საქართველოს მეცნიერებათა აკადემიის მაცნე. მათემატიკა

PROCEEDINGS

of the

GEORGIAN ACADEMY OF SCIENCES.

MATHEMATICS

Volume 1
Number 3
June 1993

თბილისი * TBILISI

PROCEEDINGS საქართველოს
of the GEORGIAN ACADEMY OF SCIENCES .
მეცნიერებათა აკადემიის მოამბე.
MATHEMATICS მათემატიკა

Editorial Office: A.Razmadze Mathematical Institute of the Georgian Academy of Sciences, Rukhadze St.1, Tbilisi 380093
Republic of Georgia

The *Proceedings of the Georgian Academy of Sciences. Mathematics* is issued bimonthly as of February 1993.

Subscriptions and orders for publications should be addressed to the Editorial Offic

Published by "Metsniereba", D.Gamrekeli St.19, Tbilisi 380060

გამომცემლობა „მეცნიერება", თბილისი, 380060, დ.გამრეკელის ქ.19

საქართველოს მეცნ. აკადემიის სტამბა, თბილისი, 380060, დ.გამრეკელის ქ.19

Typeset by TEX

Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993), No. 3, 257-262

SOME OPEN PROBLEMS ABOUT THE SOLUTIONS OF THE DELAY DIFFERENCE EQUATION $x_{n+1} = A/x_n^2 + 1/x_{n-k}^p$

M.ARCIERO, G.LADAS AND S.W.SCHULTZ

ABSTRACT. We discuss the dynamics of the positive solutions of the delay difference equation in the title for some special values of the parameters A, p and k and we pose a conjecture and two open problems.

რეზიუმე. ნაშრომში განხილულია სათაურში მოყვანილი დაგვიანებული სხვაობიანი განტოლების დადებითი ამონახსნების დინამიკა A, p და k პარამეტრების ზოგიერთი კერძო მნიშვნელობისათვის. ჩამოყალიბებულია ერთი ჰიპოთეზა და დასმულია ორი ამოცანა.

1. Introduction. Consider the difference equation

$$x_{n+1} = \frac{A}{x_n^2} + \frac{1}{\sqrt{x_{n-1}}}, \quad n = 0, 1, \dots, \tag{1}$$

where $A \in (0, \infty)$ and the initial conditions x_{-1} and x_0 are arbitrary positive numbers. The following conjecture is predicted by computer simulations.

2. Conjecture. *Let $\bar{x}$ denote the unique positive equilibrium of Eq.* (1).

(a) *Show that when*

$$0 < A < \frac{15}{4} \tag{2}$$

the positive equilibrium of Eq. (1) *is globally asymptotically stable.*

(b) *Show that when*

$$A > \frac{15}{4} \tag{3}$$

there exists a periodic cycle with period two which is asymptotically stable.

1991 *Mathematics Subject Classification.* 39A12.

With the use of a computer one can easily experiment with difference equations and one can easily discover that such equations possess fascinating properties with a great deal of structure and regularity. Of course all computer observations and predictions must also be proven analytically. Therefore this is a fertile area of research, still in its infancy, with deep and important results which require our attention.

For some developments on the global behavior of solutions of delay difference equations the reader is referred to the forthcoming monograph by Kocic and Ladas [2]. See also [1] and [3].

Although we are unable to establish the above conjecture, we have proven the following result.

Theorem 1. (a) *Assume that* (2) *holds. Then the positive equilibrium* $\bar{x}$ *of Eq.* (1) *is locally asymptotically stable.*

(b) *Assume that* (3) *holds. Then Eq.* (1) *has a periodic solution with period two.*

Proof. (a) Set $\varrho = \sqrt{\bar{x}}$. Then the linearized equation of Eq. (1) about $\bar{x}$ is

$$y_{n+1} + \frac{2A}{\varrho^6} y_n + \frac{1}{2\varrho^3} y_{n-1} = 0, \quad n = 0, 1, \ldots . \tag{4}$$

From the well-known Schur-Cohn criterion, Eq. (4) is asymptotically stable provided that

$$\frac{2A}{\varrho^6} < 1 + \frac{1}{2\varrho^3} < 2. \tag{5}$$

Note that ϱ satisfies the equation

$$\varrho^2 = \frac{A}{\varrho^4} + \frac{1}{\varrho}. \tag{6}$$

Hence $\varrho > 1$ and (5) is satisfied if and only if

$$2A < \varrho^6 + \frac{1}{2}\varrho^3 = A + \frac{3}{2}\varrho^3,$$

that is,

$$\varrho > \left(\frac{2A}{3}\right)^{1/3}. \tag{7}$$

Set

$$f(t) = t^6 - t^3 - A$$

and observe that

$$f(t) < 0 \quad \text{if} \quad 0 < t < \varrho$$

and

$$f(t) > 0 \quad \text{if} \quad t > \varrho.$$

Hence (7) is equivalent to $f\left(\left(\frac{2A}{3}\right)^{1/3}\right) < 0$; that is

$$A < \frac{15}{4}.$$

(b) Eq. (1) has a periodic solution of the form

$$\{p, q, p, q, \dots\} \quad \text{or} \quad \{q, p, q, p, \dots\}$$

if and only if

$$p = \frac{A}{q^2} + \frac{1}{\sqrt{p}} \quad \text{and} \quad q = \frac{A}{p^2} + \frac{1}{\sqrt{q}}. \tag{8}$$

Set $x = \sqrt{p}$ and $y = \sqrt{q}$. Then the system of algebraic equations (8) is equivalent to

$$\left.\begin{aligned} x^2 &= \frac{A}{y^4} + \frac{1}{x} \\ y^2 &= \frac{A}{x^4} + \frac{1}{y} \end{aligned}\right\} \quad \text{with} \quad x, y > 0. \tag{9}$$

Set

$$\xi = x + y, \quad \eta = xy \quad \text{and} \quad \zeta = \eta^3.$$

Then x and y are the roots of the quadratic equation

$$\lambda^2 - \xi\lambda + \eta = 0$$

and these roots are real, positive, and distinct if and only if

$$\xi, \eta \in (0, \infty) \quad \text{and} \quad \eta < \frac{1}{4}\xi^2. \tag{10}$$

Cancel the denominators in (9), then multiply the first equation by x and the second by y, equate the terms x^4y^4, and divide by $x - y$. This leads to

$$A\xi = \eta(\xi^2 - \eta). \tag{11}$$

Cancel the denominators in (9), subtract and then divide by $x - y$. This yields

$$\eta^3 = -A + \xi(\xi^2 - 2\eta). \tag{12}$$

Subtract from the first equation in (9), the second, and use (12) to obtain

$$\xi = \frac{(A-1)\eta^3 + A^2}{\eta^4}. \tag{13}$$

By substituting (13) into (11) we find

$$G(\zeta) = \zeta^3 + (A-1)\zeta^2 + A^2(2-A)\zeta - A^4 = 0. \tag{14}$$

Note that

$$G(z) < 0 \text{ if } z < \zeta \text{ and } G(z) > 0 \text{ if } z > \zeta. \tag{15}$$

In view of (10) and (13) we obtain

$$4\zeta^3 < (A-1)^2\zeta^2 + 2A^2(A-1)\zeta + A^4$$

and so by using (14) we find

$$H(\zeta) = (A+3)(A-1)\zeta^2 + 2A^2(3-A)\zeta - 3A^4 > 0.$$

The positive root of this quadratic equation is $\zeta = 3A^2/(A+3)$ and so $H(\zeta) > 0$ if and only if $G(3A^2/(A+3)) < 0$, that is

$$A > \frac{15}{4}.$$

The proof of the theorem is complete. ■

3. Open problems. A related difference equation is

$$x_{n+1} = \frac{a}{x_n^2} + \frac{1}{x_{n-1}}, \quad n = 0, 1, \dots, \tag{16}$$

where $a \in (0, \infty)$ and $x_{-1}, x_0 \in (0, \infty)$.

One can show that the following result holds.

Theorem 2. *The following statements are true:*

(a) *The unique positive equilibrium $\bar{x}$ of Eq.* (16) *is locally asymptotically stable if*

$$a < 2\sqrt{3} \tag{17}$$

and unstable if

$$a > 2\sqrt{3}. \tag{18}$$

(b) *When* (18) *holds, Eq.* (16) *has a periodic cycle with period two,* $\{p, q, p, q, \dots\}$.

Furthermore

$$p = \frac{a + \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2} \quad \text{and} \quad q = \frac{a - \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2},$$

Open problem 1. (a) For what values of a is the positive equilibrium $\bar{x}$ of Eq. (16) globally asymptotically stable?

(b) For what values of a is the periodic cycle $\{p, q, p, q, \dots\}$ of Eq. (16) asymptotically stable? What is its basin of attraction?

Eqs. (1) and (16) are special cases of the delay difference equation

$$x_{n+1} = \frac{A}{x_n^2} + \frac{1}{x_{n-k}^p} \quad n = 0, 1, \dots \tag{19}$$

where

$$A, p \in (0, \infty) \quad \text{and} \quad k \in \{0, 1, \dots\}$$

and the initial conditions $x_{-k}, \dots, x_0$ are arbitrary positive numbers.

Open problem 2. (a) Obtain conditions on A, p and k under which the positive equilibrium of Eq. (19) is globally asymptotically stable.

(b) Obtain conditions on A, p and k under which Eq. (19) has periodic cycles of period two. Under what conditions on A, p and k are these periodic cycles stable? What is the basin of attraction?

(c) Do there exist periodic cycles of period greater than two?

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(Received 12.04.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993) No 3, 263-271

ON CONVERGENCE SUBSYSTEMS OF ORTHONORMAL SYSTEMS

G. BARELADZE

ABSTRACT. It is proved that for any sequence $\{R_k\}_{k=1}^{\infty}$ of real numbers satisfying

$$R_k \geq k \ \ (k \geq 1) \ \ \text{and} \ \ R_k = o(k \log_2 k), \ \ k \to \infty,$$

there exists a orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0;1)$, such that none of its subsystems $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ with $n_k \leq R_k$ $(k \geq 1)$ is a convergence subsystem.

რეზიუმე. მტკიცდება, რომ ნამდვილ რიცხვთა ყოველი $\{R_k\}_{k=1}^{\infty}$ მიმდევრობისათვის, რომელიც აკმაყოფილებს პირობებს

$$R_k \geq k \ \ (k \geq 1) \ \ \text{და} \ \ R_k = o(k \log_2 k), \ \ k \to \infty,$$

მოიძებნება ისეთი ორთონორმირებული სისტემა $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0;1)$, რომლის არცერთი $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ ქვესისტემა, სადაც $n_k \leq R_k$ $(k \geq 1)$, არ არის კრებადობის ქვესისტემა.

Let $\{\varphi_n(x)\}$ be an orthonormal system (ONS) on $(0;1)$. It is called a convergence system if the series $\sum c_n \varphi_n(x)$ is convergent almost everywhere whenever the sequence $\{c_n\}$ of real numbers satisfies $\sum c_n^2 < \infty$.

It is well-known [1] that not every ONS $\{\varphi_n(x)\}$ is a convergence system. However [2], [3], each of them contains some convergence subsystem $\{\varphi_{n_k}(x)\}$. A question was formulated later [4] whether there exists a common estimate of growth rate of numbers n_k in the class of all ONS. B.S.Kashin [5] answered this question in the affirmative: one can determine a sequence of positive numbers $\{R_k\}$ such that from any ONS it is possible to choose a convergence subsystem $\{\varphi_{n_k}\}$ with $n_k \leq R_k$, $1 \leq k < \infty$. In the same paper [5] the problem of finding $\{R_k\}$ with a minimal admissible growth order is formulated and the hypothesis $R_k = k^{1+\varepsilon}$ $(\varepsilon > 0)$ is conjectured. G.A.Karagulyan [6] proved that one can take $R_k = \lambda^k$, $\lambda > 1$. However, this upper estimate is rougher than the one expected in [5].

1991 *Mathematics Subject Classification.* 42C20.

In this paper we shall give the proof of the theorem providing the lower estimate for $\{R_k\}$.

Theorem. *For any sequence* $\{R_k\}_{k=1}^{\infty}$ *of real numbers satisfying*

$$R_k \geq k \ \ (k \geq 1) \ \ \text{and} \ \ R_k = o(k \log_2 k), \ \ k \to \infty, \tag{1}$$

there exists an ONS $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0;1)$, *such that none of its subsystems* $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ *with* $n_k \leq R_k$ $(k \geq 1)$ *is a convergence subsystem.*

Several lemmas are needed to prove this theorem.

Lemma 1 (H.Rademacher [7]). *For any ONS* $\{\psi_n(x)\}_{n=1}^{N}$, $x \in (0;1)$, *and any collection of real numbers* $\{c_n\}_{n=1}^{N}$

$$\int_0^1 \left(\max_{1 \leq j \leq N} \left| \sum_{n=1}^{j} c_n \psi_n(x) \right| \right)^2 dx \leq c \log_2^2 (N+1) \sum_{n=1}^{N} c_n^2, \quad 1 \leq N < \infty.$$ [1]

Lemma 2. *For any* $N \geq 1$ *there exists an ONS*

$$\psi(N) := \left\{ \psi_n^N(x) \right\}_{n=1}^{N}, \quad x \in (0;1),$$

satisfying, for any collection of natural numbers $1 \leq n_1 < n_2 < \cdots < n_m \leq N$ $(1 \leq m \leq N)$, *the inequality*

$$\int_0^1 \max_{1 \leq j \leq m} \left| \sum_{k=1}^{j} \psi_{n_k}^N(x) \right| dx \geq c \frac{\log_2 N}{\sqrt{N}} m. \tag{2}$$

Proof. We shall assure that the requirements of this lemma are satisfied by the ONS usually used in the proof of the Menshov-Rademacher theorem. The functions $\psi_n^N(x)$, $1 \leq n \leq N$, belonging to this ONS (see [8], p.295) have, in particular, the following properties:

(i)

$$\psi_n^N(x) := \begin{cases} \frac{c\sqrt{N}}{s-n}, & x \in \left(\frac{s-1}{2N}; \frac{2s-1}{4N} \right), \\ \frac{c\sqrt{N}}{n-s}, & x \in \left(\frac{2s-1}{4N}; \frac{s}{2N} \right), \quad 1 \leq s \leq N, \ s \neq n; \\ 0, & x \in \left(\frac{n-1}{2N}; \frac{n}{2N} \right); \end{cases}$$

[1] Here and in what follows c denotes positive absolute constants which, in general, may differ from one equality (inequality) to another.

(ii) $\psi_n^N(x)$ is constant on each of the intervals

$$\left(\frac{s-1}{4N};\frac{s}{4N}\right),\quad 2N+1\le s\le 4N;$$

(iii)

$$\int_0^1 \psi_n^N(x)\,dx=0;$$

(iv) $\psi_n^N(x)$ is extended from $(0;1)$ onto $(-\infty;\infty)$ with period 1.

Denote $\delta_s:=\left(\frac{s-1}{2N};\frac{2s-1}{4N}\right)$, $1\le s\le N$. When $x\in\delta_s$, because of (i) we have $\psi_p^N(x)\ge 0$ for $1\le p\le s$ and $\psi_p^N(x)\le 0$ for $s\le p\le N$ $(1\le s\le N)$. Therefore for fixed numbers $1\le n_1<n_2<\cdots<n_m\le N$ $(1\le m\le N)$ and for each $x\in\delta_s$ $(1\le s\le N)$ we obtain

$$\sum_{k=1}^m\left|\psi_{n_k}^N(x)\right|=\sum_{k:n_k\le s}+\sum_{k:n_k>s}=\left|\sum_{k:n_k\le s}\psi_{n_k}^N(x)\right|+$$
$$+\left|\sum_{k:n_k>s}\psi_{n_k}^N(x)\right|\le 3\max_{1\le j\le m}\left|\sum_{k=1}^j\psi_{n_k}^N(x)\right|.$$

Hence

$$\int_{\delta_s}\max_{1\le j\le m}\left|\sum_{k=1}^j\psi_{n_k}^N(x)\right|dx\ge\frac{1}{3}\int_{\delta_s}\sum_{k=1}^m\left|\psi_{n_k}^N(x)\right|dx=$$
$$=\frac{c}{\sqrt{N}}\sum_{\substack{k:1\le k\le m,\\ n_k\ne s}}\frac{1}{|n_k-s|},\quad 1\le s\le N;$$

$$\int_0^1\max_{1\le j\le m}\left|\sum_{k=1}^j\psi_{n_k}^N(x)\right|dx\ge\sum_{s=1}^N\int_{\delta_s}\max_{1\le j\le m}\left|\sum_{k=1}^j\psi_{n_k}^N(x)\right|dx\ge$$
$$\ge\frac{c}{\sqrt{N}}\sum_{s=1}^N\sum_{\substack{k:1\le k\le m,\\ n_k\ne s}}\frac{1}{|n_k-s|}=$$
$$=\frac{c}{\sqrt{N}}\sum_{k=1}^m\sum_{\substack{s:1\le s\le N,\\ s\ne n_k}}\frac{1}{|n_k-s|}\ge c\frac{\log_2 N}{\sqrt{N}}m.\ \blacksquare$$

Remark 1. For any positive integer Q, $\{\psi_n^N(Qx)\}_{n=1}^N$ is an ONS on $(0;1)$ also satisfying the inequality (2).

Remark 2. Let N_0, N_1, Q_0, Q_1, p be positive integers. If $Q_1=4pN_0Q_0$, then functions belonging to different collections $\{\psi_n^{N_0}(Q_0x)\}_{n=1}^{N_0}$ and $\{\psi_n^{N_1}(Q_1x)\}_{n=1}^{N_1}$ are mutually orthogonal and pairwise stochastically independent on $(0;1)$.

Both conclusions follow readily from (i)–(iv).

Lemma 3. *Suppose a function $f \in L^1_{(0;1)}$ is not equivalent to zero and*

$$A := \Big\{x \in (0;1) \,:\, |f(x)| > \frac{1}{2}\|f\|_{L^1_{(0;1)}}\Big\}.$$

Then

$$\operatorname{mes} A \geq \|f\|^2_{L^1_{(0;1)}} \Big/ 4\|f\|^2_{L^2_{(0;1)}}. \tag{3}$$

Proof. Indeed, we obtain (using Hölder's inequality)

$$\|f\|_{L^1_{(0;1)}} = \int_A |f(x)|\,dx + \int_{(0;1)\setminus A} |f(x)|\,dx \leq$$

$$\leq (\operatorname{mes} A)^{1/2} \cdot \|f\|_{L^2_{(0;1)}} + \frac{1}{2}\|f\|_{L^1_{(0;1)}},$$

which immediately implies (3). ■

Lemma 4. *Let N, Q, m $(1 \leq m \leq N)$ be positive integers. Then for any collection of natural numbers $1 \leq n_1 < n_2 < \cdots < n_m \leq N$*

$$\operatorname{mes}\Big\{x \in (0;1) \,:\, \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j} \psi^N_{n_k}(Qx)\Big| > \frac{J}{2}\Big\} \geq c\frac{m}{N},$$

where $\psi^N_n(x) \in \psi(N)$, $1 \leq n \leq N$ (see Lemma 2), and

$$J := \int_0^1 \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j} \psi^N_{n_k}(Qx)\Big|\,dx.$$

Proof. By Lemmas 1 and 2 (see also Remark 1) we have

$$\int_0^1 \Big(\max_{1\leq j\leq m}\Big|\sum_{k=1}^{j} \psi^N_{n_k}(Qx)\Big|\Big)^2 dx \leq cm\log_2^2(m+1),$$

$$J^2 \geq c\frac{m^2}{N}\log_2^2 N; \quad 1 \leq m \leq N < \infty.$$

Thus, applying Lemma 3 , we obtain

$$\operatorname{mes}\Big\{x \in (0;1) \,:\, \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j} \psi^N_{n_k}(Qx)\Big| > \frac{J}{2}\Big\} \geq$$

$$\geq c\frac{m}{N} \cdot \frac{\log_2^2 N}{\log_2^2(m+1)} \geq c\frac{m}{N}. \ ■$$

Proof of the theorem. If the conditions (1) are fulfilled, then $R_k = k\log_2 k/\varepsilon(k)$, $2 \leq k < \infty$, where $\varepsilon(k)$ tends to infinity. Moreover, without loss of generality, it can be assumed that

a) $R_1 = 1$;
b) $\varepsilon(k)$ is a nondecreasing sequence of positive integers;
c) the sets $\Delta_m := \{k : \varepsilon(k) = m\}$ have the form $(\nu_{m-1}; \nu_m] \cap \mathbb{N}$ $(m = 1, 2, \dots)$, where $\nu_0 = 0$, $\log_2 \log_2 \log_2 \nu_m = p_m$ $(m \geq 1)$ and $\{p_m\}_{m=1}^{\infty}$ is some increasing sequence of positive integers.

In particular, for $k \in \Delta_m$ $(m \geq 2)$ we have

$$\begin{gathered} \log_2 k > \log_2 \nu_{m-1} = 2^{2^{p_{m-1}}} \geq 2^{2^{m-1}} > m = \varepsilon(k), \\ \varepsilon(k \log_2 k) \leq \varepsilon(\nu_m \log_2 \nu_m) < \varepsilon(\nu_{m+1}) = m + 1 \leq 2\varepsilon(k). \end{gathered} \tag{4}$$

Denote

$$T_1 := 0, \quad T_k := 2^{[k \log_2 k \log_2 \log_2 k]}, \quad 2 \leq k < \infty, \tag{5}$$

and

$$E_m := \{k : R_k \leq T_m\}, \quad 2 \leq m < \infty$$

(here $[x]$ is the integer part of the number x).

Since the function $\varphi(x) = x\varepsilon(x)/\log_2 x$ increases on $(e; \infty)$, taking into account (4) and (5), we have, for $m > \nu_1$,

$$\begin{aligned} E_m &= \{1; 2\} \cup \Big\{k \geq 3 : R_k \leq T_m\Big\} = \\ &= \{1; 2\} \cup \Big\{k \geq 3 : \frac{k \log_2 k}{\varepsilon(k)} \leq T_m\Big\} = \\ &= \{1; 2\} \cup \Big\{k \geq 3 : \varphi\Big(\frac{k \log_2 k}{\varepsilon(k)}\Big) \leq \varphi(T_m)\Big\} \supset \\ &\supset \{1; 2\} \cup \Big\{k \geq 3 : 2k \leq \varphi(T_m)\Big\} = \\ &= \Big\{k \geq 1 : k \leq \frac{1}{2}\varphi(T_m)\Big\}. \end{aligned}$$

Therefore for a large m $(m \geq m_1 > \nu_1)$

$$|E_m| > \frac{1}{2}\varphi(T_m) - 1 \geq \frac{1}{3} \frac{T_m \varepsilon(T_m)}{\log_2 T_m}, \tag{6}$$

$$|E_{m+1}| \geq \frac{1}{3} \frac{T_{m+1}\varepsilon(T_{m+1})}{\log_2 T_{m+1}} \geq \frac{1}{3} \frac{T_{m+1}}{T_m \log_2 T_{m+1}} \cdot T_m \geq 2T_m; \tag{7}$$

Because of

$$\begin{aligned} \sum_{k \in \Delta_m} \frac{1}{\log_2 T_k} &= \sum_{k \in \Delta_m} \frac{1}{[k \log_2 k \log_2 \log_2 k]} > \\ &> \frac{1}{8}(p_m - p_{m-1}) \geq \frac{1}{8} \qquad (m \geq 2) \end{aligned}$$

we can select a subsequence $T_{q_k} \equiv \tilde{T}_k$ $(1 \le k < \infty;\ \tilde{T}_1 = 0,\ \tilde{T}_2 = \nu_1)$ such that

$$\frac{1}{2m^2} < \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 \tilde{T}_k} \equiv \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} \le \frac{1}{m^2}, \quad m \ge 2, \tag{8}$$

and hence

$$\sum_{k:q_k \in \Delta_m} \frac{\varepsilon(\tilde{T}_k)}{\log_2 \tilde{T}_k} = \sum_{k:q_k \in \Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} \ge$$

$$\ge \sum_{k:q_k \in \Delta_m} \frac{\varepsilon(q_k)}{\log_2 T_{q_k}} = m \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} > \frac{1}{2m}, \quad m \ge 2. \tag{9}$$

Let

$$\begin{gathered} N_m := \tilde{T}_m - \tilde{T}_{m-1}, \\ Q_2 := 1, \quad Q_{m+1} := 4N_m Q_m, \quad 2 \le m < \infty. \end{gathered} \tag{10}$$

Consider the orthonormal collections

$$\left\{\psi_n^{N_m}(Q_m x)\right\}_{n=1}^{N_m}, \quad 2 \le m < \infty,$$

and construct with their aid the desired ONS $\{\varphi_n(x)\}_{n=1}^{\infty}$ as

$$\varphi_n(x) := \psi_k^{N_m}(Q_m x), \tag{11}$$

where $n \in (\tilde{T}_{m-1}; \tilde{T}_m]$, $k = n - \tilde{T}_{m-1}$, $2 \le m < \infty$, $x \in (0;1)$ (the orthonormality follows from (10) and Remark 2).

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers with $k \le n_k \le R_k$, $1 \le k < \infty$. We set

$$\begin{aligned} &G_m := \left\{k : \tilde{T}_{m-1} < n_k \le \tilde{T}_m\right\}, \\ &M_m := |G_m|, \\ &a_n := \left((1 + M_m)\log_2 \tilde{T}_m\right)^{-1/2} \quad \text{for } n \in (\tilde{T}_{m-1}; \tilde{T}_m];\ m \ge 2. \end{aligned}$$

On account of (8)

$$\sum_{k=1}^{\infty} a_{n_k}^2 = \sum_{m=2}^{\infty} \sum_{k:\tilde{T}_{m-1} < n_k \le \tilde{T}_m} a_{n_k}^2 = \sum_{m=2}^{\infty} \frac{M_m}{(1 + M_m)\log_2 \tilde{T}_m} <$$

$$< \sum_{k=2}^{\infty} \frac{1}{\log_2 T_{q_k}} = \sum_{m=2}^{\infty} \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} < \infty.$$

It is thus sufficient to show that the series

$$\sum_{k=1}^{\infty} a_{n_k}\varphi_{n_k}(x) \tag{12}$$

diverges on some set of positive measure.

Note that

$$G_m = \{k : n_k \le \widetilde{T}_m\}\backslash\{k : n_k \le \widetilde{T}_{m-1}\} \supset$$
$$\supset \{k : R_k \le \widetilde{T}_m\}\backslash\{k : k \le \widetilde{T}_{m-1}\}, \quad m \ge 2.$$

Consequently in view of (6) and (7)

$$\begin{aligned} M_m &\ge |\{k : R_k \le \widetilde{T}_m\}| - |\{k : k \le \widetilde{T}_{m-1}\}| = \\ &= |\{k : R_k \le T_{q_m}\}| - T_{q_{m-1}} > \\ &> |E_{q_m}| - \frac{1}{2}|E_{q_{m-1}+1}| \ge \frac{1}{2}|E_{q_m}| \ge \\ &\ge \frac{1}{6}\,\frac{\widetilde{T}_m\varepsilon(\widetilde{T}_m)}{\log_2 \widetilde{T}_m}, \quad m \ge m_1. \end{aligned} \tag{13}$$

Hence by (11) and Lemma 2

$$\begin{aligned} \widetilde{J}_m := \int_0^1 \max_{\widetilde{T}_{m-1} < j \le \widetilde{T}_m} \Bigg| \sum_{k : \widetilde{T}_{m-1} < n_k \le j} a_{n_k}\varphi_{n_k}(x) \Bigg| dx &\ge \\ \ge c \cdot \frac{1}{\sqrt{(1 + M_m)\log_2 \widetilde{T}_m}} \cdot \frac{\log_2 N_m}{\sqrt{N_m}} \cdot M_m &\ge \\ \ge c\sqrt{\frac{M_m \log_2 \widetilde{T}_m}{\widetilde{T}_m}} \ge c\sqrt{\varepsilon(\widetilde{T}_m)}, \quad m \ge m_1&, \end{aligned}$$

and therefore

$$\lim_{m\to\infty} \widetilde{J}_m = \infty. \tag{14}$$

If A_m denotes the set

$$\Bigg\{x \in (0;1) : \max_{\widetilde{T}_{m-1} < j \le \widetilde{T}_m} \Bigg| \sum_{k : \widetilde{T}_{m-1} < n_k \le j} a_{n_k}\varphi_{n_k}(x) \Bigg| > \frac{1}{2}\widetilde{J}_m \Bigg\}, \quad m \ge 2,$$

then by (11), Lemma 4, (13) and (9)

$$\operatorname{mes} A_m \geq \frac{M_m}{N_m} > \frac{M_m}{\tilde{T}_m} > \frac{1}{6}\frac{\varepsilon(\tilde{T}_m)}{\log_2 \tilde{T}_m}, \quad m \geq m_1;$$

$$\sum_{m=2}^{\infty} \operatorname{mes} A_m \geq \frac{1}{6}\sum_{k=m_1}^{\infty} \frac{\varepsilon(\tilde{T}_k)}{\log_2 \tilde{T}_k} = \frac{1}{6}\sum_{k:q_k \geq q_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} >$$

$$> \frac{1}{6}\sum_{k:q_k > \nu_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \frac{1}{6}\sum_{m=1+m_1}\ \sum_{k:q_k \in \Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \infty. \qquad (15)$$

It is easy to verify (see (10),(11), Remark 2) that $\{A_m\}_{m=2}^{\infty}$ is a sequence of stochastically independent sets. Therefore by (15) and the Borel-Cantelli lemma

$$\operatorname{mes}\Big(\limsup_{m\to\infty} A_m\Big) = 1.$$

Hence we conclude because of (14) and the definition of sets A_m that the series (12) diverges almost everywhere on (0;1). ■

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(Received 25.02.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993) No 3, 273-279

A SECOND-ORDER NONLINEAR PROBLEM WITH TWO-POINT AND INTEGRAL BOUNDARY CONDITIONS

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Abstract. The paper gives sufficient conditions for the existence and nonuniqueness of monotone solutions of a nonlinear ordinary differential equation of the second order subject to two nonlinear boundary conditions one of which is two-point and the other is integral. The proof is based on an existence result for a problem with functional boundary conditions obtained by the author in [6].

რეზიუმე. მეორე რიგის არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებისათვის განხილულია ერთი სასაზღვრო ამოცანა ორწერტილოვანი და ინტეგრალური პირობებით. ავტორის 6 შრომაში მიღებული შედეგების გამოყენებით გამოკვლეულია ამ ამოცანის მონოტონური ამონახსნის არსებობისა და არაერთადერთობის საკითხი.

The present paper is concerned with the theory of nonlinear boundary value problems for equations with ordinary derivatives, see e.g. [1–4], and is closely related to [5, 6]. We deal here with the solvability of a certain essentially nonlinear second-order problem.

The following notation is used:

R is the set of all real numbers;

$[a, b]$ denotes a closed interval where a differential equation is considered, $-\infty < a < b < +\infty$;

C^0 denotes the space of all continuous functions;

C^1 is the space of all continuously differentiable functions;

L_1 denotes the space of all Lebesgue measurable functions with integrable absolute value;

AC stands for the space of all absolutely continuous functions;

CL_1^2 is the space of all $x(\cdot) \in C^1$ such that $\dot{x}(\cdot) \in AC$.

1991 *Mathematics Subject Classification.* 34B15, 34K10.

3

We consider the existence of monotone solutions of the boundary value problem

$$\ddot{x} = f(t, x, \dot{x}), \quad t \in [a, b], \tag{1}$$

$$\omega(x(a), x(b)) = 0, \tag{2}$$

$$\int_a^b \varphi(|\dot{x}(\tau)|)d\tau = g. \tag{3}$$

The solution $x(\cdot) \in CL_1^2([a,b], \mathbf{R})$ should satisfy equation (1) almost everywhere. Assume that the function $f : [a,b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfies the Carathéodory conditions, i.e. $f(t, x_0, x_1)$ is measurable in t for any fixed numbers x_0, x_1 and is continuous in x_0, x_1 for almost every fixed t. Assume also that $|f(t, x_0, x_1)| \leq M$ for almost all t and all x_0, x_1, the constant M is positive, the number $g \in \mathbf{R}$ is fixed, the functions $\omega : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $\varphi : [0, \infty) \to \mathbf{R}$ are continuous, $\omega(s_1, s_2)$ is nondecreasing in each of the arguments s_1, s_2 and is strictly increasing at least in one of the two arguments, the set of pairs s_1, s_2 that satisfy equality $\omega(s_1, s_2) = 0$ is nonempty, the function $\varphi(z)$ strictly increases and

$$\lim_{z \to +\infty} \varphi(z) = +\infty.$$

For example, if $\varphi(z) = z$ then the boundary condition (3) fixes L_1-norm of the derivative of the unknown function. And in the case $\varphi(z) = \sqrt{1 + z^2}$ the equality (3) fixes the length of the curve which is the graph of the solution $x(t)$, $t \in [a, b]$. Let us note also that the equality $x(a) = g_0$, where $g_0 \in \mathbf{R}$ is a number, can be considered as the simplest special case of (2). Thus, the boundary conditions (2), (3) can describe, in particular, a curve with a fixed length emanating from a given initial point.

Denote

$$A_\varphi = \int_0^{b-a} \varphi(M\tau)d\tau.$$

Theorem. *If $g \geq A_\varphi$ then every solution of boundary value problem (1)–(3) is strictly monotone, and there exist at least one increasing and at least one decreasing solutions.*

The above theorem was previously announced by the author, cf. Proposition 2 in [5]. The proof will be given below.

Remark 1. If $g < A_\varphi$ then problem (1)–(3) may have no monotone solutions. This is the case, for example, if $f \equiv M > 0$.

Remark 2. A similar theorem is valid also for an equation with deviating arguments and for condition (3) where $g = g(x(\cdot))$ is a nonlinear functional.

The proof of Theorem employs (and illustrates) the following existence result for a boundary value problem of the form

$$\ddot{x}(t) = F(x(\cdot))(t), \quad t \in [a,b], \tag{4}$$

$$B_0(x(\cdot)) = B_1(\dot{x}(\cdot)) = 0. \tag{5}$$

The solution $x(\cdot) \in CL_1^2 = CL_1^2([a,b], \mathbf{R})$ satisfies equation (4) almost everywhere. The mappings

$$F : C^1 \to L_1, \quad B_0 : CL_1^2 \to \mathbf{R}, \quad B_1 : AC \to \mathbf{R}$$

are assumed to be continuous. Let us fix a closed set of functions $A \subset CL_1^2$. Denote $A^{(k)} = \{x^{(k)}(\cdot) : x(\cdot) \in A\}$. Assume that the family of functions A satisfies $A^{(2)} = L_1$.

