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ON CONJUGACY OF HIGH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

T.CHANTURIA †

ABSTRACT. It is shown that the differential equation

$$u^{(n)} = p(t)u$$

where $n \geq 2$ and $p : [a,b] \to \mathbb{R}$ is a summable function, is not conjugate in the segment $[a,b]$, if for some $l \in \{1,\dots,n-1\}$, $\alpha \in]a,b[$ and $\beta \in]\alpha,b[$ the following inequalities

$$n \geq 2 + \frac{1}{2}(1+(-1)^{n-l}), \quad (-1)^{n-l}p(t) \geq 0 \quad \text{for} \quad t \in [a,b],$$

$$\int_\alpha^\beta (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)!\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)},$$

hold.

რეზიუმე. ნაჩვენებია, რომ დიფერენციალური განტოლება

$$u^{(n)} = p(t)u,$$

სადაც $n \geq 2$, ხოლო $p : [a,b] \to \mathbb{R}$ ჯამებადი ფუნქციაა, არის $[a,b]$ სეგმენტზე ოსცილაციური, თუ რაიმე $l \in \{1,\dots,n-1\}$, $\alpha \in]a,b[$ და $\beta \in]\alpha,b[$ რიცხვებისათვის დაცულია უტოლობები

$$n \geq 2 + \frac{1}{2}(1+(-1)^{n-l}), \quad (-1)^{n-l}p(t) \geq 0, \quad \text{როცა} \quad t \in [a,b],$$

$$\int_\alpha^\beta (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)!\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)}.$$

Consider the differential equation

$$u^{(n)} = p(t)u, \tag{1}$$

where $n \geq 2$, $p \in L_{loc}(I)$ and $I \subset \mathbb{R}$ is an interval.

The following definitions will be used below.

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Equation (1) is said to be conjugate in I if there exists a nontrivial solution of this equation with at least n zeroes (each zero counted accordingly to its multiplicity) in I.

Let $l \in \{1, \dots, n-1\}$. Equation (1) is said to be $(l, n-l)$ conjugate in I if there exists a nontrivial solution u of this equation satisfying

$$u^{(i)}(t_1) = 0 \quad (i = 0, \dots, l-1),$$
$$u^{(i)}(t_2) = 0 \quad (i = 0, \dots, n-l-1),$$

with $t_1, t_2 \in I$ and $t_1 < t_2$.

Suppose first that $-\infty < a < b < +\infty$ and $p \in L([a, b])$.

Lemma. *Let $a < \alpha < \beta < b$. Then the Green's function G of the problem*

$$u^{(n)}(t) = 0 \quad \text{for } t \in [a, b],$$
$$u^{(j)}(a) = 0 \quad (j = 0, \dots, l-1),$$
$$u^{(j)}(b) = 0 \quad (j = 0, \dots, n-l-1),$$

satisfies the inequality

$$(-1)^{n-l} G(t, s) >$$
$$> \frac{(b-\beta)(\alpha-a)(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times$$
$$\times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for } \alpha \le t < s \le \beta. \tag{2}$$

Proof. The function G can be written in the form

$$G(t, s) = \begin{cases} \sum\limits_{i=n-l+1}^{n} (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for} \quad a \le s < t \le b, \\ -\sum\limits_{i=1}^{n-l} (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for} \quad a \le t \le s \le b, \end{cases}$$

where

$$x_i(t) = \frac{(t-a)^{n-i}(b-t)^{i-1}}{(i-1)!(b-a)^{n-i}}.$$

It is easy to verify that for any fixed $s \in]a, b[$ the function

$$\frac{(-1)^{n-l} G(\cdot, s)}{x_{n-l}(\cdot) x_{l+1}(s)}$$

decreases on $]a,b[$ and the function

$$\frac{(-1)^{n-l}G(\cdot,s)}{x_{n-l+1}(\cdot)x_l(s)}$$

increases on $]a,b[$. Thus

$$(-1)^{n-l}G(t,s) \geq (-1)^{n-l}G(s,s)\frac{x_{n-l}(t)}{x_{n-l}(s)} \quad \text{for} \quad t \leq s. \tag{3}$$

Taking into account that

$$(-1)^{n-l}G(s,s) = (-1)^{n-l-1}\sum_{i=1}^{n-l}(-1)^{i-1}x_i(s)x_{n-i+1}(s) =$$
$$= \frac{(s-a)^{n-1}(b-s)^{n-1}}{(b-a)^{n-1}}\sum_{i=1}^{n}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!}$$

and

$$\frac{x_{n-l}(t)}{x_{n-l}(s)} = \frac{(t-a)^l(b-t)^{n-l-1}}{(s-a)^l(b-s)^{n-l-1}},$$

from the inequality (3) we deduce

$$(-1)^{n-l}G(t,s) \geq$$
$$\geq \frac{(s-a)^{n-l-1}(b-s)^l(t-a)^l(b-t)^{n-l-1}}{(b-a)^{n-1}}\sum_{i=1}^{n-l}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} >$$
$$> (b-\beta)(\alpha-a)\frac{(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times$$
$$\times \sum_{i=1}^{n-l}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for} \quad \alpha \leq t < s \leq \beta. \quad \blacksquare$$

Theorem 1. *Let* $l \in \{1,\dots,n-1\}$,

$$n \geq 2 + \frac{1+(-1)^{n-l}}{2} \quad \textit{and} \quad (-1)^{n-l}p(t) \geq 0 \quad \textit{for} \quad t \in [a,b]. \tag{4}$$

If, in addition, there exist $\alpha, \beta \in]a,b[$ *such that* $a < \alpha < \beta < b$ *and*

$$\int_\alpha^\beta (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)!\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)}, \tag{5}$$

then the equation (1) *is* $(l, n-l)$ *conjugate in* $[a,b]$.

