Let ψ, σ and K have the same meaning as in Lemma 3 and let $g \in C([0, \infty), (0, \infty))$ be a nondecreasing function. Then $\lambda \in C([b, \infty, [0, \infty))$ is an upper solution of (5) if there is a function*

$\rho \in C([b,\infty),[0,\infty))$ *such that*

$$\lambda(t) = \sigma(t) + \int_b^t K(t,s)\rho(s)\,ds, \quad b \le t < \infty, \tag{9}$$

and

$$0 \le \omega(t) \le \frac{\rho(t)}{g(\|\psi\| + \lambda(t))}, \quad b \le t < \infty. \tag{10}$$

Proof. By combining (9), (10) we get that λ determined by (9) is an upper solution of (5) in $[b,\infty)$. ■

Remark 1. Similarly, the necessary and sufficient condition for $\lambda \in C([b,\infty,[0,\infty))$ to be solution of (5) is that there exists a function $\rho \in C([b,\infty,[0,\infty))$ such that (9) is fulfilled and

$$\omega(t) = \frac{\rho(t)}{g(\|\psi\| + \lambda(t))}, \quad b \le t < \infty.$$

§ 2. The Main Theorem

The main theorem reads as follows.

Theorem 1. *Let* $\psi \in H_{p-1}$, $f \in C([b,\infty) \times H_{p-1} \times \cdots \times h_0, R^d)$. *Let, further,* $\omega \in C([b,\infty),[0,,\infty))$, $g \in C([0,\infty),[0,\infty))$ *be a non-dereasing function and let*

(ix) $|f(t,\chi_t,\dots,\chi_t^{(p-1)}| \le \omega(t)g(\|\chi_t\|)$

for each $(t,\chi) \in [b,\infty) \times M^*$, *where*

$$M^* = \Big\{x \in X^* : \sum_{k=0}^{p-1} \left|x^{(k)}(t) - \psi^{(k)}(0)\right| \le \varphi(t) \ \textit{for}\ t \ge b \ \textit{and}$$
$$x_b^{(k)} = \psi^{(k)},\ k = 0,\dots,p-1\Big\},$$

and φ *is a solution of the equation* (5) *where the functions* σ *and* K *are determined by* (3) *and* (4), *respectively.*

Then the problem (1), (2) *has a solution* x *lying on* M^* *and the set* F^* *is a compact* R_δ*-set in the space* X^*.

Proof. Consider the set

$$M = \Big\{x \in X : \sum_{k=0}^{p-1} \left|x^{(k)}(t) - \psi^{(k)}(0)\right| \le \varphi(t) \ \text{for}\ t \ge b \ \text{and}$$
$$x^{(k)}(b) = \psi^{(k)}(0),\ k = 0,\dots,p-1\Big\}.$$

Clearly, the restriction $P : X^* \to X$ determined by $P(x) = x\big|_{[b,\infty)}$ is a homeomorphism of M^* onto M. Let the map $T : m \to X$ be determined by

$$T(x)(t) = \sum_{k=0}^{p-1} \frac{\psi^{(k)}(0)}{k!}(t-b)^k + \\ + \int_b^t \frac{(t-s)^{p-1}}{(p-1)!} f[s,(P^{-1}x)_s,\dots,(P^{-1}x)_s^{(p-1)}]\,ds, \qquad (11) \\ x \in M, \quad t \geq b,$$

where P^{-1} is the inverse of $P\big|_{M^*}$. Then $F^* = P^{-1}(F)$, where F is the set of all fixed points of the map T. Since the homeomorpic image of the compact R_δ-set is again a compact R_δ-set, it is sufficient to prove that F is a compact R_δ-set in the space X. This will be done by using Lemma 2 where we put $r_k = \psi^{(k)}(0)$, $k = 0,\dots,p-1$. Due to (ix) the maps $T_n : M \to X$ determined by

$$T_n(x)(t) = \begin{cases} \sum_{k=0}^{p-1} \frac{\psi^{(k)}(0)}{k!}(t-b)^k \text{ for } b \leq t \leq b + \frac{1}{n}, \\ \sum_{k=0}^{p-1} \frac{\psi^{(k)}(0)}{k!}(t-b)^k + \int_b^{t-\frac{1}{n}} \frac{(t-\frac{1}{n}-s)^{p-1}}{(p-1)!} \times \\ \times f[s,(P^{-1}x)_s,\dots,(P^{-1}x)_s^{(p-1)}]\,ds \\ \text{for } b + \frac{1}{n} \leq t < \infty \text{ and } x \in M \end{cases} \qquad (12)$$

are, together with T, compact and, again by (ix),

$$\sum_{l=0}^{p-1} \left| \big(T_n(x)\big)^{(l)}(t) - \big(T(x)\big)^{(l)}(t) \right| \leq \\ \leq \begin{cases} \sum_{l=0}^{p-1} \int_b^t \frac{(t-s)^{p-1-l}}{(p-1-l)!}\omega(s)g(\|(p^{-1}x)_s\|)ds, \quad b \leq t \leq b+\frac{1}{n}, \\ \sum_{l=0}^{p-1} \Big\{ \int_b^{t-\frac{1}{n}} \frac{(t-s)^{p-1-l}-(t-\frac{1}{n}-s)^{p-1-l}}{(p-1-l)!}\omega(s)g(\|(p^{-1}x)_s\|)ds + \\ + \int_{t-\frac{1}{n}}^t \frac{(t-s)^{p-1-l}}{(p-1-l)!}\omega(s)g(\|(p^{-1}x)_s\|)\Big\}ds, \quad b+\frac{1}{n} \leq t < \infty. \end{cases}$$

By Lemma 3 φ is nondecreasing in $[b,\infty)$ and therefore $\|(P^{-1}x)_s\| \leq \psi + \varphi(s)$ for each $b \leq s < \infty$, $x \in M$. Hence, using also (4), we get

$$\sum_{l=0}^{p-1} \left| \big(T_n(x)\big)^{(l)}(t) - \big(T(x)\big)^{(l)}(t) \right| \leq \varphi_n(t), \quad t \geq b, \quad x \in M,$$

where

$$\varphi_n(t) = \begin{cases} \int_b^t K(t,s)\omega(s)g(\|\psi\| + \varphi(s))ds, & b \le t \le b + \frac{1}{n}, \\ \int_b^t K(t,s)\omega(s)g(\|\psi\| + \varphi(s))ds - \int_b^{t-\frac{1}{n}} K(t-\frac{1}{n}, s)\omega(s)\times \\ \times(\|\psi\| + \varphi(s))ds, & b + \frac{1}{n} \le t < \infty, \ n \in N. \end{cases}$$

Clearly, $\varphi_n \in C([b,\infty),[0,\infty))$ and the relations $\varphi_{n+1}(t) \le \varphi_n(t)$, $\lim_{n\to\infty}\varphi_n(t) = 0$ can be proved for each $t \ge b$. Therefore these functions satisfy the assumptipons (v), (vi) of Lemma 2.

Further,

$$\sum_{l=0}^{p-1}\left|\left(T_n(x)\right)^{(l)}(t) - \varphi^{(l)}(0)\right| \le \varphi_{*n}(t), \quad t \ge b, \quad x \in M,$$

where

$$\varphi_{*n} = \begin{cases} \sigma(t), \quad b \le t \le b + \frac{1}{n}, \\ \sigma(t) + \int_b^{t-\frac{1}{n}} K(t - \frac{1}{n}, s)\omega(s)g(\|\psi\| + \varphi(s))ds, \\ \qquad\qquad b + \frac{1}{n} \le t < \infty, \end{cases}$$

$n \in N$, σ and K are the functions introduced by (3) and (4). The functions φ_{*n} are nonnegative continuous functions on $[b,\infty)$ and, by virtue of (5) we obtain $\varphi_n(t) + \varphi_{*n}(t) = \varphi(t)$, $t \ge b$, $n \in N$. Hence the assumption (vii) of Lemma 2 is satisfied, too. Thus it remains for us to show that the assumption (viii) holds and then Lemma 2 will imply the statement of Theorem 1.

Let $n \in N$ be arbitrary but fixed. If $x, y \in M$, $x \ne y$, then there exists a $t_0 \in [b,\infty)$ such that $x(t_0) \ne y(t_0)$. If $b \le t_0 \le b + \frac{1}{n}$, then, taking into account (12), we obtain $x(t_0) - T_n(x)(t_0) \ne y(t_0) - T_n(y)(t_0)$. In the other case there is a $t_1 \ge b + \frac{1}{n}$ such that $t_1 = \sup\{\tau > b : x(t) = y(t) \text{ for } b \le t < \tau\}$. Then there exists $t_0 \in (t_1, t_1 + \frac{1}{n})$ such that $x(t_0) \ne y(t_0)$. By (12) we now have

$$T_n(x)(t_0) = \sum_{k=0}^{p=1}\frac{\psi^{(k)}(0)}{k!}(t_0 - b)^k + \int_b^{t_0-\frac{1}{n}}\frac{(t_0 - \frac{1}{n} - s)^{p-1}}{(p-1)!}\times$$

$$\times f[s,(P^{-1}x)_s,\dots,(P^{-1}x)_s^{(p-1)}]ds = \sum_{k=0}^{p-1}\frac{\psi^{(k)}(0)}{k!}(t_0 - b)^k +$$

$$+\int_b^{t_0-\frac{1}{n}}\frac{(t_0 - \frac{1}{n} - s)^{p-1}}{(p-1)!}f[s,(P^{-1}y)_s,\dots,(P^{-1}y)_s^{(p-1)}]ds =$$

$$= T_n(y)(t_0)$$

and thus $x(t_0) - T_n(x)(t_0) \neq y(t_0) - T_n(y)(t_0)$, which we were to prove. ∎

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ON THE MODIFIED BOUNDARY VALUE PROBLEM OF DE LA VALLÉE-POUSSIN FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The sufficient conditions of the existence, uniqueness and correctness of the solution of the modified boundary value problem of de la Vallée-Poussin have been found for a nonlinear ordinary differential equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}),$$

where the function f has nonitegrable singularities with respect to the first argument.

რეზიუმე. დადგენილია ვალე-პუსენის მრავალწერტილოვანი სასაზღვრო ამოცანის მოდიფიკაციის ამოხსნადობის, ცალსახად ამოხსნადობის და კორექტულობის საკმარისი პირობები

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)})$$

არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებისათვის, სადაც f ფუნქციას პირველი არგუმენტის მიმართ გააჩნია არაინტეგრებადი განსაკუთრებულებანი.

§ 1. STATEMENT OF THE MAIN RESULTS

In this paper for an ordinary differential equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1.1}$$

we shall consider the boundary value problem

$$u^{(k-1)}(t_i) = 0 \quad (k = 1, \dots, n_i;\ i = 1, \dots, m),^1 \tag{1.2$_1$}$$

$$\sup\Big\{(t-a)^{l-1-\lambda_1}(b-t)^{l-1-\lambda_2}|u^{(l-1)}(t)| : a < t < b\Big\} < +\infty, \tag{1.2$_2$}$$

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[1]Here $u^{(i)}(a)$ ($u^{(i)}(b)$) denotes the right (left) limit of the function $u^{(i)}$ at the point a (b).

where $n \geq 2$, $l \in \{1, \dots, n\}$, $m \in \{2, \dots, n\}$, $-\infty < a = t_1 < \dots < t_m = b < +\infty$, $n_i \in \{1, \dots, n-1\}$ $(i = 1, \dots, m)$, $\sum_{i=1}^m n_i = n$, $\lambda_1 \in]n_1 - 1, n_1[$, $\lambda_2 \in]n_m - 1, n_m[$, $I_m = [a,b] \setminus \{t_1, \dots, t_m\}$ and the function $f : I_m \times \mathbb{R}^n \to \mathbb{R}$ satisfies the Caratheodori conditions on each compactum contained in $I_m \times \mathbb{R}^n$.