Proposition 1. *Let M, N be fixed numbers. Consider the following conditions:*

a) *if* $\| x(\cdot) \|_{C^1} \leq N$ *then* $|F(x(\cdot))(t)| \leq M$ *for almost all* t,

b) *if* $x(\cdot) \in A$ *satisfies* (5) *and* $|\ddot{x}(t)| \leq M$ *almost everywhere then* $\| x(\cdot) \|_{C^1} \leq N$,

c) *if* $x(\cdot) \in A$ *and almost everywhere* $|\ddot{x}(t)| \leq M$ *then there exist a unique number* $c_0 \in \mathbf{R}$ *such that*

$$B_0(x(\cdot) + c_0) = 0, \quad x(\cdot) + c_0 \in A,$$

and a unique number $c_1 \in \mathbf{R}$ *that satisfies*

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(\cdot) + c_1 \in A^{(1)}.$$

Conditions a), b), c) *imply that problem* (4), (5) *has at least one solution in* A.

A more general version of Proposition 1 was proven by the author in [6].

We need also the following simple auxiliary result.

Proposition 2. *Let the above given assumptions on* φ *hold. If a function* $u : [a,b] \to \mathbf{R}$ *satisfies the Lipschitz condition with the coefficient* M *and vanishes at least at one point then*

$$\int_a^b \varphi(|u(\tau)|)d\tau \leq \int_0^{b-a} \varphi(M\tau)d\tau.$$

Here the equality holds only for the following four functions

$$u = \pm M(t-a), \quad u = \pm M(b-t).$$

Proof of Proposition 2. Let $u(s) = 0$ for some $s \in [a,b]$. Then $|u(t)| \leq M|t-s|$. Thus

$$\int_a^b \varphi(\,|u(\tau)|\,)d\tau \leq \int_a^b \varphi(M|\tau - s|\,)d\tau. \tag{6}$$

Denote the right-hand side of (6) by $\Psi(s)$. We have

$$\Psi(s) = \int_a^s \varphi(M(s-\tau)\,)d\tau + \int_s^b \varphi(M(\tau - s)\,)d\tau =$$
$$\int_0^{s-a} \varphi(M\tau)d\tau + \int_0^{b-s} \varphi(M\tau)d\tau.$$

And so, the derivative

$$\frac{d\Psi}{ds} = \varphi(M(s-a)) - \varphi(M(b-s))$$

is negative for $a \leq s < \frac{1}{2}(a+b)$ and positive for $\frac{1}{2}(a+b) < s \leq b$. Consequently, the value $\Psi(a) = \Psi(b) = \int\limits_0^{b-a} \varphi(M\tau)d\tau$ is the maximum of $\Psi(s)$ for $s \in [a,b]$, which is attained only at the ends of the interval. The desired inequality is proven. If the right- and left-hand sides of this inequality are equal then s equals either a or b, and besides that (6) turns to equality. Taking into account strict monotonicity of φ we come to the conclusion that $|\,u(t)\,| \leq M|t-s|$ also turns to equality. Thus, either $u = \pm M(t-a)$, or $u = \pm M(b-t)$. ∎

Proof of Theorem. The boundary value problem (1)–(3) is a special case of problem (4), (5). Really, it suffices to assume

$$F(x(\cdot))(t) = f(t, x(t), \dot{x}(t)),$$
$$B_0(x(\cdot)) = \omega(x(a), x(b)),$$
$$B_1(u(\cdot)) = \int_a^b \varphi(\,|u(\tau)|\,)d\tau - g.$$

The mappings $F : C^1 \to L_1$, $B_0 : C^0 \to \mathbf{R}$, $B_1 : C^0 \to \mathbf{R}$ are continuous. Denote by A_+ the set of all monotone nondecreasing functions in CL_1^2 and by A_- the set of all nonincreasing ones. The sets A_+, A_- are closed in CL_1^2, and $A_+^{(2)} = A_-^{(2)} = L_1$. Condition a) of Proposition 1 holds obviously. Let us verify condition b). Let

$x(\cdot) \in CL_1^2$ satisfy (2), (3), and $|\ddot{x}(t)| \leq M$ be true almost everywhere. Since $\varphi(z) \to +\infty$ as $z \to +\infty$ there exists a number r that satisfies $\varphi(r) > (b-a)^{-1}g$. If we suppose that $|\dot{x}(t)| \geq r$ for all t then

$$\int_a^b \varphi(|\dot{x}(\tau)|)d\tau \geq (b-a)\varphi(r) > g,$$

which contradicts (3). Consequently, $|\dot{x}(s)| < r$ for at least one s. Consider some l_1, l_2 such that $\omega(l_1, l_2) = 0$. Let us show that

$$\min\{l_1, l_2\} \leq x(\sigma) \leq \max\{l_1, l_2\} \tag{7}$$

for some σ. Really, if (7) does not hold for any $\sigma \in [a, b]$ then due to continuity of $x(t)$ two cases are possible. Either $x(t) > \max\{l_1, l_2\}$ for all t, or $x(t) < \min\{l_1, l_2\}$ for all t. Monotonicity of ω implies that $\omega(x(a), x(b)) > \omega(l_1, l_2) = 0$ in the first case, and $\omega(x(a), x(b)) < 0$ in the second case. It follows from (2) that neither of the two cases can take place. The existence of the numbers r, s, σ named above and the inequality $|\ddot{x}(t)| \leq M$ imply boundedness of $\|x(\cdot)\|_{C^1}$. Let us verify now condition c) for $A = A_+$ and for $A = A_-$. The function $x(\cdot)$ being fixed, the number c_0 is defined uniquely by the equality $B_0(x(\cdot) + c_0) = 0$ due to the properties of the real function

$$B_0(x(\cdot) + c) = \omega(x(a) + c, x(b) + c)$$

of the argument c. Really, the function is continuous and strictly increasing. It suffices to show that this function takes both positive and negative values. As above, we fix l_1, l_2 for which $\omega(l_1, l_2) = 0$. Then for $c > \max\{l_1 - x(a), l_2 - x(b)\}$ we have $\omega(x(a) + c, x(b) + c) > \omega(l_1, l_2) = 0$, and for $c < \min\{l_1 - x(a), l_2 - x(b)\}$ we obtain $\omega(x(a) + c, x(b) + c) < \omega(l_1, l_2) = 0$. So, the desired properties of the function $B_0(x(\cdot) + c)$ are established. We have only to note that $x(\cdot) + c$ for a fixed c is monotone in the same sense as $x(\cdot)$. Consider now c_1. Assume that $x(\cdot) \in CL_1^2$ and almost everywhere $|\ddot{x}(t)| \leq M$. Continuity of φ implies that the function

$$\Phi(c) = \int_a^b \varphi(|\dot{x}(\tau) + c|)d\tau$$

is also continuous. With the help of Proposition 2 we obtain the following. For $c \in (-\infty, -\max_t \dot{x}(t)]$ we have $\dot{x}(t) + c \leq 0$, and the function $\Phi(c)$ strictly decreases taking values from $+\infty$ to a number not larger than A_φ; and if $c \in [-\min_t \dot{x}(t), +\infty)$ then $\dot{x}(t) + c \geq 0$, and the function $\Phi(c)$ strictly increases taking values from a number not larger than A_φ to $+\infty$. A conclusion follows that if $x(\cdot) \in CL_1^2$

and almost everywhere $|\ddot{x}(t)| \leq M$ then there exists a unique c_1 that satisfies

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(t) + c_1 \geq 0.$$

Similarly, conditions

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(t) + c_1 \leq 0$$

also define a unique c_1. Thus, condition c) is valid. It follows from Proposition 1 that boundary value problem (1)–(3) is solvable in A_+ and in A_-. Now we have to show that every solution $x(t)$ of this problem is strictly monotone. If the derivative $\dot{x}(t)$ does not vanish then all its values have the same sign, and so $x(t)$ is obviously monotone. Let now the derivative $\dot{x}(t)$ vanish at least at one point. Using Proposition 2 we obtain

$$\int_a^b \varphi(\,|\dot{x}(\tau)|\,)d\tau \leq A_\varphi. \tag{8}$$

Taking into account the inequality $A_\varphi \leq g$ and boundary condition (3) we see that the two values in (8) are equal. Employing again Proposition 2 we conclude that either $\dot{x}(t) = \pm M(t-a)$, or $\dot{x}(t) = \pm M(b-t)$. And since $M \neq 0$ the function $x(t)$ is strictly monotone. Theorem is proven. Let us note that Theorem can be proven also basing on results of [7]. ■

In conclusion we verify Remark 1. Assume $f \equiv M > 0, \quad g < A_\varphi$. We need to show that problem (1)–(3) has no monotone solutions. The equation (1) takes the form $\ddot{x} = M$. And since $M \neq 0$ we obtain $\dot{x}(t) = M(t-\gamma)$. Thus, $x(t)$ can be monotone only if $\gamma \leq a$ or $\gamma \geq b$. Let us consider these two cases separately. If $\gamma \leq a$ then $|\dot{x}(\tau)| = M(\tau-\gamma) \geq M(\tau-a)$ for $\tau \in [a,b]$, and (3) implies $g \geq \int\limits_a^b \varphi(M(\tau-a))d\tau = A_\varphi > g$. This contradiction shows that the inequality $\gamma \leq a$ does not hold. Similarly, if $\gamma \geq b$ then $|\dot{x}(\tau)| = M(\gamma-\tau) \geq M(b-\tau)$ for $\tau \in [a,b]$ and thus $g \geq \int\limits_a^b \varphi(M(b-\tau))d\tau = A_\varphi > g$. And so, the case $\gamma \geq b$ is not possible either. Remark 1 is verified.

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(Received 03.05.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993), No. 3, 281-297

TWO-DIMENSIONAL PROBLEMS OF STATIONARY FLOW OF A NONCOMPRESSIBLE VISCOUS FLUID IN THE CASE OF OZEEN'S LINEARIZATION

T. BUCHUKURI AND R. CHICHINADZE

ABSTRACT. Two-dimensional boundary value problems of flow of a viscous micropolar fluid are investigated in the case of linearization by Ozeen's method.

რეზიუმე. პოტენციალისა და ინტეგრალურ განტოლებათა მეთოდებით გამოკვლეულია ბლანტი უკუმში მიკროპოლარული სითხის დინების ორგანზომილებიანი სასაზღვრო ამოცანები ოზეენის გაწრფივებისას.

1. Basic Equations. A system of equations of motion of a noncompressible micropolar fluid was obtained in 1964 by Condiff and Dahler [1] and, independently, in 1966 by Eringen. It is a generalization of the classical system of Navier–Stokes for micropolar fluids. In real life we observe such properties in fluids containing polymer particles as admixtures. When fluids of this kind flow along the body, surface friction is 30 to 35% less than in the case of flow of fluids without polymer admixtures [2]. It is impossible to predict such effects by the classical theory of Navier–Stokes, but a fairly good explanation can be found within the framework of the theory of micropolar fluids.

We consider a two-dimensional model of stationary flow of a micropolar fluid. A system of the basic equations then has the form

$$
\begin{aligned}
&\operatorname{div}\tilde{v} = 0,\\
&(\mu+\alpha)\Delta\tilde{v}_1 + 2\alpha\frac{\partial\tilde{\omega}}{\partial x_2} - \frac{\partial p}{\partial x_1} + \rho F_1 = \rho\sum_{k=1}^{2}\tilde{v}_k\frac{\partial\tilde{v}_1}{\partial x_k},\\
&(\mu+\alpha)\Delta\tilde{v}_2 - 2\alpha\frac{\partial\tilde{\omega}}{\partial x_1} - \frac{\partial p}{\partial x_2} + \rho F_2 = \rho\sum_{k=1}^{2}\tilde{v}_k\frac{\partial\tilde{v}_2}{\partial x_k},\\
&\gamma\Delta\tilde{\omega} - 4\alpha\tilde{\omega} + 2\alpha\left(\frac{\partial\tilde{v}_2}{\partial x_1} - \frac{\partial\tilde{v}_1}{\partial x_2}\right) + \rho F_3 = \mathcal{I}\sum_{k=1}^{2}\tilde{v}_k\frac{\partial\tilde{\omega}}{\partial x_k},
\end{aligned}
\tag{1}
$$

1991 *Mathematics Subject Classification.* 76A05, 35Q20.

4

where $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$ is the velocity vector, $\tilde{F} = (F_1, F_2)$ is the mass force, p is the pressure, ρ is the density; μ, α, γ, $\mathcal{I}$ are positive constants. In the two-dimensional case the microrotation and mass moment vectors have one component each and are denoted by $\tilde{\omega}$ and F_3, respectively.

Since the system (1) is nonlinear, we come across certain difficulties during its investigation. On the other hand, to solve many problems of applied nature it is sufficient to consider a linearized variant of the system (1). The equations of Navier-Stokes can be linearized by two well-known methods: that of Stokes and that of Ozeen. When using Stokes' method of linearization, nonlinear terms are totally discarded. This method yields satisfactory results for small $\tilde{v}$ and $\tilde{\omega}$ (note that in this case nonlinear terms are small values of higher order). However, if the fluid flow velocity $\tilde{v}$ is not a small value, this model leads to an essential error. In particular, the effects predicted by this method when a fluid flows along a solid body do not agree with experimental data.

A lesser error is obtained in the case of linearization by Ozeen's method consisting in the following: it is assumed that fluid flow differs but little from flow along the x_1-axis with the constant velocity v_0. Then we set

$$\tilde{v}_k = v_0\delta_{k1} + v_k, \quad k = 1,2, \quad \tilde{\omega} = \omega,$$

where v_k, $k = 1,2$, ω are small values; δ_{kj} is the Kronecker symbol.

On substituting these values in (1), we obtain an Ozeen-linearized system of equations of stationary flow of a micropolar fluid in the two-dimensional case:

$$\begin{aligned}
&\operatorname{div} v = 0,\\
&(\mu+\alpha)\Delta v_1 + 2\alpha\frac{\partial\omega}{\partial x_2} - \frac{\partial p}{\partial x_1} + \rho F_1 = \eta_1\frac{\partial v_1}{\partial x_1},\\
&(\mu+\alpha)\Delta v_2 - 2\alpha\frac{\partial\omega}{\partial x_1} - \frac{\partial p}{\partial x_2} + \rho F_2 = \eta_1\frac{\partial v_2}{\partial x_1},\\
&\gamma\Delta\omega - 4\alpha\omega + 2\alpha\Big(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\Big) + \rho F_3 = \eta_2\frac{\partial\omega}{\partial x_1}.
\end{aligned} \tag{2}$$

Here $\eta_1 = \rho v_0$, $\eta_2 = \mathcal{I} v_0$.

The system (2) can be rewritten in the matrix form if we introduce the notation

$$L(\partial_x) = \left\| \begin{matrix} (\mu+\alpha)\Delta & 0 & 2\alpha\frac{\partial}{\partial x_2} \\ 0 & (\mu+\alpha)\Delta & -2\alpha\frac{\partial}{\partial x_1} \\ -2\alpha\frac{\partial}{\partial x_2} & 2\alpha\frac{\partial}{\partial x_1} & \gamma\Delta - 4\alpha \end{matrix} \right\|,$$

$$G(\partial_x) = \left\| \begin{matrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ 0 \end{matrix} \right\|, \quad \eta = \left\| \begin{matrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{matrix} \right\|,$$

$$(v_1, v_2, \omega) \equiv u = (u_1, u_2, u_3) = \left\| \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \right\|, \quad F = (F_1, F_2, F_3) = \left\| \begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix} \right\|.$$

Now the system (2) takes the form

$$\begin{gathered} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ L(\partial_x)u - G(\partial_x)p + \rho F = \eta\frac{\partial u}{\partial x_1}. \end{gathered} \qquad (3)$$

Alongside with the system (3), we will also consider its conjugate

$$\begin{gathered} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ L(\partial_x)u - G(\partial_x)p + \rho F = -\eta\frac{\partial u}{\partial x_1}, \end{gathered} \qquad (4)$$

which is obtained from (3) if we replace η_1 and η_2 by $-\eta_1$ and $-\eta_2$, respectively.

Let us formulate the boundary value problems for (2). Denote by D^+ a finite domain in the Euclidean two-dimensional space $\mathbb{R}^2$ bounded with a piecewise-smooth contour S. Let $D^- \equiv \mathbb{R}^2 \backslash (D^+ \cup S)$. Denote by $n(y) = (n_1(y), n_2(y))$ the unit normal at a point $y \in S$, external with respect to the domain D^+.

The pair (u,p), where $u = (v_1, v_2, \omega)$, will be called regular in D^+ if $u \in \mathbb{C}^2(D^+) \cap \mathbb{C}^1(\bar{D}^+)$, $p \in \mathbb{C}^1(D^+) \cap \mathbb{C}(\bar{D}^+)$.

The pair (u,p) will be called regular in D^- if $u \in \mathbb{C}^2(D^-) \cap \mathbb{C}^1(\bar{D}^-)$, $p \in \mathbb{C}^1(D^-) \cap \mathbb{C}(\bar{D}^-)$ and the conditions

$$u(x) = O\left(|x|^{-\frac{1}{2}}\right), \quad p(x) = o(1) \qquad (5)$$

are fulfilled in the neighbourhood of $|x| = \infty$.

The boundary value problems for the system (3) are formulate as follows:

Problem (I)$^\pm$. In the domain $D^\pm$ find a regular solution of the system (3) by the boundary condition

$$\lim_{D^\pm \ni x \to y \in S} u(x) = f(y). \tag{6}$$

Problem (II)$^\pm$. In the domain $D^\pm$ find a regular solution of the system (3) by the boundary condition

$$\lim_{D^\pm \ni x \to y \in S} \Big[P(\partial_x, n(y))u(x) - \frac{1}{2}n_1(y)\eta u(x) - N(y)p(x)\Big] = f(y), \tag{7}$$

where $f = (f_1, f_2, f_3)$ is a given vector on S,

$$P(\partial_x, n(y)) = \|P_{ij}(\partial_x, n(y))\|_{3\times 3}, \tag{8}$$

$$P_{ij}(\partial_x, n(y)) = (\mu+\alpha)n_j(y)\frac{\partial}{\partial x_i} + (\mu+\alpha)\delta_{ij}\sum_{k=1}^{2} n_k(y)\frac{\partial}{\partial x_k}, \quad i,j = 1,2;$$

$$P_{i3}(\partial_x, n(y)) = 2\alpha(i-1)n_1(y) + 2\alpha(i-2)n_2(y), \quad i = 1,2;$$

$$P_{3j}(\partial_x, n(y)) = 0, \quad j = 1,2;$$

$$P_{33}(\partial_x, n(y)) = \gamma\sum_{k=1}^{2} n_k(y)\frac{\partial}{\partial x_k};$$

$$N_i(y) = (N_1(y), N_2(y), N_3(y)), \quad N_i(y) = n_i(y), \ i = 1,2; \ N_3(y) = 0.$$

The boundary value problems $(\tilde{\text{I}})^\pm$ and $(\tilde{\text{II}})^\pm$ for the conjugate system (4) are formulated similarly. In that case the boundary condition of Problem (I)$^\pm$ coincides with (6), while the boundary condition of Problem (II)$^\pm$ is obtained from (7) if we replace η_1 and η_2 by $-\eta_1$ and $-\eta_2$, respectively.

Remark. Our previous thorough treatment of the boundary value problems of stationary flow of a micropolar fluid under Ozeen's linearization in the three-dimensional case is given in [3]. Since investigations of the two-dimensional problems are mostly the same, we will dwell on only the part differing from the three-dimensional case.

2. On Fundamental solutions. The fundamental solutions of the system (1) are found from the relations

$$
\begin{aligned}
&\frac{\partial \overset{(m)}{v}_1}{\partial x_1}+\frac{\partial \overset{(m)}{v}_2}{\partial x_2}=0,\\
&(\mu+\alpha)\Delta \overset{(m)}{v}_1-\eta_1\frac{\partial \overset{(m)}{v}_1}{\partial x_1}+2\alpha\frac{\partial \overset{(m)}{\omega}}{\partial x_2}-\frac{\partial \overset{(m)}{p}}{\partial x_1}+\overset{(m)}{a}_1\,\delta(x)=0,\\
&(\mu+\alpha)\Delta \overset{(m)}{v}_2-\eta_1\frac{\partial \overset{(m)}{v}_2}{\partial x_1}-2\alpha\frac{\partial \overset{(m)}{\omega}}{\partial x_1}-\frac{\partial \overset{(m)}{p}}{\partial x_2}+\overset{(m)}{a}_2\,\delta(x)=0,\\
&\gamma\Delta\overset{(m)}{\omega}-4\alpha\overset{(m)}{\omega}-\eta_2\frac{\partial \overset{(m)}{\omega}}{\partial x_1}-2\alpha\frac{\partial \overset{(m)}{v}_1}{\partial x_2}+2\alpha\frac{\partial \overset{(m)}{v}_2}{\partial x_1}+\overset{(m)}{b}\,\delta(x)=0,
\end{aligned}
\tag{9}
$$

where $\delta(x)$ is the Dirac distribution,

$$
\overset{(m)}{a}_k=2\delta_{km},\quad \overset{(m)}{b}=0,\quad k,m=1,2;
$$

$$
\overset{(3)}{a}_k=0,\quad k=1,2,\quad \overset{(3)}{b}=2.
$$

Assume that the fundamental solutions $(\overset{(m)}{v}_1,\overset{(m)}{v}_2,\overset{(m)}{\omega},\overset{(m)}{p})$, $m=1,2,3$, satisfy the conditions

$$
\lim_{|x|\to\infty}\Big(\overset{(m)}{v}_1(x),\overset{(m)}{v}_2(x),\overset{(m)}{\omega}(x),\overset{(m)}{p}(x)\Big)=0. \tag{10}
$$

Then $\overset{(m)}{v}_1$, $\overset{(m)}{v}_2$, $\overset{(m)}{\omega}$ and $\overset{(m)}{p}$ are gradually increasing distributions in $\mathbb{R}^2$, and, on subjecting the system (9) to the Fourier transformation, we obtain

$$
\begin{aligned}
&\xi_1\widehat{\overset{(m)}{v}}_1+\xi_2\widehat{\overset{(m)}{v}}_2=0,\\
&i\xi_1\widehat{\overset{(m)}{p}}-(\mu+\alpha)|\xi|^2\widehat{\overset{(m)}{v}}_1+i\eta_1\xi_1\widehat{\overset{(m)}{v}}_1-2i\alpha\xi_2\widehat{\overset{(m)}{\omega}}+\overset{(m)}{a}_1=0,\\
&i\xi_2\widehat{\overset{(m)}{p}}-(\mu+\alpha)|\xi|^2\widehat{\overset{(m)}{v}}_2+i\eta_1\xi_1\widehat{\overset{(m)}{v}}_2+2i\alpha\xi_1\widehat{\overset{(m)}{\omega}}+\overset{(m)}{a}_2=0,\\
&-\gamma|\xi|^2\widehat{\overset{(m)}{\omega}}-4\alpha\widehat{\overset{(m)}{\omega}}+i\eta_2\xi_1\widehat{\overset{(m)}{\omega}}+2i\alpha\xi_2\widehat{\overset{(m)}{v}}_1-2i\alpha\xi_1\widehat{\overset{(m)}{v}}_2+\overset{(m)}{b}=0.
\end{aligned}
\tag{11}
$$

Here $|\xi|\equiv(\xi_1^2+\xi_2^2)^{1/2}$, $\widehat{f}$ denotes the Fourier transform of the gradually increasing distribution f:

$$
\widehat{f}(\xi)=\int_{\mathbb{R}^2}e^{i\xi\cdot x}f(x)\,dx,\quad \xi=(\xi_1,\xi_2)\in\mathbb{R}^2,
$$

$$
|\xi|\equiv(\xi_1^2+\xi_2^2)^{1/2}.
$$

Solving the system (11) with respect to $\widehat{\overset{(m)}{v}}_1$, $\widehat{\overset{(m)}{v}}_2$, $\widehat{\overset{(m)}{\omega}}$, $\widehat{\overset{(m)}{p}}$, we have

$$
\begin{aligned}
\widehat{\overset{(m)}{p}} &= \frac{i}{|\xi|^2}\xi\cdot \overset{(m)}{a},\\
\widehat{\overset{(m)}{v}}_1 &= \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{\Phi(\xi)}\overset{(m)}{a}_1 - \frac{2i\alpha\xi_2}{\Phi(\xi)}\overset{(m)}{b} -\\
&- \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{|\xi|^2\Phi(\xi)}\Big(\xi\cdot \overset{(m)}{a}\Big)\xi_1,\\
\widehat{\overset{(m)}{v}}_2 &= \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{\Phi(\xi)}\overset{(m)}{a}_2 + \frac{2i\alpha\xi_1}{\Phi(\xi)}\overset{(m)}{b} -\\
&- \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{|\xi|^2\Phi(\xi)}\Big(\xi\cdot \overset{(m)}{a}\Big)\xi_2,\\
\widehat{\overset{(m)}{\omega}} &= \frac{(\mu+\alpha)|\xi|^2 - i\eta_1\xi_1}{\Phi(\xi)}\overset{(m)}{b} +\\
&+ \frac{2i\alpha}{\Phi(\xi)}\Big(\overset{(m)}{a}_1\xi_2 - \overset{(m)}{a}_2\xi_1\Big), \quad m = 1,2,3,
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\Phi(\xi) = \gamma(\mu+\alpha)|\xi|^4 - i[\eta_1\gamma + \eta_2(\mu+\alpha)]\xi_1|\xi|^2 -\\
- \eta_1\eta_2\xi_1^2 + 4\mu\alpha|\xi|^2 - 4i\alpha\eta_1\xi_1.
\end{aligned}
$$

Denote by $\overset{(m)}{v}_1$, $\overset{(m)}{v}_2$, $\overset{(m)}{\omega}$, $\overset{(m)}{p}$, $m = 1,2,3$, the inverse Fourier transforms of $\widehat{\overset{(m)}{v}}_1$, $\widehat{\overset{(m)}{v}}_2$, $\widehat{\overset{(m)}{\omega}}$, $\widehat{\overset{(m)}{p}}$, respectively, and by Γ and Q the matrices

$$
\begin{gathered}
\Gamma = \|\Gamma_{ik}\|_{3\times 3},\\
\Gamma_{ik} = \overset{(k)}{v}_i, \quad i = 1,2, \quad k = 1,2,3; \quad \Gamma_{3k} = \overset{(k)}{\omega};\\
Q = \|Q_k\|_{3\times 1},\\
Q_k = \overset{(k)}{p}, \quad k = 1,2; \quad Q_3 = 0.
\end{gathered}
\tag{13}
$$

Γ and Q will be called the matrices of fundamental solutions of the system (1). Their Fourier transform is

$$\widehat{Q}(\xi) = \Big(\frac{2i\xi_1}{|\xi|^2}, \frac{2i\xi_2}{|\xi|^2}, 0\Big), \tag{14}$$

$$\widehat{\Gamma}(\xi) = \left\| \begin{matrix} 2A(\xi)\Big(1-\frac{\xi_1^2}{|\xi|^2}\Big) & -2A(\xi)\frac{\xi_1\xi_2}{|\xi|^2} & -2B(\xi)\xi_2 \\ -2A(\xi)\frac{\xi_1\xi_2}{|\xi|^2} & 2A(\xi)\Big(1-\frac{\xi_2^2}{|\xi|^2}\Big) & 2B(\xi)\xi_1 \\ 2B(\xi)\xi_2 & -2B(\xi)\xi_1 & \frac{2}{\Phi(\xi)}\Big((\mu+\alpha)|\xi|^2-i\eta_1\xi_1\Big) \end{matrix} \right\|, \tag{15}$$

where

$$A(\xi) = \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{\Phi(\xi)}, \quad B(\xi) = \frac{2i\alpha}{\Phi(\xi)}.$$

Calculating the inverse Fourier transform of (14), we get

$$Q(x) = \Big(\frac{1}{\pi}\frac{x_1}{|x|^2}, \frac{1}{\pi}\frac{x_2}{|x|^2}, 0\Big). \tag{16}$$

Though the inverse transform of $\widehat{\Gamma}$ is not expressed in terms of elementary functions, we can nevertheless obtain asymptotic representations of the fundamental matrix Γ in the neighbourhood of the points $|x| = 0$ and $|x| = \infty$, which is convenient in investigating the boundary value problems.

Represent $\widehat{\Gamma}$ in the form

$$\widehat{\Gamma}(\xi) = \widehat{\Gamma}^{(0)}(\xi) + \widehat{\Gamma}^{(1)}(\xi),$$

where

$$\widehat{\Gamma}^{(0)}(\xi) = \left\| \begin{matrix} \frac{2\xi_1^2}{(\mu+\alpha)|\xi|^4} & -\frac{2\xi_1\xi_2}{(\mu+\alpha)|\xi|^4} & 0 \\ -\frac{2\xi_1\xi_2}{(\mu+\alpha)|\xi|^4} & \frac{2\xi_2^2}{(\mu+\alpha)|\xi|^4} & 0 \\ 0 & 0 & \frac{2}{\gamma|\xi|^2} \end{matrix} \right\| \tag{17}$$

and the elements of $\widehat{\Gamma}^{(1)}$ are written as

$$\widehat{\Gamma}_{ij}^{(1)}(\xi) = \sum_{3\le 2k-|\alpha|\le 4} a_{k,\alpha}^{ij}\frac{|\xi|^\alpha}{|\xi|^{2k}} + \varphi_{ij}(\xi), \quad i,j = 1,2,3. \tag{18}$$

Here $a_{k,\alpha}^{ij}$ are some constants, φ_{ij} are functions admitting, in the neighbourhood of $|x| = \infty$, the estimate

$$\varphi_{ij}(\xi) = O(|\xi|^{-5}).$$

Taking (18) into account, it can be proved [3] that, $\Gamma_{ij}^{(1)}$ satisfies, in the neighbourhood of $|x| = 0$, the conditions

$$\begin{aligned} \Gamma_{ij}^{(1)}(x) &= O(1), \quad i,j = 1,2,3; \\ \partial^\alpha \Gamma_{ij}^{(1)}(x) &= O\Big(|x|^{1-|\alpha|}\ln|x|\Big), \quad i,j = 1,2,3. \end{aligned} \tag{19}$$

Performing the inverse Fourier transform of (17), we obtain

$$\Gamma^{(0)}(x) = \begin{Vmatrix} -\frac{1}{4\pi(\mu+\alpha)}\left(\ln|x| + \frac{x_1^2}{|x|^2}\right) & -\frac{x_1x_2}{4\pi(\mu+\alpha)|x|^2} & 0 \\ -\frac{x_1x_2}{4\pi(\mu+\alpha)|x|^2} & -\frac{1}{4\pi(\mu+\alpha)}\left(\ln|x| + \frac{x_2^2}{|x|^2}\right) & 0 \\ 0 & 0 & -\frac{1}{\pi\gamma}\ln|x| \end{Vmatrix}. \tag{20}$$

Thus we have

Theorem 1. *The fundamental matrix* Γ *is represented as*

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)}, \tag{21}$$

where $\Gamma^{(0)}$ *is defined from* (20), $\Gamma^{(1)} \in \mathbb{C}^\infty(\mathbb{R}\setminus\{0\})$ *and satisfies, in the neighbourhood of the point* $|x| = 0$, *the conditions* (19).

To obtain the necessary representation of Γ in the neighbourhood of $|x| = \infty$ we represent $\hat{\Gamma}$ as the sum

$$\hat{\Gamma}(\xi) = \hat{\Gamma}^{(\infty)}(\xi) + \hat{\Gamma}^{(2)}(\xi), \tag{22}$$

where

$$\hat{\Gamma}^{(\infty)}(\xi) = \frac{2}{(\mu|\xi|^2 - i\eta_1\xi_1)} \begin{Vmatrix} \frac{\xi_2^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{i\xi_2}{2} \\ -\frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_1^2}{|\xi|^2} & \frac{i\xi_1}{2} \\ \frac{i\xi_2}{2} & -\frac{i\xi_1}{2} & \frac{i\eta_1\xi_1}{4\mu} \end{Vmatrix}. \tag{23}$$

Then, taking (15) into account, it can be proved that the components of $\hat{\Gamma}^{(2)}$ satisfy the estimates

$$\begin{aligned} \left|\partial^\alpha \hat{\Gamma}_{ij}^{(2)}(\xi)\right| &\le \frac{c}{(|\xi|^2 + |\xi_1|)^{|\alpha|}}, \quad |\xi| \le 1, \ |\alpha| \le 2, \ i,j = 1,2, \\ \left|\partial^\alpha \hat{\Gamma}_{3j}^{(2)}(\xi)\right| &\le \frac{c|\xi|}{(|\xi|^2 + |\xi_1|)^{|\alpha|}}, \quad |\xi| \le 1, \ |\alpha| \le 2, \ j = 1,2,3, \\ \left|\partial^\alpha \hat{\Gamma}_{i3}^{(2)}(\xi)\right| &\le \frac{c|\xi|}{(|\xi|^2 + |\xi_1|)^{|\alpha|}}, \quad |\xi| \le 1, \ |\alpha| \le 2, \ i = 1,2,3, \end{aligned} \tag{24}$$

from which we obtain the estimates of $\Gamma^{(2)}$ in the neighbouhood of the point $|x| = \infty$ [3]:

$$\begin{aligned}
&\partial^\alpha \widehat{\Gamma}^{(2)}_{ij}(x) \le o\Big(|x|^{-1}\Big), \quad |\alpha| \ge 0, \;\; i,j = 1,2,3, \\
&\partial^\alpha \widehat{\Gamma}^{(2)}_{3j}(x) \le o\Big(|x|^{-2}\Big), \quad |\alpha| \ge 1, \;\; j = 1,2,3, \\
&\partial^\alpha \widehat{\Gamma}^{(2)}_{i3}(x) \le o\Big(|x|^{-2}\Big), \quad |\alpha| \ge 1, \;\; i = 1,2,3.
\end{aligned} \tag{25}$$

Let us now calculate the inverse Fourier transform of $\widehat{\Gamma}^{(\infty)}$. We obtain

$$\begin{aligned}
\Gamma^{(\infty)}_{11}(x) &= \frac{1}{2\pi\mu}\Big(K_0(m|x|) - \frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1} - \frac{1}{2\pi m\mu}\frac{x_1}{|x|^2}, \\
\Gamma^{(\infty)}_{12}(x) &= \Gamma^{(\infty)}_{21}(x) = -\frac{1}{2\pi\mu}\frac{x_2}{|x|}K_0'(m|x|)e^{mx_1} - \frac{1}{2\pi m\mu}\frac{x_2}{|x|^2}, \\
\Gamma^{(\infty)}_{22}(x) &= \frac{1}{2\pi\mu}\Big(K_0(m|x|) + \frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1} + \frac{1}{2\pi m\mu}\frac{x_1}{|x|^2}, \\
\Gamma^{(\infty)}_{13}(x) &= -\Gamma^{(\infty)}_{31}(x) = \frac{m}{2\pi\mu}\frac{x_2}{|x|}K_0'(m|x|)e^{mx_1}, \\
\Gamma^{(\infty)}_{23}(x) &= -\Gamma^{(\infty)}_{32}(x) = -\frac{m}{2\pi\mu}\Big(K_0(m|x|) + \frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1}, \\
\Gamma^{(\infty)}_{33}(x) &= -\frac{\eta_1 m}{4\pi\mu^2}\Big(K_0(m|x|) + \frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1},
\end{aligned} \tag{26}$$

where K_0 is the MacDonald function of zero order:

$$K_0(t) = \int_0^\infty e^{-t\,\mathrm{ch}\,\eta}\,d\eta,$$
$$m = \frac{\eta_1}{2\mu}.$$

The equalities (22)–(26) imply the validity of

Theorem 2. *The fundamental matrix Γ is represented in the form*

$$\Gamma = \Gamma^{(\infty)} + \Gamma^{(2)}, \tag{27}$$

where the components $\Gamma^{(\infty)}$ are written as (26) and $\Gamma^{(2)}$ admits the estimates (25) in the neighbourhood of the point $|x| = \infty$.

It is not difficult to obtain the following asymptotic representation in the neighbourhood of the infinity of the MacDonald function and

5

its derivative:

$$K_0(t) = \frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 - \frac{1}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big),$$

$$K_0'(t) = -\frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 + \frac{3}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big),$$

$$K_0''(t) = \frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 + \frac{7}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big).$$

Hence we obtain the asymptotic representations of the components of the matrix $\Gamma^{(\infty)}$ in the neighbourhood of the point $|x| = \infty$:

$$\begin{aligned}
\Gamma_{11}^{(\infty)}(x) &= \frac{(|x| + x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi m}|x|^{3/2}} - \frac{1}{2\pi m\mu}\frac{x_1}{|x|^2} + O\Big(|x|^{-3/2}\Big),\\
\Gamma_{12}^{(\infty)}(x) = \Gamma_{21}^{(\infty)}(x) &= \frac{x_2 e^{-m(|x|-x_1)}}{2\sqrt{2\pi m\mu}|x|^{3/2}} - \frac{1}{2\pi m\mu}\frac{x_2}{|x|^2} + O\Big(|x|^{-3/2}\Big),\\
\Gamma_{13}^{(\infty)}(x) = -\Gamma_{31}^{(\infty)}(x) &= -\frac{\sqrt{m}}{2\sqrt{2\pi}}\frac{x_2 e^{-m(|x|-x_1)}}{|x|^{3/2}} + O\Big(|x|^{-3/2}\Big),\\
\Gamma_{22}^{(\infty)}(x) &= \frac{(|x| - x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi m}|x|^{3/2}} + \frac{1}{2\pi m\mu}\frac{x_1}{|x|^2} + O\Big(|x|^{-3/2}\Big),\\
\Gamma_{23}^{(\infty)}(x) = -\Gamma_{32}^{(\infty)}(x) &= -\frac{\sqrt{m}(|x| - x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi}\mu|x|^{3/2}} + O\Big(|x|^{-3/2}\Big),\\
\Gamma_{33}^{(\infty)}(x) &= -\frac{\eta_1\sqrt{m}(|x| - x_1)e^{-m(|x|-x_1)}}{4\sqrt{2\pi}\mu^2|x|^{3/2}} + O\Big(|x|^{-3/2}\Big).
\end{aligned} \tag{28}$$

In particular, (28) implies

$$\Big|\Gamma_{kj}^{(\infty)}(x)\Big| \le c_1\frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + c_2|x|^{-3/2}, \quad k,j = 1,2,3. \tag{29}$$

In a similar manner we can obtain the estimates for the derivatives of $\Gamma_{kj}^{(\infty)}$ as well:

$$\begin{gathered}
\Big|\partial^\alpha\Gamma_{kj}^{(\infty)}(x)\Big| \le c_1\frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + c_2|x|^{-2},\\
|\alpha| \ge 1, \quad k,j = 1,2,3.
\end{gathered} \tag{30}$$

Remark. As follows from (25), (29), in the case of Ozeen's linearization the fundamental matrix has order $O(|x|^{-1/2})$ at infinity, but, as shown in [4], the fundamental matrix of the system obtained in the case of Stokes' linearization has order $O(\ln|x|)$ at infinity, i.e. it is unbounded. Therefore the properties of solutions of the external boundary value problems are different in the two cases.

We also note that no such difference between the fundamental solutions is observed for Stokes' and Ozeen's linearizations. In that case both fundamental solutions have order $O(|x|^{-1})$ at infinity [3], [5].

3. Regular Solution Representation Formulas. The Uniqueness Theorems. Let D^+ be a finite domain in $\mathbb{R}^2$ bounded by the piecewise-smooth contour S; (u,p) and (u',p') be regular pairs satisfying the conditions

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2}=0.$$

If, besides, $L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\in L_1(D^+)$, $L(\partial_x)u'-\eta\frac{\partial u'}{\partial x_1}-G(\partial_x)p'\in L_1(D^+)$, then the following identities of the Green formula type are fulfilled:

$$\int_{D^+}\Big[u\Big(L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\Big)+E(u,u)\Big]dx=$$
$$=\int_S\Big[u\Big(P(\partial_y,n)u-\frac{1}{2}n_1\eta u-Np\Big)\Big]^+d_yS, \tag{31}$$

$$\int_{D^+}\Big[u'\Big(L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\Big)-$$
$$-u\Big(L(\partial_x)u'+\eta\frac{\partial u'}{\partial x_1}-G(\partial_x)p\Big)\Big]dx=$$
$$=\int_S\Big[u'\Big(P'(\partial_y,n)u-\frac{1}{2}n_1\eta u-Np\Big)-$$
$$-u\Big(P(\partial_y,n)u'+\frac{1}{2}n_1\eta u'-Np'\Big)\Big]^+d_yS, \tag{32}$$

where

$$E(u,u')=(\mu+\alpha)\sum_{i,j=1}^{2}v_{ij}v'_{ji}+(\mu-\alpha)\sum_{i,j=1}^{2}v_{ji}v'_{ij}+\gamma\sum_{i=1}^{2}\omega_i\omega_i',$$

$$v_{ij}=\frac{\partial v_j}{\partial x_i}+(j-i)\omega,\quad i,j=1,2;\quad \omega_i=\frac{\partial\omega}{\partial x_i},\quad i=1,2.$$

$E(u,u')$ is an analogue of the energetic form. In particular,

$$E(u,u')=(\mu+\alpha)(v_{12}^2+v_{21}^2)+2(\mu-\alpha)v_{12}v_{21}+$$
$$+2\mu(v_{11}^2+v_{22}^2)+\gamma(\omega_1^2+\omega_2^2). \tag{33}$$

One can easily verify that for the form (33) to be positive definite with respect to v_{ij}, ω_i, it is necessary and sufficient that the conditions

$$\mu > 0, \quad \alpha > 0, \quad \gamma > 0 \tag{34}$$

be fulfilled.

If we use the area potentials

$$U(F,x) = \frac{1}{2}\int_{D^\pm} \rho\Gamma(x-y)F(y)\,dy,$$
$$q(F,x) = \frac{1}{2}\int_{D^\pm} \rho Q(x-y)F(y)\,dy,$$

where $F = (F_1, F_2, F_3) \in \mathbb{C}^{0,h}(D^\pm)$, $0 < h \leq 1$, and is the finite vector in the case of D^-, then the boundary value problems for the systems (3) and (4) can be reduced to the corresponding problems for the homogeneous systems [3]

$$\begin{gathered} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ L(\partial_x)u - G(\partial_x)p - \eta\frac{\partial u}{\partial x_1} = 0 \end{gathered} \tag{35}$$

and

$$\begin{gathered} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ L(\partial_x)u - G(\partial_x)p + \eta\frac{\partial u}{\partial x_1} = 0. \end{gathered} \tag{36}$$

Therefore in the sequel we will consider the boundary value problems only for the homogeneous systems (35) and (36).

Like in the three-dimensional case [3], from (32) we can obtain the following formula for representation of solutions of the system (3) in the domain D^+:

$$\begin{aligned} u(x) = \frac{1}{2}\int_S \Gamma(x-y)\Big[P(\partial_y,n)u(y) - \frac{1}{2}n_1\eta u(y) - N(y)p(y)\Big]^+ d_yS \, - \\ -\frac{1}{2}\int_S \Big[P(\partial_y,n)\Gamma'(x-y) + \frac{1}{2}n_1\eta\Gamma'(x-y) - \\ -N(y) * Q(x-y)\Big]' u^+(y)d_yS, \end{aligned} \tag{37}$$

$$p(x) = \frac{1}{2}\int_S Q(x-y)\Big[P(\partial_y,n)u(y) - \frac{1}{2}n_1\eta u(y) - N(y)p(y)\Big]^+ d_yS -$$
$$-\frac{1}{2}\int_S \Big[P(\partial_y,n)Q(x-y) + \frac{1}{2}n_1\eta Q(x-y) -$$
$$-\eta_1 Q(x-y)N(y)\Big]u^+(y)d_yS. \qquad (38)$$

Here the prime above the matrix denotes the operation of transposition and

$$N(y) * Q(z) = \begin{Vmatrix} n_1(y)Q_1(z) & n_1(y)Q_2(z) \\ n_2(y)Q_1(z) & n_2(y)Q_2(z) \end{Vmatrix}.$$

Taking the properties (29) and (30) of the matrix of fundamental solutions into account and repeating the reasoning from [3], one can prove that the formulas (37), (38) are also valid for regular solutions in the domain D^-.

Now let us prove that the equality

$$\int_{D^-} E(u,u)\,dx = -\int_S u^-\Big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\Big]^- d_yS \qquad (39)$$

is fulfilled for the regular solution (u,p) of the homogeneous system (35) in D^-.

To this effect we have to use the formula (31) in the domain $D^- \cap B(0,R)$, where $B(0,R)$ is the circle of radius R centred at $x=0$ and containing the domain D^+. Recalling that the pair (u,p) is the solution of the system (35), we obtain

$$\int_{D^-\cap B(0,R)} E(u,u)\,dx = -\int_S u^-\Big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\Big]^- d_yS + \mathcal{I}(R),$$

where

$$\mathcal{I}(R) = \int_{\partial B(0,R)} u\Big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\Big]\,d_yS. \qquad (40)$$

Obviously, to prove (39) it is enough to prove

$$\lim_{R\to\infty} \mathcal{I}(R) = 0. \qquad (41)$$

From (16), (25), (29) and the formula of representation of regular solutions in the domain D^- it follows that in the neighbourhood of infinity

$$\big|\partial^\alpha u(x)\big| = c_1\frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + O(|x|^{-1}),\ \ |\alpha| \geq 0,$$
$$|p(x)| = O(|x|^{-1}).$$

Taking these estimates into account in (40) and passing to the limit, we obtain (41). The equality (39) is proved.

Let us turn to proving the uniqueness theorems. In particular, we will prove

Theorem 3. *Each solution of the homogeneous problem* $(\mathrm{I})_0^+$ *has the form*

$$u = 0, \qquad p = p_0, \tag{42}$$

where p_0 *is an arbitrary constant. The homogeneous Problems* $(\mathrm{II})_0^+$, $(\mathrm{I})_0^-$ *and* $(\mathrm{II})_0^-$ *can have only the trivial solutions* $u = 0$, $p = 0$.

Proof. Let (u, p) be a solution of anyone of the considered problems. Then by virtue of (31) and (39)

$$\int_{D^\pm} E(u,u)\, dx = 0.$$

Therefore $E(u,u) = 0$, $x \in D^+$ $(x \in D^-)$. Hence on account of (34)

$$\frac{\partial v_j}{\partial x_i} + (j-i)\omega = 0, \quad \frac{\partial \omega}{\partial x_i} = 0, \quad i,j = 1,2.$$

The general solution of the resulting system has the form

$$v_1 = ax_2 + b_1, \quad v_2 = -ax_1 + b_2, \quad \omega = a. \tag{43}$$

Since for the external domain the regular solution must vanish at infinity, the external homogeneous boundary value problems have trivial solutions. In the case of Problem $(\mathrm{I})_0^+$, (43) also implies that $v_1 = v_2 = \omega = 0$; substituting these value in the equation (35) we obtain

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = 0,$$

whence it follows that $p = const.$

Next, let us consider Problem $(\mathrm{II})_0^+$. By virtue of (43) and the homogeneous boundary condition we have $a = 0$, i.e. $v_i = b_i$. Then (35) yields $p = p_0$. Considering again the boundary condition, we obtain

$$\eta_1 n_1 b_k + 2p_0 n_k = 0, \quad k = 1,2.$$

Since the boundary of D^+ is not the straight line, we conclude that $b_1 = b_2 = p_0 = 0$. ■

4. Reduction of the Boundary Value Problems to Integral Equations. Like in the three-dimensional case, we introduce the simple-layer potentials

$$V(\varphi)(x) = \int_S \Gamma(x-y)\varphi(y)\,d_yS, \quad a(\varphi)(x) = \int_S Q(x-y)\varphi(y)\,d_yS$$

and the double-layer potentials

$$W(\varphi)(x) = \int_S \Big[P(\partial_y, n)\Gamma'(x-y) + \frac{1}{2}n_1\eta\Gamma'(x-y) -$$
$$- N(y) * Q(x-y)\Big]'\varphi(y)d_yS,$$
$$b(\varphi)(x) = \int_S \Big[P(\partial_y, n)Q(x-y) + \frac{1}{2}n_1\eta Q(x-y) -$$
$$- \eta_1 Q_1(x-y)N(y)\Big]\varphi(y)d_yS.$$

Denote by $\tilde{V}(\varphi)$, $\tilde{a}(\varphi)$, $\tilde{W}(\varphi)$, $\tilde{b}(\varphi)$ the potential obtained from $V(\varphi)$, $a(\varphi)$, $W(\varphi)$, $b(\varphi)$ after replacing η_1 and η_2 by $-\eta_1$ and $-\eta_2$, respectively (note that $\tilde{a}(\varphi)$ coincides with $a(\varphi)$).

We will seek for the solution of Problem $(\mathrm{I})^{\pm}$ $[(\tilde{\mathrm{I}})^{\pm}]$ in the form of double-layer potentials

$$u(x) = W(\varphi)(x), \quad p(x) = b(\varphi)(x)$$
$$\Big[u(x) = \tilde{W}(\varphi)(x), \quad p(x) = \tilde{b}(\varphi)(x)\Big]$$

and the solution of Problem $(\mathrm{II})^{\pm}$ $[(\tilde{\mathrm{II}})^{\pm}]$ in the form of single-layer potentials

$$u(x) = V(\psi)(x), \quad p(x) = a(\psi)(x)$$
$$\Big[u(x) = \tilde{V}(\varphi)(x), \quad p(x) = a(\varphi)(x)\Big].$$

Then, as was done in [3], we obtain, for the densities φ, ψ, the singular integral equations

$$\mp\varphi(z) + \int_S \Big[P(\partial_y, n)\Gamma'(z-y) + \frac{1}{2}n_1\eta\Gamma'(z-y) -$$
$$- N(y) * Q(y-z)\Big]'\varphi(y)d_yS = f(z), \qquad (\mathrm{I})^{\pm}$$

$$\mp\varphi(z) + \int_S \Big[P(\partial_y, n)\Gamma(z-y) - \frac{1}{2}n_1\eta\Gamma(z-y) - \\ - N(y) * Q(y-z)\Big]'\varphi(y)d_yS = f(z), \qquad (\tilde{\mathrm{I}})^\pm$$

$$\pm\psi(z) + \int_S \Big[P(\partial_y, n)\Gamma(z-y) - \frac{1}{2}n_1\eta\Gamma(z-y) - \\ - N(z) * Q(y-z)\Big]\psi(y)d_yS = f(z), \qquad (\mathrm{II})^\pm$$

$$\pm\psi(z) + \int_S \Big[P(\partial_y, n)\Gamma'(z-y) + \frac{1}{2}n_1\eta\Gamma'(z-y) - \\ - N(z) * Q(y-z)\Big]\psi(y)d_yS = f(z). \qquad (\tilde{\mathrm{II}})^\pm$$

The investigation of these equations leads to the following result [3, 6]:

Theorem 4. *If* $S \in \mathcal{L}_{k+1}(h')$, $f \in \mathbb{C}^{k,h}(S)$, $0 < h < h' \leq 1$, $k = 1, 2, \dots,$ *then the Fredholm theorems hold for the pairs of equations* $(\mathrm{I})^+$ *and* $(\tilde{\mathrm{II}})^-$, $(\mathrm{I})^-$ *and* $(\tilde{\mathrm{II}})^+$, $(\mathrm{II})^+$ *and* $(\tilde{\mathrm{I}})^-$, $(\mathrm{II})^-$ *and* $(\tilde{\mathrm{I}})^+$ *in the space* $\mathbb{C}^{h,k}(S)$.

5. Existence Theorems for the Boundary Value Problems. In this paragraph we present the existence theorems for the boundary value problems. Their proofs are left out because they repeat the ones given in [3].

Theorem 5. *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{1,h}(S)$, $0 < h < h' \leq 1$, *and* f *satisfies the condition*

$$\int_S Nf\,dS = 0, \qquad (44)$$

then Problem $(\mathrm{I})^+$ *has a regular solution. Moreover, if the condition*

$$\int_S p\,dS = 0$$

is fulfilled, then this solution is unique.

We observe that the condition (44) is not only sufficient, but also necessary for the existence of a regular solution of Problem $(\mathrm{I})^+$.

Theorem 6. *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{1,h}(S)$, $0 < h < h' \leq 1$, *then Problem* $(\mathrm{I})^-$ *has a unique regular solution.*

Theorem 7. *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{0,h}(S)$, $0 < h < h' \leq 1$, *then Problem* $(\mathrm{II})^+$ *has a unique regular solution.*

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(Received 18.03.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993) No 3, 299-309

STURM'S THEOREM FOR EQUATIONS WITH DELAYED ARGUMENT

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ABSTRACT. Sturm's type theorems on separation of zeros of solutions are proved for the second order linear differential equations with delayed argument.

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1. INTRODUCTION

In this article the distribution of zeros of solutions is investigated for the following differential equation with delayed arguments

$$x''(t) + \sum_{i=1}^{m} p_i(t)x(h_i(t)) = 0, \quad t \in [0, +\infty), \tag{1}$$

where p_i are locally summable nonnegative functions and h_i are nonnegative measurable functions for $i = 1, \dots, m$.

The classical result of Sturm is the following: if x_1 and x_2 are linearly independent solutions of the ordinary differential equation

$$x''(t) + p(t)x(t) = 0, \quad t \in [0, +\infty),$$

then between two adjacent zeros of x_1 there is one and only one zero of x_2. This article deals with the extension of the Sturm's theorem to equation (1) with delayed argument.

The first result of this type was obtained by N.V.Azbelev [1]. Namely, if for almost all $t \in [0, +\infty)$ there is at most one zero of each nontrivial solution of equation (1) on the interval $[h(t), t]$, where $h(t) = \min_{i=1,\dots,m} h_i(t)$, then Sturm's theorem holds for equation (1), i.e. the interval $[h(t), t]$ must be "small enough". The generalization of this result of N.V.Azbelev to the "neutral" equation

$$x''(t) - \sum_{j=1}^{n} q_j(t)x''(g_j(t)) + \sum_{i=1}^{m} p_i(t)x(h_i(t)) = 0, \quad t \in [0, +\infty),$$

was obtained in [3]. Our approach assumes that $[h_1(t), h_m(t)]$ is "small enough" for almost all $t \in [0, +\infty)$ (note that we consider the case $h_1(t) \leq \cdots \leq h_m(t) \leq t$ in this article). Namely, if a solution has zero on $[h_1(t), h_m(t)]$, then its derivative has no zero on this interval.

Note the close result of S.M.Labovsky [10] for equation (1) in the case $m = 1$ and another version of Sturm's separation theorem proposed by Yu.I.Domshlak [4,5].

It is known [1] that the space of solutions of equation (1) is two-dimensional, the Wronskian

$$W(t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}$$

of a fundamental system u, v of the solution (1) can vanish, zeros of $W(t)$ do not depend on a fundamental system, $W(0)$ is not equal to zero. Nonvanishing of Wronskian selects the class of homogeneous equations such that each of them is equivalent to a corresponding ordinary differential equation. In this case each nontrivial solution of equation (1) can have only finite number of zeros on any finite interval, moreover, all zeros are simple. It is also known [1] that nonvanishing of the Wronskian is equivalent to the validity of Sturm's theorem about separation of zeros.

The important part of this article concerns with estimates of the distance between adjacent zeros. These results are usually connected with Sturm's comparison theorem. Note in this connection the following investigations [1,4-9,11,12].

We reduce the question about lower bounds for the distance between adjacent zeros and between zero of a solution and zero of its derivative to an estimation of the spectral radius of the corresponding completely continuous operator in the space of continuous functions, i.e. to the well-known problem of functional analysis.

Our interest in the lower bounds of this distance is connected with the problem of existence and uniqueness of a solution of boundary value problems. For example, if $b-a$ is less than the distance between adjacent zeros of solutions of (1), then the boundary value problem

$$x''(t) + \sum_{i=1}^{m} p_i(t)x(h_i(t)) = f(t), \quad t \in [0, +\infty),$$
$$x(a) = A, \quad x(b) = B,$$

has for each A, B, $f(t)$ the unique solution.

2. Main Results

Let $\lambda_{\nu\mu}$ be the smallest positive characteristic number of the operator $F_{\nu\mu} : C_{[\nu,\mu]} \to C_{[\nu,\mu]}$ ($C_{[\nu,\mu]}$ is the space of continuous functions $x : [\nu, \mu] \to R$) which is defined by

$$(F_{\nu\mu}x)(t) = -\int_{\nu}^{\mu} G_{\nu\mu}(t,s) \sum_{i=1}^{m} p_i(s)x(h_i(s))\gamma(\nu, h_i(s))ds,$$

where

$$\gamma(\nu, h_i(s)) = 0 \text{ if } h_i(s) < \nu, \quad \gamma(\nu, h_i(s)) = 1 \text{ if } h_i(s) \geq \nu, \quad (2)$$

$$G_{\nu\mu}(t,s) = \begin{cases} -(\mu - t)(s - \nu)/(\mu - \nu) & \text{for } \nu \leq s \leq t \leq \mu, \\ -(t - \nu)(\mu - s)/(\mu - \nu) & \text{for } \nu \leq t < s \leq \mu, \end{cases}$$

$G_{\nu\mu}(t,s)$ is the Green's function of the boundary value problem

$$x''(t) = f(t), \quad t \in [\nu, \mu], \quad x(\nu) = 0, \quad x(\mu) = 0.$$

It is clear that the operator $F_{\nu\mu}$ is positive.

Theorem 1. *Let*

1) *the functions h_i be nondecreasing and the inequalities $h_i(t) \leq h_{i+1}(t)$ hold for $i = 1, \dots, m-1$ and almost all $t \in [0, +\infty)$;*

2) *the functions p_{i+1}/p_i be nondecreasing for $i = 1, \dots, m-1$;*

3) *at least one of the following inequalities be fulfilled*

$$\operatorname*{ess\,sup}_{s \in [h_1(t), h_m(t)]} \sum_{i=1}^{m} p_i(s)[h_m(t) - h_1(t)]^2 < 2, \quad (3)$$

$$[h_m(t) - h_1(t)] \int_{h_1(t)}^{h_m(t)} \sum_{i=1}^{m} p_i(s)ds \leq 1, \quad (4)$$

for almost all $t \in [0, +\infty)$.

Then

a) *$W(t)$ doesn't vanish for $t \in [0, +\infty)$;*

b) *if ν and μ are two zeros of some nontrivial solution x of equation (1), then $\lambda_{\nu\mu} \leq 1$;*

c) *there is one and only one zero of the derivative of a nontrivial solution between any two adjacent zeros of this nontrivial solution.*

Examples: the condition 2) of Theorem 1 is fulfilled for the following cases:

1) if $m = 2$, p_1 is nonincreasing and p_2 is nondecreasing;
2) if $p_i(t) = a_i f(t)$, where $a_i = \text{const}$, $i = 1, \dots, m$;
3) if $p_i(t) = a_i t^i$, $i = 1, \dots, m$;
4) if $p_i(t) = a_i t + b_i$ ($a_i > 0$, $b_i > 0$), where b_i/a_i are nonincreasing for $i = 1, \dots, m$.

The condition that the functions p_{i+1}/p_i are nondecreasing for $i = 1, \dots, m$ is essential, as the following example shows.

Example 1. The function

$$x(t) = \begin{cases} 1 - t^2, & 0 \le t \le 2, \\ 0,01t^2 - 4,04t + 5,04, & 2 < t \le 210, \\ 2(t - 5239,5)^2, & 210 < t \end{cases}$$

has a multiple zero at the point $t = 5239,5$. This function is the solution of the equation

$$x''(t) + p_1(t)x(h_1(t)) + x(h_2(t)) = 0,$$

where

$$h_1(t) = \begin{cases} 0, & 0 \le t \le 2, \\ 0,9, & t > 2, \end{cases} \qquad h_2(t) = \begin{cases} 0, & 0 \le t \le 2, \\ 1,1, & t > 2, \end{cases}$$

$$p_1(t) = \begin{cases} 1, & 0 \le t \le 210, \\ (21 + 32/10059)/19, & t > 210. \end{cases}$$

It is clear that $W(5239,5) = 0$.

The following fact follows from Theorem 1.

Corollary. *If $m = 1$ and h_1 is nondecreasing, then the assertions* a),b),c) *of Theorem* 1 *are fulfilled.*

The condition that the functions h_i are nondecreasing is essential, as the example of N.V.Azbelev [1] shows.

Let $R_{\nu\mu}(t, s)$, $Q_{\nu\mu}(t, s)$ be Green's functions of the boundary value problems

$$x''(t) = f(t), \quad t \in [\nu, \mu], \quad x(\nu) = 0, \quad x'(\mu) = 0,$$
$$x''(t) = f(t), \quad t \in [\nu, \mu], \quad x'(\nu) = 0, \quad x(\mu) = 0$$

respectively. It is clear that

$$R_{\nu\mu}(t,s) = \begin{cases} \nu - s & \text{if } \nu \le s \le t \le \mu, \\ \nu - t & \text{if } \nu \le t < s \le \mu, \end{cases}$$

$$Q_{\nu\mu}(t,s) = \begin{cases} t - \mu & \text{if } \nu \le s \le t \le \mu, \\ s - \mu & \text{if } \nu \le t < s \le \mu. \end{cases}$$

Define the operators $R_{\nu\mu}, Q_{\nu\mu} : C_{[\nu,\mu]} \to C_{[\nu,\mu]}$ by the formulas

$$(R_{\nu\mu}x)(t) = \int_\nu^\mu R_{\nu\mu}(t,s)\sum_{i=1}^m p_i(s)x(h_i(s))\gamma(\nu,h_i(s))ds,$$

$$(Q_{\nu\mu}x)(t) = \int_\nu^\mu Q_{\nu\mu}(t,s)\sum_{i=1}^m p_i(s)x(g_i(s))ds,$$

here g_i $(i = 1,\dots,m)$ are measurable functions such that $\nu \le g_i(t) \le \mu$. Let $r_{\nu\mu}$, $q_{\nu\mu}$ be the smallest positive characteristic numbers of the operators $R_{\nu\mu}$, $Q_{\nu\mu}$ respectively.

Theorem 2. *Let the conditions* 1), 2) *of Theorem* 1 *be fulfilled,* $r_{h_1(t)h_m(t)} > 1$ *for almost all* $t \in [0,+\infty)$, $q_{h_1(t)h_m(t)} > 1$ *for almost all* $t \in [0,+\infty)$ *and all possible functions* g_i *such that* $g_i(s) \in [h_1(t), h_m(t)]$ *for* $s \in [h_1(t), h_m(t)]$, $i = 1,\dots,m$. *Then the assertions* a),b),c) *of Theorem* 1 *are fulfilled.*

Remark. The inequalities $r_{h_1(t)h_m(t)} > 1$ and $q_{h_1(t)h_m(t)} > 1$ for $t \in [0,+\infty)$ guarantee that a solution of equation (1), having zero on the interval $[h_1(t), h_m(t)]$ has no zero of its derivative on this interval.

3. Proofs

We start with some auxiliary results.

Lemma 1. *Let* α *be a zero of the nontrivial solution* x *of equation* (1), β *be a zero of its derivative such that* $x(t) > 0$ *for* $t \in (\alpha,\beta)$, $\alpha < \beta$. *Then there exists a set* $e \subset (\alpha,\beta)$ *of positive measure such that* $\sum_{i=1}^m p_i(t)x(h_i(t)) > 0$ *for* $t \in e$.

Proof of Lemma 1. Let we have on the contrary, $\sum_{i=1}^m p_i(t)x(h_i(t)) \le 0$ for almost all $t \in [\alpha,\beta]$. By the theorem of Lagrange there exists $d \in (\alpha,\beta)$ such that $x'(d) > 0$. $x'(\beta) = x'(d) + \int_d^\beta x''(s)ds = x'(d) - \int_d^\beta \sum_{i=1}^m p_i(s)x(h_i(s))ds > 0$, that contrast the assumption: $x'(\beta) = 0$. ∎

Lemma 2. *Let*

1) *y be a nondecreasing function in the interval $[a,b]$;*

2) *$a \le h_1(t) \le h_2(t) \le \dots \le h_m(t) \le b$ for almost all $t \in [c,d] \in [a,b]$, h_i be nondecreasing for $i = 1,\dots,m$;*

3) *the functions p_{i+1}/p_i be nondecreasing for $i = 1,\dots,m-1$.*

Then from the existence of a set $e \subset [c_1,d_1] \subset [c,d)$ such that $\operatorname{mes}(e) > 0$ and $\sum_{i=1}^m p_i(t)y(h_i(t)) > 0$ for $t \in e$, it follows that $\sum_{i=1}^m p_i(t)y(h_i(t)) > 0$ for almost all $t \in [d_1,d]$.

Proof of Lemma 2. Let k be a number such that $y(h_i(t)) \ge 0$ for $t \in e$, $i \ge k$. By the condition we have the inequality $-\sum_{i=1}^{k-1} p_i(t)y(h_i(t)) < \sum_{i=k}^m p_i(t)y(h_i(t))$ for $t \in e$.

For all $i = 1,\dots,m$ $y(h_i(t))$ are nondecreasing since y and h_i are nondecreasing. Using the condition 3) we obtain for $t \in e$ and r such that $t + r \in [d_1,d]$:

$$-\sum_{i=1}^{k-1} p_i(t+r)y(h_i(t+r)) = -\sum_{i=1}^{k-1}(p_i(t+r)/p_i(t))p_i(t)y(h_i(t+r)) \le$$

$$\le -(p_{k-1}(t+r)/p_{k-1}(t))\sum_{i=1}^{k-1} p_i(t)y(h_i(t+r)) <$$

$$< (p_{k-1}(t+r)/p_{k-1}(t))\sum_{i=k}^{m} p_i(t)y(h_i(t+r)) \le$$

$$\le \sum_{i=k}^{m} p_i(t+r)y(h_i(t+r)). \quad \blacksquare$$

Lemma 3. *Let $[\alpha,\beta] \subset [\nu,\mu]$. Then*

1) *if $\lambda_{\nu\mu} > 1$, then $\lambda_{\alpha\beta} > 1$;*
if $\lambda_{\alpha\beta} \le 1$, then $\lambda_{\nu\mu} \le 1$;

2) *if $r_{\nu\mu} > 1$, then $r_{\alpha\beta} > 1$;*
if $r_{\alpha\beta} \le 1$, then $r_{\nu\mu} \le 1$;

3) *if $q_{\nu\mu} > 1$ for each collection of functions such that $g_i(t) \in [\nu,\mu]$, $t \in [\nu,\mu]$, $i = 1,\dots,m$, then $q_{\alpha\beta} > 1$ each collection of functions such that $\bar{g}_i(t) \in [\alpha,\beta]$, $t \in [\alpha,\beta]$, $i = 1,\dots,m$;*

if there exists a collection of functions such that $\bar{g}_i(t) \in [\alpha,\beta]$, $i = 1,\dots,m$, and $q_{\alpha\beta} \le 1$, then there exists a collection of functions $g_i(t) \in [\nu,\mu]$, $t \in [\nu,\mu]$, $i = 1,\dots,m$, such that $q_{\nu\mu} \le 1$.

Proof of Lemma 3. Assertion 1) is proved in [1] and the proof of Assertion 2) is analogous, therefore we prove only Assertion 3).

Let us take an arbitrary collection of functions $\bar{g}_i(t) \in [\alpha,\beta]$, $t \in [\alpha,\beta]$, $i = 1,\dots,m$, and denote

$$g_i(t) = \begin{cases} \bar{g}_i(t) & \text{for } t \in [\alpha,\beta], \\ \alpha & \text{for } t \in [\alpha,\beta]. \end{cases}$$

By condition $q_{\nu\mu} > 1$ for this collection g_i, $i = 1,\dots,m$, the equation $x = Q_{\nu\mu}x + 1$ has a positive solution $v = \lim_{n\to\infty} x_n$, where $x_0 = 1$, $x_{n+1} = Q_{\nu\mu}x_n + 1$. Since $Q_{\nu\mu}(t,s) \geq Q_{\alpha\beta}(t,s)$ for $t,s \in [\alpha,\beta]$, then $v \geq Q_{\alpha\beta}v + 1$. Now, by the theorem about integral inequalities [1], we have $q_{\alpha\beta} > 1$ for this collection $\bar{g}_i$, $i = 1,\dots,m$. The first part of the assertion 3) is proved.

The second part of the assertion 2) can be deduced from the first part. ■

Proof of Theorem 2. Let x be a nontrivial solution of the equation (1). Let us consider the case $x(0) > 0$, $x'(0) \geq 0$ (the case $x(0) \geq 0$, $x'(0) < 0$ can be considered analogously).

Denote by β_1 the first zero of the solution's derivative, by α_1 the first zero of the solution x ($\beta_1 < \alpha_1$ by our assumption). If β_1 or α_1 doesn't exist, then the theorem is trivial.

Let us show that there exists a collection $g_i(t)$, $i = 1,\dots,m$, such that $q_{\beta_1\alpha_1} \leq 1$. Really, x satisfies the following equation

$$x(t) = -\int_{\beta_j}^{\alpha_k} Q_{\beta_j\alpha_k}(t,s)\sum_{i=1}^{m} p_i(s)x(h_i(s))\gamma(\beta_j,h_i(s))ds - $$
$$-\int_{\beta_j}^{\alpha_k} Q_{\beta_j\alpha_k}(t,s)\sum_{i=1}^{m} p_i(s)x(h_i(s))[1-\gamma(\beta_j,h_i(s))]ds$$

for $t \in [\beta_1,\alpha_1]$, where $k = j = 1$ and γ is defined by (2).

Rewrite the equality (4) in the following form

$$x(t) = -\int_{\beta_1}^{\alpha_1} Q_{\beta_1\alpha_1}(t,s)\sum_{i=1}^{m} p_i(s)x(g_i(s))ds,$$

where the functions g_i such that $g_i(t) \in [\beta_1,\alpha_1]$.

Existence of these functions g_i, $i = 1,\dots,m$, follows from the next arguments. Since $x''(t) = -\sum_{i=1}^{m} p_i(t)x(h_i(t)) < 0$, then x' is not increasing. Therefore $x(\beta_1) = \max_{t\in[0,\alpha_1]} x(t)$ and the set of values of function x on the interval $[0,\beta_1]$ is included in the set of values of the function x on the interval $[\beta_1,\alpha_1]$, hence the solution g_i of the functional equation $x(h_i(t)) = x(g_i(t))$, $t \in [\beta_1,\alpha_1]$ exists.