Problem (1.1), (1.2_1) is the well-known boundary value problem of de la Vallée-Poussin and has been studied with sufficient thoroughness both when f is integrable with respect to the first argument on $[a,b]$ (see, for example, [2] and References from [5]) and when f has nonintegrable singularities at the points $t_1, \dots, t_m$ (see, for example, [5]–[7]). However, in the works devoted to the study of problem (1.1), (1.2_1) it is assumed that

$$\int_a^b (t-a)^{n-n_1-1}(b-t)^{n-n_m-1} f_r^*(t)dt < +\infty \quad \text{for} \quad r > 0, \tag{1.3}$$

where

$$f_r^*(t) = \max\Big\{\Big|f\Big(t, \prod_{i=1}^m |t-t_i|^{n_{i1}} x_1, \dots, \prod_{i=1}^m |t-t_i|^{n_{in}} x_n\Big)\Big| : \sum_{k=1}^n |x_k| \leq r\Big\},$$

$$n_{ik} = \begin{cases} n_i - k + 1 & \text{for } k \leq n_i, \\ 0 & \text{for } k > n_i. \end{cases}$$

This assumption is not casual. The matter is that if condition (1.3) is not fulfilled, then problem (1.1), (1.2_1) is not, generally speaking, uniquely solvable even in the simplest case. For example, given the boundary condition (1.2_1), the equation

$$u^{(n)} = \frac{(-1)^n \delta}{(t-a)^n} u$$

has an infinite number of solutions for $n_1 = 1$ and any sufficiently small $\delta > 0$.

Therefore to provide the solution uniqueness we have to introduce an additional and, of course, natural condition such as, for example, (1.2_2). This condition is natural because if (1.3) is fulfilled, then (1.2_1) yields (1.2_2), i.e., problem (1.1), (1.2_1), (1.2_2) coincides with the problem of de la Vallée-Poussin (1.1), (1.2_1). However, if (1.3) is not fulfilled, then, as follows from the above example, this is not so.

Problem (1.1), (1.2_1), (1.2_2) is the generalization of the boundary value problem of de la Vallée-Poussin (1.1), (1.2_1) and has been studied in [14][2] for the linear differential equation

$$u^{(n)} = \sum_{k=1}^{l} p_k(t)u^{(k-1)} + q(t), \tag{1.4}$$

where $p_k : I_m \to \mathbb{R}$ $(k = 1, \dots, l), q :]a, b[\to \mathbb{R}$.

In this paper we shall establish the sufficient conditions for the existence, uniqueness and correctness of the solution of problem (1.1), (1.2_1), (1.2_2). Note that the solution of this problem is sought for in the class of functions $u :]a, b[\to \mathbb{R}$ absolutely continuous with $u^{(k)}$ $(k = 1, \dots, n-1)$ inside $]a, b[$.[3]

The following notation will be used

$$\nu_{kl} = |(k-1-\lambda)\cdots(l-2-\lambda)| \quad (k = 1, \dots, l-1), \quad \nu_{ll}(\lambda) = 1;$$

$$\sigma_{k,\lambda_1,\lambda_2}(t) = (t-a)^{\lambda_1-k+1}(b-t)^{\lambda_2-k+1} \prod_{i=2}^{m-1} |t - t_i|^{n_{ik}};\,^{4}$$

$$\sigma_{k,n}(t) = (t-a)^{n-k}(b-t)^{n-k} \prod_{i=2}^{m-1} |t - t_i|^{n_{ik}};$$

$$\mathbb{R} =]-\infty, +\infty[; \quad \mathbb{R}_+ = [0, +\infty[;$$

$\mathbb{R}^p$, where p is a natural number, is a p-dimensional Euclidean space;

$\mathbb{C}_{loc}^{n-1}(]a, b[; \mathbb{R})$ is a set of functions $v :]a, b[\to \mathbb{R}$ which are continuous, with $v^{(k)}$ $(k = 1, \dots, n-1)$, inside $]a, b[$;

$\widetilde{\mathbb{C}}_{loc}^{n-1}(]a, b[; \mathbb{R})$ is a set of functions $v :]a, b[\to \mathbb{R}$ which are absolutely continuous, with $v^{(k)}$ $(k = 1, \dots, n-1)$ inside $]a, b[$;

$L([a, b]; I)$, where $I \subset \mathbb{R}$, is a set of functions $v : [a, b] \to I$ which are Lebesgue integrable on $[a, b]$;

$L_{loc}(]a, b[; I)$ is a set of functions $v :]a, b[\to I$ which are Lebesgue integrable inside $]a, b[$;

$L_{\alpha,\beta}(]a, b[; I)$, where $\alpha \geq 0$, $\beta \geq 0$, is a set of measurable functions $v :]a, b[\to I$ satisfying the condition

$$|v(\cdot)|_{\alpha,\beta} = \sup\left\{(t-a)^{\alpha}(b-t)^{\beta}\left|\int_{\frac{a+b}{2}}^{t} v(\tau)d\tau\right| : a < t < b\right\} < +\infty;$$

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[2] See also [10]–[13].

[3] I.e., on each segment contained in $]a, b[$.

[4] $\prod_{i=2}^{m-1} |t - t_i|^{n_{ik}}$ will denote unity when $m = 2$.

$\mathbb{K}_{loc}(I \times \mathbb{R}^p; \mathbb{R})$, where $I \in]a,b[$ is a measurable set and p a natural number, is a set of functions $g : I \times \mathbb{R}^p \to \mathbb{R}$ satisfying the Caratheodori conditions on each compactum contained in $I \times \mathbb{R}^p$;

$\mathbb{K}^0_{loc}(I \times \mathbb{R}^p; \mathbb{R})$ is a set of functions $g : I \times \mathbb{R}^p \to \mathbb{R}$ such that $g(\cdot, x_1(\cdot), \dots, x_p(\cdot)) : I \to \mathbb{R}$ is a measurable function for any continuous vector function $(x_1, \dots, x_p) : I \to \mathbb{R}^p$;

$\mathcal{D}_{\lambda_1,\lambda_2}(u_0; r)$, where $r \in \mathbb{R}_+$, $u_0 \in \mathbb{C}^{n-1}_{loc}(]a,b[;\mathbb{R})$ is a set of vectors $(x_1, \dots, x_l) \in \mathbb{R}^l$ satisfying the condition

$$\inf\left\{ \sum_{k=1}^{l} \left| x_k - \frac{u_0^{(k-1)}(t)}{\sigma_{k,\lambda_1,\lambda_2}(t)} \right| : t \in I_m \right\} \le r;$$

$W_{\lambda_1,\lambda_2}(u_0; r)$ is a set of functions $u \in \mathbb{C}^{n-1}_{loc}(]a,b[;\mathbb{R})$ satisfying the condition

$$\sum_{k=1}^{l} \frac{|u^{(k-1)}(t) - u_0^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} \le r \quad \text{for} \quad a < t < b;$$

$M([\alpha,\beta] \times \mathbb{R}_+; \mathbb{R}_+)$, where $a \le \alpha < \beta \le b$, is a set of functions $\omega \in \mathbb{K}_{loc}([\alpha,\beta] \times \mathbb{R}_+; \mathbb{R}_+)$ which are nondecreasing with respect to the second argument and satisfy the condition

$$\omega(t, 0) = 0 \quad \text{for} \quad \alpha \le t \le \beta.$$

Throughout the paper it will be assumed that

$$f \in \mathbb{K}_{loc}(I_m \times \mathbb{R}^n; \mathbb{R})$$

and the solution of problem (1.1), (1.2_1), (1.2_2) will be sought for in the class $\widetilde{\mathbb{C}}^{n-1}_{loc}(]a,b[;\mathbb{R})$.

Some definitions will be given.

Definition 1.1. Let $n_0 \in \{1, \dots, n-1\}$ and $\lambda \in]n_0 - 1, n_0[$. A vector function $(h_1, \dots, h_l)$ with measurable components $h_k :]a,b[\to \mathbb{R}$ $(k = 1, \dots, l)$ is said to belong to the set $S^+(a,b;n,n_0;\lambda)$ $(S^-(a,b;n,n_0;\lambda))$ if there exists $\alpha \in]a,b[$ such that we have the inequality

$$\limsup_{t \to a} \frac{(t-a)^{l-1-\lambda}}{(n-l)!} \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_t^{\alpha} (\tau - t)^{n-l} (\tau - a)^{\lambda-k+1} |h_k(\tau)| d\tau < 1$$

$$\left[\limsup_{t \to b} \frac{(b-t)^{l-1-\lambda}}{(n-l)!} \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_{\alpha}^{t} (t - \tau)^{n-l} (b - \tau)^{\lambda-k+1} |h_k(\tau)| d\tau < 1 \right]$$

in the case $l \in \{n_0+1,\dots,n\}$ and the inequality

$$\limsup_{t\to a}\frac{(t-a)^{l-1-\lambda}}{(n-n_0-1)!(n_0-l)!}\times$$

$$\times\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda)}\int_a^t(t-s)^{n_0-l}\int_s^\alpha(\tau-s)^{n-n_0-1}(\tau-a)^{\lambda-k+1}|h_k(\tau)|d\tau ds<1$$

$$\Big[\limsup_{t\to b}\frac{(b-t)^{l-1-\lambda}}{(n-n_0-1)!(n_0-l)!}\times$$

$$\times\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda)}\int_t^b(s-t)^{n_0-l}\int_\alpha^s(s-\tau)^{n-n_0-1}(b-\tau)^{\lambda-k+1}|h_k(\tau)|d\tau ds<1\Big]$$

in the case $l\in\{1,\dots,n_0\}$.

Definition 1.2. Let

$$\sigma_{k,\lambda_1,\lambda_2}(\cdot)p_{jk}(\cdot)\in L_{loc}(]a,b[;\mathbb{R})\quad (j=1,2;\ k=1,\dots,l),$$

$$p_{1k}(t)\le p_{2k}(t)\quad\text{for}\quad a<t<b\ \ (k=1,\dots,l),$$

$$(p_1^*,\dots,p_l^*)\in S^+(a,b;n,n_1;\lambda_1)\cap S^-(a,b;n,n_m;\lambda_2),$$

where $p_k^*(t)=\max\{|p_{1k}(t)|,|p_{2k}(t)|\}$ $(k=1,\dots,l)$ and, moreover, under the boundary conditions (1.2_1), (1.2_2) the differential equation

$$u^{(n)}=\sum_{k=1}^{l}p_k(t)u^{(k-1)}\tag{1.4_0}$$

have no nontrivial solution no matter what measurable functions $p_k:\ ]a,b[\to\mathbb{R}$ $(k=1,\dots,l)$ satisfying the inequalities

$$p_{1k}(t)\le p_k(t)\le p_{2k}(t)\quad\text{for}\quad a<t<b\ \ (k=1,\dots,l)\tag{1.5}$$

are. Then the vector function $(p_{11},\dots,p_{1l};p_{21},\dots,p_{2l})$ is said to belong to the class $V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2)$.