It is obvious that for this collection of functions $g_i, i = 1, \ldots, m$, we have $q_{\alpha_1\beta_1} \leq 1$.

We show that $x'(\alpha_1) < 0$. Indeed, by the theorem of Lagrange there exists $d \in (\beta_1, \alpha_1)$ such that $x(\alpha_1) - x(\beta_1) = x'(d)(\alpha_1 - \beta_1)$. From here $x'(d) < 0$ and $x'(\alpha_1) = x'(d) + \int_d^{\alpha_1} x''(t)dt < 0$.

Let β_2 be the first zero of the solution x after α_1. By Lemma 1 there exists a set $e \in [\alpha_1, \beta_1]$ with $\text{mes}(e) > 0$ such that $x''(t) = -\sum_{i=1}^m p_i(t)x(h_i(t)) > 0$ for $t \in e$. From here it follows that $h_m(t) > \alpha_1$ for almost all $t \geq \beta_2$. Since $q_{h_1(t)h_m(t)} > 1$, independently of collection of functions g_i, $i = 1, \ldots, m$ we obtain by Lemma 3 that $h_1(t) \geq \beta_1$ for almost all $t \geq \beta_2$.

Next, show that $r_{\alpha_1\beta_2} \leq 1$. Indeed, on the interval $[\alpha_1, \beta_2]$ the solution x of equation (1) satisfies the following integral equation

$$x(t) = -\int_{\alpha_k}^{\beta_j} R_{\alpha_k\beta_j}(t,s)\sum_{i=1}^m p_i(s)x(h_i(s))\gamma(\alpha_k, h_i(s))ds -$$
$$-\int_{\alpha_k}^{\beta_j} R_{\alpha_k\beta_j}(t,s)\sum_{i=1}^m p_i(s)x(h_i(s))[1 - \gamma(\alpha_k, h_i(s))]ds,$$

where $k = 1$, $j = 2$.

Taking $v(t) = -x(t)$, we obtain the inequality $v(t) \leq (R_{\alpha_k\beta_j}v)(t)$ for $k = 1$, $j = 2$. By the theorem about the integral inequalities (see, for example, [1]) we obtain $r_{\alpha_1\beta_2} \leq 1$.

Denote α_2 the first zero of the solution x after β_2 (if the solution x hasn't a second zero α_2, then Theorem 2 is trivial).

If there exist $d \in (\beta_2, \alpha_2)$ such that $h_1(t) \geq \alpha_1$ for almost all $t \geq d$, then $x(h_i(t)) \leq 0$ for almost all $t \in [d, \alpha_2]$, hence $x''(t) > 0$ for almost all $t \in [d, \alpha_2]$.

If $\beta_1 \leq h_1(t) < \alpha_1$, then by Lemma 2, with the use of the condition $r_{h_1(t)h_m(t)} > 1$ and Lemma 3, we can conclude that $x''(t) > 0$ for $t \in [\beta_2, d]$.

By the theorem of Lagrange, there exists $c \in (\beta_2, \alpha_2)$ such that $x(\alpha_2) - x(\beta_2) = x'(c)(\alpha_2 - \beta_2)$, this implies $x'(c) > 0$ and $x'(\alpha_2) = x'(c) + \int_c^{\alpha_2} x''(t)dt > 0$. It means that $\alpha_2 < \beta_3$ (we denote β_3 the first zero of derivative of the solution x after α_2). Now we show that $q_{\beta_2\alpha_2} \leq 1$ for some collection g_i, $i = 1, \ldots, m$. On the interval $[\beta_2, \alpha_2]$ the solution x of equation (1) satisfies equation (4) for $j = k = 2$.

Rewrite this equation in the following form:

$$x(t) = -\int_{\beta_j}^{\alpha_k} Q_{\beta_j\alpha_k}(t,s)\sum_{i=1}^{m} p_i(s)x(g_i(s))\gamma(\alpha_{k-1},h_i(s))ds -$$
$$-\int_{\beta_j}^{\alpha_k} Q_{\beta_j\alpha_k}(t,s)\sum_{i=1}^{m} p_i(s)x(h_i(s))[1-\gamma(\alpha_{k-1},h_i(s))]ds$$

where $k = j = 2$, $g_i(t) \in [\beta_2, \alpha_2]$ such that $\gamma(\alpha_1, h_i(t)) = x(g_i(t))$ for $t \in [\beta_2, \alpha_2]$. The collection g_i, $i = 1, \ldots, m$ exists since $x''(t) \geq 0$ for $t \in [\beta_2, \alpha_2]$. For, since x' isn't decreasing on the interval $[\beta_2, \alpha_2]$, therefore $|x(\beta_2)| = \max\limits_{\alpha_1 \leq t \leq \alpha_2} |x(t)|$ and the set of values of the function x on the interval $[\alpha_1, \beta_2]$ is included in the set of values of x on the interval $[\beta_2, \alpha_2]$.

It follows from [1] that $q_{\beta_2\alpha_2} \leq 1$.

The inequality $\lambda_{\alpha_1\alpha_2} \leq 1$ is proved analogously to $r_{\alpha_1\beta_2} \leq 1$.

If α_m is the last zero of the nontrivial solution, then repeating the arguments for $j, k = 3, 4, 5, \ldots, m$, we obtain the proof of the theorem. If the solution x has an infinite number of zeros, then the sequence α_k of zeros is unbounded. Indeed, we have proved that $\lambda_{\alpha_k\alpha_{k+1}} \leq 1$, this implies that [1] $(\alpha_{k+1} - \alpha_k)\int_{\alpha_k}^{\alpha_{k+1}} \sum_{i=1}^{m} p_i(t)dt > 4$ and, consequently, the increasing sequence α_k cannot be bounded from above.

It is clear that all zeros of the solution x belong to this sequence α_k. In this case the repetition of our arguments completes the proof of the assertions a) and b) of Theorem 2.

The assertion c) follows from the following argument. For each j we have proved that $\operatorname{sign} x(t) = \operatorname{sign} \sum_{i=1}^{m} p_i(t)x(h_i(t))$ for $t \in (\beta_j, \alpha_j)$, this implies $x'(t) = x'(\beta_j) - \int_{\beta_j}^{t} \sum_{i=1}^{m} p_i(s)x(h_i(s))ds \neq 0$ for $t \in (\beta_j, \alpha_j)$. ■

Proof of Theorem 1. Theorem 1 can be obtained as a corollary of Theorem 2.

Indeed, from the theorem about the integral inequalities [1] we have the following. If there exists a continuous positive function v such that $v(t) > Q_{\nu\mu}v)(t)$ $(v(t) > (R_{\nu\mu}v)(t))$ for $t \in (\nu, \mu)$, then $q_{\nu\mu} > 1$ $(r_{\nu\mu} > 1)$. Substituting $v = 1$, we obtain that the condition (4) guarantees the inequalities $r_{h_1(t)h_m(t)} > 1$, $q_{h_1(t)h_m(t)} > 1$.

If there exists a positive function v such that

$$\varphi(t) = v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\gamma(v, h_i(t)) \leq 0, \quad \int_{\nu}^{\mu} \varphi(t)dt < 0,$$
$$v'(\nu) = 0, \; v(\mu) = 0 \;\; (v(\nu) = 0, \; v'(\mu) = 0),$$

then $q_{\nu\mu} > 1$ ($r_{\nu\mu} > 1$) [2]. Substituting $v(t) = (t - 2\nu + \mu)(\mu - t)$ ($v(t) = (t - \nu)(2\mu - \nu - t)$), we conclude that inequality (3) implies the inequality $q_{h_1(t)h_m(t)} > 1$ ($r_{h_1(t)h_m(t)} > 1$). ■

ACKNOWLEDGEMENT.

The research was supported in part by grant from the Ministry of Science and the "MA-AGARA"-special project for absorption of new immigrants, in the Department of Mathematics, Technion.

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(Received 4.01.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993) No 3, 311-321

ON THE CARDINALITY OF A SEMI-ALGEBRAIC SET

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ABSTRACT. It is shown that the cardinality of a finite semi-algebraic subset over a real closed field can be computed in terms of signatures of effectively constructed quadratic forms.

რეზიუმე. ნაჩვენებია, რომ ნამდვილ ჩაკეტილ ველზე მოცემული სასრული ნახევრად ალგებრული ქვესიმრავლის სიმძლავრე შეიძლება გამოითვალოს ეფექტურად აგებული კვადრატული ფორმების სიგნატურათა მეშვეობით.

1. The problem under consideration may be described as follows. Let X be a semi-algebraic set over an ordered field K [1]

$$X = \{f_i = 0,\ g_j > 0;\ i \in I,\ j \in J\} \subset K^n, \tag{1}$$

where I and J are some finite sets of indices, and suppose we are a priori guaranteed that X is finite (e.g. it is a part of the zero-set of a non-degenerate polynomial endomorphism). Now the problem is how to estimate its cardinality in some reasonable way without solving any equations.

More formally, there are given f_i, g_j belonging to the ring K_n of polynomials in n variables with coefficients from K and we want to find effectively (by means of some algebraic operations over coefficients of these polynomials) the cardinality $\#X$, i.e. the number of elements in X (geometrically distinct or counted with the multiplicities).

Similar problems for the case when $K = \mathbb{R}$ is the usual field of reals often arise in applications [2] and they are well-studied [3]. We will show below that a number of general results may be formulated in terms which are valid in the context of real closed fields. We will not treat the problem in full even for reals preferring to exclude variuos possible degenerations. In fact, cases considered below are principal in the sense that most of reasonable situations may be reduced to them.

1991 *Mathematics Subject Classification.* 14G30, 58C27.

From now on we always suppose K to be a real closed field and all points of X to be simple in the sense of the algebraic geometry (i.e. having the multiplicity 1). Thus we are going to deal, in fact, with the number of geometrically distinct points.

We will consider two important cases: when X is the zero-set of a non-degenerate endomorphism (i.e. $\#I = n$ and f_i define a proper endomorphism of $\bar{K}^n$, where $\bar{K} = K(\sqrt{-1})$ is the algebraic closure of K), and when one has no inequalities (i.e. $J = \varnothing$).

In the first case the solution may be obtained by means of a suitable modification of the classical signature method going back to Hermite and Jacobi [3] which was outlined in [4] and then thoroughly studied in the Candidate Dissertation of T.Aliashvili for the field of reals (see [5]). The proposed generalisation is based on the existence of a purely algebraic definition of the Grothendieck residue symbol [6].

The same approach is also valid in the second case but better results may be obtained by means of more sophisticated algebraical tools used by G.Khimshiashvili [7], also by D.Eisenbud and H.Levine [8], and developed later in [9] and [10]. This enables us to get rid of the multiplicity one assumption, which seems impossible in the framework of the signature method.

In fact, some other approaches, e.g., the so-called Newton polygon method developed in the works of A.G.Khovansky [11], are possible, but the author has never seen any published results of that kind. Moreover, it seems that the named method does not in principle enable one to consider the case when inequalities are really present in the definition of X.

2. Consider now a set X of the type (1) and let f_j define a nondegenerate polynomial endomorphism $\bar{f} : \bar{K}^n \to \bar{K}^n$ with simple roots. Nondegeneracy here means as usual the absence of "roots at the infinity", that is, the "leaders" (homogeneous forms of the highest degree $\deg f_i$) f_i^* have no nontrivial common roots in $\bar{K}^n$ [2] (for $K = \mathbb{R}$ this is equivalent to f being proper).

The Bezout theorem for real closed fields [12] implies that $\bar{f}$ has exactly $N = \prod \deg f_j$ roots in $\bar{K}^n$ so that we have

$$\bar{f}^{-1}(0) = \{z_0, z_1 \ldots, z_{N-1}\} \quad \text{with} \quad z_i \neq z_j \quad \text{for} \quad i \neq j.$$

Without loss of generality we may assume that the first coordinates of roots are pairwise distinct and in such a case we say that the endomorphism is separable. This condition may always be verified effectively in terms of resultants and one can always reduce the problem to this case by performing not more than N(N-1)/2 rotations of the coordinate system.

Write now every root in the form $z_j = (u_j, z'_j)$ with the first coordinate singled out and introduce an auxiliary quadratic form on K^N which depends on an arbitrary $g \in K_n$:

$$Q^g_f(\xi) = \sum_{j=0}^{N-1} g(z_j)(\xi_0 + u_j\xi_1 + \cdots + +u_j^{N-1}\xi_{N-1})^2. \qquad (2)$$

It is easy to verify that all coefficients of this form belong to K because here we have a complete analogy with the case of reals. More precisely, the roots which do not belong to K^n appear in conjugated pairs with respect to the natural "complex conjugation" operation in $\bar{K}$, which implies the assertion.

Recall that one can define as usual the rank $\mathrm{rk}\, Q^g_f$ and the signature $\mathrm{sig}\, Q^g_f$ of the form Q^g_f [1].

The following result provides a multidimensional analogue of the Sturm algorithm [1] and enables one to solve the problem for $\#J = 1$, i.e. for domains of the type $\{g > 0\}$.

Theorem 1. *If $f : K^n \to K^n$ is a separable polynomial endomorphism over a real closed field K and $g \in K_n$, then the rank and signature of the form* (2) *satisfy the relations:*

$$N - \mathrm{rk}\, Q^g_f = \#\left(f^{-1}(0) \cap g^{-1}(0)\right), \qquad (3)$$

$$\mathrm{sig}\, Q^g_f = \#\left[f^{-1}(0) \cap \{g > 0\}\right] - \#\left[f^{-1}(0) \cap \{g < 0\}\right]. \qquad (4)$$

Denoting by Q_f the form (2) for $g \equiv 1$, we are able to derive some corollaries.

Corollary 1. *Under conditions of the theorem the form Q_f is non-degenerate and one has:*

$$\mathrm{sig}\, Q_f = \# f^{-1}(0) \qquad (5)$$

Corollary 2. *Under the same conditions for $X = f^{-1}(0) \cap \{g > 0\}$ one has:*

$$\#X = \left(\mathrm{sig}\, Q_f + \mathrm{sig}\, Q^g_f + \mathrm{rk}\, Q^g_f - N\right)/2 \qquad (6)$$

Using some simple combinatorics we may also increase the number of inequalities determining X.

Corollary 3. *If besides $f^{-1}(0) \cap g_j^{-1}(0) = \varnothing$ holds for every $j \in J$, then*

$$\#X = \Big(\sum_{\alpha} \operatorname{sig} Q_f^{\alpha}\Big) / 2^{\#J},$$

where a runs through all multiindices of the form $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $1 \leq k \leq \#J$ and $\alpha_1 < \cdots < \alpha_k$, and $Q_f^{\alpha} = Q_f^{g_\alpha}$ with $g_\alpha = g_{\alpha_1} \cdot \ldots \cdot g_k$.

For the sake of simplicity we have excluded here degenerations connected with the presence of roots on boundaries of domains $\{g_j > 0\}$.

Before presenting the proof of the theorem let us explain why it gives a solution of our problem. It suffices to show that coefficients of the form (2) may be computed by a finite sequence of rational operations over coefficients of f_j and g.

After trivial modifications of the formula (2) it is easy to see that the coefficients c_{ij} in the standard presentation of the form $Q_f^g(\xi) = \sum c_{ij}\xi_i\xi_j$ are expressed algebraically in terms of the so-called mixed Newton sums of roots

$$S_\alpha(f) = \sum_{j=0}^{N-1} (z_j^1)^{\alpha_1} \cdot \ldots \cdot (z_j^n)^{\alpha_n}, \tag{7}$$

where $\alpha \in (\mathbb{Z}_+)^n$, $z_j = (z_j^1, \ldots, z_j^n)$, $j = 0, 1, \ldots, N-1$.

In fact, certain sums S_α may be easily computed using iterated resultants. For example, this is so for small $|\alpha|$ and for "pure" Newton sums with only one nonzero α_j and this enables one already to provide the separation of roots, which was the original classical problem [3]. There are some hints in [3] about such a possibility but without any details and with a remark that this is not a universal method. For $n = 2$ the storage of easily computable Newton sums was described by T.Aliashvili who has also shown using the Hilbert theorem on invariants that all Newton sums may be computed algebraically in this case [4], [5]. Unfortunately, this approach is not constructive and it meets with serious difficulties for arbitrary n.

For $K = \mathbb{R}$ a radical tool for suppressing this difficulties is provided by an ingenious algebraic device called the Grothendieck residue symbol [6, 13]. It was shown in [7] and [8] how this residue serves to compute the topological degree of a smooth map-germ and so we naturally used it in our situation. In fact, here we need the global variant of this notion which was outlined in [6] and further investigated in [13].

For the sake of completeness we recall that the global residue of a polynomial $g \in \mathbf{R}_n$ with respect to a nondegenerate endomorphism

$f \in (\mathbb{R}_n)^n$ is defined by the integral

$$\mathrm{Res}_f g = \frac{1}{(2\pi i)^n} \int_{\Gamma_R} \frac{g(z)}{f_1(z) \cdot \ldots \cdot f_n(z)} \, d\mu, \tag{8}$$

where the cycle $\Gamma_R = \{z \in \mathbb{C}^n : |f_j(z)| = R \text{ for } j = 1, \ldots, n\}$ is defined for sufficiently large $R > 0$, its orientation is induced by the differential form $d(\arg f_1) \wedge \cdots \wedge d(\arg f_n)$ and the integral is taken with respect to the usual Lebesque measure.

This integral doesn't depend on R and vanishes on the ideal (f) generated by the components of f in $\mathbb{R}_n$ [13]. Moreover, if all roots of f are simple, one has the relation

$$\mathrm{Res}_f g = \sum_{z \in f^{-1}(0)} \frac{g(z)}{J_f(z)}, \tag{9}$$

where $J_f(z) = \det(\partial f_j / \partial z_k)(z)$ is the Jacobian of f.

Now it is clear that in the situation of Theorem 1 we have

$$S_\alpha(f) = \mathrm{Res}_f(J_f e_\alpha), \tag{10}$$

where $\alpha \in (\mathbb{Z}_+)^n$, $e_\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ is a monomial in $\mathbb{R}_n$.

Consequently, it remains to show that $\mathrm{Res}_f g$ itself can be computed by the coefficients of f and g. This circumstance should not seem strange because the global residue is the sum of local residues [13] and the latter are known to be algebraically computable but, of course, a technical difficulty arises here because we cannot assume the positions z_j of local residues are known.

Nevertheless, it turns out that the situation can be saved by means of a clever use of the transformation formula for global residues [13]. In fact, one may always reduce the problem to the case of the so-called "pure powers" $f_j = z_j^{k_j}$, where it is trivialized. The necessary transformation can be obtained from the so-called Hefer decomposition of polynomials f_j and all the procedures rise up to algorithms. We have no space here for presenting details which may be borrowed from [13], but for future generalizations it is important to notice that the main point was just the transformation formula. Thus we conclude that the Newton sums can be also computed algebraically, which gives a principal solution of our problem.

Now we observe that the same arguments can be used also for an arbitrary real closed field K because there exists a pure algebraic definition of the Grothendieck residue symbol [6] which generalizes (8) and possesses the same functorial properties.

As was already mentioned, a straightforward analysis of the residue computation in [13] shows that it uses, in fact, only the formal properties of residues and therefore also extends to the general case.

Thus we arrive to the following conclusion which complements Theorem 1 and completes the desired solution.

Theorem 2. *Under the conditions of Theorem* 1 *coefficients of the form* Q_f^g *may be algebraically expressed through coefficients of given polynomials.*

Now it is time to return to the proof of Theorem 1 which proceeds as follows. Write first $\bar{f}^{-1}(0)$ in the form

$$\bar{f}^{-1}(0) = \{x_1, \dots, x_r, z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k\},$$

where $x_1, \dots, x_r \in K^n$, $z \notin K^n$, $r + 2k = N$, which is always possible in virtue of the observation following the formula (2).

Define now a linear transformation T in K^N by the formulas

$$\begin{cases} \eta_j = \xi_0 + \xi_1 u_j + \cdots + \xi_{N-1} u_j^{N-1}, & j = 1, \dots, r; \\ \eta_{r+j} = \mathrm{Re}(\xi_0 + \cdots + \xi_{N-1} u_{r+j}^{N-1}), & j = 1, \dots, k; \\ \eta_{r+k+j} = \mathrm{Im}(\xi_0 + \cdots + \xi_{N-1} u_{r+j}^{N-1}), & j = 1, \dots, k. \end{cases}$$

Evidently, this transformation diagonalizes our form, this immediately implying (3) and (4). It remains to verify that this is a genuine change of coordinates, that is its determinant is nonzero. Anyone who is fond of linear algebra can easily compute it by reducing it to a Vandermonde of the first coordinates which is nonzero due to the separability of f. Another way is to observe that the form Q_f becomes nondegenerate; hence $\mathrm{rk}\, T \geq N$, which again finishes the proof. ■

All corollaries become immediate now. We have only to introduce numbers m_α of roots belonging to $U_\alpha = \cap\{g_{\alpha_k} > 0\}$ and to sum up all relations (4) for functions g_α, which terminates all the numbers m except the required $m_{1,\dots,n} = \#X$.

Turning again to the proof of Theorem 2 we shall also point out that there were two nearly equivalent possibilities of deducing the general case from the case of real numbers. Firstly, one can mimic the algorithm from [13] referring to the properties of the general notion from [6]. Secondly, one can directly define the global residue $\mathrm{Res}_f g$ by the formula presented in [13], page 60, and verify that it possesses all necessary properties forcing it to coincide with the residue from [6].

In both cases details are routine and we have omitted them. In fact, the shortest though a little mistifying way is concerned with the Zaidenberg–Tarski principle [12], which makes all these troubles unnecessary as soon as a formula for Res_f is proven for reals so that the proof of our generalization becomes complete.

Nevertheless, we preferred to recall the analytical formula for the global residue having in mind an effective algorithm for dealing with the problem in practice, which is by no means available by the Zaidenberg–Tarski yoga.

A number of curious questions arise here. For example, one can try to estimate the computational complexity of corresponding algorithms and compare it with that of the cylindrical decomposition method from [14]. When $g = J_f$ and we are dealing with the topological degree of f necessary estimates were obtained by T.Aliashvili [5] and they witness in favour of the approach outlined above.

3. Let now X be an affine algebraic subset of K^n, that is a set of the type (1) with $J = \varnothing$. We are going to describe an ano ther solution of our problem also valid without assuming that all points of X are simple.

As is well known, every such subset may be represented as a hypersurface $X = \{F = 0\}$ with $F = f_1^2 + \cdots + f_{\#I}^2$ so that we may assume that X is a hypersurface consisting of a finite number of points. At first glance such an object may seem unusual but the point is that for a hypersurface one can always compute its Euler characteristic in a pure algebraic form as in [7, 9, 10]. In our situation the Euler characteristic simply reduces to the number of geometrically distinct points so that we become able to give a very concise solution of our problem.

The discussion below can be adapted for arbitrary real closed fields, but this requires some caution and additional work so that we consider here only the case when $K = \mathbb{R}$.

Recall that we deal with the usual Euler characteristic $\chi(X)$ which is the alternated sum of homology groups ranks (Betti numbers) of a topological space X under consideration. We write $\deg_p f$ for the local degree of an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ in an isolated preimage of the origin $p \in f^{-1}(0)$ which is defined as the topological degree of a mapping $\hat{f} = f/\|f\| : S_\varepsilon^{n-1}(p) \to S_1^{n-1}(0) = S^{n-1}$.

All the results below are based on the following formulas obtained by the author in [7].

Theorem 3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a polynomial with an isolated singularity at the origin. Then for a sufficiently small enough $\lambda > 0$ one has*

$$\chi\Big(\{F < \lambda\} \cap B_\delta^n\Big) = 1 + (-1)^{n-1} \deg_0(\operatorname{grad} F), \tag{11}$$

where B^n_δ is a ball of the small radius δ.

If the polynomial F is homogeneous then for sufficiently small $\lambda, \delta > 0$ one has also the equality

$$\chi\Big(\{F \geq 0\} \cap S^{n-1}_\delta\Big) = 1 + (-1)^{n-1} \deg_0(\text{grad}\, F). \qquad (12)$$

These results are local but there is a natural link with the global ones provided by the projectivization.

With this in mind, suppose that $f_1, \ldots, f_p$ are homogeneous polynomials of the degrees $d_1 \leq d_2 \leq \cdots \leq d_p$, respectively. Then, besides X, they also define a projective algebraic variety V_f in $\mathbb{RP}^{n-1}$ which can also be determined by a single homogeneous polynomial $f = \sum f_j^2 \cdot \|x\|^{2(d_p - d_j)}$ of the degree $2d_p$, where $\|x\|$ is the usual euclidean norm of $x \in \mathbb{R}^n$. Now it is not difficult to tie together the invariants of X with those of its projectivization using the following lemma established in [10].

Lemma. *Under these conditions for all $\lambda \neq 0$ the polynomial $F_\lambda = f^2 - \lambda^2(\sum x_j^{2d+2})$ has an isolated singularity at 0 and for a sufficiently small $|\lambda|$ the real hypersurface $\{F_\lambda = 0\}$ does not have singularities inside small balls and is transversal to their boundaries . Moreover, denoting*

$$Z = \Big\{x \in S^{n-1}_1 : f(x) = 0\Big\}, \;\; Z_\lambda = \Big\{x \in S^{n-1}_1 : F_\lambda(x) \leq 0\Big\},$$

one has that $Z_\lambda \backslash Z$ is diffeomorphic to $Y_\lambda \times (0, \lambda]$, where $Y_\lambda = \{x \in S^{n-1}_1 : F_\lambda(x) = 0\}$.

Collecting together these observations, we are able to obtain the final result.

Theorem 4. *Let $f_1, \ldots, f_p \in \mathbb{R}_n$ be real polynomials of degrees not exceeding d. Suppose that they have only a finite number M of real common zeroes. Set*

$$h_i(x_0, x_1, \ldots, x_n) = x_0^{d+1} f_i(x_1/x_0, \ldots, x_n/x_0),$$

$$H = \sum_{i=1}^{p} h_i^2 - \sum_{k=0}^{n} x_k^{2d+4}.$$

Then H has an algebraically isolated critical point at the origin and the following equality holds:

$$M = \Big[(-1)^n - \deg_0(\text{grad}\, H)\Big]/2. \qquad (13)$$

Proof. Evidently, all polynomials h_j are homogeneous of the degree $d+1$, which enables us to transplant considerations on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and use the lemma.

With this purpose we introduce a subset $Y = S^n \cap \{h_1 = \cdots = h_p = 0\}$ and observe that $Y = Y_+ \cup Y_- \cup S^{n-1}$ with $Y_\pm = \{x \in S^n : \pm x_0 > 0, h_1(x) = \cdots = h_p(x) = 0\}$.

Evidently, Y_+ and Y_- are homeomorphic to X so that we obtain $\chi(Y) = 2\chi(X) - \chi(S^{n-1})$ or, equivalently, $\chi(X) = [\chi(Y) + \chi(S^{n-1})]/2$ and it remains to compute $\chi(Y)$, which is already possible using (11) for H.

Working with homology with integer coefficients, in virtue of the Lefshetz duality [2] we obtain

$$\begin{aligned}\chi(S\backslash Y) &= \sum(-1)^k \operatorname{rk} H_k(S\backslash Y) = \sum(-1)^k \operatorname{rk} H_{n-k-1}(S,Y) = \\ &= (-1)^{n+1}\chi(S,Y) = (-1)^{n+1}[\chi(S) - \chi(Y)] = \\ &= (-1)^n\chi(Y) + (-1)^{n+1} + 1.\end{aligned}$$

On the other hand, applying the lemma to H instead of F one gets

$$S\backslash Y = \big(S \cap \{F_\lambda \le 0\}\big) \cup \big(S \cap \{F_\lambda \le 0\}\big).$$

The first set is fibred in virtue of the lemma and the second one cannot contain any points of Y because there we have $\sum_j x_j^{2d+4} > 0$. Consequently, we obtain

$$\begin{aligned}\chi(S\backslash Y) &= \chi\big(\{F_\lambda = 0\} \cap S\big) + \chi\big(\{F_\lambda \ge 0\} \cap S\big) - \chi\big(\{F_\lambda = 0\} \cap S\big) = \\ &= \chi\big(\{F_\lambda \ge 0\}\big) = 1 + (-1)^{n+1} \deg_0(\operatorname{grad} F_\lambda).\end{aligned}$$

This naturally implies

$$\chi(Y) = (-1)^n\big[(-1)^n + (-1)^{n+1}\deg_0(\operatorname{grad} F_\lambda)\big] = 1 - \deg_0(\operatorname{grad} F_\lambda).$$

Now, our lemma yields that the family F provides an admissible homotopy with $F_1 = H$ so that we may put $\lambda = 1$ and get

$$\chi(Y) = 1 - \deg_0(\operatorname{grad} H),$$

which immediately gives (13) and finishes the proof. ∎

Granted formula (13) we have only to observe that the local topological degree is algebraically computable being, in fact, equal to the signature of an effectively constructible quadratic form on the coordinate algebra of a given mapping [7, 8]. Thus, we obtain another way of computing $\#X$, which turns out to be more convenient and effective than the method of §2.

One could now combine our results with those on algorithmic computation of the local degree in order to estimate the computational complexity of this method. We shall not pursue this topic here but rather make several remarks in conclusion.

An interesting open problem is to generalize all these things for arbitrary real closed fields. In fact, most of necessary algebraical and topological notions are also available in the general case. One has only to obtain a formula expressing the local Euler characteristic in terms of the local topological degree as in [7]. The author feels that a portion of the semi-algebraic topology in the spirit of [15] should be helpful here. One could also try to combine this with the discussion of real singularities in [16].

Some concrete results become more or less immediate now. For example, one can directly verify a result of R.Thom stating that the number of cusps of a stable smooth mapping from the real projective plane into real plane is always odd because such maps may be approximated by rational ones given by ratios of polynomials of even degrees for which the result follows directly from the formula (11). Perhaps, some other "oddity results" may be obtained in a similar manner.

One can also give a closed algebraical formula for the number of cusps of a polynomial Whitney mapping (stable mapping of $\mathbb{R}^2$ in itself) which complements recent results of K.Aoki and T.Fukuda [17] and provides sharp estimates for such numbers.

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(Received 15.01.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993), No. 3, 323-339

ON SOME PROPERTIES OF MULTIPLE CONJUGATE TRIGONOMETRIC SERIES

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ABSTRACT. We have obtained the estimate in the terms of partial and mixed moduli of continuity of deviation of Cesáro $(C, \boldsymbol{\alpha})$ means ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > -1$, $i = \overline{1,n}$) of the sequence of rectangular partial sums of n-multiple ($n > 1$) conjugate trigonometric series from n-multiple truncated conjugate function. This estimate implies the result on the m_λ-convergence ($\lambda \geq 1$) of $(C, \boldsymbol{\alpha})$ means ($\alpha_i > 0$, $i = \overline{1,n}$), provided that the essential conditions are imposed on the partial moduli of continuity. Finally, it is shown, that the m_λ-convergence cannot be replaced by ordinary convergence.

რეზიუმე. სტატიაში მიღებულია n-ჯერადი ($n > 1$) შეუღლებული ტრიგონომეტრიული მწკრივის მართკუთხა კერძო ჯამების მიმდევრობის ჩეზაროს $(C, \boldsymbol{\alpha})$ საშუალოების ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > -1$, $i = \overline{1,n}$) ე.წ. n-ჯერადი წაკვეთილი შეუღლებული ფუნქციისგან გადახრის შეფასება კერძო და შერეული უწყვეტობის მოდულების ტერმინებში. ამ შეფასებიდან უწყვეტობის კერძო მოდულებზე არსებითი პირობების დადების შემთხვევაში გამომდინარეობს შედეგი $(C, \boldsymbol{\alpha})$ საშუალოების ($\alpha_i > 0$, $i = \overline{1,n}$) m_λ-კრებადობის შესახებ. დაბოლოოს, ნაჩვენებია, რომ m_λ-კრებადობა არ შეიძლება შეიცვალოს ჩვეულებრივი კრებადობით.

1. Let $f \in L([-\pi;\pi]^n)$, $n \in \mathbb{N}$, $n > 1$, be a function, 2π-periodic in each variable, $\sigma_n[f]$ its n-multiple trigonometric Fourier series, and $\bar{\sigma}_n[f]$ its conjugate series with respect to n variables (see, e.g., [1]).

We set

$$\mathbf{m} = (m_1, \dots, m_n) \ \ (m_i \in \mathbb{N},\ i = \overline{1,n});$$

$$\mathbf{x} = (x_1, \dots, x_n) \ \ (x_i \in \mathbb{R},\ i = \overline{1,n});$$

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \ \ (\alpha_i \in \mathbb{R},\ \alpha_i > -1,\ i = \overline{1,n});\ \ \mathbf{M} = \{1, 2, \dots, n\}.$$

1991 *Mathematics Subject Classification.* 42B08.

By M_j we denote the set of all subsets of M with j elements, by $M^{(A)}$ a set $M\backslash A$ $(A \subset M)$, and by $M_j^{(A)}$ the set of all subsets of $M^{(A)}$ with j elements.

For $B = \{i_i, \dots, i_k\} \subset M$ we define $m(B) = \{1/m_{i_1}, \dots, 1/m_{i_k}\}$ and the truncated conjugate function with respect to the corresponding variables

$$\bar{f}_{m(B)}(\mathrm{x}) =$$

$$= \frac{1}{(-2\pi)^k} \int_{1/m_{i_1}}^{\pi} \cdots \int_{1/m_{i_k}}^{\pi} \Big(\Delta_{k,s_{i_k}}\Big(\Delta_{k-1,s_{i_{k-1}}}\Big(\dots\Big(\Delta_{1,s_{i_1}}(f;\mathrm{x})\Big)\dots\Big)\Big)\Big) \times$$

$$\times \prod_{j=1}^{k} \operatorname{ctg}\frac{s_{i_j}}{2} ds_{i_1} \dots ds_{i_k}, \quad \bar{f}_{\mathrm{m}}(\mathrm{x}) = \bar{f}_{m(M)}(\mathrm{x}),$$

where

$$\Delta_{i,h}(f,\mathrm{x}) = f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) -$$

$$- f(x_1, \dots, x_{i-1}, x_i - h, x_{i+1}, \dots, x_n), \quad i = \overline{1,n}.$$

For $f \in L^q([-\pi;\pi]^n)$, $1 \le q \le +\infty$ $(L^\infty = C)$, we consider its mixed modulus of continuity

$$\omega_B\Big(m(B); f\Big)_{L^q} =$$

$$= \sup_{|h_1|<1/m_{i_1}, \dots, |h_k|<1/m_{i_k}} \Big\|\Delta_k^{h_k}\Big(\Delta_{k-1}^{h_{k-1}}\Big(\dots\Big(\Delta_1^{h_1}(f;\mathrm{x})\Big)\dots\Big)\Big)\Big\|_{L^q},$$

where

$$\Delta_i^h(f;\mathrm{x}) = f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n).$$

Let $\bar{\sigma}_{\mathrm{m}}^{\alpha}(\mathrm{x}; f)$ be Cesáro means of $\bar{\sigma}_n[f]$.