1.1. Existence and Uniqueness Theorems

The General Case.

Theorem 1.1. *Let the following inequalities be fulfilled on* $]a,b[\times\mathbb{R}^n$:

$$\Big|f(t,x_1,\dots,x_n)-\sum_{k=1}^{l}p_k(t,x_1,\dots,x_n)x_k-q_0(t)\Big|\le$$

$$\le q\Big(t,\sum_{k=1}^{l}\frac{|x_k|}{\sigma_{k,\lambda_1,\lambda_2}(t)}\Big)\tag{1.6}$$

and

$$p_{1k}(t) \leq p_k(t, x_1, \dots, x_n) \leq p_{2k}(t) \quad (k = 1, \dots, l), \tag{1.7}$$

where

$$(p_{11}, \dots, p_{1l}; p_{21}, \dots, p_{2l}) \in V(t_1, \dots, t_m; n_1, \dots, n_m; \lambda_1, \lambda_2), \tag{1.8}$$
$$p_k \in \mathbb{K}^0_{loc}(I_m \times \mathbb{R}^n; \mathbb{R}) \quad (k = 1, \dots, l), \quad q_0 \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}),$$

the function $q :]a, b[\times \mathbb{R}_+ \to \mathbb{R}_+$ *is nondecreasing with respect to the second argument,* $q(\cdot, \varrho) \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}_+)$ *for any* $\varrho \geq 0$, *and*

$$\lim_{\varrho \to +\infty} \frac{|q(\cdot, \varrho)|_{n-1-\lambda_1, n_1-1-\lambda_2}}{\varrho} = 0. \tag{1.9}$$

Then problem (1.1), (1.2_1), (1.2_2) *is solvable.*

Corollary 1.1. *Let the following inequality be fulfilled:*

$$|f(t, x_1, \dots, x_n) - q_0(t)| \leq \sum_{k=1}^{l} p_k(t)|x_k| + \\ + q\Big(t, \sum_{k=1}^{l} \frac{|x_k|}{\sigma_{k,\lambda_1,\lambda_2}(t)}\Big), \tag{1.10}$$

where

$$q_0 \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}),$$
$$(-p_1, \dots, -p_l; p_1, \dots, p_l) \in V(t_1, \dots, t_m; n_1, \dots, n_m; \lambda_1, \lambda_2), \tag{1.11}$$

and the function $q :]a, b[\times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy the conditions of Theorem* 1.1. *Then problem* (1.1), (1.2_1), (1.2_2) *is solvable.*

For the case when the right-hand side of equation (1.1) is independent of the last $n - l$ arguments, i.e.,

$$u^{(n)} = f(t, u, \dots, u^{(l-1)}) \tag{1.1'}$$

we have

Theorem 1.2. *Let*

$$f(\cdot, 0, \dots, 0) \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R})$$

and the function f have partial derivatives with respect to phase arguments belonging to $\mathbb{K}_{loc}(I_m \times \mathbb{R}^l; \mathbb{R})$. Let, moreover, the following inequalities be fulfilled on $]a, b[\times\mathbb{R}^l$:

$$p_{1k}(t) \le \frac{\partial f(t, x_1, \dots, x_l)}{\partial x_k} \le p_{2k}(t) \quad (k = 1, \dots, l), \tag{1.12}$$

and

$$(p_{11}, \dots, p_{1l}; p_{21}, \dots, p_{2l}) \in V(t_1, \dots, t_m; n_1, \dots, n_m; \lambda_1, \lambda_2).$$

Then problem (1.1′), (1.2$_1$), (1.2$_2$) *has the unique solution.*

The case $l = 1$.

In this subsection we shall consider the problem

$$u^{(n)} = f(t, u), \tag{1.13}$$

$$u^{(k-1)}(t_i) = 0 \quad (k = 1, \dots, n_i;\ i = 1, \dots, m), \tag{1.14$_1$}$$

$$\sup\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)| : a < t < b\} < +\infty \tag{1.14$_2$}$$

assuming that

$$f \in \mathbb{K}_{loc}(I_m \times \mathbb{R}; \mathbb{R}).$$

For any $r \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R})$ we introduce the notation

$$\varrho_{\lambda_1,\lambda_2}(r) = \operatorname{vrai\,max}\left\{\frac{|u_0(r)(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)} : a < t < b\right\},$$

where $u_0(r)(\cdot)$ is the unique solution of the equation

$$u^{(n)} = r(t)$$

satisfying the boundary conditions (1.14$_1$), (1.14$_2$) (see, for example, Proposition 2.3).

Theorem 1.3. *Let the following inequality be fulfilled on* $]a, b[\times\mathbb{R}$:

$$|f(t, x) - q_0(t)| \le \frac{r(t)}{\sigma_{1,\lambda_1,\lambda_2}(t)}|x| + q\left(t, \frac{|x|}{\sigma_{1,\lambda_1,\lambda_2}(t)}\right), \tag{1.15}$$

where

$$q_0, r \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}),$$

and the function $q :]a, b[\times\mathbb{R}_+ \to \mathbb{R}_+$ *be nondecreasing with respect to the second argument,* $q(\cdot, \varrho) \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}_+)$ *for any* $\varrho \ge 0$ *and*

$$\lim_{\varrho \to +\infty} \frac{|q(\cdot, \varrho)|_{n-1-\lambda_1, n-1-\lambda_2}}{\varrho} = 0.$$

Let, moreover,

$$\varrho_{\lambda_1,\lambda_2}(r) < 1. \tag{1.16}$$

Then problem (1.13), (1.14_1), (1.14_2) *is solvable.*

Corollary 1.2. *Let* $m = 2$ *and the following inequality be fulfilled on* $]a, b[\times\mathbb{R}$:

$$|f(t,x) - q_0(t)| \le \frac{r_0}{(t-a)^n(b-t)^n}|x| + q\Big(t, \frac{|x|}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}}\Big),$$

where the functions q_0 *and* q *satisfy the conditions of Theorem* 1.3, *and the number* r_0 *the inequality*

$$\begin{aligned} r_0 < \Big[&n_1!n_2!(\lambda_1 - n_1 + 1)(\lambda_2 - n_2 + 1)(n_1 - \lambda_1)(n_2 - \lambda_2)(b-a)^n\Big] \times \\ &\times [2n_1(\lambda_2 - n_2 + 1)(n_1 - \lambda_1)(\lambda_1 - n_1 + 1 + n_2 - \lambda_2) + \\ &+ 2n_2(\lambda_1 - n_1 + 1)(n_2 - \lambda_2)(\lambda_2 - n_2 + 1 + n_1 - \lambda_1)]^{-1}. \end{aligned} \tag{1.17}$$

Then problem (1.13), (1.14_1), (1.14_2) *is solvable.*

For the two-point boundary value problem

$$u'' = f(t,u), \tag{1.18}$$

$$u(a) = u(b) = 0, \tag{1.19_1}$$

$$\sup\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)| : a < t < b\} < +\infty, \tag{1.19_2}$$

where $0 < \lambda_1, \lambda_2 < 1$, and for the three-point boundary value problem

$$u''' = f(t,u), \tag{1.20}$$

$$u(a) = u(t_0) = u(b) = 0, \tag{1.21_1}$$

$$\sup\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)| : a < t < b\} < +\infty, \tag{1.21_2}$$

where $a < t_0 < b$, $0 < \lambda_1, \lambda_2 < 1$, from Theorem 1.3 we obtain

Corollary 1.3. *Let the following inequality be fulfilled on* $]a, b[\times\mathbb{R}$:

$$\begin{aligned} |f(t,x) - q_0(t)| \le r_0\Big(&\frac{\lambda_1(1-\lambda_1)}{(t-a)^2} + \frac{2\lambda_1\lambda_2}{(t-a)(b-t)} + \frac{\lambda_2(1-\lambda_2)}{(b-t)^2}\Big)|x| + \\ &+q\Big(t, \frac{|x|}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}}\Big), \end{aligned}$$

where $r_0 \in]0, 1[$, *and the functions* q_0 *and* q *satisfy the conditions of Theorem* 1.3 *for* $n = m = 2$, $n_1 = n_2 = 1$. *Then problem* (1.18), (1.19_1), (1.19_2) *is solvable.*

Corollary 1.4. *Let the following inequality be fulfilled on* $]a,b[\times\mathbb{R}$*:*

$$|f(t,x)-q_0(t)|\le r_0\Big|\frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}-\frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)}+$$
$$+\frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)}-\frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)}-\frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2}-$$
$$-\frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2}-\frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3}\Big|\,|x|+$$
$$+q\Big(t,\frac{|x|}{(t-a)^{\lambda_1}|t-t_0|(b-t)^{\lambda_2}}\Big),$$

where $r_0\in]0,1[$, *and the functions* q_0 *and* q *satisfy the conditions of Theorem* 1.3 *for* $n=m=3$, $n_1=n_2=n_3=1$. *Then problem* (1.20), (1.21_1), (1.21_2) *is solvable.*

Theorem 1.4. *Let*

$$f(\cdot,0)\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$$

and the following inequality be fulfilled on $]a,b[\times\mathbb{R}$*:*

$$|f(t,x)-f(t,y)|\le\frac{r(t)}{\sigma_{1,\lambda_1,\lambda_2}(t)}|x-y|, \tag{1.22}$$

where the function $r\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ *satisfies condition* (1.16). *Then problem* (1.13), (1.14_1), (1.14_2) *has the unique solution* u *and*

$$\operatorname{vrai\,max}\left\{\frac{|u_j(t)-u(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)}:a<t<b\right\}\to 0\quad for\ j\to+\infty, \tag{1.23}$$

where $u_0(t)\equiv 0$, *and for each natural number* j *the function* u_j *is a solution of the equation*

$$u_j^{(n)}(t)=f(t,u_{j-1}(t)), \tag{1.24}$$

satisfying the boundary conditions (1.14_1), (1.14_2).

Corollary 1.5. *Let* $m=2$,

$$f(\cdot,0)\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$$

and the following inequality be fulfilled on $]a,b[\times\mathbb{R}$*:*

$$|f(t,x)-f(t,y)|\le\frac{r_0}{(t-a)^n(b-t)^n}|x-y|,$$

where r_0 is a number satisfying (1.17). *Then problem* (1.13), (1.14_1), (1.14_2) *has the unique solution u and*

$$\operatorname{vrai\,max}\left\{\frac{|u_j(t)-u(t)|}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}} : a<t<b\right\}\to 0 \quad \textit{for } j\to+\infty,$$

where $u_0(t)\equiv 0$, and for each natural number j the function u_j is a solution of problem (1.24), (1.14_1), (1.14_2).

Corollary 1.6. *Let*

$$f(\cdot,0)\in L_{1-\lambda_1,1-\lambda_2}(]a,b[;\mathbb{R})$$

and the following inequality be fulfilled on $]a,b[\times\mathbb{R}$:

$$|f(t,x)-f(t,y)|\le r_0\left(\frac{\lambda_1(1-\lambda_1)}{(t-a)^2}+\frac{2\lambda_1\lambda_2}{(t-a)(b-t)}+\frac{\lambda_2(1-\lambda_2)}{(b-t)^2}\right)|x-y|,$$

where $r_0\in]0,1[$. Then problem (1.18), (1.19_1), (1.19_2) *has the unique solution u and*

$$\operatorname{vrai\,max}\left\{\frac{|u_j(t)-u(t)|}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}} : a<t<b\right\}\to 0 \quad \textit{for } j\to+\infty,$$

where $u_0(t)\equiv 0$, and for each natural number j the function u_j is a solution of the equation

$$u_j''(t)=f(t,u_{j-1}(t)),$$

satisfying the boundary conditions (1.19_1), (1.19_2).