In the sequel by $A, B, A_1, B_1, C(\alpha), C(\beta), C(\alpha,\beta)$, etc. we will denote, in general, different positive constants.

Finally, we set

$$\lambda(k,\beta) = \begin{cases} k^{-\beta}, & -1 < \beta < 0, \\ \ln(k+1), & \beta = 0, \\ 1, & \beta > 0, \quad (k = 1, 2, \dots). \end{cases}$$

In the present paper we give the estimate of the deviation of n-multiple Cesáro means of the sequence of rectangular partial sums of $\bar{\sigma}_n[f]$ from $\bar{f}_{\mathrm{m}}$ in the norm of L^q, $q \in [1; +\infty]$ $(L^\infty = C)$, in terms of partial and mixed moduli of continuity of f. This result generalizes the corresponding result of L.Zhizhiashvili (see [1]).

From this estimate ensues the result on the Cesáro summability of the sequence of rectangular partial sums of $\bar{\sigma}_n[f]$ and then the correctness of this result is shown.

2. The following is true:

Theorem 1. *If* $f \in L^q([-\pi;\pi]^n)$, $q \in [1;+\infty]$ $(L^\infty = C)$, *then*

$$\|\bar{\sigma}^{\alpha}_{\mathrm{m}}(\mathrm{x};f) - \bar{f}_{\mathrm{m}}(\mathrm{x})\|_{L^q} \leq C(\alpha)\left\{\sum_{k=1}^{n-1}\sum_{B\in M_k}\omega_B(m(B);f)_{L^q}\times\right.$$

$$\times \prod_{\substack{j=1\\ \{i_1,\dots,i_k\}\in M_k}}^{k} \lambda(m_{i_j},\alpha_{i_j}) + \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\sum_{A\in M_i}\sum_{B\in M_k^{(A)}}\omega_B(m(B);\bar{f}_{m(A)})_{L^q}\times$$

$$\left.\times \prod_{\substack{j=1\\ \{t_1,\dots,t_k\}\in M_k^{(A)}}}^{k} \lambda(m_{t_j},\alpha_{t_j}) + \prod_{i=1}^{n}\lambda(m_i,\alpha_i)\omega_M(m(M);f)_{L^q}\right\}. \quad (1)$$

Proof. For simplicity, we will prove the theorem in the case $n = 2$ which is typical. We will use the method of L.Zhizhiashvili ([1], p. 160-191), this method proving to be true for $n \geq 3$.

Let $\omega(\delta_1,\delta_2;f)_{L^p}$ be the mixed modulus of continuity of the function $f(x,y)$, $x,y \in \mathbb{R}$, with respect to two variables. By $\omega_i(\delta;f)_{L^q}$ $(i \in \mathbb{N}$, $\delta \in \mathbb{R}^+)$, as usual, we denote the modulus of continuity of f in L^q with respect to the corresponding variable. Let $m,n \in \mathbb{N}$, $\alpha,\beta \in \mathbb{R}$, $\alpha,\beta > -1$, $\bar{f}_{m,n}(x,y)$ be the two-dimensional truncated conjugate function and $\bar{\sigma}^{\alpha,\beta}_{mn}(x,y;f)$ be the Cesáro means of the double conjugate series $\bar{\sigma}_2[f]$.

We have ([1], p.187-188):

$$\bar{\sigma}^{\alpha,\beta}_{mn}(x,y;f) = 1/\pi^2\int_0^{\pi/m}\int_0^{\pi/n}\psi_{x,y}(s,t;f)\bar{K}^{\alpha}_m(s)\bar{K}^{\beta}_n(t)\,dt\,ds +$$

$$+ 1/(2\pi^2)\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{t}{2}\bar{K}^{\alpha}_m(s)\,dt\,ds +$$

$$+ 1/\pi^2\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\bar{K}^{\alpha}_m(s)H^{\beta}_{n,1}(t)\,dt\,ds +$$

$$+ 1/\pi^2\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\bar{K}^{\alpha}_m(s)H^{\beta}_{n,2}(t)\,dt\,ds +$$

$$+ 1/(2\pi^2)\int_{\pi/m}^{\pi}\int_0^{\pi/n}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{s}{2}\bar{K}^{\beta}_n(t)\,dt\,ds+$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{0}^{\pi/n} \psi_{x,y}(s,t;f) H_{m,1}^{\alpha}(s) \bar{K}_n^{\beta}(t)\, dt\, ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{0}^{\pi/n} \psi_{x,y}(s,t;f) H_{m,2}^{\alpha}(s) \bar{K}_n^{\beta}(t)\, dt\, ds +$$

$$+ 1/(4\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) \operatorname{ctg}\frac{s}{2} \operatorname{ctg}\frac{t}{2}\, dt\, ds +$$

$$+ 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) \operatorname{ctg}\frac{s}{2} H_{n,1}^{\beta}(t)\, dt\, ds +$$

$$+ 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) \operatorname{ctg}\frac{s}{2} H_{n,2}^{\beta}(t)\, dt\, ds +$$

$$+ 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) \operatorname{ctg}\frac{t}{2} H_{m,1}^{\alpha}(s)\, dt\, ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) H_{m,1}^{\alpha}(s) H_{n,1}^{\beta}(t)\, dt\, ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) H_{m,1}^{\alpha}(s) H_{n,2}^{\beta}(t)\, dt\, ds +$$

$$+ 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) \operatorname{ctg}\frac{t}{2} H_{m,2}^{\alpha}(s)\, dt\, ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) H_{m,2}^{\alpha}(s) H_{n,1}^{\beta}(t)\, dt\, ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f) H_{m,2}^{\alpha}(s) H_{n,2}^{\beta}(t)\, dt\, ds =$$

$$= \sum_{j=1}^{16} P_{mn}^{(j)}(x,y;f), \tag{2}$$

where

$$\begin{aligned} \psi_{x,y}(s,t;f) = f(x+s, y+t) - f(x-s, y+t) - \\ - f(x+s, y-t) + f(x-s, y-t) \end{aligned} \tag{3}$$

and $H_{m,1}^{\alpha}(s)$, $H_{m,2}^{\alpha}(s)$ are the summands of the conjugate Fejér kernel $\bar{K}_m^{\alpha}(s)$ (see [2], pp.157-160)

$$\bar{K}_m^{\alpha}(s) = \frac{1}{2}\operatorname{ctg}\frac{s}{2} + H_{m,1}^{\alpha}(s) + H_{m,2}^{\alpha}(s), \tag{4}$$

$$H_{m,1}^{\alpha}(s) = -\frac{\cos\left((m + \frac{1}{2} + \frac{\alpha}{2})s - \frac{\pi\alpha}{2}\right)}{A_m^{\alpha}(2\sin\frac{s}{2})^{1+\alpha}}. \tag{5}$$

Besides,

$$|\bar{K}_m^\alpha(s)| \le C(\alpha)m, \quad |s| \le \pi; \tag{6}$$

$$|H_{m,2}^\alpha(s)| \le \frac{C(\alpha)}{m^2 s^3}, \quad \pi/m \le |s| \le \pi, \quad m > 1. \tag{7}$$

The estimate (7) is more precise than the estimate (2.1.14) in [1] and it can be proved by arguments analogous to those in [2], pp.157-160.

From (2) we obtain

$$\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f) - \bar{f}_{mn}(x,y) = \bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f) - P_{mn}^{(8)}(x,y;f) = \\ = \sum_{k=1}^{16}{}' P_{mn}^{(k)}(x,y;f), \tag{8}$$

where $'$ indicates that the eighth member is omitted (the replacement of $1/m$ $(1/n)$ in $\bar{f}_{mn}(x,y)$ by π/m (π/n) does not matter).

In the sequel we will use the inequality (see [3], p.179)

$$\left\{\int_a^b \left|\int_c^d f(x,y)\,dy\right|^q dx\right\}^{1/q} \le \int_c^d \left\{\int_a^b |f(x,y)|^q\,dx\right\}^{1/q} dy, \\ 1 \le q < +\infty. \tag{9}$$

Taking into account (6) and (9), we obtain

$$\|P_{mn}^{(1)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\Big\{\omega_1(1/m;f)_{L^q} + \omega_2(1/n;f)_{L^q}\Big\}. \tag{10}$$

It is easy to see, that

$$P_{mn}^{(2)}(x,y;f) = -\frac{1}{\pi}\int_0^{\pi/m}\Big[\bar{f}_n^{(2)}(x+s,y) - \bar{f}_n^{(2)}(x-s,y)\Big]\bar{K}_m^\alpha(s)\,ds,$$

$$P_{mn}^{(5)}(x,y;f) = -\frac{1}{\pi}\int_0^{\pi/n}\Big[\bar{f}_m^{(1)}(x,y+t) - \bar{f}_m^{(1)}(x,y-t)\Big]\bar{K}_n^\beta(t)\,dt,$$

where

$$\bar{f}_m^{(1)}(x,y) = -\frac{1}{2\pi}\int_{\pi/m}^{\pi}[f(x+s,y) - f(x-s,y)]\operatorname{ctg}\frac{s}{2}\,ds, \tag{11}$$

$$\bar{f}_n^{(2)}(x,y) = -\frac{1}{2\pi}\int_{\pi/n}^{\pi}[f(x,y+t) - f(x,y-t)]\operatorname{ctg}\frac{t}{2}\,dt. \tag{12}$$

Hence, using again (6) and (9), we obtain:

$$\|P_{mn}^{(2)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_1(1/m;\bar{f}_n^{(2)})_{L^q}; \tag{13}$$

$$\|P_{mn}^{(5)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_2(1/n;\bar{f}_m^{(1)})_{L^q}. \tag{14}$$

Now, it is easy to see that for estimation of $P^{(3)}_{mn}(x,y;f)$ it suffices to estimate the integral

$$I(m,n)=n^{-\beta}\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\bar{K}^{\alpha}_m(s)\omega_\beta(t)\cos nt\,dt\,ds,$$

where

$$\omega_\beta(t)=\frac{\cos\frac{1+\beta}{2}t}{(\sin\frac{t}{2})^{1+\beta}}.$$

We have

$$2I(m,n)=$$

$$=n^{-\beta}\Big\{\int_0^{\pi/m}\int_{\pi/n}^{\pi}\Big[\psi_{x,y}(s,t;f)-\psi_{x,y}(s,t+\pi/n;f)\Big]\times$$

$$\times\bar{K}^{\alpha}_m(s)\omega_\beta(t)\cos nt\,dt\,ds+\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t+\pi/n;f)\Big[\omega_\beta(t)-$$

$$-\omega_\beta(t+\pi/n)\Big]\bar{K}^{\alpha}_m(s)\cos nt\,dt\,ds+$$

$$+\int_0^{\pi/m}\int_{\pi-\pi/n}^{\pi}\psi_{x,y}(s,t+\pi/n;f)\bar{K}^{\alpha}_m(s)\omega_\beta(t+\pi/n)\cos nt\,dt\,ds-$$

$$-\int_0^{\pi/m}\int_0^{\pi/n}\psi_{x,y}(s,t+\pi/n;f)\bar{K}^{\alpha}_m(s)\omega_\beta(t+\pi/n)\cos nt\,dt\,ds\Big\}.$$

Now we note that (see [1], p.56, (2.1.18))

$$|\omega_\beta(t)-\omega_\beta(t+\pi/n)|\le C(\beta)/(nt^{2+\beta}),\quad \frac{\pi}{n}\le t\le\pi. \tag{15}$$

(6) and (15) yield

$$\|I(m,n)\|_{L^q}\le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;f)_{L^q}$$

and hence

$$\|P^{(3)}_{mn}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;f)_{L^q}. \tag{16}$$

Analogously

$$\|P^{(6)}_{mn}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}. \tag{17}$$

Furthermore, using (6),(7) and (9), we obtain

$$\|P^{(k)}_{mn}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\Big\{\omega_1(1/m;f)_{L^q}+\omega_2(1/n;f)_{L^q}\Big\}\quad (k=4,7,16). \tag{18}$$

Analogously, taking into account that

$$P^{(9)}_{mn}(x,y;f) = -\frac{1}{\pi}\int_{\pi/n}^{\pi}\left[\bar{f}^{(1)}_m(x,y+t) - \bar{f}^{(1)}_m(x,y-t)\right]H^{\beta}_{n,1}(t)\,dt,$$

$$P^{(11)}_{mn}(x,y;f) = -\frac{1}{\pi}\int_{\pi/m}^{\pi}\left[\bar{f}^{(2)}_n(x+s,y) - \bar{f}^{(2)}_n(x-s,y)\right]H^{\alpha}_{m,1}(s)\,ds,$$

we can prove

$$\|P^{(9)}_{mn}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;\bar{f}^{(1)}_m)_{L^q}; \tag{19}$$

$$\|P^{(11)}_{mn}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(m,\alpha)\omega_1(1/m;\bar{f}^{(2)}_n)_{L^q}. \tag{20}$$

Using the same arguments and applying (6), (7) and (9), we can prove

$$\|P^{(10)}_{mn}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_2(1/n;\bar{f}^{(1)}_m)_{L^q}; \tag{21}$$

$$\|P^{(14)}_{mn}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_1(1/m;\bar{f}^{(2)}_n)_{L^q}. \tag{22}$$

Now we observe that the following lemma holds true (see [1], p.160, Lemma 10):

Lemma 1. *Let* $f \in L^q([-\pi;\pi]^2)$, $1 \le q \le +\infty$, *and*

$$\phi_{x,y}(s,t;f) = f(x+s,y+t) + f(x-s,y+t) + f(x+s,y-t) + \\ + f(x-s,y-t) - 4f(x,y). \tag{23}$$

Then

$$\left\| m^{-\alpha}n^{-\beta}\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\phi_{x,y}(s,t;f)\omega_\alpha(s)\omega_\beta(t)\,\begin{matrix}\sin\\ \cos\end{matrix}\,ms\,\begin{matrix}\sin\\ \cos\end{matrix}\,nt\,dt\,ds\right\|_{L^q} =$$

$$= O\Big\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q} + \lambda(m,\alpha)\omega_1(1/m;f)_{L^q} + \\ +\lambda(n,\beta)\omega_2(1/n;f)_{L^q}\Big\}.$$

There is another lemma in [1] (see p.171, Lemma 11), which can be corrected by means of (7) as follows:

Lemma 2. *For $f \in L^q([-\pi;\pi]^2)$, $1 \le q \le +\infty$, we have*

$$\left\| m^{-\alpha} \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \phi_{x,y}(s,t;f)\omega_\alpha(s) H^{\beta}_{n,2}(t) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} ms\,dt\,ds \right\|_{L^q} =$$
$$= O\left\{\lambda(m,\alpha)\omega_1(1/m;f)_{L^q} + \omega_2(1/n;f)_{L^q}\right\};$$
$$\left\| n^{-\beta} \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \phi_{x,y}(s,t;f)\omega_\beta(t) H^{\alpha}_{m,2}(s) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} nt\,dt\,ds \right\|_{L^q} =$$
$$= O\left\{\lambda(n,\beta)\omega_2(1/n;f)_{L^q} + \omega_1(1/m;f)_{L^q}\right\}.$$

The lemmas and (3), (5), (7) and (23) yield

$$\left\| P^{(12)}_{mn}(x,y;f) + P^{(13)}_{mn}(x,y;f) + P^{(15)}_{mn}(x,y;f) \right\|_{L^q} \le$$
$$\le C(\alpha,\beta)\left\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q} + \lambda(m,\alpha)\omega_1(1/m;f)_{L^q} + \right.$$
$$\left. +\lambda(n,\beta)\omega_2(1/n;f)_{L^q}\right\}. \tag{24}$$

Finally, (2)–(24) yield

$$\left\| \bar\sigma^{\alpha,\beta}_{mn}(x,y;f) - \bar f_{mn}(x,y) \right\|_{L^q} \le$$
$$\le C(\alpha,\beta)\left\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q} + \lambda(m,\alpha)\omega_1(1/m;f)_{L^q} + \right.$$
$$+\lambda(n,\beta)\omega_2(1/n;f)_{L^q} + \lambda(m,\alpha)\omega_1(1/m;\bar f^{(2)}_n)_{L^q} +$$
$$\left. +\lambda(n,\beta)\omega_2(1/n;\bar f^{(1)}_m)_{L^q}\right\}, \tag{25}$$

which is the formula (1) in the case $n = 2$. ■

Corollary. *If $f \in C([-\pi;\pi]^n)$ $(n \ge 2)$ and*

$$\omega_i(\delta;f)_C = o\left(1/\ln^{n-1}(1/\delta)\right) \quad (i = \overline{1,n}) \tag{26}$$

as $\delta \to 0+$, then

$$\lim_{\mathbf{m}_\lambda \to \infty} \|\bar\sigma^{\boldsymbol\alpha}_{\mathbf{m}}(\mathbf{x};f) - \bar f_{\mathbf{m}}(\mathbf{x})\|_C = 0,$$

$\boldsymbol\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_i > 0$ $(i = \overline{1,n})$, $\lambda \ge 1$.

Now we will prove that the m_λ-summability in the corollary is essential. Namely, the following result holds true:

Theorem 2. *There exists a function $f \in C([-\pi;\pi]^n)$ which satisfies (26) and*

$$\overline{\lim_{\mathbf{m}\to\infty}} |\bar\sigma^{\boldsymbol\alpha}_{\mathbf{m}}(\mathbf{0};f) - \bar f_{\mathbf{m}}(\mathbf{0})| = +\infty, \tag{27}$$

$\boldsymbol\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_i > 0$ $(i = \overline{1,n})$, $\mathbf{0} = (0,0,\dots,0)$.

Proof. We will prove the theorem for $n = 2$, this case being quite typical. First, let $\alpha, \beta \in (0;1)$. We set

$$m_k = 2^{2^{2^k}}, \quad n_k = m_k^{m_k^{m_k^{m_k}}}, \quad (k \in \mathbb{N}); \tag{28}$$

$$\ln^{(2)} x = \ln \ln x, \quad \ln^{(k)} x = \ln \ln^{(k-1)} x, \quad k \in \mathbb{N}, \quad k \geq 3;$$

$$g_k(x) = \begin{cases} \frac{1}{\ln(1/(x-\pi/(6m_k)))\ln^{(2)}(1/(x-\pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \leq \frac{11\pi}{60m_k}, \\ \frac{1}{\ln(1/(\pi/(5m_k)-x))\ln^{(2)}(1/(\pi/(5m_k)-x))}, & \frac{11\pi}{60m_k} \leq x < \frac{\pi}{5m_k}, \\ 0, & x \in (0;\pi)\setminus\left(\frac{\pi}{6m_k}; \frac{\pi}{5m_k}\right). \end{cases}$$

Furthermore,

$$h(x) = \sum_{k=1}^{\infty} g_k(x);$$

$$p(y) = \begin{cases} \frac{1}{\ln(2\pi/y)\ln^{(2)}(2\pi/y)}, & y \in (0; \frac{\pi}{2}], \\ \frac{1}{\ln(2\pi/(\pi-y))\ln^{(2)}(2\pi/(\pi-y))}, & y \in [\frac{\pi}{2}; \pi); \end{cases}$$

$$f(x,y) = \begin{cases} h(x)p(y), & (x,y) \in (0;\pi)^2, \\ 0, & (x,y) \in [-\pi;\pi]^2 \setminus (0;\pi)^2. \end{cases}$$

Finally, outside the square $[-\pi;\pi]^2$, we extend the function f by periodicity with the period 2π in each variable. It is easy to see that f satisfies (26).

From now on we set $m = 2m_k$, $n = n_k + 1$. We have

$$\begin{aligned} \bar{\sigma}_{mn}^{\alpha,\beta}(0,0;f) - \bar{f}_{mn}(0,0) &= 1/\pi^2 \int_0^{\pi} \int_0^{\pi} f(x,y)\bar{K}_m^{\alpha}(x)\bar{K}_n^{\beta}(y)\,dy\,dx - \\ &-1/(4\pi^2) \int_{1/m}^{\pi} \int_{1/n}^{\pi} f(x,y) \operatorname{ctg}\frac{x}{2} \operatorname{ctg}\frac{y}{2}\,dy\,dx = \\ &= 1/\pi^2 \int_0^{1/m} \int_0^{1/n} f(x,y)\bar{K}_m^{\alpha}(x)\bar{K}_n^{\beta}(y)\,dy\,dx + \\ &+1/\pi^2 \int_0^{1/m} \int_{1/n}^{\pi} f(x,y)\bar{K}_m^{\alpha}(x)\bar{K}_n^{\beta}(y)\,dy\,dx + \\ &+1/\pi^2 \int_{1/m}^{\pi} \int_0^{1/n} f(x,y)\bar{K}_m^{\alpha}(x)\bar{K}_n^{\beta}(y)\,dy\,dx + \end{aligned}$$

$$+1/\pi^2 \int_{1/m}^{\pi} \int_{1/n}^{\pi} f(x,y)\Big(\bar{K}_m^\alpha(x)\bar{K}_n^\beta(y) - 1/4 \operatorname{ctg}\frac{x}{2}\operatorname{ctg}\frac{y}{2}\Big)\, dy\, dx =$$
$$= \sum_{j=1}^{4} R_j(m,n). \tag{29}$$

Obviously,

$$R_1(m,n) = o(1) \quad (m,n \to \infty). \tag{30}$$

Then,

$$R_2(m,n) = 1/\pi^2 \int_0^{1/m} \int_{1/n}^{\pi} f(x,y)\bar{K}_m^\alpha(x)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\, dy\, dx -$$
$$- 1/\pi^2 \int_0^{1/m} \int_{1/n}^{\pi} f(x,y)\bar{K}_m^\alpha(x) H_n^\beta(y)\, dy\, dx =$$
$$= R_2'(m,n) + R_2''(m,n), \tag{31}$$

where

$$H_n^\alpha(t) = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_\nu^{\alpha-1}\,\frac{\cos(\nu+1/2)t}{2\sin(t/2)}. \tag{32}$$

The following estimates hold true (see [2], (5.12)):

$$|\bar{K}_n^\alpha(t)| \le n, \quad |t| \le \pi; \tag{33}$$
$$|H_n^\alpha(t)| \le C(\alpha) n^{-\alpha} t^{-(\alpha+1)}, \quad 1/n \le |t| \le \pi. \tag{34}$$

Now we have

$$\int_{1/n}^{\pi} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\, dy = \int_{1/n}^{\pi/2} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\, dy +$$
$$+ \int_{\pi/2}^{\pi} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\, dy = U_1(n) + U_2(n). \tag{35}$$

Now,

$$U_1(n) = -C_1 \int_{1/n}^{\pi/2} \frac{d(2\pi y)}{2\pi y \ln(2\pi y)\ln|\ln(2\pi y)|} \le C_2 \ln^{(3)} n \le$$
$$\le C_3 m_k \ln m_k; \tag{36}$$
$$|U_2(n)| \le M. \tag{37}$$

(31) and (35)–(37) yield

$$|R_2'(m,n)| \le \frac{C}{\ln m_{k+1} \ln^{(2)} m_{k+1}}\, m_k \ln m_k. \tag{38}$$

As to $R_2''(m,n)$, we have

$$|R_2''(m,n)| \leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}} \int_0^{1/m} \int_{1/n}^{\pi} \frac{m}{n^\beta y^{\beta+1}}\, dy\, dx \leq$$
$$\leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}}. \tag{39}$$

(31), (38) and (39) yield

$$|R_2(m,n)| \leq \frac{C(\beta)}{\ln m_{m+1} \ln^{(2)} m_{k+1}} m_k \ln m_k. \tag{40}$$

Analogously

$$R_3(m,n) = R_3'(m,n) + R_3''(m,n). \tag{41}$$

Now,

$$|R_3'(m,n)| \leq \frac{C}{\ln n \ln^{(2)} n} \ln m, \tag{42}$$

$$|R_3''(m,n)| \leq \frac{C(\alpha)}{\ln n \ln^{(2)} n}. \tag{43}$$

(41)–(43) yield

$$|R_3(m,n)| = o(1) \quad (m,n \to \infty). \tag{44}$$

Now let us consider $R_4(m,n)$. We break it into 4 parts as follows

$$R_4(m,n) = \Bigg(\int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} + \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} + \int_{1/m^\tau}^{\pi} \int_{1/n}^{1/n^\tau} +$$
$$+ \int_{1/m^\tau}^{\pi} \int_{1/n^\tau}^{\pi} \Bigg) 1/\pi^2 f(x,y) \Big(\bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) -$$
$$- \frac{1}{4} \operatorname{ctg} \frac{x}{2} \operatorname{ctg} \frac{y}{2} \Big)\, dy\, dx = \sum_{j=1}^{4} I_j(m,n), \tag{45}$$

where $1/2 \leq \tau < 1$.

We have

$$|I_4(m,n)| \leq C(\alpha,\beta) \int_{1/m^\tau}^{\pi} \int_{1/n^\tau}^{\pi} \Bigg(\frac{1}{m^\alpha x^{\alpha+1} y} +$$
$$+ \frac{1}{n^\beta y^{\beta+1} x} + \frac{1}{m^\alpha n^\beta x^{\alpha+1} y^{\beta+1}} \Bigg)\, dy\, dx.$$

It is easy to see that

$$|I_4(m,n)| = o(1) \quad (m,n \to \infty). \tag{46}$$

Now we estimate $I_2(m,n)$.

$$I_2(m,n) = \sum_{k=1}^{3} Q_k(m,n). \tag{47}$$

$$|Q_2(m,n)| = \left| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} f(x,y) \frac{-H_n^\beta(y)}{2\,\mathrm{tg}(x/2)}\, dy\, dx \right| \le$$

$$\le C(\beta) \max|f| \ln m \frac{n^{\beta\tau}}{n^\beta}$$

and

$$|Q_2(m,n)| = o(1) \quad (m,n \to \infty); \tag{48}$$

$$|Q_3(m,n)| =$$

$$= \left| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} f(x,y) H_m^\alpha(x) H_n^\beta(y)\, dy\, dx \right| = o(1) \tag{49}$$

$$(m,n \to \infty).$$

Next we will show that $|Q_1(m,n)| \to +\infty$ as $m,n \to \infty$. We have $\frac{1}{m} < \frac{\pi}{6m_k} < \frac{\pi}{5m_k} < \frac{1}{m^\tau}$. Therefore

$$Q_1(m,n) = 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} h(x)(-H_m^\alpha(x))\, dx \times$$

$$\times \int_{1/n^\tau}^{\pi} p(y) \frac{1}{2\,\mathrm{tg}(y/2)}\, dy = Q_1^{(1)}(m) Q_1^{(2)}(n). \tag{50}$$

Since $m = 2m_k$, we have $\cos(i+1/2)x > 0$ for $i = 0,1,\dots,m$ and $x \in [\frac{\pi}{6m_k}; \frac{\pi}{5m_k}]$. Hence $(-H_m^\alpha(x)) < 0$ (see (32)). Therefore we have

$$|Q_1^{(1)}(m)| = 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} H_m^\alpha(x) h(x)\, dx \ge$$

$$\ge \frac{C}{A_m^\alpha} \int_{21\pi/120m_k}^{23\pi/120m_k} \sum_{i=0}^{m} A_i^{\alpha-1} \frac{\cos\frac{9\pi}{20}}{x} h(x)\, dx \ge$$

$$\ge \frac{C}{A_m^\alpha} A_m^\alpha \frac{1}{\ln m_k \ln^{(2)} m_k} \int_{21\pi/120m_k}^{23\pi/120m_k} \frac{dx}{x} \ge \frac{C}{\ln m_k \ln^{(2)} m_k}. \tag{51}$$

As to $Q_1^{(2)}(n)$, we have (analogously to (36))

$$Q_1^{(2)}(n) \ge C m_k \ln m_k. \tag{52}$$

(50)–(52) imply

$$|Q_1(m,n)| \ge \frac{C}{\ln m_k \ln^{(2)} m_k} m_k \ln m_k. \tag{53}$$

(28), (40) and (53) yield

$$|Q_1(m,n)| - R_2(m,n) \to +\infty \tag{54}$$

as $m,n \to \infty$ (for $m = 2m_k$, $n = n_k + 1$).

From (47)–(49) and (54) we obtain

$$|I_2(m,n)| - R_2(m,n) \to +\infty \quad (m,n \to \infty). \tag{55}$$

Now let us consider $I_3(m,n)$. As in the case of $I_2(m,n)$, we have

$$I_3(m,n) = \sum_{\nu=1}^{3} J_\nu(m,n). \tag{56}$$

Then

$$|J_1(m,n)| = 1/\pi^2 \left| \int_{1/m^\tau}^{\pi} h(x)(-H_m^\alpha(x))\,dx \int_{1/n}^{1/n^\tau} \frac{p(y)}{2\,\mathrm{tg}(y/2)}\,dy \right| \le$$

$$\le C(\alpha)\frac{m^{\alpha\tau}}{m^\alpha}(\ln^{(3)} n^\tau - \ln^{(3)} n) = \frac{C(\alpha)}{m^{\alpha(1-\tau)}} \ln \frac{\ln^{(2)} n^\tau}{\ln^{(2)} n}; \tag{57}$$

$$|J_2(m,n)| = \left| 1/\pi^2 \int_{1/m^\tau}^{\pi} \frac{h(x)}{2\,\mathrm{tg}(x/2)}\,dx \int_{1/n}^{1/n^\tau} p(y)(-H_n^\beta(y))\,dy \right| \le$$

$$\le \frac{C(\beta)\ln m}{\ln n \ln^{(2)} n} \int_{1/n}^{1/n^\tau} \frac{dy}{n^\beta y^{\beta+1}}; \tag{58}$$

$$|J_3(m,n)| = \left| 1/\pi^2 \int_{1/m^\tau}^{\pi} \int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y)\,dy\,dx \right| \le$$

$$\le C(\alpha,\beta)\frac{m^{\alpha\tau}}{m^\alpha}\frac{n^\beta}{n^\beta}. \tag{59}$$

(56)–(59) yield

$$I_3(m,n) = o(1) \quad (m,n \to \infty). \tag{60}$$

Now we consider $I_1(m,n)$. As above,

$$I_1(m,n) = \sum_{j=1}^{3} T_j(m,n). \tag{61}$$

We have

$$|T_1(m,n)| = 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)(-H_m^\alpha(x))\frac{1}{2}\operatorname{ctg}\frac{y}{2}\,dy\,dx\right| \le$$
$$\le \frac{C_1(\alpha)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\alpha)}{\ln n_k \ln^{(2)} n_k}\,\frac{m^\alpha}{m^\alpha}\ln n; \tag{62}$$

$$|T_2(m,n)| = 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)(-H_n^\beta(y))\frac{1}{2}\operatorname{ctg}\frac{x}{2}\,dy\,dx\right| \le$$
$$\le \frac{C_1(\beta)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\beta)}{\ln n_k \ln^{(2)} n_k}\,\frac{n^\beta}{n^\beta}\ln m; \tag{63}$$

$$|T_3(m,n)| = 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y)\,dy\,dx\right| \le$$
$$\le \frac{C_1(\alpha,\beta)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\alpha,\beta)}{\ln n_k \ln^{(2)} n_k}\,\frac{m^\alpha n^\beta}{m^\alpha n^\beta}. \tag{64}$$

(61)–(64) yield

$$I_1(m,n) = o(1) \quad (m,n\to\infty) \tag{65}$$

(we remind once more that $m = 2m_k$, $n = n_k + 1$).

Finally, (29), (30), (44)–(46), (55), (60) and (65) prove the Theorem 2 in the case $n = 2$ and $\alpha,\beta \in (0;1)$.

For $\alpha = 1$ we have

$$H_n^1(t) = \frac{\sin(n+1)t}{(n+1)(2\sin\frac{t}{2})^2} \tag{66}$$

and an estimate analogous to (34) holds true.

Now we consider the case when $\alpha > 1$. Using the method represented in [4], p.507, we obtain

$$H_n^\alpha(t) = \frac{1}{A_n^\alpha 2\sin\frac{t}{2}}\operatorname{Re}\left\{\frac{e^{i(n+1/2)t}}{(1-e^{-it})^\alpha} - e^{-\frac{1}{2}it}\sum_{m=1}^{d} A_n^{\alpha-m}(1-e^{-it})^{-m} - \right.$$
$$\left. - \sum_{m=n+1}^{\infty} A_m^{\alpha-d-1} e^{-i(m-n-1/2)t}(1-e^{-it})^{-d}\right\}; \tag{67}$$

where $d = [\alpha]$.

Taking the real part we obtain that the first term of the finite sum is 0. Therefore if $[\alpha] = 1$ we apply once more the Abel transformation to the infinite sum in (67) and obtain

$$H_n^\alpha(t) = \frac{\cos((n+\frac{1}{2}+\frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha(2\sin\frac{t}{2})^{1+\alpha}} - \frac{(1-\alpha)\alpha\cos(t/2)}{8(n+1)(n+\alpha)(\sin\frac{t}{2})^3} - $$
$$- \frac{1}{A_n^\alpha}\sum_{m=1}^{\infty} A_{m+n+1}^{\alpha-3}\frac{\sin(m-1)t}{(2\sin\frac{t}{2})^3}. \tag{68}$$

Then, again, we have an estimate analogous to (34), which enables us to fulfil the proof. Namely,

$$|H_n^\alpha(t)| \le \frac{C_1(\alpha)}{n^\alpha t^{\alpha+1}} + \frac{C_2(\alpha)}{n^2t^3}. \tag{69}$$

If $[\alpha] = 2$ without further transformation we obtain

$$H_n^\alpha(t) = \frac{\cos((n+\frac{1}{2}+\frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha(2\sin\frac{t}{2})^{1+\alpha}} + \frac{(\alpha-1)\alpha\cos(t/2)}{8(n+\alpha-1)(n+\alpha)(\sin\frac{t}{2})^3} + $$
$$+ \frac{1}{A_n^\alpha}\sum_{m=1}^{\infty} A_{m+n}^{\alpha-3}\frac{\cos(m-3/2)t}{(2\sin\frac{t}{2})^3} \tag{70}$$

and, again, (69) holds true.

Analogous equations may be obtained if $[\alpha] \ge 3$.

Now, when $\alpha = 1$ and $\beta = 1$, we use (66). If $\alpha = 1$ and $\beta < 1$ (or vice versa), we use (34) and (66). If $\alpha = 1$ and $\beta > 1$ (or vice versa), we use (66) and (67) (for the corresponding d). If $\alpha > 1$ and $\beta < 1$ (or vice versa), we use again (67) (for the corresponding d) and (34).

In the n-dimensional case we define f as follows. We set

$$m_k = 2^{2^{2^k}}, \quad (k \in \mathbb{N});$$
$$g_k(x) =$$
$$= \begin{cases} \frac{1}{\ln^{n-1}(1/(x-\pi/(6m_k)))\ln^{(n)}(1/(x-\pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \le \frac{11\pi}{60m_k}, \\ \frac{1}{\ln^{n-1}(1/(\pi/(5m_k)-x))\ln^{(n)}(1/(\pi/(5m_k)-x))}, & \frac{11\pi}{60m_k} \le x < \frac{\pi}{5m_k}, \\ 0, & x \in (0;\pi)\setminus\left(\frac{\pi}{6m_k};\frac{\pi}{5m_k}\right). \end{cases}$$

Again

$$h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then

$$p(x_2,\dots,x_n)=$$
$$=\frac{1}{\ln^{n-1}\frac{(2\pi)^{n-1}}{\prod_{i=2}^{n}(\pi/2-|\pi/2-x_i|)}\ln^{(n)}\frac{(2\pi)^{n-1}}{\prod_{i=2}^{n}(\pi/2-|\pi/2-x_i|)}} \tag{71}$$

for $(x_2,\dots,x_n)\in(0;\pi)^{n-1}$.