Corollary 1.7. *Let*

$$f(\cdot,0)\in L_{2-\lambda_1,2-\lambda_2}(]a,b[;\mathbb{R})$$

and the following inequality be fulfilled on $]a,b[\times\mathbb{R}$:

$$|f(t,x)-f(t,y)|\le r_0\left|\frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}-\frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)}+\right.$$
$$+\frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)}-\frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)}-\frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2}-$$
$$\left.-\frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2}-\frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3}\right||x-y|,$$

where $r_0\in]0,1[$. Then problem (1.20), (1.21_1), (1.21_2) *has the unique solution u and*

$$\operatorname{vrai\,max}\left\{\frac{|u_j(t)-u(t)|}{(t-a)^{\lambda_1}|t-t_0|(b-t)^{\lambda_2}} : a<t<b\right\}\to 0 \quad \textit{for } j\to+\infty,$$

where $u_0(t) \equiv 0$, and for each natural number j the function u_j is a solution of the equation

$$u_j'''(t) = f(t, u_{j-1}(t)),$$

satisfying the boundary conditions (1.21_1), (1.21_2).

Remark 1.1. In Corollaries 1.3, 1.4, 1.6 and 1.7 the condition

$$r_0 \in]0,1[\tag{1.25}$$

cannot be replaced by the equality

$$r_0 = 1. \tag{1.26}$$

1.2. Continuous Dependence of Solutions of the Right-Hand Side of the Equation

In this subsection we shall consider the boundary value problem

$$u^{(n)} = f(t, u, u', \dots, u^{(l-1)}), \qquad (1.1')$$

$$u^{(k-1)}(t_i) = 0 \quad (k = 1, \dots, n_i;\ i = 1, \dots, m), \qquad (1.1')$$

$$\sup\Big\{(t-a)^{l-1-\lambda_1}(b-t)^{l-1-\lambda_2}|u^{(l-1)}(t)| : a < t < b\Big\} < +\infty, \qquad (1.2_2)$$

assuminmg that

$$f,\ \frac{\partial f}{\partial x_k} \in \mathbb{K}_{loc}(I_m \times \mathbb{R}^l; \mathbb{R}) \quad (k = 1, \dots, l)$$

and give the sufficient conditions for its solutions to be stable with respect to small perturbations of the right-hand side of the equation (1.1′).

Definition 1.3. Let u_0 be a solution of problem (1.1′), (1.2_1), (1.2_2) and r be a positive number. It will be said that u_0 is r-stable with respect to small perturbations of the right-hand side of equation (1.1′) if for any $\varepsilon \in]0, r[$, $\alpha \in]a, t_2[$, $\beta \in]t_{m-1}, b[$, $(x_{10}, \dots, x_{l0}) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0; r)$ and $\omega \in M([\alpha,\beta] \times \mathbb{R}_+; \mathbb{R}_+)$ there exists $\delta > 0$ such that for any function $\eta \in \mathbb{K}_{loc}(I_m \times \mathbb{R}^l;\mathbb{R})$ satisfying the conditions

$$\left|\int_\alpha^t \eta(\tau,\sigma_{1,\lambda_1,\lambda_2}(\tau)x_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(\tau)x_l)d\tau\right| \le \delta$$

$$\text{for } \alpha \le t \le \beta,\ (x_1,\dots,x_l) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0;r),$$

$$|\eta(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_l) -$$

$$-\eta(t,\sigma_{1,\lambda_1,\lambda_2}(t)y_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(t)y_l)| \le \omega\Big(t,\sum_{k=1}^l |x_k-y_k|\Big)$$

$$\text{for } \alpha \le t \le \beta,\ (x_1,\dots,x_l) \text{ and } (y_1,\dots,y_l) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0;r),$$

$$(t-a)^{n-1-\lambda_1}\left[\int_t^\alpha \eta^*_{\lambda_1,\lambda_2}(\tau;u_0;r)\,d\tau + \left|\int_t^\alpha \eta_0(\tau)d\tau\right|\right] \le \delta$$

$$\text{for } a < t \le \alpha,$$

$$(b-t)^{n-1-\lambda_2}\left[\int_\beta^t \eta^*_{\lambda_1,\lambda_2}(\tau;u_0;r)\,d\tau + \left|\int_\beta^t \eta_0(\tau)d\tau\right|\right] \le \delta$$

$$\text{for } \beta \le t < b,$$

where

$$\eta_0(t) = \eta(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_{10},\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_{l0}),$$

$$\eta^*_{\lambda_1,\lambda_2}(t;u_0;r) = \sup\{|\eta(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_l) - \eta_0(t)| : (x_1,\dots,x_l)\mathcal{D}_{\lambda_1,\lambda_2}(u_0,r)\},$$

the equation

$$u^{(n)} = f(t,u,\dots,u^{(l-1)}) + \eta(t,u,\dots,u^{(l-1)})$$

has at the least one solution in $W_{\lambda_1,\lambda_2}(u_0;r)$ and every such solution is also contained in $W_{\lambda_1,\lambda_2}(u_0;\varepsilon)$.

Definition 1.4. It will be said that the solution u_0 of problem $(1.1')$, (1.2_1), (1.2_2) is stable with respect to small perturbations of the right-hand side of equation $(1.1')$ if it is r-stable for any $r > 0$.

Theorem 1.5. *Let u_0 be a solution of problem* $(1.1')$, (1.2_1), (1.2_2), $r > 0$ *and the following inequalities be fulfilled on* $]a,b[\times\mathcal{D}_{\lambda_1,\lambda_2}(u_0;r)$:

$$p_{1k}(t) \le \frac{\partial f(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_l)}{\partial x_k} \le p_{2k}(t) \quad (k=1,\dots,l), \tag{1.27}$$

where

$$(p_{11},\dots,p_{1l};p_{21},\dots,p_{2l})\in V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2). \quad (1.8)$$

Then u_0 is r-stable with respect to small perturbations of the right-hand side of equation $(1.1')$.

Theorem 1.6. *Let*

$$f(\cdot,0,\dots,0)\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$$

and the following inequalities be fulfilled on $]a,b[\times\mathbb{R}^l$*:*

$$p_{1k}(t)\le\frac{\partial f(t,x_1,\dots,x_l)}{\partial x_k}\le p_{2k}(t)\quad (k=1,\dots,l),$$

where p_{jk} $(j=1,2;\ k=1,\dots,l)$ satisfy condition (1.8). *Then problem* $(1.1')$, (1.2_1), (1.2_2) *has the unique solution u_0 and, moreover, this solution is stable with respect to small perturbations of the right-hand side of equation* $(1.1')$.

§ 2. Auxiliary Propositions

Proposition 2.1. *Let*

$$(p_{11},\dots,p_{1l};p_{21},\dots,p_{2l})\in V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2).$$

Then there exists a positive number ϱ_0 such that for any $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ and measurable functions $p_k:]a,b[\to\mathbb{R}$ $(k=1,\dots,l)$ satisfying inequalities (1.5) *an arbitrary solution u of problem* (1.4), (1.2_1), (1.2_2) *admits the estimate*

$$|u^{(k-1)}(t)|\le\varrho_0\sigma_{k,\lambda_1,\lambda_2}(t)|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}$$
$$\text{for}\quad a<t<b\quad (k=1,\dots,l).$$

Proposition 2.2. *Let*

$$\sigma_{k,\lambda_1,\lambda_2}(\cdot)p_k(\cdot)\in L_{loc}(]a,b[;\mathbb{R})\quad (k=1,\dots,l) \quad (2.1)$$

and

$$(p_1,\dots,p_l)\in S^+(a,b;n,n_1;\lambda_1)\cap S^-(a,b;n,n_m;\lambda_2). \quad (2.2)$$

Then for problem (1.4), (1.2_1), (1.2_2) *to be uniquely solvable for each $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ it is necessary and sufficient that the corresponding homogeneous problem* (1.4_0), (1.2_1), (1.2_2) *have the trivial solution only.*

Proposition 2.3. *Let*

$$p_k(t) = \frac{g_{1k}(t)}{(t-a)^{n-k+1}} + \frac{g_{2k}(t)}{(b-t)^{n-k+1}} + p_{0k}(t) \quad (k = 1, \dots, l),$$

where

$$\sigma_{k,n}(\cdot)p_{0k}(\cdot) \in L([a,b];\mathbb{R}) \quad (k = 1, \dots, l),$$

and $g_{1k}, g_{2k} : [a,b] \to \mathbb{R}$ $(k = 1, \dots, l)$ *are continuous functions satisfying the inequalities*

$$\sum_{k=1}^{l} \frac{|g_{1k}(a)|}{\nu_{kl}(\lambda_1)\nu_{ln+1}(\lambda_1)} < 1, \quad \sum_{k=1}^{l} \frac{|g_{2k}(b)|}{\nu_{kl}(\lambda_2)\nu_{ln+1}(\lambda_2)} < 1.$$

Then for problem (1.4), (1.2_1), (1.2_2) *to be uniquely solvable for each* $q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ *it is necessary and sufficient that the corresponding homogeneous problem* (1.4_0), (1.2_1), (1.2_2) *have the trivial solution only.*

In the rest of this paragraph it is assumed that $p_k :]a,b[\to \mathbb{R}$ $(k = 1, \dots, l)$ are the fixed functions satisfying conditions (2.1), (2.2) and problem (1.4_0), (1.2_1), (1.2_2) has the trivial solution only. Then by Proposition 2.2 problem (1.4), (1.2_1), (1.2_2) has the unique solution for each $q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$. The operator $\mathcal{G} : L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R}) \to \widetilde{C}^{n-1}_{loc}(]a,b[;\mathbb{R})$ that puts the solution $u(t) = \mathcal{G}(q)(t)$ of problem (1.4), (1.2_1), (1.2_2) into the correspondence with each $q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ will be called the Green operator of problem (1.4_0), (1.2_1), (1.2_2).

Proposition 2.4. *There exists a positive number* ϱ_0 *such that for any* $q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ *we have the inequalities*

$$\left|\frac{d^{k-1}\mathcal{G}(q)(t)}{dt^{k-1}}\right| \le \varrho_0 \sigma_{k,\lambda_1,\lambda_2}(t)|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}$$
$$\text{for } a < t < b \quad (k = 1, \dots, l)$$

and

$$\left|\frac{d^{n-1}\mathcal{G}(q)(t)}{dt^{n-1}} - \frac{d^{n-1}\mathcal{G}(q)(s)}{ds^{n-1}}\right| \le \int_s^t p^*(\tau)d\tau + \left|\int_s^t q(\tau)d\tau\right|$$
$$\text{for } a < s \le t < b,$$

where

$$p^*(t) = \varrho_0 |q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} \sum_{k=1}^{l} |p_k(t)|\sigma_{k,\lambda_1,\lambda_2}(t).$$

Proposition 2.5. *Let*

$$q, q_j \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a,b[;\mathbb{R}) \quad (j = 1, 2, \dots),$$

$$\lim_{j\to+\infty} \int_{\frac{a+b}{2}}^{t} q_j(\tau)d\tau = \int_{\frac{a+b}{2}}^{t} q(\tau)d\tau \quad \textit{uniformly inside }]a,b[$$

and

$$\limsup_{j\to+\infty} |q_j(\cdot) - q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} < +\infty.$$

Then

$$\lim_{j\to+\infty} \frac{d^{k-1}\mathcal{G}(q_j)(t)}{dt^{k-1}} = \frac{d^{k-1}\mathcal{G}(q)(t)}{dt^{k-1}} \quad \textit{uniformly inside }]a,b[.$$

The proofs of Propositions 2.1-2.5 are given in [14] [5] (see Lemma 1, Theorems 1 and 5, Corollaries 1 and 4).

To conclude this paragraph let us consider a quasilinear differential equation

$$u^{(n)} = \sum_{k=1}^{l} p_k(t)u^{(k-1)} + q(t, u, \dots, u^{(n-1)}), \tag{2.3}$$

where $p_k : I_m \to \mathbb{R}$ $(k = 1, \dots, l)$ are measurable functions satisfying (2.1), (2.2),

$$q \in K_{loc}(]a,b[\times\mathbb{R}^n;\mathbb{R}). \tag{2.4}$$

Proposition 2.6. *Let problem* (1.4_0), (1.2_1), (1.2_2) *have the trivial solution only. Let, moreover, there exist functions* $q_0 \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a,b[;\mathbb{R})$ *and* $q^* \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a,b[;\mathbb{R}_+)$ *such that the inequality*

$$|q(t, x_1, \dots, x_n) - q_0(t)| \le q^*(t) \tag{2.5}$$

is fulfilled on $]a,b[\times\mathbb{R}^n$. *Then problem* (2.3), (1.2_1), (1.2_2) *is solvable.*

Proof. Let $\mathcal{G}$ be the Green operator of problem (1.4_0), (1.2_1), (1.2_2). By Proposition 2.4 there exists $\varrho_0 > 0$ such that the inequalities

$$\left|\frac{d^{k-1}\mathcal{G}(\tilde{q})(t)}{dt^{k-1}}\right| \le \varrho_0 \sigma_{k,\lambda_1,\lambda_2}(t) |\tilde{q}(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \quad \text{for } a < t < b \ (k = 1, \dots, l) \tag{2.6}$$

[5]See also [12] and [13]

and

$$\left|\frac{d^{n-1}\mathcal{G}(\tilde q)(t)}{dt^{n-1}}-\frac{d^{n-1}\mathcal{G}(\tilde q)(s)}{ds^{n-1}}\right|\le$$

$$\le\varrho_0|\tilde q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\sum_{k=1}^{l}\int_s^t|p_k(\tau)|\sigma_{k,\lambda_1,\lambda_2}(\tau)d\tau+\left|\int_s^t\tilde q(\tau)d\tau\right| \quad (2.7)$$

$$\text{for } a<s\le t<b$$

are fulfilled for any $\tilde q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$.

Let $\mathbb{C}^{n-1}(]a,b[;\mathbb{R})$ be a topologic space of $n-1$ times continuously differentiable functions $u:]a,b[\to\mathbb{R}$, where under the convergence of the sequence $(u_i)_{i=1}^{+\infty}$ we mean the uniform convergence of the sequences $(u_i^{(k)})_{i=1}^{+\infty}$ $(k=0,\dots,n-1)$ inside $]a,b[$.

It is assumed that

$$\varrho=\varrho_0\Big[|q_0(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}+|q^*(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\Big],$$

$$p^*(t)=\varrho\sum_{k=1}^{l}|p_k(t)|\sigma_{k,\lambda_1,\lambda_2}(t)+|q_0(t)|+q^*(t),$$

and A is a set of all elements u of the space $\mathbb{C}^{n-1}(]a,b[;\mathbb{R})$ satisfying the inequalities

$$|u^{(k-1)}(t)|\le\varrho\sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for } a<t<b \ (k=1,\dots,l),$$

$$|u^{(n-1)}(t)-u^{(n-1)}(s)|\le\int_s^t p^*(\tau)d\tau \quad \text{for } \ a<s\le t<b.$$

It is obvious that A is a convex set. On the other hand, by the Arcela–Askoli lemma it immediately follows that A is a compactum.

The operator $\tilde{\mathcal{G}}$ is given on A as follows:

$$\tilde{\mathcal{G}}(u)(t)=\mathcal{G}(q(\cdot,u(\cdot),\dots,u^{(n-1)}(\cdot)))(t) \quad \text{for } \ u\in A.$$

According to (2.5)–(2.7), for any $u\in A$ the function $\tilde u(\cdot)=\tilde{\mathcal{G}}(u)(\cdot)$ satisfies the inequalities

$$|\tilde u^{(k-1)}(t)|\le\varrho_0|q(\cdot,u(\cdot),\dots,u^{(n-1)}(\cdot))|_{n-1-\lambda_1,n-1-\lambda_2}\sigma_{k,\lambda_1,\lambda_2}(t)\le$$
$$\le\varrho\sigma_{k,\lambda_1,\lambda_2}(t) \ \ a<t<b \ \ \text{for } a<t<b \ (k=1,\dots,l),$$

$$|\tilde u^{(n-1)}(t)-\tilde u^{(n-1)}(s)|\le\varrho\sum_{k=1}^{l}\int_s^t|p_k(\tau)|\sigma_{k,\lambda_1,\lambda_2}(\tau)\,d\tau+$$

$$+\left|\int_s^t q(\tau,u(\tau),\dots,u^{(n-1)}(\tau))d\tau\right|\le\int_s^t p^*(\tau)d\tau \ \ \text{for } a<s\le t<b.$$

Therefore the operator $\tilde{\mathcal{G}}$ maps A into itself. On the other hand, by Proposition 2.5 condition (2.4) guarantees the continuity of the operator $\tilde{\mathcal{G}}$. According to the Chauder–Tikhonov theorem [4], $\tilde{\mathcal{G}}$ has at the least one fixed point. Therefore there exists a function u such that

$$u(t) = \mathcal{G}(q(\cdot, u(\cdot), \dots, u^{(n-1)}(\cdot)))(t) \quad \text{for } a < t < b.$$

Hence it is clear that u is the solution of problem (2.3), (1.2_1), (1.2_2). ■

§ 3. Proof of the Main Results

Proof of Theorem 1.1. Let ρ_0 be a positive number for which Proposition 2.1 is valid. According to (1.9) there exists $\rho^* > 0$ such that

$$l\rho_0\Big[|q(\cdot,\rho)|_{n-1-\lambda_1, n-1-\lambda_2} + |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}\Big] < \rho \quad \text{for} \quad \rho \geq \rho^*. \tag{3.1}$$

Let

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \rho^*, \\ 2 - \dfrac{s}{\rho^*} & \text{for } \rho^* \leq s \leq 2\rho^*, \\ 0 & \text{for } s \geq 2\rho^* \end{cases} \tag{3.2}$$

and $\varepsilon_0 = \frac{a+b}{4}$. Assume that for each j

$$\varepsilon_j(t) = \begin{cases} 0 & \text{for } t \in]a, a + \frac{\varepsilon_0}{j}[\cup]b - \frac{\varepsilon_0}{j}, b[, \\ 1 & \text{for } t \in [a + \frac{\varepsilon_0}{j}, b - \frac{\varepsilon_0}{j}], \end{cases} \tag{3.3}$$

$$q_j(t, x_1, \dots, x_n) = \varepsilon_j(t)\chi\Big(\sum_{i=1}^{l} \frac{|x_i|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\Big)\Big[f(t, x_1, \dots, x_n) - \\ - \sum_{k=1}^{l} p_{1k}(t)x_k - q_0(t)\Big] + q_0(t) \tag{3.4}$$

and consider the differential equation

$$u^{(n)} = \sum_{k=1}^{l} p_{1k}(t)u^{(k-1)} + q_j(t, u, \dots, u^{(n-1)}) \tag{3.5}$$

for an arbitrary natural number j.

Taking into account inequalities (1.6), (1.7), from (3.2)–(3.4) we obtain on $]a,b[\times\mathbb{R}^n$

$$|q_j(t, x_1, \dots, x_n) - q_0(t)| \leq q_j^*(t),$$

where

$$q_j^*(t) = \varepsilon_j(t)q(t,2\rho^*) + 2\rho^*\varepsilon_j(t)\sum_{k=1}^{l}|p_{2k}(t) - p_{1k}(t)|\sigma_{k,\lambda_1,\lambda_2}(t).$$

By (1.8) and (3.3) it is easy to ascertain that

$$q_j^* \in L([a,b];\mathbb{R}_+).$$

Therefore, according to Proposition 2.6, Problem (3.5), (1.2_1), (1.2_2) has a solution u_j.

By (3.4) and (3.5) it is obvious that

$$u_j^{(n)}(t) = \sum_{k=1}^{l}\tilde{p}_{kj}(t)u_j^{(k-1)}(t) + \tilde{q}_j(t), \tag{3.6}$$

where

$$\tilde{p}_{kj}(t) = p_{1k}(t) + \varepsilon_j(t)\chi\Big(\sum_{i=1}^{l}\frac{|u_j^{(i-1)}(t)|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\Big)\times$$
$$\times\Big[p_k(t,u_j(t),\dots,u_j^{(n-1)}(t)) - p_{1k}(t)\Big],$$
$$\tilde{q}_{kj}(t) = \varepsilon_j(t)\chi\Big(\sum_{i=1}^{l}\frac{|u_j^{(i-1)}(t)|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\Big)\Big[f(t,u_j(t),\dots,u_j^{(n-1)}(t)) -$$
$$-\sum_{k=1}^{l}p_k(t,u_j(t),\dots,u_j^{(n-1)}(t))u_j^{(k-1)}(t) - q_0(t)\Big] + q_0(t).$$

On the other hand, (1.6) and (1.7) imply

$$p_{1k}(t) \le \tilde{p}_{kj}(t) \le p_{2k}(t) \quad \text{for } a<t<b \ (k=1,\dots,l) \tag{3.7}$$

and

$$|\tilde{q}_j(t) - q_0(t)| \le q(t,\rho_j) \quad \text{for } a<t<b, \tag{3.8}$$

where

$$\rho_j = \operatorname{vrai\,max}\Big\{\sum_{i=1}^{l}\frac{|u_j^{(i-1)}(t)|}{\sigma_{i,\lambda_1,\lambda_2}(t)} : a<t<b\Big\}. \tag{3.9}$$

In view of conditions (1.8), (3.7) and the choice of the number ρ_0 we have

$$|u_j^{(i-1)}(t)| \le \rho_0|\tilde{q}_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\sigma_{i,\lambda_1,\lambda_2}(t)$$
$$\text{for } a<t<b \ (i=1,\dots,l).$$

Hence, taking into account (3.8) and (3.9), we find

$$\rho_j \leq l\rho_0\Big[|q_0(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} + |q(\cdot,\rho_j)|_{n-1-\lambda_1,n-1-\lambda_2}\Big].$$

Consequently, by (3.1)

$$\rho_j \leq \rho^*.$$

According to this inequality, for each j (3.2), (3.6)–(3.9) imply

$$\chi\Big(\sum_{i=1}^{l}\frac{|u_j^{(i-1)}(t)|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\Big) = 1 \quad \text{for} \quad a<t<b, \tag{3.10}$$

$$|\tilde{q}_j(t) - q_0(t)| \leq q(t,\rho^*) \quad \text{for} \quad a<t<b, \tag{3.11}$$

$$|u_j^{(k-1)}(t)| \leq \rho^*\sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for } a<t<b \; (k=1,\dots,l) \tag{3.12}$$

and

$$|u_j^{(n)}(t) - q_0(t)| \leq p^*(t) \quad \text{for} \quad a<t<b, \tag{3.13}$$

where

$$p^*(t) = \rho^*\sum_{k=1}^{l}\Big(|p_{1k}(t)| + |p_{2k}(t)|\Big)\sigma_{k,\lambda_1,\lambda_2}(t) + q(t,\rho^*)$$

belongs to the class $L_{loc}(]a,b[;\mathbb{R}_+)$.