And, finally

$$f(x_1,\dots,x_n)=\begin{cases} h(x_1)p(x_2,\dots,x_n), & (x_1,\dots,x_n)\in(0;\pi)^n,\\ 0, & (x_1,\dots,x_n)\in[-\pi;\pi]^n\backslash(0;\pi)^n.\end{cases}$$

Outside $[-\pi;\pi]^n$ we extend the function f by periodicity with the period 2π in each variable.

We observe, that functions of the $p(x_2,\dots,x_n)$-type were for the first time introduced and applied in the works of L.Zhizhiashvili (see [1], [5]). ■

Remark 1. *For the function* $f(x_1,\dots,x_n)$ *a stronger condition than* (26) *holds true, namely,*

$$\omega_i(\delta;f)_C\le\frac{C(f,n)}{\ln^{n-1}(1/\delta)\ln^{(n)}(1/\delta)},\quad i=\overline{1,n}.$$

Remark 2. *Results analogous to Theorem* 1, *the corollary and Theorem* 2 *hold true for the n-multiple Abel–Poisson summability method.*

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(Received 25.02.1993,
revized version 13.05.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993) No 3, 341-353

ON SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Sufficient conditions for the solvability of two-point boundary value problems for the system $x_i' = f_i(t, x_1, x_2)$ $(i = 1, 2)$ are given, where f_1 and $f_2 : [a_1, a_2] \times R^2 \to R$ are continuous functions.

რეზიუმე. დადგენილია ორწერტილოვან სასაზღვრო ამოცანათა ამოხსნადობის საკმარისი პირობები $x_i' = f_i(t, x_1, x_2)$ $(i = 1, 2)$ სახის დიფერენციალურ განტოლებათა სისტემისათვის, სადაც f_1 და $f_2 : [a_1, a_2] \times R^2 \to R$ უწყვეტი ფუნქციებია.

1. Statement of the Problems and Formulation of the Main Results

Consider the system of ordinary differential equations

$$x_i' = f_i(t, x_1, x_2) \quad (i = 1, 2) \tag{1.1}$$

with boundary conditions

$$\lambda_{i1} x_1(a_i) + \lambda_{i2} x_2(a_i) + g_i(x_1, x_2) = 0 \quad (i = 1, 2) \tag{1.2}$$

or

$$\lambda_{i1} x_1(a_i) + \lambda_{i2} x_2(a_i) + h_i(x_1(a_i), x_2(a_i)) = 0 \quad (i = 1, 2), \tag{1.3}$$

where $-\infty < a_1 < a_2 < +\infty$, $\lambda_{ij} \in R$ $(i, j = 1, 2)$, the functions $f_i : [a_1, a_2] \times R^2 \to R$, $h_i : R^2 \to R$ $(i = 1, 2)$ are continuous and $g_i : C([a_1, a_2]; R^2) \to R$ $(i = 1, 2)$ are the continuous functionals.

The problems of the forms (1.1),(1.2) and (1.1),(1.3) have been studied earlier in [1-10]. In the present paper new criteria for solvability of these problems are established which have the nature of one-sided restrictions imposed on f_1 and f_2.

We use the following notation:

1991 *Mathematics Subject Classification.* 34B15.

R is the set of all real numbers; $R_+ = [0, +\infty[$,
$D = [a_1, a_2] \times R^2$,
$D_1 = [a_1, a_2] \times (R\backslash\{0\}) \times R$; $D_2 = [a_1, a_2] \times R \times (R\backslash\{0\})$,
$C(A, B)$ is the set of continuous maps from A to B.

A solution of the system (1.1) is sought in the class of continuously differentiable vector-functions $(x_1, x_2) : [a_1, a_2] \to R^2$.

1.1. Problem (1.1), (1.2). We shall study the problem (1.1),(1.2) in the case when

$$(-1)^i\sigma \cdot \lambda_{i1}\lambda_{i2} > 0 \quad (i = 1, 2),$$

and the functionals g_1 and g_2 satisfy the inequality

$$|g_1(x, y)| + |g_2(x, y)| \le l,$$

on $C([a_1, a_2]; R^2)$, where $\sigma \in \{-1, 1\}$ and $l \in R_+$.

Theorem 1.1. *Suppose that*

$$\sigma[f_1(t, x, y) - p_{11}(t, x, y)x - p_{12}(t, x, y)y]\,\mathrm{sgn}\, y \ge -q_0 \quad \text{for } (t, x, y) \in D_2, \tag{1.4}$$

$$\sigma[f_2(t, x, y) - p_{21}(t, x, y)x - p_{22}(t, x, y)y]\,\mathrm{sgn}\, x \ge -q_0 \quad \text{for } (t, x, y) \in D_1, \tag{1.5}$$

$$\sigma f_1(t, 0, y)\,\mathrm{sgn}\, y \ge 0 \quad \text{for } a_1 \le t \le a_2, \ |y| \ge r_0, \tag{1.6}$$

$$\sigma f_2(t, x, 0)\,\mathrm{sgn}\, x \ge 0 \quad \text{for } a_1 \le t \le a_2, \ |x| \ge r_0, \tag{1.7}$$

where $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12} and $p_{21} : D \to R$ are continuous bounded functions and q_0, r_0 are positive constants. Then the problem (1.1.), (1.2) *has at least one solution.*

Corollary 1.1. *Let the inequalities*

$$\sigma f_1(t, x, y)\,\mathrm{sgn}\, y \ge p_0(|y| - |x|) - q_0, \tag{1.8}$$

$$\sigma f_2(t, x, y)\,\mathrm{sgn}\, x \ge p_0(|x| - |y|) - q_0, \tag{1.9}$$

hold on D, where p_0 and q_0 are positive constants. Then the problem (1.1), (1.2) *is solvable.*

Theorem 1.2. *Suppose that*

$$\sigma[f_1(t,x,y) - p_{11}(t,x,y)x]\operatorname{sgn} y \geq 0 \quad \text{for } a_1 \leq t \leq a_2,\ \mu xy > 0, \tag{1.10}$$

$$\sigma[f_1(t,x,y) - p_{11}(t,x,y)x - p_{12}(t,x,y)y]\operatorname{sgn} y \geq -q_0 \quad \text{for } a_1 \leq t \leq a_2,\ \mu xy < 0, \tag{1.11}$$

$$\sigma[f_2(t,x,y) - p_{22}(t,x,y)y]\operatorname{sgn} x \geq -q(x) \quad \text{for } a_1 \leq t \leq a_2,\ \mu xy > 0, \tag{1.12}$$

$$\sigma[f_2(t,x,y) - p_{21}(t,x,y)x - p_{22}(t,x,y)y]\operatorname{sgn} x \geq -q_0 \quad \text{for } a_1 \leq t \leq a_2,\ \mu xy < 0 \tag{1.13}$$

and the inequality (1.7) *holds, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, $p_{12}, p_{21} : D \to R$ *are continuous bounded functions,* $q \in C(R; R_+)$, $\mu \in \{-1, 1\}$ *and* q_0, r_0 *are positive constants. Then the problem* (1.1), (1.2) *has at least one solution.*

Theorem 1.3. *Suppose that*

$$\sigma[f_1(t,x,y) - p_{11}(t,x,y)x]\operatorname{sgn} y \geq -q_0 \quad \text{for } a_1 \leq t \leq a_2,\ \mu xy < 0, \tag{1.14}$$

$$\sigma[f_2(t,x,y) - p_{22}(t,x,y)y]\operatorname{sgn} x \geq -q(x) \quad \text{for } (t,x,y) \in D, \tag{1.15}$$

and the inequalities (1.7) *and* (1.10) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$ *are continuous bounded functions,* $q \in C(R, R_+)$, $\mu \in \{-1, 1\}$ *and* r_0, q_0 *are positive constants. Then the problem* (1.1), (1.2) *has at least one solution.*

1.2. The Problem (1.1), (1.3). We shall study the problem (1.1),(1.3) in the case when

$$(-1)^i \sigma \lambda_{i1} \lambda_{i2} \geq 0, \quad |\lambda_{i1}| + |\lambda_{i2}| \neq 0 \quad (i = 1, 2)$$

and

$$\sup\{|h_i(x,y)| : (-1)^i \sigma xy > 0\} < +\infty \quad (i = 1, 2),$$

where $\sigma \in \{-1, 1\}$.

Theorem 1.4. *Let the inequalities* (1.4)–(1.7) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, $p_{12}, p_{21} : D \to R$ *are continuous bounded functions and* q_0, r_0 *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Corollary 1.2. *Let the inequalities* (1.8) *and* (1.9) *hold on* D, *where* p_0 *and* q_0 *are positive constants. Then the problem* (1.1), (1.3) *has one solution.*

Consider as an example the boundary value problem

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + g_{11}(t)x_2^{2k_1+1} + \\ &+ g_{12}(t)|x_1|^{n_1}x_2^{2m_1+1} + q_1(t), \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + g_{21}(t)x_1^{2k_2+1} + \\ &+ g_{22}(t)|x_2|^{n_2}x_1^{2m_2+1} + q_2(t), \end{aligned} \tag{1.16}$$

$$x_2(a_1) = h_1(x_1(a_1)), \quad x_2(a_2) = h_2(x_1(a_2)), \tag{1.17}$$

where $n_i,\ k_i,\ m_i \in \{1,2,3,\dots\}$ $(i = 1,2)$, $p_{ij},\ g_{ij},\ q_i \in C([a_1,a_2];R)$ $(i,j = 1,2)$, $h_i \in C(R;R)$ $(i = 1,2)$. It follows from Corollary 1.2 that if for some $\sigma \in \{-1,1\}$ and $r \in R_+$ the inequalities

$$\sigma g_{i1}(t) > 0, \quad \sigma g_{i2}(t) \geq 0 \quad \text{for} \quad a_1 \leq t \leq a_2 \quad (i = 1,2)$$

and

$$(-1)^i h_i(x)\,\mathrm{sgn}\, x \leq 0 \quad \text{for} \quad |x| \geq r \quad (i = 1,2) \tag{1.18}$$

hold, then the problem (1.16),(1.17) has at least one solution. Therefore, the problem (1.16),(1.17) is solvable in the resonance case, i.e. in the case when the corresponding homogeneous problem

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 \end{aligned}, \qquad x_2(a_1) = 0, \quad x_2(a_2) = 0$$

has a nontrivial solution.

Theorem 1.5. *Let the inequalities* (1.7), *and* (1.10)-(1.13) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, $p_{12}, p_{21} : D \to R$ *are continuous bounded functions,* $q \in C(R, R_+)$, $\mu \in \{-1,1\}$ *and* r_0, q_0 *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Theorem 1.6. *Let the inequalities* (1.7), (1.10), (1.14) *and* (1.15) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$ *are continuous bounded functions,* $q \in C(R; R_+)$, $\mu \in \{-1,1\}$ *and* r_0, q_0 *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Consider as an example the system

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 - \sigma\mu(|x_1x_2| - \mu x_1x_2)^n x_1^{2k+1} + \\ + \left[\frac{|x_1x_2| - \mu x_1x_2}{|x_1x_2|+1}\right]^m q(t), \tag{1.19}$$
$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + f_1(x_1)\cdot f_2(x_2),$$

where $\mu \in \{-1,1\}$, m, n, $k \in \{1,2,3,\dots\}$, p_{ij}, $q \in C([a_1,a_2];R)$ $(i,j=1,2)$, $f_1 : R \to R$ is a continuous function and $f_2 : R \to R$ is a continuous bounded function. It follows from Theorem 1.6 that if for some $\sigma \in \{-1,1\}$ and $r \in R_+$ the inequalities (1.18) and

$$\sigma p_{12}(t) \geq 0, \quad \sigma p_{21}(t) \geq 0 \quad \text{for} \quad a_1 \leq t \leq a_2$$

hold, then the problem (1.19),(1.17) is solvable.

2. Some Auxiliary Statements

In this section we shall give some lemmas on a priori estimates of the solutions of the system

$$x_i' = p_{i1}(t,x_1,x_2)x_1 + p_{i2}(t,x_1,x_2)x_2 + q_i(t,x_1,x_2) \\ (i=1,2), \tag{2.1}$$

where $q_1 : D_2 \to R$, $q_2 : D_1 \to R$ are continuous functions and $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12}, $p_{21} : D \to R$ are continuous functions bounded by a positive number p_0.

Lemma 2.1. *Suppose that*

$$q_1(t,x,y)\operatorname{sgn} y \geq -q_0, \quad q_2(t,x,y)\operatorname{sgn} x \geq -q_0 \\ \text{for} \quad (t,x,y) \in D, \tag{2.2}$$

$$q_1(t,0,y)\operatorname{sgn} y > -p_{12}(t,0,y)|y| \\ \text{for} \quad a_1 \leq t \leq a_2, \quad |x| \geq r_0, \tag{2.3}$$

$$q_2(t,x,0)\operatorname{sgn} x > -p_{21}(t,x,0)|x| \\ \text{for} \quad a_1 \leq t \leq a_2, \quad |y| \geq r_0, \tag{2.4}$$

where r_0 and q_0 are positive constants. Suppose, moreover, that an absolutely continuous vector-function $(x_1,x_2) : [a_1,a_2] \to R$ satisfies the system (2.1) *almost everywhere and the conditions*

$$\text{either} \quad (-1)^i x_1(a_i)x_2(a_i) \leq 0 \\ \text{or} \quad |x_1(a_i)| + |x_2(a_i)| \leq c \quad (i=1,2). \tag{2.5}$$

Then the estimate

$$|x_1(t)| + |x_2(t)| \le (c + r_0 + 2q_0(a_2 - a_1)) \exp[4p_0(a_2 - a_1)]$$
$$\text{for } a_1 \le t \le a_2 \tag{2.6}$$

holds.

Proof. Let $t_0 \in]a_1, a_2[$. Suppose that $(-1)^k x_1(t_0)x_2(t_0) > 0$, where $k \in \{1,2\}$. Then either

$$(-1)^k x_1(t)x_2(t) > 0 \quad \text{for} \quad \min\{t_0, a_k\} \le t \le \max\{t_0, a_k\},$$
$$|x_1(a_k)| + |x_2(a_k)| \le c$$

or $t_1 \in [\min\{t_0, a_k\}, \max\{t_0, a_k\}]$ can be found such that

$$(-1)^k x_1(t)x_2(t) > 0 \quad \text{for} \quad \min\{t_0, t_1\} < t < \max\{t_0, t_1\},$$
$$x_1(t_1)x_2(t_1) = 0. \tag{2.7}$$

In the case when (2.7) holds, the inequality

$$x_1'(t_1)x_2(t_1) + x_2'(t_1)x_1(t_1) \le 0$$

together with (2.3) and (2.4) implies

$$|x_1(t_1)| + |x_2(t_1)| \le r_0.$$

Therefore, if $(-1)^k x_1(t_0)x_2(t_0) > 0$, then $t_1 \in [\min\{t_0, a_k\}, \max\{t_0, a_k\}]$ can be found such that

$$(-1)^k x_1(t)x_2(t) > 0 \quad \text{for} \quad \min\{t_0, t_1\} < t < \max\{t_0, t_1\},$$
$$|x_1(t_1)| + |x_2(t_1)| \le c + r_0. \tag{2.8}$$

Integrating the sum $x_1'(t) + (-1)^k x_2'(t)$ from t_1 to t and taking into consideration (2.1),(2.2) and (2.8) we easily see that

$$|x_1(t)| + |x_2(t)| \le c + r_0 + 2q_0(a_2 - a_1) - (-1)^k 4p_0 \int_{t_1}^{t} [|x_1(\tau)| + |x_2(\tau)|] d\tau$$
$$\text{for} \quad \min\{t_0, t_1\} < t < \max\{t_0, t_1\}.$$

Applying the Gronwall-Bellman lemma, we obtain that the estimate (2.6) holds for $t = t_0$.

Suppose now that $x_1(t_0)x_2(t_0) = 0$. Then either a sequence $(t_n)_{n=1}^{+\infty}$, $t_n \in]a_1, a_2[$ $n \in \{1, 2, 3, \dots\}$ can be found such that

$$\lim_{n \to +\infty} t_n = t_0, \quad x_1(t_n)x_2(t_n) \ne 0 \quad n \in \{1, 2, \dots\} \tag{2.9}$$

or for some $\varepsilon \in]0, \min(b - t_0, t_0 - a)[$

$$x_1(t)x_2(t) = 0 \quad \text{for} \quad t_0 - \varepsilon < t < t_0 + \varepsilon. \tag{2.10}$$

If (2.9) is true, then as it already was shown above, the estimate (2.6) holds for $t = t_n$ $n \in \{1, 2, \dots\}$. Hence (2.6) holds for $t = t_0$ also. And if (2.10) is true, then according to (2.3) and (2.4) we obtain from the equality

$$x_1'(t_0)x_2(t_0) + x_2'(t_0)x_1(t_0) = 0$$

that $|x_1(t_0)| + |x_2(t_0)| \le r_0$. ■

Lemma 2.2. *Suppose that*

$$q_1(t,x,y)\operatorname{sgn} y \ge -p_{12}(t,x,y)|y| \quad \text{for } a_1 < t < a_2,\ xy > 0, \tag{2.11}$$

$$q_1(t,x,y)\operatorname{sgn} y \ge -q_0 \quad \text{for } a_1 < t < a_2,\ xy < 0, \tag{2.12}$$

$$q_1(a_1,0,y)\operatorname{sgn} y > p_{12}(a_1,0,y)|y| \quad \text{for } |y| \ge r_0, \tag{2.13}$$

$$q_2(t,x,y)\operatorname{sgn} x \ge -q(x) - p_{21}(t,x,y)|x| \quad \text{for } a_1 < t < a_2,\ xy > 0, \tag{2.14}$$

$$q_2(t,x,y)\operatorname{sgn} x \ge -q_0 \quad \text{for } a_1 < t < a_2,\ xy < 0 \tag{2.15}$$

and (2.4) *holds, where* $q \in C(R; R_+)$ *and* r_0 *and* q_0 *are positive constants. Then for any absolutely continuous vector-function* (x_1, x_2) : $[a_1, a_2] \to R$ *satisfying the system* (2.1) *and the conditions* (2.5), *the estimate*

$$|x_1(t)| + |x_2(t)| \le 2(c + r_0 + 2q_0(a_2 - a_1) + \max\{q(x) : |x| \le \le (c + r_0)\exp[p_0(a_2 - a_1)]\})\exp[4p_0(a_2 - a_1)] \quad \text{for } a_1 \le t \le a_2 \tag{2.16}$$

holds.

Proof. Let $t_0 \in]a_1, a_2[$. Suppose first $x_1(t_0)x_2(t_0) > 0$. Then either

$$x_1(t)x_2(t) > 0 \quad \text{for } t_0 < t \le a_2 \quad |x_1(a_2)| + |x_2(a_2)| \le c$$

or $t_1 \in]t_0, a_2[$ can be found such that

$$x_1(t)x_2(t) > 0 \quad \text{for } t_0 < t < t_1, \quad x_1(t_1)x_2(t_1) = 0. \tag{2.17}$$

If (2.17) is true, then according to (2.11) from (2.1) we have

$$|x_1(t)|' \ge -p_0|x_1(t)| \quad \text{for } t_0 < t < t_1. \tag{2.18}$$

Therefore if $x_1(t_1) = 0$, then $x_1(t) \equiv 0$ for $t_0 < t < t_1$ which contradicts (2.17). So $x_2(t_1) = 0$, and hence $x_2'(t_1)\operatorname{sgn} x_1(t_1) \le 0$. From this according to (2.4) we see that $|x_1(t_1)| \le r_0$.

Thus if $x_1(t_0)x_2(t_0) > 0$, then $t_1 \in]t_0, a_2]$ can be found such that

$$x_1(t)x_2(t) > 0 \quad \text{for } t_0 < t < t_1, \quad |x_1(t_1)| + |x_2(t_1)| \le c + r_0.$$

By virtue of the above-said and from (2.18) we easily find that

$$|x_1(t)| \le (c + r_0)\exp[p_0(a_2 - a_1)] \quad \text{for} \quad t_0 \le t \le t_1. \tag{2.19}$$

According to (2.11),(2.14) and (2.19) the second of the equalities (2.1) implies

$$|x_2'(t)| \ge -p_0|x_2(t)| - q(x_1(t)) \quad \text{for} \quad t_0 < t < t_1$$

and

$$|x_2(t)| \le (c + r_0 + \max\{q(x) : |x| \le (c + r_0)\}\exp[p_0(a_2 - a_1)]) \times \\ \times \exp[p_0(a_2 - a_1)] \quad \text{for} \quad t_0 \le t \le t_1.$$

Therefore the estimate (2.16) holds for $t = t_0$.

Suppose now that $x_1(t_0)x_2(t_0) < 0$. Then either

$$x_1(t)x_2(t) \le 0, \;\; x_2(t) \ne 0 \;\; \text{for} \;\; a_1 < t < t_0, \;\; |x_1(a_1)| + |x_2(a_1)| \le c,$$

or

$$x_1(t)x_2(t) \le 0, \;\; x_2(t) \ne 0 \;\; \text{for} \;\; a_1 < t < t_0, \;\; x_1(a_1) = 0, \tag{2.20}$$

or $t_1 \in [a_1, t_0[$ can be found such that

$$x_1(t)x_2(t) \le 0, \;\; x_2(t) \ne 0 \;\; \text{for} \;\; t_1 < t < t_0, \;\; x_2(t_1) = 0. \tag{2.21}$$

If (2.20) ((2.21)) is true, then according to (2.13) ((2.4)) we obtain from the inequality $x_1'(a_1)\operatorname{sgn} x_2(a_1) \le 0$ $(x_2'(t_1)\operatorname{sgn} x_1(t_1) \le 0)$ that $|x_2(a_1)| \le r_0$ $(|x_1(t_1)| \le r_0)$.

Thus if $x_1(t_0)x_2(t_0) < 0$, then $t_1 \in [a_1, t_0[$ can be found such that

$$x_1(t)x_2(t) \le 0, \;\; x_2(t) \ne 0 \;\; \text{for} \;\; t_1 < t < t_0, \\ |x_1(t_1)| + |x_2(t_1)| \le c + r_0. \tag{2.22}$$

Integrating the difference of the equalities (2.1) from t_1 to t, taking into consideration (2.12),(2.15) and applying the Gronwall-Bellman lemma, we see that the estimate (2.16) holds for $t = t_0$.

Consider, at least, the case when $x_1(t_0)x_2(t_0) = 0$. Then either a sequence $(t_n)_{n=1}^{+\infty}$, $t_n \in]a_1, a_2[$, $n \in \{1, 2, 3, \dots\}$ can be found such that (2.9) holds or for some $\varepsilon \in]0, \min(b - t_0, t_0 - a)[$ (2.10) is valid. Suppose that (2.10) is true. Then either $x_1(t_0) = x_2(t_0) = 0$ or

$$x_1(t_0) \ne 0, \quad x_2(t_0) = 0 \tag{2.23}$$

or

$$x_2(t_0) \ne 0, \quad x_1(t_0) = 0. \tag{2.24}$$

Let (2.23) be fulfilled. Then $\varepsilon_1 \in]0, \varepsilon[$ can be found such that

$$x_1(t) \neq 0, \quad x_2(t) = 0 \quad \text{for} \quad t_0 - \varepsilon_1 < t < t_0.$$

According to (2.4) from the equality $x_2'(t_0) \operatorname{sgn} x_1(t_0) = 0$ we have that $|x_1(t_0)| \leq r_0$. Therefore, the estimate (2.16) is true for $t = t_0$.

Let (2.24) be fulfilled. Then $\varepsilon_1 \in]0, \varepsilon]$ can be found such that

$$x_2(t) \neq 0, \quad x_1(t) = 0 \quad \text{for} \quad t_0 - \varepsilon_1 < t < t_0.$$

Put

$$\alpha = \inf\{\tau \in]a_1, t_0[: x_1(t) \equiv 0, \quad x_2(t) \neq 0 \quad \text{for} \quad \tau < t < t_0\}.$$

If $\alpha = a_1$, then according to (2.13) from the equality $x_1'(a_1) \operatorname{sgn} x_2(a_1) = 0$ we find that $|x_2(a_1)| \leq r_0$. And if $\alpha > a_1$, then either

$$x_1(\alpha) = x_2(\alpha) = 0$$

or $\varepsilon_0 \in]0, \alpha - a_1[$ can be found such that

$$x_1(t)x_2(t) < 0 \quad \text{for} \quad \alpha - \varepsilon_0 \leq t < \alpha.$$

Since $x_1(\alpha - \varepsilon_0)x_2(\alpha - \varepsilon_0) < 0$, as it was already shown above, $t_1 \in [a_1, \alpha - \varepsilon_0[$ can be found such that (2.22) holds.

Thus, if (2.24) is valid, then $t_1 \in [a_1, t_0[$ can be found such that (2.22) is true.

Integrating the difference of the equalities (2.1) from t_1 to t, taking into consideration (2.12),(2.15) and applying the Gronwall-Bellman lemma, we see that the estimate (2.16) is true for $t = t_0$. ■

The proof of the following lemma is quite analogous.

Lemma 2.3. *Suppose that*

$$q_1(t, x, y) \operatorname{sgn} y \geq -p_{12}(t, x, y)|y| - q_0 \quad \textit{for} \quad a_1 < t < a_2, \quad xy < 0,$$
$$q_2(t, x, y) \operatorname{sgn} x \geq -q(x) - p_{21}(t, x, y)|x| \quad \textit{for} \quad (t, x, y) \in D,$$

and the conditions (2.4), (2.11) *and* (2.13) *hold, where* $q \in C(R; R_+)$ *and* r_0, q_0 *are positive constants. Then for any absolutely continuous vector-function* $(x_1, x_2) : [a_1, a_2] \to R$ *satisfying the system* (2.1) *and conditions* (2.5), *the estimate* (2.16) *holds.*

3. Proof of the Main Results

We shall carry out the proof only in the case $\sigma = \mu = 1$, since the general case by the change of variables

$$\bar{x}_1(t) = -\sigma\mu x_1\left(\sigma\mu t + \frac{1-\sigma\mu}{2}\right),$$
$$\bar{x}_2(t) = -\sigma x_2\left(\sigma\mu t + \frac{1-\sigma\mu}{2}\right)$$

can be reduced to this one.

Proof of Theorem 1.1. Assume first that instead of (1.6) and (1.7) the conditions

$$f_1(t,0,y)\operatorname{sgn} y > 0 \quad \text{for} \quad a_1 \le t \le a_2, \quad |y| \ge r_0,$$
$$f_2(t,x,0)\operatorname{sgn} x > 0 \quad \text{for} \quad a_1 \le t \le a_2, \quad |x| \ge r_0,$$

are fulfilled.

Put

$$\eta = \sum_{i,j=1}^{2} |\lambda_{ij}|^{-1}, \quad p_0 = \sup\{|p_{ij}(t,x,y)| : i,j = 1,2,\ (t,x,y) \in D\},$$
$$r_1 = 1 + (\eta l + r_0 + 2q_0(a_2 - a_1))\exp[4p_0(a_2 - a_1)], \tag{3.1}$$
$$\chi(\tau) = \begin{cases} 1 & \text{for } 0 \le \tau \le r_1 \\ 2 - \frac{\tau}{r_1} & \text{for } r_1 < \tau < 2r_1 \\ 0 & \text{for } \tau \ge 2r_1 \end{cases}, \tag{3.2}$$
$$q_i(t,x,y) = \chi(|x| + |y|)[f_i(t,x,y) - p_{i1}(t,x,y)x - p_{i2}(t,x,y)y]$$
$$(i = 1,2) \quad \text{for} \quad (t,x,y) \in D, \tag{3.3}$$
$$\tilde{q}_1(t,x,y) = \chi(|x| + |y|)[f_1(t,x,y) - y] \quad \text{for} \quad (t,x,y) \in D,$$
$$\tilde{q}_2(t,x,y) = \chi(|x| + |y|)[f_2(t,x,y) - x] \quad \text{for} \quad (t,x,y) \in D,$$
$$\tilde{g}_i(x,y) = \chi(\|x\|_C + \|y\|_C)g_i(x,y) \quad (i = 1,2)$$
$$\text{for} \quad x,y \in C([a_1,a_2];R), \tag{3.4}$$

where $\|p\|_C = \max\{|p(t)| : t \in [a_1,a_2]\}$, and consider the boundary value problem

$$x_1' = x_2 + \tilde{q}_1(t,x_1,x_2),$$
$$x_2' = x_1 + \tilde{q}_2(t,x_1,x_2), \tag{3.5}$$
$$\lambda_{i1}x_1(a_i) + \lambda_{i2}x_2(a_i) + \tilde{g}_i(x_1,x_2) = 0 \quad (i = 1,2). \tag{3.6}$$

According to Theorem 2.1 from [2], the problem (3.5),(3.6) has at least one solution (x_1, x_2). It is easy to see that (x_1, x_2) is a solution of the system

$$x_i' = \tilde{p}_{i1}(t, x_1, x_2)x_1 + \tilde{p}_{i2}(t, x_1, x_2)x_2 + q_i(t, x_1, x_2) \quad (i = 1, 2), \tag{3.7}$$

where

$$\tilde{p}_{ij}(t, x, y) = 1 + (p_{ij}(t, x, y) - 1)\chi(|x| + |y|) \quad (i, j = 1, 2, \ i \neq j),$$
$$\tilde{p}_{ii}(t, x, y) = p_{ii}(t, x, y)\chi(|x| + |y|) \quad (i = 1, 2). \tag{3.8}$$

In view of (3.2),(3.4) and (3.6) we have

$$\sum_{i=1}^{2} |\lambda_{i1}x_1(a_i)| + |\lambda_{i2}x_2(a_i)| \leq l.$$

from which we get that the solution (x_1, x_2) of the system (3.7) satisfies the conditions

$$\text{either } x_1(a_1)x_2(a_1) > 0 \text{ or } |x_1(a_1)| + |x_2(a_1)| \leq \eta l$$

and

$$\text{either } x_1(a_2)x_2(a_2) < 0 \text{ or } |x_1(a_2)| + |x_2(a_2)| \leq \eta l.$$

According to Lemma 2.1 and (3.1) we have

$$|x_1(t)| + |x_2(t)| \leq r_1 \quad \text{for } a_1 \leq t \leq a_2. \tag{3.9}$$

This estimate together with (3.2)-(3.4) and (3.6)-(3.8) implies that (x_1, x_2) is a solution of the problem (3.1),(3.2). Moreover, (3.9) holds.

Consider now the case when (1.6) and (1.7) are fulfilled. According to what has been proved above, for any natural n the system of the differential equations

$$x_1' = f_1(t, x_1, x_2) + \frac{x_2}{n(1 + |x_2|)}$$
$$x_2' = f_2(t, x_1, x_2) + \frac{x_1}{n(1 + |x_1|)}$$

has the solution (x_{1n}, x_{2n}) satisfying the boundary conditions (1.2) and the inequality

$$|x_{1n}(t)| + |x_{2n}(t)| \leq r_1 \quad \text{for } a_1 \leq t \leq a_2.$$

It is clear that the sequences of functions $(x_{in})_{n=1}^{+\infty}$ $(i = 1, 2)$ are uniformly bounded and equicontinuous on $[a_1, a_2]$. Therefore, without loss of generality, we can assume that they are uniformly convergent. Putting

$$x_i(t) = \lim_{n\to+\infty} x_{in}(t) \quad \text{for} \quad a_1 \le t \le a_2 \quad (i = 1,2)$$

it is easy to see that (x_1, x_2) is a solution of the problem (1.1),(1.2). ■

The proofs of the other theorems are quite analogous to the one of Theorem 1.1. The difference is that instead of Lemma 2.1 one has to apply Lemma 2.2 in proving Theorems 1.2, 1.5 and Lemma 2.3 in proving Theorems 1.3, 1.6.

Applying Theorems 1.1 and 1.4 in the case when

$$p_{11}(t,x,y) = \begin{cases} p_{11}(t)\,\mathrm{sgn}(xy) & \text{for} \quad x \neq 0 \\ p_{11}(t) & \text{for} \quad x = 0 \end{cases}$$

$$p_{22}(t,x,y) = \begin{cases} p_{22}(t)\,\mathrm{sgn}(xy) & \text{for} \quad y \neq 0 \\ p_{22}(t) & \text{for} \quad y = 0 \end{cases}$$

one can easily be convinced in the validity of Corollaries 1.1 and 1.2.

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(Received 21.10.1992)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993), No. 3, 355-364

LIMIT BEHAVIOR OF SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. A classification of classes of equivalent linear differential equations with respect to ω-limit sets of their canonical representatives is introduced. Some consequences of this classification to the oscillatory behavior of solution spaces are presented.

რეზიუმე. შემოღებულია წრფივი დიფერენციალური განტოლებების კლასიფიკაცია მათი ω-ზღვრული სიმრავლეების მიხედვით, რის საფუძველზეც გამოკვლეულია ამონახსნთა სივრცეების ოსცილაციური თვისებები.

1. INTRODUCTION

Many authors dealt with the behavior of solutions of differential equations to the (mostly right) end of the interval of definition – the limit behavior (often considered for the independent variable tending to ∞). Asymptotic, oscillatory and other qualitative properties of solutions of linear differential equations were intensively studied e.g. by N.V.Azbelev and Z.B.Caljuk [1], J.H.Barrett [2], G.D.Birkhoff [3], O.Borůvka [4], W.A.Coppel [5], M.Greguš [6], G.B.Gustafson [7], M.Hanan [8], I.T.Kiguradze and T.A.Chanturia [9], G.Sansone [14], C.A.Swanson [15], and many others.

The aim of this paper is to introduce a certain classification of the limit behavior of solutions of linear differential equations, a classification which is invariant with respect to the most general pointwise transformations of these equations. This classification has natural consequences to the oscillatory and asymptotic behavior of solutions. The main tool is based on the geometric approach introduced in [11] which enables us to convert some "non-compact" problems into "compact" ones. This method was applied for solving some open problems

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1991 *Mathematics Subject Classification.* 34A26, 34A30, 34C05, 34C10, 34C11, 34C20.

[12], and it has recently been explained systematically in detail together with other methods and results concerning linear differential equations in the monograph [13].

2. Background and preliminary results

Let $C^n(I)$ denote the set of all functions defined on an open interval $I \subseteq \mathbb{R}$ with continuous derivatives up to and including the order n. For $n \geq 2$, let $\mathcal{L}_n$ stand for all ordinary linear differential equations of the form

$$P_n \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \quad \text{on} \quad I,$$

I being an open interval of the reals, p_i are real continuous functions defined on I for $i = 0, 1, \ldots, n-1$, i.e. $p_i \in C^0(I)$, $p_i : I \to \mathbb{R}$.

Consider $Q_n \in \mathcal{L}_n$,

$$Q_n \equiv z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \quad \text{on} \quad J.$$

We say that the equation P_n is globally equivalent to the equation Q_n if there exist two functions,

$$f \in C^n(J), \quad f(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and}$$
$$h \in C^n(J), \quad h'(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and} \quad h(J) = I,$$

such that whenever $y : I \to \mathbb{R}$ is a solution of P_n then

$$z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J, \tag{1}$$

is a solution of Q_n.

Let $\mathbf{y}(x) = (y_1(x), \ldots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation P_n considered as a column vector function or as a curve in n-dimensional euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \ldots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^T$ denotes an n-tuple of linearly independent solutions of the equation Q_n, then the global transformation (1) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x)) \tag{1'}$$

or, for an arbitrary regular constant $n \times n$ matrix A,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x)) \tag{1''}$$

expressing only that another n-tuple of linearly independent solutions of the same equation Q_n is taken.

Denote the n-tuple $\mathbf{v} = (v_1, \dots, v_n)^T$,

$$\mathbf{v}(x) := \mathbf{y}(x)/\|\mathbf{y}(x)\|,$$

where $\|\mathbf{y}(x)\| := (y_1^2(x) + \cdots + y_n^2(x))^{1/2}$ is the euclidean norm of $\mathbf{y}$ in $\mathbb{E}_n$. It was shown (see [11] or [13]) that $\mathbf{v} \in C^n(I)$, $\mathbf{v} : I \to \mathbb{E}^n$, and the Wronski determinant of $\mathbf{v}$ is different from zero on I. Of course, $\|\mathbf{v}(x)\| = 1$, i.e. $\mathbf{v} \in \mathbb{S}_{n-1}$, where $\mathbb{S}_{n-1}$ is the unit sphere in $\mathbb{E}_n$. Denote by T_n the differential equation from $\mathcal{L}_n$ which has this $\mathbf{v}$ as its n-tuple of linearly independent solutions. Evidently T_n is globally equivalent to P_n. Moreover (see again [11] or [13]), if

$$\mathbf{u}(s) := \mathbf{v}(g(s)),$$

where the function g satisfies

$$g(s) : J \to I \subseteq \mathbb{R}, \quad g(J) = I, \quad |(g^{-1}(x))'| = \|\mathbf{v}'(x)\|$$

for the inverse g^{-1} to g, and hence $g \in C^n(J)$, $g'(s) \neq 0$ on J, we have $\|\mathbf{u}'(s)\| = 1$, i.e. this $\mathbf{u}$ is the length reparametrization of the curve $\mathbf{v}$. Of course, $\|\mathbf{u}(s)\| = \|\mathbf{v}(g(s))\| = 1$. If R_n denotes the differential equation admitting $\mathbf{u}$ as its n-tuple of linearly independent solutions on $J \subseteq \mathbb{R}$, then the above considered equation P_n is globally equivalent both to equation T_n and to R_n; equation R_n is also called the canonical equation of the whole class of equations from $\mathcal{L}_n$ globally equivalent to P_n. Canonical equations are characterized by admitting n-tuples of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1, \qquad \|\mathbf{u}'(s)\| = 1;$$

for more details see [13].

The following result describes the connection between the behavior of curves $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{u}$ and the zeros of solutions of the corresponding equations P_n, T_n and R_n, see [11] or [13].

Proposition 1. *Let P_n, T_n and R_n be equations from $\mathcal{L}_n$, and let $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{u}$ denote their n-tuples of linearly independent solutions defined as above. For an arbitrary nonzero constant vector $\mathbf{c} = (c_1, \dots, c_n)^T$, the solution $\mathbf{c}^T\mathbf{y}(x)$ of the equation P_n has the zero at x_0 if and only if the hyperplane*

$$H(\mathbf{c}) \equiv c_1\xi_1 + \cdots + c_n\xi_n = 0 \quad \text{in} \quad \mathbb{E}_n$$

intersects the curve $\mathbf{y}$ at the point of the parameter x_0.

Moreover, the solution $\mathbf{c}^T\mathbf{v}(x)$ of the equation T_n has the zero at x_0 if and only if the great circle $H(\mathbf{c}) \cap \mathbb{S}_{n-1}$ intersects the curve $\mathbf{v}$ at the point of the parameter x_0. And the solution $\mathbf{c}^T\mathbf{u}(s)$ of the equation R_n

has the zero at $s_0 = g^{-1}(x_0)$ *if and only if the great circle* $H(\mathrm{c}) \cap \mathbb{S}_{n-1}$ *intersects the curve* u *at the point of the parameter* s_0.

In each of the above cases, the order of contact corresponds to the multiplicity of zero.

3. Classification of ω-limit behavior

We have seen that a class of globally equivalent equations from $\mathcal{L}_n$ is characterized by curve $\mathrm{v} \in \mathbb{S}_{n-1}$, having coordinates in C^n with the nonvanishing wronskian. Since the sphere $\mathbb{S}_{n-1}$ is compact, the ω-limit set of v, denoted by $\omega(\mathrm{v})$, is nonempty, closed and connected, see e.g. [10]. Exactly one from the following cases occurs:

a_1 : : $\omega(\mathrm{v})$ is a point $\mathrm{p} \in \mathbb{S}_{n-1}$, i.e. a connected subset of the intersection of a 1-dimensional subspace with $\mathbb{S}_{n-1}$;

a_2 : : $\omega(\mathrm{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_2)$, where $\mathbb{E}_2$ is a 2-dimensional subspace of $\mathbb{E}_n$, and the case a_1 is not valid;

. . . :

a_i : : $\omega(\mathrm{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_i)$, where $\mathbb{E}_i$ is an i-dimensional subspace of $\mathbb{E}_n$, and neither from the above cases is valid;

. . . :

a_{n-1} : : $\omega(\mathrm{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_{n-1})$, and neither from the above cases holds;

a_n : : neither from the above cases is valid.

We will consider also the following subcases of the cases a_i for $i = 1, \dots, n$:

a_i^0 : : if the case a_i is valid and $\omega(\mathrm{v}) \subseteq \mathbb{S}_{n-1}^0$, where $\mathbb{S}_{n-1}^0$ is an *open hemisphere* of $\mathbb{S}_{n-1}$.

Evidently the case a_1 coincides with a_1^0.

4. Main result

Theorem. *Consider an equation* P_n *from* $\mathcal{L}_n$*; let* T_n *and* R_n *be equations defined as in §2, and* y, v *and* u *denote their* n*-tuples of linearly independent solutions. Let* $\omega(\mathrm{v})$ *and* $\omega(\mathrm{u})$ *be the* ω*-limit sets of* v *and* u*, respectively. If, for some* i*, the case* a_i *is valid for* v *(or for* u*), then the same case holds for every equation globally equivalent to* P_n*. Moreover, if the subcase* a_i^0 *is valid for some* i*, then the same subcase is true for every equation globally equivalent to* P_n*.*

Proof. Suppose first that the case a_i is valid for $P_n \in \mathcal{L}_n$. First it means that $\omega(\mathrm{v}) \subseteq \mathbb{S}_{n-1} \cap \mathbb{E}_i$ for $\mathrm{v} := \mathrm{y}/\|\mathrm{y}\|$. Then for each z,

$$\mathrm{z}(t) := Af(t) \cdot \mathrm{y}(h(t)),$$

obtained by a global transformation (1″), we have

$$\omega(\mathbf{z}/\|\mathbf{z}\|) = \omega(Af \cdot \mathbf{y}(h)/\|Af \cdot \mathbf{y}(h)\|) \subseteq S_{n-1} \cap (A\mathbb{E}_i),$$

where $A\mathbb{E}_i$ is again an i-dimensional subspace of $\mathbb{E}_n$. Moreover, if $\omega(\mathbf{z}/|\mathbf{z}|) \subseteq S_{n-1} \cap (A\mathbb{E}_j)$ for some $j < i$, we would get the contradiction to our supposition. Hence the case a_i is valid for every equation from $\mathcal{L}_n$ globally equivalent to P_n.

Now suppose that the subcase a_i^0 is valid for P_n, that means that $\omega(\mathbf{v}) \subseteq \mathbb{S}_{n-1}^0 \cap \mathbb{E}_i$ for $\mathbf{v} := \mathbf{y}/\|\mathbf{y}\|$. Then for each $\mathbf{z}$, $\mathbf{z}(t) := Af(t) \cdot \mathbf{y}(h(t))$, we have $\omega(\mathbf{z}/\|\mathbf{z}\|) \subseteq \hat{S}_{n-1}^0 \cap (A\mathbb{E}_i)$, where $\hat{S}_{n-1}^0 = \{\mathbf{s};\ \mathbf{s} = A\mathbf{r}/\|A\mathbf{r}\|,\ \mathbf{r} \in S_{n-1}^0\}$ is again an open hemisphere in $\mathbb{E}_n$ and $A\mathbb{E}_i$ is an i-dimensional subspace of $\mathbb{E}_n$. Hence the case a_1^0 is valid for every equation from $\mathcal{L}_n$ globally equivalent to P_n. ■

Remark 1. This theorem also shows that we may speak about the above cases and subcases with respect to a given equation and not only with respect to a particular n-tuple of its solutions, because, due to an arbitrary matrix A in (1″), these cases and subcases are characterized by the properties which are invariant with respect to a choice of an n-tuple of linearly independent solutions of the considered equation.

5. Consequences

Oscillation or nonoscillation will be always considered with respect to the right end of the definition interval of a considered equation.

Corollary 1. (Oscillatory behavior of solutions). *If the case a_1 is valid for $P_n \in \mathcal{L}_n$, then there do not exist n linearly independent oscillatory solutions (for $t \to b_-$) of P_n. Moreover, there exist n linearly independent nonoscillatory solutions of P_n as ($t \to b_-$).*

Proof. Let P_n be a given equation, and $\mathbf{y}$ denote an n-tuple of its linearly independent solutions. Suppose that there exist n linearly independent oscillatory solutions of P_n. Then, due to Proposition 1, there are n great circles on $\mathbb{S}_{n-1}$, not containing a common point, each of them being intersected by $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|$, or equivalently, by $\mathbf{u}$ (see notation in §2) at points with infinitely many parameters to the right end of the interval of definition. Hence on each of these great circles there is at least one point belonging to $\omega(\mathbf{v})$ ($\omega(\mathbf{u})$). Under our assumption, the case a_1 is valid for P_n, i.e. $\omega(\mathbf{v})$ is a single point, say $\mathbf{p}$ on $\mathbb{S}_{n-1}$. Thus this point must be common to n considered circles, which is a contradiction to the linear independence of the solutions. Hence there do not exist n linearly independent oscillatory solutions of P_n.

Now choose n independent vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$ in $\mathbb{E}_n$ such that the hyperplanes $H(\mathbf{c}_i)$, $i = 1, \dots, n$ do not go through the point $\mathbf{p}$. Then each solution $\mathbf{c}_i^T \cdot \mathbf{y}(x)$ is nonoscillatory. In fact, if $\mathbf{c}_i^T \cdot \mathbf{y}(x)$ were oscillatory, then $\mathbf{y}/\|\mathbf{y}\| \cap H(\mathbf{c}_i)$ would be an infinite sequence on the great circle $\mathbb{S}_{n-1} \cap H(\mathbf{c}_i)$ that should have an accumulation point in $\omega(\mathbf{y}/\|\mathbf{y}\|) = \mathbf{p}$, contrary to our choice of the hyperplanes. ■

Corollary 2. (Asymptotic behavior of solutions). *If the case a_1 is valid for equation P_n from $\mathcal{L}_n$, then P_n admits an n-tuple $\mathbf{y}^* = (y_1^*, \dots, y_n^*)^T$ of linearly independent solutions such that*

$$\lim_{x \to b_-} \frac{y_1^*}{\sqrt{(y_1^*)^2 + \cdots + (y_n^*)^2}} = 1$$

and

$$\lim_{x \to b_-} \frac{y_i^*}{\sqrt{(y_1^*)^2 + \cdots + (y_n^*)^2}} = 0 \quad \text{for } i = 2, \dots, n.$$

Proof. In the case a_1 we have $\lim_{x \to b_-} \mathbf{y}(x)/\|\mathbf{y}(x)\| = \mathbf{p}$, $\mathbf{p}$ being a point on $\mathbb{S}_{n-1}$. Choose an n-tuple of orthonormal vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$, where $\mathbf{c}_1 := \mathbf{p}$, otherwise arbitrary. Denote by C the orthogonal matrix $(\mathbf{c}_1, \dots, \mathbf{c}_n)$. Define $y_i^* := \mathbf{c}_i^T \cdot \mathbf{y}$, i.e. $\mathbf{y}^* = C^T \cdot \mathbf{y}$. Then

$$\lim_{x \to b_-} y_1^*/\|\mathbf{y}^*\| = \lim_{x \to b_-} \frac{\mathbf{c}_1^T \mathbf{y}}{\|\mathbf{y}^T C C^T \mathbf{y}\|} = \mathbf{c}_1^T \cdot \lim_{x \to b_-} \mathbf{y}/\|\mathbf{y}\| = \mathbf{c}_1^T \cdot \mathbf{p} = \mathbf{c}_1^T \cdot \mathbf{c}_1 = 1$$

and for $i = 2, \dots, n$,

$$\lim_{x \to b_-} y_i^*/\|\mathbf{y}^*\| = \lim_{x \to b_-} \frac{\mathbf{c}_i^T \mathbf{y}}{|\mathbf{y}^T C C^T \mathbf{y}|} = \mathbf{c}_i^T \cdot \lim_{x \to b_-} \mathbf{y}/\|\mathbf{y}\| = \mathbf{c}_i^T \cdot \mathbf{p} = \mathbf{c}_i^T \cdot \mathbf{c}_1 = 0. \quad ■$$

Corollary 3. *If the second order equation*

$$y'' + p_1(x)y' + p_0(x)y = 0 \quad \text{on} \quad I = (a, b), \quad -\infty \le a < b \le \infty \qquad (2)$$

is nonoscillatory (for $x \to b_-$), then the case a_1 is valid for (2). *If the equation* (2) *is oscillatory (for $x \to b_-$), then the case a_2 holds for* (2). *The subcase a_2^0 cannot occur.*

Proof. For two linearly independent solutions y_1, y_2 of equation (2), $\mathbf{y} = (y_1, y_2)^T$, the curve $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|$ is an arc on the unit circle $\mathbb{S}_1$ in the plane $\mathbb{E}_2$. Due to Proposition 1, if equation (2) is oscillatory for $x \to b_-$, then this arc $\mathbf{v}$ infinitely many times encircles the origin (without turning points, see [13]), and hence $\omega(\mathbf{v})$ is exactly $\mathbb{S}_1$. If equation (2) is nonoscillatory for $x \to b_-$, then the arc v ends by approaching a point on $\mathbb{S}_1$, exactly its ω-limit set, and the case a_1 holds for (2). ■

Corollary 4. *If the case a_j is valid for an equation P_n for some $j > 1$, then there exist n linearly independent oscillatory solutions of P_n.*

Proof. In the case a_j for some $j > 1$, the set $\omega(\mathrm{v})$ contains two different points on $\mathbb{S}_{n-1}$, say p_1 and p_2. Evidently, there exist n hyperplanes $H(\mathrm{c}_i)$, $i = 1, \dots, n$, in $\mathbb{E}_n$ with linearly independent vectors $\mathrm{c}_1, \dots, \mathrm{c}_n$, each of them separating points p_1 and p_2 into opposite *open* halfspaces of $\mathbb{E}_n$, i.e. $\mathrm{c}_i^T \mathrm{p}_1 > 0$ and $\mathrm{c}_i^T \mathrm{p}_2 < 0$ for each $i = 1, \dots, n$. Hence, due to Proposition 1, each solution $\mathrm{c}_i^T \mathrm{y}(x)$ oscillates for $x \to b_-$, because the curve v intersects infinitely many times the hyperplane $H(\mathrm{c}_i)$ as $x \to b_-$. ■

Remark 1. As an immediate consequence of this corollary we may state:

If equation L_n does not admit n linearly independent oscillatory solutions, then the case a_1 is valid for it. In particular, if each solution of equation P_n is nonoscillatory, then the case a_1 takes place for L_n.

Corollary 5. *If, for some $i = 1, \dots, n$, the case a_i^0 is valid for equation P_n, then there exist n linearly independent nonoscillatory solutions of P_n.*

Proof. Let y denote an n-tuple of linearly independent solutions of P_n. Under our assumption, $\omega(\mathrm{y}/\|\mathrm{y}\|)$ lies inside an open hemisphere of $\mathbb{S}_{n-1}$ determined by a hyperplane $H(\mathrm{p})$. Evidently $\mathrm{p}^T\mathrm{y}(x)$ is a nonoscillatory solution. Moreover, $\omega(\mathrm{y}/\|\mathrm{y}\|)$ is closed, and hence there exists an neighbourhood N of the point $\mathrm{p} \in \mathbb{S}_{n-1}$ such that $H(\mathrm{q}) \cap \omega(\mathrm{y}/\|\mathrm{y}\|) = \emptyset$ for each $\mathrm{q} \in N$. If we take n linearly independent vectors (points) $\mathrm{q}_1, \dots, \mathrm{q}_n$ from N, then

$$y_i := \mathrm{q}_i^T \mathrm{y}, \quad i = 1, \dots, n,$$

are required nonoscillatory solutions. In fact, if one of these solutions were oscillatory, then, again due to Proposition 1, the corresponding hyperplane would intersect the curve y (or equivalently $\mathrm{y}/\|\mathrm{y}\|$) infinitely many times. Hence this hyperplane would contain at least one point in $\omega(\mathrm{y}/\|\mathrm{y}\|)$, contrary to our choice of the above hyperplanes. ■

Remark 2. Comparing Corollaries 4 and 5 we see that in the case a_i^0 with $i > 1$ for L_n, this equation admits both an n-tuple of oscillatory solutions and, at the same time, another n-tuple of nonoscillatory solutions.

Remark 3. Also other (e.g. topological) properties of $\omega(\mathrm{v})$ that are invariant with respect to the centroaffine transformations can be considered for introducing other, more detailed classifications of the classes of equivalent linear differential equations from $\mathcal{L}_n$.

6. Examples

1. The differential equation

$$y^{(n)} = 0 \quad \text{on} \quad (0, \infty)$$

has n linearly independent solutions: $x^{n-1}, x^{n-2}, \dots, 1$. For this equation the case a_1 holds, no solution is oscillatory and

$$\lim_{x\to\infty} \frac{x^{n-1}}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 1,$$

$$\lim_{x\to\infty} \frac{x^{n-2}}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 0, \dots, \lim_{x\to\infty} \frac{1}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 0,$$

in accordance with Corollary 2 and Remark 1.

2. The equation

$$y''' + 2y'' + 2y' = 0 \quad \text{on} \quad (0, \infty)$$

admits the solutions: 1, $e^{-x}\sin x$, $e^{-x}\cos x$. For this equation the case a_1 is valid. There are two linearly independent oscillatory solutions as $x \to \infty$, there are no three linearly independent oscillatory solutions. This equation admits three linearly independent nonoscillatory solutions, and

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+e^{-2x}}} = 1, \quad \lim_{x\to\infty} \frac{e^{-x}\sin x}{\sqrt{1+e^{-2x}}} = 0, \quad \lim_{x\to\infty} \frac{e^{-x}\cos x}{\sqrt{1+e^{-2x}}} = 0,$$

as Corollaries 1 and 2 state.

3. However, the equation

$$y''' - 2y'' + 2y' = 0 \quad \text{on} \quad (0, \infty)$$

admits the solutions: 1, $e^{x}\sin x$, $e^{x}\cos x$; the corresponding ω-limit set is a great circle on the sphere $\mathbb{S}_2$ in $\mathbb{E}_3$ and hence the case a_2 is valid for it. However, the subcase a_2^0 does not take place. Except of the constant solutions, each other solution is oscillatory (as $x \to \infty$), see Corollaries 4,5 and Remark 2.

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(Received 06.04.1993)

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Proceedings of the Georgian Academy of Sciences. Mathematics
1(1993), No. 3, 365-382

CONTACT PROBLEMS FOR TWO ANISOTROPIC HALF-PLANES WITH SLITS

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ABSTRACT. The problem of a stressed state in a nonhomogeneous infinite plane consisting of two different anisotropic half-planes and having slits of the finite number on the interface line is investigated. It is assumed that a difference between the displacement and stress vector values is given on interface line segments; on the edges of slits we have the following data: boundary values of the stress vector (problem of stress) or displacement vector values on the one side of slits, and stress vector values on the other side (mixed problem). Solutions are constructed in quadratures.

რეზიუმე. შეისწავლება ამოცანები დაძაბული მდგომარეობის შესახებ უსასრულო არაერთგვაროვან სიბრტყეში, რომელიც შედგენილია ორი განსხვავებული ანიზოტროპული ნახევარსიბრტყისაგან და რომელსაც გააჩნია ჭრილების სასრული რაოდენობა მასალათა გამყოფი წრფის გასწვრივ. იგულისხმება, რომ მასალათა გამყოფ უბნებზე მოცემულია გადაადგილებისა და ძაბვის ვექტორთა ზღვრული მნიშვნელობების სხვაობები, ხოლო ჭრილის ნაპირებზე – ძაბვის ვექტორის ზღვრული მნიშვნელობები (ძაბვის ამოცანა) ან გადაადგილების ვექტორის ზღვრული მნიშვნელობა ერთი მხრიდან და ძაბვის ვექტორის ზღვრული მნიშვნელობა მეორე მხრიდან (შერეული ამოცანა). ამოცანების ამონახსნები წარმოდგენილია კვადრატურებში.

In this paper, employing the methods of the potential theory and of systems of singular integral equations, we investigate problems of a stressed state in a nonhomogeneous infinite plane consisting of two anisotropic half-planes with different elastic constants and having slits on the interface line between the half-planes. The stressed state is determined giving displacement and stress vector jumps on the interface line segments, and boundary values of either the stress vector or of the displacement vector on the one side of slits, and stress vector values on the other side.

1991 *Mathematics Subject Classification.* 73C30.

The difficulty of solving the problems lies, in particular, in lengthy calculations one has to perform in order to verify certain conditions, but it can be overcome using the constants introduced by of M.O. Basheleishvili [1]. The solutions obtained are constructed in quadratures.

It should be observed that, when the half-planes are welded to each other along the interface line segments, the problem of stress was studied on the basis of the theory of functions of a complex variable in [2], where the problem is reduced to the solution of four problems of linear conjugation. The solution of the problem is not, however, simple and demands some refinement.

Formulation of Problems. Let the real x-axis be the interface line between two different anisotropic materials filling up the upper ($y > 0$) and the lower ($y < 0$) half-planes and having Hook's coefficients

$$A_{11}^{(0)},\quad A_{12}^{(0)},\quad A_{22}^{(0)},\quad A_{13}^{(0)},\quad A_{23}^{(0)},\quad A_{33}^{(0)}$$

and

$$A_{11}^{(1)},\quad A_{12}^{(1)},\quad A_{22}^{(1)},\quad A_{13}^{(1)},\quad A_{23}^{(1)},\quad A_{33}^{(1)}$$

respectively.

It is assumed that slits are located on the segments $l_p = a_p b_p$, $p = 1, 2, \ldots, n$, of the x-axis. Let $l = \cup_{p=1}^{n} l_p$ and L be the remainder part of the real axis outside slits. Denote the domain $y > 0$ by D_0, and the domain $y < 0$ by D_1.

As is known, in the absence of mass force the system of differential equations of equilibrium of an anisotropic elastic body in the domain D_j, $j = 0, 1$, looks like [3]

$$\begin{aligned}
&A_{11}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x^2} + 2A_{13}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x \partial y} + A_{33}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2} + A_{13}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2} + \\
&\qquad + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 v^{(j)}}{\partial x \partial y} + A_{23}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2} = 0, \\
&A_{13}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x^2} + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 u^{(j)}}{\partial x \partial y} + A_{23}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2} + A_{33}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2} + \\
&\qquad + 2A_{23}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x \partial y} + A_{22}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2} = 0,
\end{aligned} \tag{1}$$

where $u^{(j)}$ and $v^{(j)}$ are the Cartesian coordinates of the displacement vector.

The stressed state in the anisotropic body occupying the domain D_j is determined by three stress components $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$, which, in turn, are expressed by means of the strain components $\varepsilon_x^{(j)}$, $\varepsilon_y^{(j)}$, $\varepsilon_{xy}^{(j)}$ as follows:

$$\begin{aligned}\sigma_x^{(j)} &= A_{11}^{(j)}\varepsilon_x^{(j)} + A_{12}^{(j)}\varepsilon_y^{(j)} + A_{13}^{(j)}\varepsilon_{xy}^{(j)},\\ \sigma_y^{(j)} &= A_{12}^{(j)}\varepsilon_x^{(j)} + A_{22}^{(j)}\varepsilon_y^{(j)} + A_{23}^{(j)}\varepsilon_{xy}^{(j)},\\ \tau_{xy}^{(j)} &= A_{13}^{(j)}\varepsilon_x^{(j)} + A_{23}^{(j)}\varepsilon_y^{(j)} + A_{33}^{(j)}\varepsilon_{xy}^{(j)},\end{aligned} \tag{2}$$

where

$$\varepsilon_x^{(j)} = \frac{\partial u^{(j)}}{\partial x}, \quad \varepsilon_y^{(j)} = \frac{\partial v^{(j)}}{\partial y}, \quad \varepsilon_{xy}^{(j)} = \frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x}.$$

We will consider the following boundary-contact problems:

In the domain D_j, $j = 0, 1$, find a regular solution of system (1), i.e., determine the displacement components $u^{(j)}$, $v^{(j)}$ and the stress components $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$ when on L a difference is given between boundary values of the displacement and stress vectors[1]

$$\begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix}^+ - \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}^- = \mathbf{f}, \quad \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ - \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^- = -\boldsymbol{\varphi}, \tag{3}$$

while on the edges of slits we have either boundary values of the stress vectors

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ = -\mathbf{F}, \quad \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^- = -\boldsymbol{\Phi}, \tag{4}$$

or boundary values of the stress vector from D_0 and of the displacement vector from D_1

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ = -\mathbf{F}, \quad \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}^- = \boldsymbol{\Phi}. \tag{5}$$

The regularity of the solution of system (1) implies that: 1) this solution has continuous partial derivatives of second order in the domain D_j, $j = 0, 1$; 2) it can be continuously extended onto the whole real axis; 3) stress components $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$ which by (2) correspond to it can be continuously extended onto the whole real axis except perhaps the end points of slits in whose neighbourhoods they have an integrable singularity.

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[1]All the vectors considered are columns but they will sometimes be written as rows. To make the formulas shorter, the commonly used notations $\frac{1}{a}\mathbf{C}$ and $b\mathbf{C}$ with numbers $a \neq 0$, b and vector or matrix $\mathbf{C}$ are often replaced by $\frac{\mathbf{C}}{a}$ and $\mathbf{C}b$, respectively. The superscript $+(-)$ denotes that the boundary value of the function is taken from D_0 (D_1).

In the sequel the problem with the boundary conditions (3), (4) will be called the *problem of stress*, while the problem with the boundary conditions (3), (5) the *mixed problem*. We will investigate each of these problems separately.

Problem of Stress. It is assumed that the known vectors $\mathbf{f}$, φ, $\mathbf{F}$ and Φ satisfy the following conditions:

a) when $|x| \to \infty$

$$|x|^{\alpha}\mathbf{f}(x) \to \beta, \quad |x|^{1+\delta}\varphi(x) \to \gamma_0 \quad (\alpha > 0,\ \delta > 0), \tag{6}$$

where β, γ_0 are the constant vectors;

b) $\mathbf{f}$ belongs to the Hölder class on L (including the neighbourhood of the point at infinity);

c) $\mathbf{f}'$, φ, $\mathbf{F}$ and Φ belong to the class H^* [4];

d) the vector $\mathbf{f}$ satisfies the conditions

$$\mathbf{f}(a_p) = \mathbf{f}(b_p) = 0, \quad p = 1, 2, \dots, n; \tag{7}$$

e) stress and rotation vanish at infinity.

Like in [5], the displacement vector $(u^{(j)}, v^{(j)})$ in the domain D_j, $j = 0, 1$, will be sought as a combination of simple- and double-layer potentials

$$\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\mathbf{E}^{(j)}(k)\Big\{(\mathbf{A}^{(j)} + i\mathbf{B}^{(j)})\Big(\int_L \frac{\mathbf{f}(t)\,dt}{t - z_{kj}} + \int_l \mathbf{g}(t)\ln(t - z_{kj})\,dt\Big) - (\mathbf{C}^{(j)} + i\mathbf{D}^{(j)})\int_{-\infty}^{+\infty}\mathbf{h}(t)\ln(t - z_{kj})\,dt\Big\} - \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\mathbf{A}^{(j)}(k)\mathbf{X}^{(j)}\mathbf{P}\ln(z_{kj} + i(-1)^j), \tag{8}$$

where $\mathbf{g} = (g_1, g_2)$ and $\mathbf{h} = (h_1, h_2)$ are the unknown vectors which , in view of the fact that the logarithmic function is many-valued, must satisfy the conditions

$$\int_{l_p} \mathbf{g}(t)\,dt = 0, \quad p = 1, 2, \dots, n, \tag{9}$$

$$\int_{-\infty}^{+\infty} \mathbf{h}(t)\,dt = 0; \tag{10}$$

$\mathbf{P}$ denotes the principal vector of external force

$$\mathbf{P} = \int_{-\infty}^{+\infty}\left\{\begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+}\right\} dt; \tag{11}$$

$i = \sqrt{-1}$ is the complex unity; $z_{kj} = x + \alpha_{kj} y$, $k = 1, 2$, $j = 0, 1$, where $\alpha_{kj} = a_{kj} + i b_{kj}$ $(b_{kj} > 0)$ is the root of the characteristic equation corresponding to system (1) (as known [3], the latter equation is a fourth-degree equation with real coefficients of the form

$$a_{11}^{(j)}\alpha_j^4 - 2a_{13}^{(j)}\alpha_j^3 + (2a_{12}^{(j)} + a_{33}^{(j)})\alpha_j^2 - 2a_{23}^{(j)}\alpha_j + a_{22}^{(j)} = 0$$

and has complex roots only, $a_{ks}^{(j)}$ are the coefficients at $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\tau_{xy}^{(j)}$ when the strain components from (2) are expressed in terms of stress components); the constant two-dimensional matrices $\mathbf{A}^{(j)}(k)$ and $\mathbf{E}^{(j)}(k)$, $k = 1, 2$, $j = 0, 1$, occur in [1] when constructing the matrix of fundamental solutions of system (1) and the double-layer potential, respectively, and are written as

$$\mathbf{A}^{(j)}(k) = \begin{Vmatrix} A_k^{(j)} & B_k^{(j)} \\ B_k^{(j)} & C_k^{(j)} \end{Vmatrix}, \quad \mathbf{E}^{(j)}(k) = -\frac{i}{m_j} A_k^{(j)} \begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\sum_{k=1}^{2} \mathbf{E}^{(j)}(k) = \mathbf{E} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix},$$

$$A_k^{(j)} = -\frac{2}{\Delta^{(j)} a_{11}^{(j)}} \left\{ A_{22}^{(j)} \alpha_{kj}^2 + 2A_{23}^{(j)} \alpha_{kj} + A_{33}^{(j)} \right\} d_{kj},$$

$$C_k^{(j)} = -\frac{2}{\Delta^{(j)} a_{11}^{(j)}} \left\{ A_{33}^{(j)} \alpha_{kj}^2 + 2A_{13}^{(j)} \alpha_{kj} + A_{11}^{(j)} \right\} d_{kj},$$

$$B_k^{(j)} = \frac{2}{\Delta^{(j)} a_{11}^{(j)}} \left\{ A_{23}^{(j)} \alpha_{kj}^2 + (A_{12}^{(j)} + A_{33}^{(j)}) \alpha_{kj} + A_{13}^{(j)} \right\} d_{kj},$$

$$\Delta^{(j)} a_{11}^{(j)} = A_{22}^{(j)} A_{33}^{(j)} - (A_{12}^{(j)})^2 > 0,$$

$$d_{1j}^{-1} = (\alpha_{1j} - \bar{\alpha}_{1j})(\alpha_{1j} - \alpha_{2j})(\alpha_{1j} - \bar{\alpha}_{2j}),$$

$$d_{2j}^{-1} = (\alpha_{2j} - \alpha_{1j})(\alpha_{2j} - \bar{\alpha}_{1j})(\alpha_{2j} - \bar{\alpha}_{2j}),$$

$\Delta^{(j)}$ is the determinant of system (2) whose positiveness follows from the positiveness of potential energy; A_j, B_j, C_j, ω_j, m_j, $\varkappa_N^{(j)}$ are the above-mentioned Basheleishvili's constants [1], [5]:

$$A_j = 2i \sum_{k=1}^{2} d_{kj}, \quad B_j = 2i \sum_{k=1}^{2} \alpha_{kj}^2 d_{kj}, \quad C_j = 2i \sum_{k=1}^{2} \alpha_{kj} d_{kj},$$

$$\omega_j = b_{1j} b_{2j} - a_{1j} a_{2j} + \frac{a_{13}^{(j)}}{a_{11}^{(j)}}, \quad m_j = a_{11}^{(j)} \left[1 - \omega_j^2 (B_j C_j - A_j^2) \right],$$

$$\varkappa_N^{(j)} = \frac{\omega_j (B_j C_j - A_j^2)}{m_j},$$

they satisfy the conditions

$$\operatorname{Im} A_j = 0, \quad B_j > 0, \quad C_j > 0, \quad B_jC_j - A_j^2 > 0,$$
$$m_j > 0, \quad \varkappa_N^{(j)} > 0, \quad \omega_j a_{11}^{(j)} > 0, \quad j = 0, 1;$$

the constant matrices $\mathbf{A}^{(j)}$, $\mathbf{B}^{(j)}$, $\mathbf{C}^{(j)}$, $\mathbf{D}^{(j)}$, $\mathbf{X}^{(j)}$ in (8) ensure the fulfilment of the contact conditions (3) and have the form [5]

$$\mathbf{A}^{(0)} = \frac{1}{\Delta}\left\{\left(\frac{B_1C_1 - A_1^2}{m_1 a_{11}^{(1)}} + \varkappa_N^{(0)}\varkappa_N^{(1)}\right)\mathbf{E} + \right.$$
$$\left. + \frac{1}{m_0 m_1}\left\| \begin{matrix} B_1C_0 - A_0A_1 & A_0C_1 - A_1C_0 \\ A_0B_1 - A_1B_0 & B_0C_1 - A_0A_1 \end{matrix} \right\|\right\},$$
$$\mathbf{B}^{(0)} = -\frac{1}{\Delta}\left\{\frac{\varkappa_N^{(1)}}{m_0}\left\| \begin{matrix} A_0 & -C_0 \\ B_0 & -A_0 \end{matrix} \right\| + \frac{\varkappa_N^{(0)}}{m_1}\left\| \begin{matrix} A_1 & -C_1 \\ B_1 & -A_1 \end{matrix} \right\|\right\},$$
$$\mathbf{C}^{(0)} = \frac{\varkappa_N^{(0)} - \varkappa_N^{(1)}}{\Delta}\left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\|,$$
$$\mathbf{D}^{(0)} = \frac{1}{\Delta}\left\{\frac{1}{m_0}\left\| \begin{matrix} C_0 & A_0 \\ A_0 & B_0 \end{matrix} \right\| + \frac{1}{m_1}\left\| \begin{matrix} C_1 & A_1 \\ A_1 & B_1 \end{matrix} \right\|\right\},$$
$$\mathbf{X}^{(0)} = \frac{m_1}{\Delta^*(B_1C_1 - A_1^2)}\left\{m_1\mathbf{E} + \right.$$
$$\left. + \frac{m_0}{B_0C_0 - A_0^2}\left\| \begin{matrix} B_0C_1 - A_0A_1 & A_1B_0 - A_0B_1 \\ A_1C_0 - A_0C_1 & B_1C_0 - A_0A_1 \end{matrix} \right\|\right\},$$

where

$$\Delta = \frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} + \frac{B_1C_1 - A_1^2}{m_1 a_{11}^{(1)}} +$$
$$+ \frac{B_1C_0 + B_0C_1 - 2A_0A_1}{m_0 m_1} + 2\varkappa_N^{(0)}\varkappa_N^{(1)} > 0,$$
$$\Delta^* = \frac{m_0^2}{B_0C_0 - A_0^2} + \frac{m_1^2}{B_1C_1 - A_1^2} +$$
$$+ \frac{m_0 m_1(B_1C_0 + B_0C_1 - 2A_0A_1)}{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)} > 0,$$

the matrices $\mathbf{A}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{C}^{(1)}$, $\mathbf{D}^{(1)}$, $\mathbf{X}^{(1)}$ are obtained from $\mathbf{A}^{(0)}$, $\mathbf{B}^{(0)}$, $\mathbf{C}^{(0)}$, $\mathbf{D}^{(0)}$, $\mathbf{X}^{(0)}$ permuting the indices 0 and 1.