By virtue of (3.12) and (3.13) the sequences $(u_j^{(k-1)})_{j=1}^{+\infty}$ $(k=1,\dots,n)$ are uniformly bounded and equicontinuous inside $]a,b[$. Therefore by the Arcela–Askoli lemma it can be assumed without loss of generality that they converge uniformly inside $]a,b[$.

Let

$$u(t) = \lim_{j\to+\infty} u_j(t) \quad \text{for} \quad a<t<b.$$

Then

$$u^{(k-1)}(t) = \lim_{j\to+\infty} u_j^{(k-1)}(t) \quad \text{for} \quad a<t<b \;\; (k=1,\dots,n). \tag{3.14}$$

Taking into account (3.3), (3.6), (3.10) and (3.13), it readily follows from (3.14) that u is a solution of equation (1.1).

On the other hand, since the sequences $(u_j^{(k-1)})_{j=1}^{+\infty}$ $(k=1,\dots,n)$ are uniformly bounded inside $]a,b[$ and on account of (1.8), (3.11), (3.12) and the equalities

$$u_j^{(k-1)}(a) = 0 \;\; (k=1,\dots,n_1), \quad u_j^{(k-1)}(b) = 0 \;\; (k=1,\dots,n_m),$$

from (3.6) we find

$$|u_j^{(k-1)}(t)| \le r_1(t-a)^{\lambda_1-k+1} + \int_a^t (t-s)^{n_1-k} \int_s^{\frac{a+b}{2}} (\tau-s)^{n-n_1-1}\widetilde{p}(\tau)\,d\tau ds$$

$$\text{for } a < t \le \frac{a+b}{2}, \quad l \in \{1,\dots,n_1\} \quad (k=1,\dots,n_1)$$

and

$$|u_j^{(k-1)}(t)| \le r_2(b-t)^{\lambda_2-k+1} + \int_t^b (s-t)^{n_m-k} \int_{\frac{a+b}{2}}^s (s-\tau)^{n-n_m-1}\widetilde{p}(\tau)\,d\tau ds$$

$$\text{for } \frac{a+b}{2} \le t < b, \quad l \in \{1,\dots,n_m\} \quad (k=1,\dots,n_m),$$

where

$$\widetilde{p}(t) = p^*(t) - q(t,\rho^*),$$

and r_1 and r_2 are positive numbers not depending on j. Taking into account these estimates together with (3.12) and (3.14), we ascertain that u satisfies the boundary conditions (1.2_1),(1.2_2).

Thus u is the solution of problem (1.1),(1.2_1),(1.2_2). ■

Proof of Corollary 1.1. We set

$$\gamma(t,x_1,\dots,x_n) = \Big[f(t,x_1,\dots,x_n) - q_0(t)\Big] \times$$
$$\times \left[1 + \sum_{k=1}^{l} p_k(t)|x_k| + q\left(t, \sum_{i=1}^{l} \frac{|x_i|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\right)\right]^{-1}$$

and

$$\widetilde{p}_k(t,x_1,\dots,x_n) = \gamma(t,x_1,\dots,x_n)p_k(t)\,\mathrm{sign}\,x_k \quad (k=1,\dots,l).$$

Then

$$f(t,x_1,\dots,x_n) = \sum_{k=1}^{l} \widetilde{p}_k(t,x_1,\dots,x_n)x_k +$$
$$+ \left[1 + q\left(t, \sum_{i=1}^{l} \frac{|x_i|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\right)\gamma(t,x_1,\dots,x_n)\right] + q_0(t)$$

and

$$\widetilde{p}_k \in K^0_{loc}(I_m \times \mathbb{R}^n; \mathbb{R}). \tag{3.15}$$

On the other hand, according to (1.9) and (1.10), on $]a,b[\times\mathbb{R}^n$ we have

$$\begin{gathered}|\gamma(t,x_1,\dots,x_n)|<1,\\ -p_k(t)\le \tilde p_k(t,x_1,\dots,x_n)\le p_k(t)\quad (k=1,\dots,l)\end{gathered}\tag{3.16}$$

and

$$\Big|f(t,x_1,\dots,x_n)-\sum_{k=1}^{l}\tilde p_k(t,x_1,\dots,x_n)x_k-q_0(t)\Big|\le$$
$$\le \tilde q\Big(t,\sum_{i=1}^{l}\frac{|x_i|}{\sigma_{i,\lambda_1,\lambda_2}(t)}\Big),\tag{3.17}$$

where the function

$$\tilde q(t,\rho)=1+q(t,\rho)$$

satisfies the condition

$$\lim_{\rho\to+\infty}\frac{|\tilde q(\cdot,\rho)|_{n-1-\lambda_1,n-1-\lambda_2}}{\rho}=0.\tag{3.18}$$

By Theorem 1.1 from conditions (1.11), (3.15)–(3.18) we conclude that problem (1.1),(1.2_1),(1.2_2) is solvable. ■

Proof of Theorem 1.2. By inequalities (1.12), the equality

$$f(t,x_1,\dots,x_l)=\sum_{k=1}^{l}p_k(t,x_1,\dots,x_l)x_k+q_0(t),$$

where

$$p_k(t,x_1,\dots,x_l)=\int_0^1\frac{\partial f(t,\xi x_1,\dots,\xi x_l)}{\partial x_k}\,d\xi\quad (k=1,\dots,l),$$
$$q_0(t)=f(t,0,\dots,0),$$

yields

$$\Big|f(t,x_1,\dots,x_l)-\sum_{k=1}^{l}p_k(t,x_1,\dots,x_l)x_k-q_0(t)\Big|=0$$

and

$$p_{1k}(t)\le p_k(t,x_1,\dots,x_l)\le p_{2k}(t)\quad (k=1,\dots,l).$$

Therefore by Theorem 1.1 problem $(1.1')$,(1.2_1),(1.2_2) has a solution u_0.

Let us show that u_0 is the unique solution of this problem. Assume that u_1 is an arbitrary solution of problem $(1.1'),(1.2_1),(1.2_2)$. Set

$$u(t) = u_0(t) - u_1(t), \quad \tilde{p}_k(t; x_1, \dots, x_l; y_1, \dots, y_l) = \\ = \int_0^1 \frac{\partial f(t, \xi x_1 + (1-\xi)y_1, \dots, \xi x_l + (1-\xi)y_l)}{\partial x_k} d\xi$$

and

$$p_k(t) = \tilde{p}_k(t; u_0(t), \dots, u_0^{(l-1)}(t); u_1(t), \dots, u_1^{(l-1)}(t)).$$

By virtue of the equality

$$f(t, x_1, \dots, x_l) - f(t, y_1, \dots, y_l) = \sum_{k=1}^{l} \tilde{p}_k(t; x_1, \dots, x_l; y_1, \dots, y_l)(x_k - y_k)$$

and inequalities (1.12) u is a solution of the equation

$$u^{(n)} = \sum_{k=1}^{l} p_k(t) u^{(k-1)}, \tag{3.19}$$

satisfying the boundary conditions $(1.2_1),(1.2_2)$ and

$$p_{1k}(t) \le p_k(t) \le p_{2k}(t) \quad \text{for } a < t < b \ (k = 1, \dots, l). \tag{3.20}$$

But by (1.8) and (3.20) problem $(3.19),(1.2_1),(1.2_2)$ has the trivial solution only, i.e. $u_1(t) \equiv u_0(t)$. ■

Proof of Theorem 1.3 Set

$$p(t) = \frac{r(t)}{\sigma_{1,\lambda_1,\lambda_2}(t)} \quad \text{for } a < t < b.$$

Let us show that

$$(p) \in S^+(a, b; n, n_1; \lambda_1) \cap S^-(a, b; n, n_m; \lambda_2). \tag{3.21}$$

By virtue of (1.14_1) there exist points $t_{0n_1}, t_{0n_1+1}, \dots, t_{0n-1}$ such that

$$a < t_{0n_1} \le \dots \le t_{0n-1} < b \tag{3.22}$$

and

$$u_0^{(k)}(r)(t_{0k}) = 0 \quad (k = n_1, \dots, n-1). \tag{3.23}$$

At the same time,

$$u_0^{(n)}(r)(t) = r(t) \ge 0 \quad \text{for } a < t < b.$$

Therefore by (3.22) it is obvious that for each $k \in \{n_1, \dots, n-1\}$ the function $u_0^{(k)}(r)(\cdot)$ does not change its sign on $]a, t_{0k}[$ and, taking into account (3.23), we readily obtain

$$|u_0^{(n_1)}(r)(t)| \geq \frac{1}{(n-n_1-1)!} \int_t^{t_{0n_1}} (\tau - t)^{n-n_1-1} r(\tau)\, d\tau \tag{3.24}$$

$$\text{for} \quad a < t \leq t_{0n_1}.$$

At the same time, by (1.14_1) we have

$$|u_0(r)(t)| = \frac{1}{(n_1-1)!} \int_a^t (\tau - t)^{n_1-1} |u^{(n_1)}(r)(\tau)|\, d\tau$$

$$\text{for} \quad a < t \leq \min\{t_2, t_{0n_1}\}.$$

Using (1.16), choose $\alpha \in]a, t_2[$ such that $\alpha \leq t_{0n_1}$ and

$$\rho_{\lambda_1,\lambda_2}(r) < \frac{(b-\alpha)^{\lambda_2} \prod_{i=2}^{m-1} |\alpha - t_i|^{n_i}}{(b-a)^{\lambda_2} \prod_{i=2}^{m-1} |a - t_i|^{n_i}}. \tag{3.25}$$

By (3.24) the latter equality yields

$$|u_0(r)(t)| \geq \frac{(b-\alpha)^{\lambda_2} \prod_{i=2}^{m-1} |\alpha - t_i|^{n_i}}{(n_1-1)!(n-n_1-1)!} \times$$

$$\times \int_a^t (t-s)^{n_1-1} \int_s^\alpha (\tau - t)^{n-n_1-1} (\tau - a)^{\lambda_1} |p(\tau)|\, d\tau ds$$

$$\text{for} \ a < t \leq \alpha,$$

whence

$$\rho_{\lambda_1,\lambda_2}(r) \geq \limsup_{t \to a} \frac{|u_0(r)(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)} \geq$$

$$\geq \frac{(b-\alpha)^{\lambda_2} \prod_{i=2}^{m-1} |\alpha - t_i|^{n_i}}{(b-a)^{\lambda_2} \prod_{i=2}^{m-1} |a - t_i|^{n_i}} \limsup_{t \to a} \frac{(t-a)^{-\lambda_1}}{(n_1-1)!(n-n_1-1)!} \times$$

$$\times \int_a^t (t-s)^{n_1-1} \int_s^\alpha (\tau - t)^{n-n_1-1} (\tau - a)^{\lambda_1} |p(\tau)|\, d\tau ds.$$