It is easy to obtain

$$\mathbf{A}^{(0)} + \mathbf{A}^{(1)} = \mathbf{E}, \quad \mathbf{X}^{(0)} + \mathbf{X}^{(1)} = \mathbf{E},$$

$$\frac{m_1}{B_1C_1 - A_1^2}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\mathbf{X}^{(1)} = \frac{m_0}{B_0C_0 - A_0^2}\begin{Vmatrix} C_0 & A_0 \\ A_0 & B_0 \end{Vmatrix}\mathbf{X}^{(0)}.$$

By virtue of conditions (7) the stress components $\tau_{xy}^{(j)}$ and $\sigma_y^{(j)}$ will have the form

$$\begin{pmatrix} \tau_{xy}^{(j)} \\ \sigma_y^{(j)} \end{pmatrix} = \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathbf{E}_0^{(j)}(k)\Big\{(\mathbf{A}^{(j)} + i\mathbf{B}^{(j)})\Big(\int_l \frac{\mathbf{g}(t)\,dt}{t - z_{kj}} -$$
$$-\int_L \frac{\mathbf{f}'(t)\,dt}{t - z_{kj}}\Big) - (\mathbf{C}^{(j)} + i\mathbf{D}^{(j)})\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t - z_{kj}}\Big\} +$$
$$+\frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathbf{N}^{(j)}(k)\mathbf{X}^{(j)}\frac{\mathbf{P}}{z_{kj} + i(-1)^j}, \quad j = 0, 1, \tag{12}$$

where for $k = 1, 2$, $j = 0, 1$

$$\mathbf{E}_0^{(j)}(k) = -\frac{i}{m_j}\mathbf{N}^{(j)}(k)\begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\mathbf{N}^{(j)}(k) = \left(\varkappa_N^{(j)}\begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} - \frac{i}{m_j}\begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix}\right)\mathbf{A}^{(j)}(k),$$

$$\sum_{k=1}^{2}\mathbf{E}_0^{(j)}(k) = \varkappa_N^{(j)}\begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} - \frac{i}{m_j}\begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\sum_{k=1}^{2}\mathbf{N}^{(j)}(k) = \mathbf{E} + i\omega_j\begin{Vmatrix} -A_j & -B_j \\ C_j & A_j \end{Vmatrix}.$$

It is easy to show that the vector $(u^{(j)}, v^{(j)})$ given by formula (8) satisfies the first of the contact conditions (3).

If we now calculate the boundary values of the vector $(\tau_{xy}^{(j)}, \sigma_y^{(j)})$, $j = 0, 1$, by formula (12), then we readily obtain that at points of the real axis, except perhaps points a_p, b_p, $p = 1, 2, \dots, n$,

$$\mathbf{h}(x) = \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+} - \frac{1}{\pi}\mathrm{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(1)}(k)}{x - i}\mathbf{X}^{(1)}\mathbf{P} +$$
$$+\frac{1}{\pi}\mathrm{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(0)}(k)}{x + i}\mathbf{X}^{(0)}\mathbf{P}, \tag{13}$$

Since the difference

$$\begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+}$$

is known due to conditions (3) and (4), the vector $\mathbf{h}$ will be known, too. One can easily verify that $\mathbf{h}$ defined by equality (13) will satisfy condition (10).

Therefore the unknown vector $\mathbf{h}$ is defined by equality (13) and there remains for us to define the vector $\mathbf{g}$.

By summing up the boundary values of the vector $(\tau_{xy}^{(j)}, \sigma_y^{(j)})$ from D_0 and D_1 at points belonging to l, except perhaps end points, we obtain the following system of integral equations for the vector $\mathbf{g}$ [6]:

$$\mathbf{A}^* g(x) + \frac{\mathbf{B}^*}{\pi} \int_l \frac{\mathbf{g}(t)\,dt}{t-x} = \boldsymbol{\Omega}(x), \quad x \in l, \tag{14}$$

where

$$\mathbf{A}^* = \left(\frac{\varkappa_N^{(1)}(B_0C_0 - A_0^2)}{m_0 a_{11}^{(0)}} - \frac{\varkappa_N^{(0)}(B_1C_1 - A_1^2)}{m_1 a_{11}^{(1)}} \right) \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix},$$

$$\mathbf{B}^* = \frac{B_1C_1 - A_1^2}{m_0 m_1 a_{11}^{(1)}} \begin{Vmatrix} B_0 & -A_0 \\ -A_0 & C_0 \end{Vmatrix} + \frac{B_0C_0 - A_0^2}{m_0 m_1 a_{11}^{(1)}} \begin{Vmatrix} B_1 & -A_1 \\ -A_1 & C_1 \end{Vmatrix},$$

$$\boldsymbol{\Omega}(x) = \frac{\Delta}{2}\big(\mathbf{F}(x) + \Phi(x)\big) + \frac{\mathbf{B}^*}{\pi} \int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{1}{2}\mathbf{C}^*\mathbf{h}(x) -$$

$$- \frac{\mathbf{D}^*}{\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{h}(t)\,dt}{t-x} + \frac{\Delta}{2\pi} \operatorname{Im} \frac{\sum_{k=1}^{2} \mathbf{N}^{(0)}(k)}{x+i} \mathbf{X}^{(0)}\mathbf{P} +$$

$$+ \frac{\Delta}{2\pi} \operatorname{Im} \frac{\sum_{k=1}^{2} \mathbf{N}^{(1)}(k)}{x-i} \mathbf{X}^{(1)}\mathbf{P},$$

$$\mathbf{C}^* = \left(\frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} + \frac{B_1C_1 - A_1^2}{m_1 a_{11}^{(1)}} \right)\mathbf{E} +$$

$$+ \frac{1}{m_0 m_1} \begin{Vmatrix} B_0C_1 - C_0B_1 & 2(A_1B_0 - A_0B_1) \\ 2(A_1C_0 - A_0C_1) & B_1C_0 - B_0C_1 \end{Vmatrix},$$

$$\mathbf{D}^* = \frac{\varkappa_N^{(0)}}{m_1} \begin{Vmatrix} -A_1 & -B_1 \\ C_1 & A_1 \end{Vmatrix} + \frac{\varkappa_N^{(1)}}{m_0} \begin{Vmatrix} -A_0 & -B_0 \\ C_0 & A_0 \end{Vmatrix}.$$

The properties of the boundary data enable us to conclude that $\boldsymbol{\Omega} = (\Omega_1, \Omega_2)$ is a vector of the class H^*. The solution of system (14) should be sought for in the same class.

System (14) is of the normal type, since

$$\det(\mathbf{A}^* \pm i\mathbf{B}^*) = -\frac{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(0)} a_{11}^{(1)}}\Delta < 0.$$

As known, in construction the solution of system (14) we encounter certain difficulties and hence we have to seek for its solution by reducing it to a singular integral equation for some scalar function [7].

To this end, multiplying the first equation of system (14) by the constant M and adding to the second equation, we obtain

$$S\big(g_1(x) - Mg_2(x)\big) + \frac{1}{\pi}\int_l \frac{(B_{11}^*M + B_{21}^*)g_1(t) + (B_{12}^*M + B_{22}^*)g_2(t)}{t - x}\,dt =$$
$$= \Omega_2(x) + M\Omega_1(x), \tag{15}$$

where B_{kj}^*, $k = 1,2$, $j = 1,2$, are the elements of the matrix $\mathbf{B}^*$ and

$$S = \frac{\varkappa_N^{(1)}(B_0C_0 - A_0^2)}{m_0 a_{11}^{(0)}} - \frac{\varkappa_N^{(0)}(B_1C_1 - A_1^2)}{m_1 a_{11}^{(1)}}.$$

Next we choose a constant M such that

$$B_{12}^*M + B_{22}^* = -M(B_{11}^*M + B_{21}^*).$$

The latter relation gives, for M, the quadratic equation

$$\left(\frac{B_0(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(1)}} + \frac{B_1(B_0C_0 - A_0^2)}{m_0 m_1 a_{11}^{(0)}}\right)M^2 -$$
$$-2\left(\frac{A_0(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(1)}} + \frac{A_1(B_0C_0 - A_0^2)}{m_0 m_1 a_{11}^{(0)}}\right)M +$$
$$+\frac{C_0(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(1)}} + \frac{C_1(B_0C_0 - A_0^2)}{m_0 m_1 a_{11}^{(0)}} = 0,$$

whose discriminant is equal to $-T^2$, where

$$T^2 = \frac{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(0)} a_{11}^{(1)}}\Delta +$$
$$+\left(\frac{\varkappa_N^{(1)}(B_0C_0 - A_0^2)}{m_0 a_{11}^{(0)}} - \frac{\varkappa_N^{(0)}(B_1C_1 - A_1^2)}{m_1 a_{11}^{(1)}}\right)^2 > 0.$$

Therefore the equation has complex roots. Let us choose M such that

$$M = \frac{\dfrac{A_0(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(1)}} + \dfrac{A_1(B_0C_0 - A_0^2)}{m_0 m_1 a_{11}^{(0)}} - iT}{\dfrac{B_0(B_1C_1 - A_1^2)}{m_0 m_1 a_{11}^{(1)}} + \dfrac{B_1(B_0C_0 - A_0^2)}{m_0 m_1 a_{11}^{(0)}}}.$$

Introducing the notations $\omega = g_1 - Mg_2$ and $\Omega_0 = \Omega_2 + M\Omega_1$, we obtain, for ω, a singular integral equation of the normal type

$$S\omega(x) + \frac{T}{\pi i}\int_l \frac{\omega(t)\,dt}{t-x} = \Omega_0(x), \quad x \in l. \tag{16}$$

Let us define a character of the end points of the integration segment using the results from [4]. We have the equation

$$\gamma = \frac{1}{2\pi i}\ln\frac{S-T}{S+T} = \frac{1}{2\pi i}\ln\frac{-m_0m_1a_{11}^{(0)}a_{11}^{(1)}(S-T)^2}{(B_0C_0-A_0^2)(B_1C_1-A_1^2)\Delta} = \frac{1}{2} - i\beta,$$

$$\beta = \frac{1}{2\pi}\ln\frac{m_0m_1a_{11}^{(0)}a_{11}^{(1)}(S-T)^2}{(B_0C_0-A_0^2)(B_1C_1-A_1^2)\Delta}.$$

Therefore all end points are nonsingular. The solution of (16) is sought for in the class of unbounded functions at the end points.

A canonical solution of the corresponding Hilbert problem in the class of unbounded solutions will have the form

$$X(z) = \prod_{p=1}^{n}(z-a_p)^{-\gamma}(z-b_p)^{\gamma-1}, \quad z = x+iy,$$

where we mean the branch defined by the conditions $z^nX(z) \to 1$ as $z \to \infty$.

Since the order of the canonical solution at infinity is equal to $-n$, the index of the class of unbounded solutions is $\varkappa = n$. Hence it follows that equation (16) is always solvable in this class and the solution will be written as

$$\omega(x) = \frac{S}{S^2-T^2}\Omega_0(x) - \frac{TX^+(x)}{\pi i(S^2-T^2)}\int_l \frac{\Omega_0(t)\,dt}{X^+(t)(t-x)} +$$
$$+ \frac{T}{S-T}X^+(x)P_{n-1}(x), \quad x \in l, \tag{17}$$

where $X^+(x)$ is the boundary value of the canonical solution $X(z)$ on l from D_0; $P_{n-1}(x)$ is the polynomial of degree $n-1$ with arbitrary complex coefficients, $P_{n-1}(x) = K_0x^{n-1} + K_1x^{n-2} + \cdots + K_{n-1}$.

Obviously, having defined ω, we thereby define the vector $\mathbf{g}$ which will be unbounded near points a_p, b_p, $p = 1, 2, \ldots, n$, and will linearly depend on $2n$ arbitrary real constants $\operatorname{Re} K_j$ and $\operatorname{Im} K_j$, $j = 0, 1, 2, \ldots, n-1$,

$$\mathbf{g} = -\frac{1}{\operatorname{Im} M}\begin{pmatrix} \operatorname{Im}(\omega\bar{M}) \\ \operatorname{Im}\omega \end{pmatrix}. \tag{18}$$

Let us choose these arbitrary real constants such that the vector **g** satisfy condition (9). The latter condition gives, for the unknown constants, a system of $2n$ algebraic equations with the same number of unknowns. This system is always solvable. Indeed, the homogeneous system obtained in the case of the boundary functions $\mathbf{f} = \varphi = \mathbf{F} = \Phi = 0$ cannot have nontrivial solutions. Then, as one can easily establish by the uniqueness theorem, the original problem has only the trivial solution. Therefore the nonhomogeneous problem is always solvable uniquely.

We have thus proved

Theorem 1. *If the conditions* a), b), c), d), e) *are fulfilled, then the stress problem with the boundary-contact conditions* (3) *and* (4) *always has the unique solution to within an additive constant. The solution is given by formula* (8), *where the vector* **h** *is defined by equality* (13) *and the vector* **g** *by equalities* (18) *and* (17).

Taking into account the behaviour of Cauchy-type integrals near the ends of integration lines one can easily obtain the asymptotics of stress components at the vertices of slits in the case of concrete boundary data.

Mixed Problem. Let, the vectors $\mathbf{f}, \varphi, \mathbf{F}, \Phi$ satisfy the conditions a),b),d) of the stress problem and the conditions

c′) $\mathbf{f}', \varphi, \mathbf{F}$ belong to the class H^*, and Φ belongs to the Hölder class and has a derivative from the class H^*;

e′) the resultant vector of force applied to the lower slit edges is given and there are no stress and rotation at infinity.

Since the boundary values of $\tau_{xy}^{(0)}$ and $\sigma_y^{(0)}$ are given on the upper slit edges, it is obvious that the resultant vector of external force applied to the real axis will also be known in this problem and represantable by formula (11).

The displacement vector $(u^{(j)}, v^{(j)})$ in the domain D_j, $j = 0, 1$, will again be sought for in form (8), where the unknown vectors **g** and **h** satisfy conditons (9) and (10). It is obvious that the stress components $\tau_{xy}^{(j)}$ and $\sigma_y^{(j)}$ are represented by formula (12).

Like in the stress problem, the vector $(u^{(j)}, v^{(j)})$ here also satisfies the first of conditions (3), whereas the vector **h** will be known on L by virtue of equality (13) and conditions (3).

Therefore it remains for us to define the vectors **g** and **h** on l. To this end let us calculate the boundary value of the vector $(\tau_{xy}^{(0)}, \sigma_y^{(0)})$

on l except perhaps points a_p, b_p, $p = 1, 2, \ldots, n$. We shall have

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ = -\frac{1}{\Delta}\mathbf{A}^*\mathbf{g}(x) + \frac{\mathbf{B}^*}{\Delta}\left(\frac{1}{\pi}\int_L \frac{\mathbf{f}'(t)dt}{t-x} - \frac{1}{\pi}\int_l \frac{\mathbf{g}(t)dt}{t-x}\right) - M^*\mathbf{h}(x) -$$
$$-\frac{\mathbf{D}^*}{\pi\Delta}\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t-x} + \frac{1}{\pi}\mathrm{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(0)}(k)}{x+i}\mathbf{X}^{(0)}\mathbf{P}, \quad x \in l, \qquad (19)$$

where

$$\mathbf{M}^* = \frac{1}{\Delta}\left\{\left(\frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} + \varkappa_n^{(1)}\varkappa_N^{(0)}\right)\mathbf{E} + \right.$$
$$\left. + \frac{1}{m_0 m_1}\begin{Vmatrix} B_0C_1 - A_0A_1 & A_1B_0 - A_0B_1 \\ A_1C_0 - A_0C_1 & B_1C_0 - A_0A_1 \end{Vmatrix}\right\}.$$

Calculating now on l the boundary value of the derivative of the vector $(u^{(1)}, v^{(1)})$ with respect to x we obtain that for every x except perhaps points a_p, b_p, $p = 1, 2, \ldots, n$,

$$\left(\frac{\partial}{\partial x}\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}\right)^- = \mathbf{A}^{(1)}g(x) + \mathbf{B}^{(1)}\left(\frac{1}{\pi}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{1}{\pi}\int_l \frac{\mathbf{g}(t)\,dt}{t-x}\right) -$$
$$-\mathbf{C}^{(1)}\mathbf{h}(x) + \frac{\mathbf{D}^{(1)}}{\pi}\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t-x} -$$
$$-\frac{m_1}{\pi(B_1C_1 - A_1^2)}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\mathbf{X}^{(1)}\mathbf{P}\frac{x}{x^2+1}. \qquad (20)$$

By virtue of the boundary conditions (5) relations (19) and (20) give for the vectors g and h a system of singular integral equations

$$\begin{aligned} &\mathbf{A}^{(1)}\mathbf{g}(x) - \mathbf{C}^{(1)}\mathbf{h}(x) - \frac{\mathbf{B}^{(1)}}{\pi}\int_l \frac{\mathbf{g}(t)\,dt}{t-x} + \\ &\quad + \frac{\mathbf{D}^{(1)}}{\pi}\int_l \frac{\mathbf{h}(t)\,dt}{t-x} = \mathbf{\Omega}^{(1)}(x), \\ &\frac{\mathbf{A}^*}{\Delta}\mathbf{g}(x) + \mathbf{M}^*\mathbf{h}(x) + \frac{\mathbf{B}^*}{\pi\Delta}\int_l \frac{\mathbf{g}(t)\,dt}{t-x} + \\ &\quad + \frac{\mathbf{D}^*}{\pi\Delta}\int_l \frac{\mathbf{h}(t)\,dt}{t-x}\Big) = \mathbf{\Omega}^{(2)}(x), \quad x \in l, \end{aligned} \qquad (21)$$

where

$$\Omega^{(1)}(x) = \Phi'(x) - \frac{\mathbf{B}^{(1)}}{\pi}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{\mathbf{D}^{(1)}}{\pi}\int_L \frac{\mathbf{h}(t)\,dt}{t-x} +$$

$$+ \frac{m_1}{\pi(B_1C_1 - A_1^2)} \begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix} \mathbf{X}^{(1)}\mathbf{P}\frac{x}{x^2+1},$$

$$\Omega^{(2)}(x) = \mathbf{F}(x) + \frac{\mathbf{B}^*}{\pi\Delta}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{\mathbf{D}^*}{\pi\Delta}\int_L \frac{\mathbf{h}(t)\,dt}{t-x} +$$

$$+ \frac{1}{\pi}\operatorname{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(0)}(k)}{x+i}\mathbf{X}^{(0)}\mathbf{P}.$$

Note that due to the properties of the boundary data the vectors $\Omega^{(1)}$ and $\Omega^{(2)}$ will belong to the class H^* on l.

If we introduce the partitioned matrices

$$\mathbf{A} = \begin{Vmatrix} \mathbf{A}^{(1)} & -\mathbf{C}^{(1)} \\ \Delta^{-1}\mathbf{A}^* & \mathbf{M}^* \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} -\mathbf{B}^{(1)} & \mathbf{D}^{(1)} \\ \Delta^{-1}\mathbf{B}^* & \Delta^{-1}\mathbf{D}^* \end{Vmatrix}$$

and denote

$$\omega = (g_1, g_1, h_1, h_2), \quad \tilde{\Omega} = (\Omega^{(1)}, \Omega^{(2)}),$$

then system (21) takes the form

$$\mathbf{A}\omega(x) + \frac{\mathbf{B}}{\pi}\int_l \frac{\omega(t)\,dt}{t-x} = \tilde{\Omega}(x), \quad x \in l. \tag{22}$$

It is obvious that system (22) is a characteristic system of singular integral equations for the real vector ω with its right-hand side from the class H^*.

The condition

$$\det(\mathbf{A} \pm i\mathbf{B}) = \frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}\Delta} > 0$$

implies that system (22) is of the normal type.

Thus the theory of systems of singular integral equations in the case of open arcs [6] can be used for system (22).

According to this theory it is required to define the roots of the equation

$$\det\{\mathbf{G}^{-1}(t+0)\mathbf{G}(t-0) - \lambda\tilde{\mathbf{E}}\} = 0 \tag{23}$$

at end points a_p, b_p, $p = 1, 2, \ldots, n$, where $\mathbf{G} = (\mathbf{A} + i\mathbf{B})^{-1}(\mathbf{A} - i\mathbf{B})$ and $\tilde{\mathbf{E}}$ denotes the unit matrix of order 4.

After long and cumbersome calculations we ascertain that equation (23) has the same form at all points a_p, b_p, $p = 1, 2, \dots, n$,

$$\lambda^4 + (b-a)\lambda^3 + 2(b+a-1)\lambda^2 + (b-a)\lambda + 1 = 0, \tag{24}$$

where

$$a = \frac{4}{\Delta}\Big(1 + \omega_0 a_{11}^{(0)} \varkappa_N^{(1)}\Big)^2 \frac{B_0 C_0 - A_0^2}{m_0 a_{11}^{(0)}} > 0, \tag{25}$$

$$b = \frac{4(B_1 C_1 - A_1^2) a_{11}^{(0)}}{\Delta m_0 m_1^{(0)}} > 0. \tag{26}$$

It can be easily verified that equation (24) reduces to the following equation:

$$\Big(\lambda + \frac{1}{\lambda}\Big)^2 + (b-a)\Big(\lambda + \frac{1}{\lambda}\Big) + 2(b+a-2) = 0. \tag{27}$$

Let us investigate the roots of this equation. Note that the inequality

$$a - b - 4 = -\Big[1 - \omega_0^2 (B_0 C_0 - A_0^2)\Big] b - \Big[1 - \omega_1^2 (B_1 C_1 - A_1^2)\Big] b - $$
$$- \frac{4}{\Delta} \frac{B_0 C_1 + B_1 C_0 - 2 A_0 A_1}{m_0 m_1} < 0$$

yields $a < b + 4$.

Consider all possible cases: $a = b$, $a < b$ and $a > b$. In the first case equation (27) has only complex roots; in the second case either it has only complex roots if $b \le 4$ or it has no positive roots if $b > 4$; and, finally, in the third case it has no positive roots if $a \le 4$ and has only complex roots if $a > 4$.

Thus in all three cases the equation has not only positive roots. Therefore all end points a_p, b_p, $p = 1, 2, \dots, n$, are nonsingular.

A solution of system (22) is to be sought for in the class functions unbounded at end points a_p, b_p, $p = 1, 2, \dots, n$. Whenever equation (27) has simple complex roots, one can easily construct the solution of system (22), having first constructed the matrix of canonical solutions of the corresponding homogeneous Hilbert problem in the class of unbounded functions.

Indeed, introducing a piecewise holomorphic vector

$$\mathbf{W}(z) = \frac{1}{2\pi i} \int_l \frac{\omega(t)\, dt}{t - z}, \quad z = x + iy,$$

system (22) becomes equivalent in a certain sense to the nonhomogeneous Hilbert problem

$$\mathbf{W}^+(z) = \mathbf{G}\mathbf{W}^-(x) + \mathbf{R}(x), \quad x \in l, \tag{28}$$

where $\mathbf{R}(x) = (\mathbf{A} + i\mathbf{B})^{-1}\tilde{\mathbf{\Omega}}(x)$.

Since we are considering the case when equation (27) has only simple complex roots λ_1, λ_2, λ_3, λ_4, it is always possible to construct a nonsingular matrix $\mathbf{S}$ such that the equality $\mathbf{S}^{-1}\mathbf{G}\mathbf{S} = \mathbf{\Lambda}$ be fulfilled, where $\mathbf{\Lambda}$ is a diagonal matrix with the elements λ_1 , λ_2, λ_3, λ_4 on the main diagonal

$$\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

Using the constructed matrix $\mathbf{S}$ we rewrite the Hilbert problem (28) as

$$\Psi^+(z) = \mathbf{\Lambda}\Psi^-(x) + \mathbf{S}^{-1}\mathbf{R}(x), \quad x \in l, \tag{29}$$

where $\Psi(z) = \mathbf{S}^{-1}\mathbf{W}(z)$.

Since $\mathbf{\Lambda}$ is a diagonal matrix, we conclude that in the class of functions unbounded at end points a_p, b_p, $p = 1, 2, \ldots, n$, the matrix of canonical solutions corresponding to the homogeneous Hilbert problem (29) has the form

$$\mathbf{X}(z) = \operatorname{diag}\{X_1(z), X_2(z), X_3(z), X_4(z)\}, \tag{30}$$

where

$$X_\sigma(z) = \prod_{p=1}^{n}(z - a_p)^{-\gamma_\sigma}(z - b_p)^{\gamma_\sigma - 1}, \quad \gamma_\sigma = \frac{1}{2\pi i}\ln\lambda_\sigma, \qquad \sigma = 1, 2, 3, 4. \tag{31}$$

By $X_\sigma(z)$, $\sigma = 1, 2, 3, 4$, we mean the branch defined by the condition

$$\lim_{z\to\infty}\{z^n X_\sigma(z)\} = 1.$$

Since we seek for an unbounded solution at end points a_p, b_p, $p = 1, 2, \ldots, n$, the numbers γ_σ, $\sigma = 1, 2, 3, 4$, have to be chosen so that $0 < \operatorname{Re}\gamma_\sigma < 1$, $\sigma = 1, 2, 3, 4$.

In the particular case when the upper and lower half-planes are filled with materials for which the conditions

$$a_{11}^{(0)} = a_{11}^{(1)}, \quad B_0 = B_1, \quad C_0 = C_1, \quad A_0 = A_1, \quad \omega_0 = \omega_1$$

are satisfied, equation (24) takes the form

$$\lambda^4 + 2\frac{1 + \omega_0^2(B_0C_0 - A_0^2)}{1 - \omega_0^2(B_0C_0 - A_0^2)}\lambda^2 + 1 = 0,$$

i.e.,

$$\left(\lambda^2+\frac{1+\omega_0\sqrt{B_0C_0-A_0^2}}{1-\omega_0\sqrt{B_0C_0-A_0^2}}\right)\left(\lambda^2+\frac{1-\omega_0\sqrt{B_0C_0-A_0^2}}{1+\omega_0\sqrt{B_0C_0-A_0^2}}\right)=0.$$

Hence it is clear that in this particular case equation (24) has purely imaginary roots

$$\lambda_1=i\sqrt{\varkappa_0},\quad \lambda_2=-i\sqrt{\varkappa_0},\quad \lambda_3=\frac{i}{\sqrt{\varkappa_0}},\quad \lambda_4=-\frac{i}{\sqrt{\varkappa_0}},$$

where

$$\varkappa_0=\frac{1+\omega_0\sqrt{B_0C_0-A_0^2}}{1-\omega_0\sqrt{B_0C_0-A_0^2}}$$

and the numbers γ_σ, $\sigma=1,2,3,4$, have the form

$$\gamma_1=\frac{1}{4}-i\beta_0,\quad \gamma_2=\frac{3}{4}-i\beta_0,\quad \gamma_3=\bar{\gamma}_1,\quad \gamma_4=\bar{\gamma}_2,\quad \beta_0=\frac{1}{4\pi}\ln\varkappa_0.$$

By the general theory of systems of singular integral equations [6], from (30) and (31) it follows that the partial indices from the class of unbounded at end points solutions of the homogeneous Hilbert problem satisfy the equations $\varkappa_1=\varkappa_2=\varkappa_3=\varkappa_4=n$ and the total index $\varkappa=4n$.

Therefore the nonhomogeneous Hilbert problem (29) will always be solvable in the class of solutions unbounded at end points, and the solution will depend on $4n$ arbitrary constants

$$\Psi(z)=\frac{\mathbf{X}(z)}{2\pi i}\int_l\frac{[\mathbf{X}^+(t)]^{-1}\mathbf{S}^{-1}\mathbf{R}(t)\,dt}{t-z}+\mathbf{X}(z)\mathbf{P}(z),\qquad (32)$$

where $\mathbf{P}(z)=(P_1(z),P_2(z),P_3(z),P_4(z))$ and $P_\sigma(z)$, $\sigma=1,2,3,4$, is a polynomial of degree $n-1$ with arbitrary real coefficients.

After finding the vector Ψ, the vector $\mathbf{W}$ is defined by the equality $\mathbf{W}=\mathbf{S}\Psi$ and, finally, the solution of system (22), unbounded at end points a_p, b_p, $p=1,2,\dots,n$, is given by the formula $\omega=\mathbf{W}^+-\mathbf{W}^-$.

Thus we have found the solution of system (22) which will depend on $4n$ arbitrary real constants.

It is obvious that the vectors $\mathbf{g}$ and $\mathbf{h}$ and, accordingly, the displacement vectors will depend linearly on the same $4n$ constants.

To define these constants we have the following conditions. In the first place, the found vectors $\mathbf{g}$ and $\mathbf{h}$ should satisfy conditions (9) and (10), which gives $2n+2$ linear equations. On the other hand, from the above arguments it is clear that on the lower slit edges the displacement vector will satisfy the conditions

$$\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}^{-} = \Phi(x) + \eta_p \quad \text{on } l_p,\ p = 1, 2, \ldots, n,$$

where $\eta_1, \eta_2, \ldots, \eta_p$ are arbitrary real constant vectors.

We obtain more $2n - 2$ linear algebraic equations provided that $\eta_1 = \eta_2 = \cdots \eta_n = \eta$, where η is an arbitrary real constant vector.

Thus we have the system of $4n$ algebraic equations for defining $4n$ unknowns and, by virtue of the uniqueness theorem, we readily conclude that this system is always solvable uniquely.

Finally, we observe that the vector

$$\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} - \eta, \quad j = 0, 1,$$

is the solution of the mixed problem.

Thus we have proved

Theorem 2. *If conditions* a), b), c′), d), e′) *are fulfilled, then under the boundary-contact conditions the mixed problem* (3) *and* (5) *has the unique solution which is represented by formula* (8), *where the vector* h *is defined on* L *by equality* (13) *and the vectors* g *and* h *are defined on* l *by the solution of system* (22).

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(Received 14.12.1992)

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ავტორთა საყურადღებოდ

"საქართველოს მეცნიერებათა აკადემიის მაცნე. მათემატიკა" გამოდის 1993 წლის თებერვლიდან ორ თვეში ერთხელ. ჟურნალი აქვეყნებს შრომებს წმინდა და გამოყენებითი მათემატიკის ყველა დარგში. შრომები უნდა შეიცავდნენ ახალ შედეგებს სრული დამტკიცებებით.

გამოსაქვეყნებლად წარმოდგენილი შრომა უნდა იყოს დაბეჭდილი ორი ინტერვალით სტანდარტული (A4) ზომის ქაღალდზე. სასურველია მისი მოცულობა არ აღემატებოდეს 24 გვერდს. პირველი გვერდი უნდა იწყებოდეს შრომის სათაურით, რომელსაც ცალკე სტრიქონზე უნდა მოჰყვებოდეს ავტორის (ავტორების) გვარი (გვარები), ხოლო შემდეგ შრომის ინგლისური და ქართული რეზიუმეები, რომელთა შემდეგაც იწყება ძირითადი ტექსტი. ამ გვერდის პირველ სქოლიოში მიუთითეთ 1991 წლის მათემატიკის საგნობრივი კლასიფიკაციის (Mathematics Subject Classification) ინდექსი. ციტირებული ლიტერატურის სიის შედგენისას გთხოვთ იხელმძღვანელოთ Mathematical Reviews სტანდარტით. ლიტერატურის სიის შემდეგ მიუთითეთ ავტორის (ავტორების) მისამართი. სპეციალური მითითებანი, თუკი ეს საჭიროა, მოიყვანეთ ცალკე ფურცელზე.

რედაქციისათვის უმჯობესია, თუ შრომა წარმოდგენილი იქნება კომპიუტერის დისკზე ჩაწერილი სახით TEX(plain,LATEX, AMS-TEX, AMS-LATEX) პროგრამის გამოყენებით. ამ შემთხვევაში გთხოვთ დაიცვათ შემდეგი სტანდარტი: ნაბეჭდის ზომა თითოეულ გვერდზე (გვერდის ნომრის გარეშე) – 125×200 მმ, შრიფტის ზომა რეზიუმესთვის – 10pt (baselineskip=12pt), ხოლო ძირითადი ტექსტისთვის – 12pt (baselineskip=14pt).

ავტორი მიიღებს გამოქვეყნებული შრომის 25 ამონაბეჭდს.

ხელნაწერი ორ ეგზემპლარად წარმოადგინეთ შემდეგ მისამართზე:
თბილისი 380093, ზ.რუხაძის ქ. 1, ა.რაზმაძის სახ. მათემატიკის ინსტიტუტი,
საქართველოს მეცნიერებათა აკადემიის მაცნე. მათემატიკა.

ტელ.: 36-39-22, 36-45-95

გადაეცა წარმოებას 3.06.93; ხელმოწერილია დასაბეჭდად 28.05.93;
ქაღალდის ზომა $70 \times 108\frac{1}{16}$; ქაღალდი ოფსეტური; ბეჭდვა ოფსეტური;
პირობითი ნაბეჭდი თაბახი 11.2;
სააღრიცხვო-საგამომცემლო თაბახი 7.57;
ტირაჟი 450 ; შეკვეთა No. 689
ფასი სახელშეკრულებო

შინაარსი

CONTENTS

Vol.1, No.3 June 1993

76202