Therefore by virtue of (3.25) we obtain

$$\limsup_{t\to a} \frac{(t-a)^{-\lambda_1}}{(n_1-1)!(n-n_1-1)!} \times$$

$$\times \int_a^t (t-s)^{n_1-1} \int_s^\alpha (\tau-t)^{n-n_1-1}(\tau-a)^{\lambda_1}|p(\tau)|\,d\tau ds < 1.$$

The inequality

$$\limsup_{t\to b} \frac{(b-t)^{-\lambda_2}}{(n_m-1)!(n-n_m-1)!} \times$$

$$\times \int_t^b (s-t)^{n_m-1} \int_\beta^s (s-\tau)^{n-n_m-1}(b-\tau)^{\lambda_2}|p(\tau)|\,d\tau ds < 1,$$

where $\beta \in]t_{m-1}, b[$, is proved in a similar manner. Thus (3.21) is valid.
Now let us show that

$$(-p,p) \in V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2). \tag{3.26}$$

Due to (3.21) it it is sufficient to verify that under the boundary conditions (1.14_1),(1.14_2) the equation

$$u^{(n)} = \tilde{p}(t)u \tag{3.27}$$

has the trivial solution only for any function $\tilde{p} :]a,b[\to \mathbb{R}$ satisfying the conditions

$$\sigma_{1,\lambda_1,\lambda_2}(\cdot)\tilde{p}(\cdot) \in L_{loc}(]a,b[;\mathbb{R})$$

and

$$-p(t) \le \tilde{p}(t) \le p(t) \quad \text{for } a < t < b. \tag{3.28}$$

Let u be an arbitrary solution of problem (3.27),(1.14_1),(1.14_2) and $g(\cdot,\cdot)$ be the Green function of the equation

$$u^{(n)} = 0 \tag{3.29}$$

by the boundary conditions of de la Vallée-Poussin (1.14_1). According to the Chichkin theorem [3][6]

$$g(t,\tau)(t-t_1)^{n_1}\cdots(t-t_m)^{n_m} \ge 0 \quad \text{for } a \le t,\tau \le b.$$

Therefore

$$|u_0(r)(t)| = \int_a^b |g(t,\tau)|r(\tau)\,d\tau. \tag{3.30}$$

[6]See also [9], Lemma 4.2.

On account of (3.28), (3.30) the equality

$$u(t) = \int_a^b g(t,\tau)\tilde{p}(\tau)u(\tau)\,d\tau$$

yields

$$\gamma \le \gamma \rho_{\lambda_1,\lambda_2}(r),$$

where

$$\gamma = \operatorname{vrai\,max}\left\{\frac{|u(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)} : a < t < b\right\}.$$

Hence, taking into account (1.16), we obtain $\gamma = 0$, i.e. $u(t) \equiv 0$. Therefore (3.26) holds. Now by virtue of (1.15) we conclude that all the conditions of Corollary 1.1 are fulfilled. Therefore problem (1.13),(1.14_1),(1.14_2) is solvable. ■

Proof of Corollary 1.2. We introduce the notation

$$\eta(a,b;n_1,n_2;\lambda_1,\lambda_2) =$$
$$= \left[n_1!n_2!(\lambda_1 - n_1 + 1)(\lambda_2 - n_2 + 1)(n_1 - \lambda_1)(n_2 - \lambda_2)(b-a)^n\right] \times$$
$$\times \Big[2n_1(\lambda_2 - n_2 + 1)(n_1 - \lambda_1)(\lambda_1 - n_1 + 1 + n_2 - \lambda_2) +$$
$$+2n_2(\lambda_1 - n_1 + 1)(n_2 - \lambda_2)(\lambda_2 - n_2 + 1 + n_1 - \lambda_1)\Big]^{-1},$$
$$r(t) = \frac{r_0}{(t-a)^{n-\lambda_1}(b-t)^{n-\lambda_2}}.$$

Now condition (1.17) can be written as

$$r_0 < \eta(a,b;n_1,n_2;\lambda_1,\lambda_2). \tag{3.31}$$

By Theorem 1.3, to prove the corollary it is sufficient to show that

$$\rho_{\lambda_1,\lambda_2}(r) = \sup\left\{\frac{|u_0(r)(t)|}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}} : a < t < b\right\} < 1,$$

where $u_0(r)(\cdot)$ is the solution of the equation

$$u^{(n)} = r(t),$$

satisfying the boundary conditions (1.14_1) for $m = 2$.

As shown in [1], for $m = 2$ the Green function $g(\cdot,\cdot)$ of problem (3.29),(1.14_1) admits the estimate

$$|g(t,\tau)| \le \frac{(t-a)^{n_1-1}(b-t)^{n_2-1}(\tau-a)^{n_2-1}(b-\tau)^{n_1-1}}{(n_1-1)!(n_2-1)!(b-a)^{n-1}} \times$$
$$\times \begin{cases} \frac{(t-a)(b-\tau)}{n_1} & \text{for } t \le \tau \\ \frac{(b-t)(\tau-a)}{n_2} & \text{for } t < \tau \end{cases}.$$

Therefore from the equality

$$u_0(r)(t) = \int_a^b g(t,\tau)r(\tau)\,d\tau$$

we find

$$|u_0(r)(t)| \le \frac{r_0}{\eta(a,b;n_1,n_2;\lambda_1,\lambda_2)}(t-a)^{\lambda_1}(b-t)^{\lambda_2},$$

whence by (3.31) we obtain

$$\rho_{\lambda_1,\lambda_2}(r) < 1. \blacksquare$$

Proof of Corollary 1.3. Let

$$r(t) = r_0(t-a)^{\lambda_1}(b-t)^{\lambda_2}\left[\frac{\lambda_1(1-\lambda_1)}{(t-a)^2} + \right.$$
$$\left. + \frac{2\lambda_1\lambda_2}{(t-a)(b-t)} + \frac{\lambda_2(1-\lambda_2)}{(b-t)^2}\right].$$

Clearly, $r \in L_{1-\lambda_1,1-\lambda_2}(]a,b[;\mathbb{R}_+)$. On the other hand, it is not difficult to verify that the function

$$u_0(r)(t) = -r_0(t-a)^{\lambda_1}(b-t)^{\lambda_2}$$

is the solution of the problem

$$u'' = r(t), \quad u(a) = u(b) = 0.$$

By condition (1.25) we obtain

$$\rho_{\lambda_1,\lambda_2}(r) = r_0 < 1.$$

Thus all the conditions of Theorem 1.3 are fulfilled. $\blacksquare$

Proof of Corollary 1.4. Let

$$r(t)=r_0(t-a)^{\lambda_1}(t_0-t)(b-t)^{\lambda_2}\Big[\frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}-\frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)}+$$

$$+\frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)}-\frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)}-\frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2}-$$

$$-\frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2}-\frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3}\Big].$$

Clearly, $r\in L_{2-\lambda_1,2-\lambda_2}(]a,b[;\mathbb{R})$. On the other hand, it is not difficult to verify that the function

$$u_0(r)(t)=-r_0(t-a)^{\lambda_1}(t-t_0)(b-t)^{\lambda_2}$$

is the solution of the problem

$$u'''=r(t),\quad u(a)=u(t_0)=u(b)=0.$$

By condition (1.25) we obtain

$$\rho_{\lambda_1,\lambda_2}(r)=r_0<1.$$

Thus all the conditions of Theorem 1.3 are fulfilled. ■

Proof of Theorem 1.4. As we have ascertained in proving Theorem 1.3, condition (1.16) guarantees the fulfilment of condition (3.26), where

$$p(t)=\frac{r(t)}{\sigma_{1,\lambda_1,\lambda_2}(t)}.$$

Therefore, according to Theorem 1.2, problem (1.13),(1.14_1),(1.14_2) has the unique solution u.

From the conditions of the theorem we have

$$f(\cdot,\tilde u(\cdot))\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$$

for any $\tilde u\in C^{n-1}_{loc}(]a,b[;\mathbb{R})$ satisfying the boundary conditions (1.14_1), (1.14_2). Therefore, by Proposition 2.3, for each natural number j problem (1.24),(1.14_1),(1.14_2) has the unique solution u_j.

It is assumed that for each j

$$v_j(t)=u_j(t)-u(t). \tag{3.32}$$

Clearly, v_j satisfies the boundary conditions (1.14_1), (1.14_2),

$$\gamma_j=\operatorname{vrai}\max\left\{\frac{|v_j(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)}:a<t<b\right\}<+\infty \tag{3.33}$$

and

$$v_j^{(n)} = f(t, u_{j-1}(t)) - f(t, u(t)). \tag{3.34}$$

Repeating the reasoning from the proof of Theorem 1.3 and using the Chichkin theorem, by virtue of (1.22), (3.32)–(3.34) we obtain

$$\gamma_j \leq \gamma_{j-1}\rho_{\lambda_1,\lambda_2}(r),$$

Hence

$$\gamma_j \leq \gamma_1\rho_{\lambda_1,\lambda_2}^{j-1}(r) \quad (j = 1, 2, \dots).$$

Therefore by (1.16), (3.32), (3.33) we obtain (1.23). ■

Corollaries 1.5-1.7 are proved similarly to Corollaries 1.2–1.4, the only difference being that Theorem 1.4 is used instead of Theorem 1.3.

Remark 3.1. In Corollaries 1.3, 1.4, 1.6 and 1.7 condition (1.25) cannot be replaced by equality (1.26), since for $\lambda_1 \in]\frac{1}{2}, 1[$ problem (1.18),(1.19_1),(1.19_2), where

$$f(t, x) = -\frac{\lambda_1(1-\lambda_1)}{(t-a)^2}\, x + (t-a)^{\lambda_1 - 2},$$

and problem (1.20),(1.21_1),(1.21_2), where

$$f(t, x) = \frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}\, x + (t-a)^{\lambda_1 - 3},$$

have no solutions though all the conditions of these corollaries are fulfilled with an exception of condition (1.25) which is replaced by (1.26).

Proof of Theorem 1.5. Let us assume that the theorem is not true. Then there exist $\varepsilon \in]0, r[$, $\alpha \in]a, t_2[$, $\beta \in]t_{m-1}, b[$, $(x_{10}, \dots, x_{l0}) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0; r)$, $\omega \in M([\alpha, \beta] \times \mathbb{R}_+; \mathbb{R}_+)$ and a sequence of functions $\eta_i \in K_{loc}(I_m \times \mathbb{R}^l; \mathbb{R})$ such that

$$\left|\int_\alpha^t \eta_i(\tau, \sigma_{1,\lambda_1,\lambda_2}(\tau)x_1, \dots, \sigma_{1,\lambda_1,\lambda_2}(\tau)x_l)\, d\tau\right| \leq \frac{1}{i} \tag{3.35}$$

$$\text{for } \alpha \leq t \leq \beta, \quad (x_1, \dots, x_l) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0; r),$$

$$\Big|\eta_i(t, \sigma_{1,\lambda_1,\lambda_2}(t)x_1, \dots, \sigma_{l,\lambda_1,\lambda_2}(t)x_l) -$$

$$-\eta_i(t, \sigma_{1,\lambda_1,\lambda_2}(t)y_1, \dots, \sigma_{l,\lambda_1,\lambda_2}(t)y_l)\Big| \leq \omega\Big(\sum_{k=1}^{l} |x_k - y_k|\Big) \tag{3.36}$$

$$\text{for } \alpha \leq t \leq \beta, \quad (x_1, \dots, x_l) \text{ and } (y_1, \dots, y_l) \in \mathcal{D}_{\lambda_1,\lambda_2}(u_0; r),$$

$$(t-a)^{n-1-\lambda_1}\Big[\int_t^\alpha \eta^*_{\lambda_1,\lambda_2,i}(\tau;u_0;r)\,d\tau+\Big|\int_t^\alpha \eta_{0i}(\tau),d\tau\Big|\Big]\le\frac{1}{i} \tag{3.37}$$

for $a<t\le\alpha$,

$$(b-t)^{n-1-\lambda_2}\Big[\int_\beta^t \eta^*_{\lambda_1,\lambda_2,i}(\tau;u_0;r)\,d\tau+\Big|\int_\beta^t \eta_{0i}(\tau),d\tau\Big|\Big]\le\frac{1}{i} \tag{3.38}$$

for $\beta\le<t<b$,

where

$$\eta_{0i}(t)=\eta_i(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_{10},\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_{l0}),$$

$$\eta^*_{\lambda_1,\lambda_2,i}(t;u_0;r)=$$

$$=\sup\Big\{|\eta_i(t,\sigma_{1,\lambda_1,\lambda_2}(t)x_1,\dots,\sigma_{l,\lambda_1,\lambda_2}(t)x_l)-\eta_{0i}|:$$

$$(x_1,\dots,x_l)\in\mathcal{D}_{\lambda_1,\lambda_2}(u_0;r)\Big\},$$

and for each i the equation

$$u^{(n)}=f(t,u,\dots,u^{(l-1)})+\eta_i(t,u,\dots,u^{(l-1)}) \tag{3.39}$$

either has no solution contained in $W_{\lambda_1,\lambda_2}(u_0;r)$ or has at least one solution contained in $W_{\lambda_1,\lambda_2}(u_0;r)\backslash W_{\lambda_1,\lambda_2}(u_0;\varepsilon)$.

It is assumed that

$$p_k(t,x_1,\dots,x_l)=\int_0^1\frac{\partial f(t,u_0(t)+\xi x_1,\dots,u_0^{(l-1)}(t)+\xi x_l)}{\partial x_k}\,d\xi$$

$$(k=1,\dots,l), \tag{3.40}$$

$$\chi(s)=\begin{cases}1 & \text{for } 0\le s\le r,\\ 2-\frac{s}{r} & \text{for } r<s<2r, ,\\ 0 & \text{for } s\ge 2r,\end{cases} \tag{3.41}$$

$$\tilde p_k(t,x_1,\dots,x_l)=p_k\Big(t,x_1\chi\Big(\sum_{j=1}^l\frac{|x_j|}{\sigma_{j,\lambda_1,\lambda_2}(t)}\Big),\dots,x_l\chi\Big(\sum_{j=1}^l\frac{|x_j|}{\sigma_{j,\lambda_1,\lambda_2}(t)}\Big)\Big)$$

$$(k=1,\dots,l), \tag{3.42}$$

$$\tilde\eta_i(t,x_1,\dots,x_l)=$$

$$=\eta_i\Big(t,u_0(t)+x_1\chi\Big(\sum_{j=1}^l\frac{|x_j|}{\sigma_{j,\lambda_1,\lambda_2}(t)}\Big),\dots,u_0^{(l-1)}(t)+$$

$$+x_l\chi\Big(\sum_{j=1}^l\frac{|x_j|}{\sigma_{j,\lambda_1,\lambda_2}(t)}\Big)\Big)\quad(k=1,\dots,l). \tag{3.43}$$

Let us consider the equation

$$v^{(n)} = \sum_{k=1}^{l} \widetilde{p}_k(t, v, \dots, v^{(l-1)}) v^{(k-1)} + \widetilde{\eta}_i(t, v, \dots, v^{(l-1)}) \quad (3.44)$$

for each i. By (1.37), (3.40)–(3.43) ascertain that the following inequalities are fulfilled on $]a, b[\times \mathbb{R}^l$:

$$p_{1k}(t) \le \widetilde{p}_k(t, x_1, \dots, x_l) \le p_{2k}(t) \quad (k = 1, \dots, l) \quad (3.45)$$

and

$$|\widetilde{\eta}_i(t, x_1, \dots, x_l) - \eta_{0i}(t)| \le q_i(t), \quad (3.46)$$

where

$$q_i(t) = \begin{cases} \omega\Big(t, r + \sum_{j=1}^{l} \frac{|u_0^{(j-1)}(t)|}{\sigma_{j,\lambda_1,\lambda_2}(t)} + |x_{k0}|\Big) & \text{for } \alpha \le t \le \beta, \\ \eta^*_{\lambda_1,\lambda_2}(t; u_0; r) & \text{for }]a, b[\setminus[\alpha, \beta]. \end{cases}$$

From (3.37), (3.38) we readily obtain

$$\eta_{0i}, q_i \in L_{n-1-\lambda_1, n-1-\lambda_2}(]a, b[; \mathbb{R}) \quad (i = 1, 2, \dots).$$

However, since condition (1.8), by virtue of Theorem 1.1 equation (3.44) has, for each natural number i, at the least one solution satisfying (1.2_1),(1.2_2).

Let v_0 be an arbitrary solution of problem (3.44),(1.2_1),(1.2_2). Then either

$$\operatorname{vrai\,max}\Big\{ \sum_{k=1}^{l} \frac{|v_0^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b \Big\} > r,$$

or

$$\operatorname{vrai\,max}\Big\{ \sum_{k=1}^{l} \frac{|v_0^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b \Big\} \le r.$$

If the latter inequality is fulfilled, then in view of (3.40)-(3.43) and the equality

$$f(t, u_0(t) + x_1, \dots, u_0^{(l-1)}(t) + x_l) - f(t, u_0(t), \dots, u_0^{(l-1)}(t)) =$$
$$= \sum_{k=1}^{l} p_k(t, x_1, \dots, x_l) x_k$$

it is obvious that the function

$$u(t) = u_0(t) + v_0(t)$$

is a solution of problem (3.39),(1.2_1),(1.2_2) contained in $W_{\lambda_1,\lambda_2}(u_0;r)$. However, by our assumption, in the case under consideration problem (3.39),(1.2_1),(1.2_2) has at the least one solution $\widetilde{u}$ such that $\widetilde{u} \in W_{\lambda_1,\lambda_2}(u_0;r)\setminus W_{\lambda_1,\lambda_2}(u_0;\varepsilon)$.

It is obvious that the function

$$v(t) = \widetilde{u}(t) - u_0(t)$$

is a solution of problem (3.44),(1.2_1),(1.2_2) satisfying the inequality

$$\operatorname{vrai\,max}\left\{\sum_{k=1}^{l}\frac{|v^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b\right\} > \varepsilon.$$

Thus, for each natural number i problem (3.44),(1.2_1),(1.2_2) has the solution v_i satisfying the condition

$$\operatorname{vrai\,max}\left\{\sum_{k=1}^{l}\frac{|v_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b\right\} > \varepsilon. \tag{3.47}$$

Let ρ_0 be the number from Proposition 2.1. Then in view of (1.8) and (3.45) the inequality

$$|v_i^{(k-1)}(t)| \le \rho_0|\widetilde{\eta}_i(\cdot, v_i(\cdot), \dots, v_i^{(l-1)}(\cdot))|_{n-1-\lambda_1,n-1-\lambda_2}\sigma_{k,\lambda_1,\lambda_2}(t)$$
$$\text{for} \quad a < t < b \;\; (k = 1,\dots,l) \tag{3.48}$$

is fulfilled for each i.

By (3.37), (3.38) and (3.46) we have

$$\eta^* = \sup\left\{|\widetilde{\eta}(\cdot, v_i(\cdot), \dots, v_i^{(l-1)}(\cdot))|_{n-1-\lambda_1,n-1-\lambda_2} : i = 1,2,\dots\right\} \le$$
$$\le \sup\left\{|\eta_{0i}(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} + |q_i(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} : i = 1,2,\dots\right\} < +\infty.$$

Using (3.45), (3.46), (3.48) and the latter inequality, from (3.44) we have

$$\sum_{k=1}^{l}|v_i^{(k-1)}(t)| \le \rho^*, \quad |v_i^{(n)} - \eta_{0i}(t)| \le p^*(t)$$
$$\text{for} \quad \alpha \le t \le \beta \;\; (i = 1,2,\dots), \tag{3.49}$$

where

$$\rho^* = \eta^*\rho_0 \max\left\{\sum_{k=1}^{l}\sigma_{k,\lambda_1,\lambda_2}(t) : \alpha \le t \le \beta\right\},$$

$$p^*(t) = \eta^*\rho_0 \sum_{k=1}^{l}\Big(|p_{1k}(t)| + |p_{2k}(t)|\Big)\sigma_{k,\lambda_1,\lambda_2}(t) +$$

$$+\omega\left(t,r+\sum_{j=1}^{l}\left(\frac{|u_0^{(j-1)}(t)|}{\sigma_{j,\lambda_1,\lambda_2}(t)}+|x_{j0}|\right)\right)$$

and

$$p^*\in L([\alpha,\beta];\mathbb{R}).$$

Moreover, since

$$\max\left\{\left|\int_s^t \eta_{0i}(\tau)\,d\tau\right| : \alpha\le s\le t\le\beta\right\}\to 0 \quad\text{for}\quad i\to+\infty,$$

it follows from (3.49) that the sequences $(v_i^{(k-1)})_{i=1}^{+\infty}$ $(k=1,\dots,l)$ are uniformly bounded and equicontinuous on $[\alpha,\beta]$. Therefore, by Lemma 3.1 [8], conditions (3.35), (3.36), (3.41) and (3.43) imply

$$\gamma_i=\max\left\{\left|\int_\alpha^t \tilde\eta_i(\tau,v_i(\tau),\dots,v_i^{(l-1)}(\tau))\,d\tau\right| : \alpha\le t\le\beta\right\}\to 0 \quad\text{for } i\to+\infty. \tag{3.50}$$

In view of (3.37),(3.38),(3.46) and (3.50)

$$|\tilde\eta_i(\cdot,v_i(\cdot),\dots,v_i^{(l-1)}(\cdot))|_{n-1-\lambda_1,n-1-\lambda_2}\le$$
$$\le\left(2\gamma_j(b-a)^{2n-2-\lambda_1-\lambda_2}+\frac{1}{i}\left[(b-a)^{n-1-\lambda_1}+(b-a)^{n-1-\lambda_2}\right]\right)\to 0$$
$$\text{for}\quad i\to+\infty$$

and now from (3.48) we find

$$\operatorname{vrai\,max}\left\{\sum_{k=1}^{l}\frac{|v_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a<t<b\right\}\to 0 \quad\text{for } i\to+\infty.$$

But this contradicts (3.47), which proves the theorem. ■

Theorem 1.6 immediately follows from Theorems 1.2 and 1.5.

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ავტორთა საყურადღებოდ

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CONTENTS

Vol.1, No. 4 August 1993

76202