Note that in the case when $n = 2$ the analogous results are given in [3,5].

Proof. Put $p(t)=0$ for $t>b$ and consider equation (1) in the interval $[a,+\infty[$. For any $\gamma>a$ let u_γ be the solution of (1) satisfying

$$u_\gamma^{(i)}(a)=0 \qquad (i=0,\dots,l-1),$$
$$u_\gamma^{(i)}(\gamma)=0 \qquad (i=0,\dots,n-l-2),$$
$$\sum_{i=0}^{n-1}|u_\gamma^{(i)}(a)|=1, \quad \max\{u_\gamma(t): a\le t\le\gamma\}>0.$$

Suppose now that in spite of the statement of the theorem equation (1) is not $(l,n-l)$ conjugate in $[a,b]$.

Note that if $\gamma\in]a,b]$, then $u_\gamma(t)>0$ for $t\in]a,\gamma[$ and $(-1)^{n-l-1}\times\times u_\gamma^{(n-l-1)}(\gamma)>0$. Indeed, if it is not so, there exists $t_0\in]a,\gamma[$ such that $u_\gamma(t_0)=0$. Let $\gamma_0=\inf\{\gamma>>a: u_\gamma(t)=0$ for a certain $t\in]a,\gamma[\}$. Then $u_{\gamma_0}(t)>0$ for $t\in]a,\gamma_0[$ and

$$u_{\gamma_0}^{(i)}(a)=0 \qquad (i=0,\dots,l-1),$$
$$u_{\gamma_0}^{(i)}(\gamma_0)=0 \qquad (i=0,\dots,n-l-1),$$

which contradicts our assumption.

Let $\gamma^0=\sup\{\gamma>b: u_\gamma(t)>0 \text{ for } t\in]a,\gamma[\}$. Consider first the case when $\gamma^0=+\infty$. There exists the sequence $\{\gamma_k\}_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty}\gamma_k=+\infty, \qquad \lim_{k\to+\infty}u_{\gamma_k}(t)=u_0(t)$$

where u_0 is the solution of equation (1). Show that

$$u_0(t)>0 \quad \text{for} \quad t>a. \tag{6}$$

It is clear that $u_0(t)\ge0$ for $t>a$. If now $u_0(t_*)=0$ for some $t_*>a$, then for any k large enough the function u'_{γ_k} will have at least one zero in $]a,\gamma_k[$. Taking into account the multiplicities of zeroes of u_{γ_k} in a and γ_k, it is easy to show that $u_{\gamma_k}^{(n-1)}$ has at least two zeroes in $]a,\gamma_k[$. Hence $u_{\gamma_k}^{(n)}$ changes sign in this interval and this is impossible.

Thus inequality (6) is proved. This inequality and the results of [1] imply that there exist $l_0\in\{1,\dots,n\}$ ($l-l_0$ is even) and $t_1>b$ such that

$$\begin{aligned} u_0^{(i)}(t)>0 \quad &\text{for} \quad t\ge t_1 \quad (i=0,\dots,l_0-1),\\ (-1)^{i+l_0}u_0^{(i)}(t)\ge0 \quad &\text{for} \quad t\ge t_1 \quad (i=l_0,\dots,n). \end{aligned} \tag{7}$$

Clearly,

$$\begin{aligned}&(-1)^{i+l_0}u_0^{(i)}(t)\geq 0 \quad \text{for} \quad t\geq a \quad (i=l_0,\dots,n),\\ &(-1)^{i+l_0}u_0^{(i)}(a)>0 \qquad\qquad\quad (i=l_0,\dots,n-1).\end{aligned} \tag{8}$$

Hence $l\in\{1,\dots,l_0\}$.

Suppose that $l<l_0$. Then for any k large enough we have $\gamma_k>t_1$, $u_{\gamma_k}^{(i)}(t_1)>0$
$(i=0,\dots,l_0-1)$. This means that the function $u_{\gamma_k}^{(i)}$ has at least one zero in $]t_1,\gamma_k[$. Taking into account the multiplicity of zero in γ_k, it is easy to see that $u_{\gamma_k}^{(n-1)}$ has at least two zeroes in $]t_1,\gamma_k[$, and $u_{\gamma_k}^{(n)}$ changes sign in this interval. But this is impossible. Thus $l=l_0$.

As $l=l_0$, inequalities (7) and (8) imply

$$\begin{aligned}(-1)^{i+l}u_0^{(i)}(t)\geq 0 \quad &\text{for} \quad t\geq a \quad (i=l,\dots,n),\\ u_0^{(i)}>0 \quad &\text{for} \quad t>a \quad (i=0,\dots,l-1).\end{aligned}$$

Let

$$v(t)=u_0^{(l-1)}(t)-\sum_{j=l}^{n-1}\frac{(-1)^{j-l}}{(j-l+1)!}(t-a)^{j-l+1}u_0^{(j)}(t),$$

then

$$v'(t)=\frac{(-1)^{n-l}}{(n-l)!}(t-a)^{n-l}u_0^{(n)}(t)\geq 0 \quad \text{for} \quad t\geq a.$$

Hence

$$\begin{aligned}u_0^{(l-1)}(t)&\geq\sum_{j=l}^{n-1}\frac{(-1)^{j-l}(t-a)^{j-l+1}u_0^{(j)}(t)}{(j-l+1)!}\geq\\ &\geq\frac{(t-a)(-1)^{n-l}}{(n-l)!}\int_t^{+\infty}(s-a)^{n-l-1}p(s)u_0(s)ds \quad \text{for} \quad t\geq a.\end{aligned} \tag{9}$$

Denote

$$\rho_i(t)=iu_0^{(l-i)}(t)-(t-a)u_0^{(l-i+1)}(t) \quad \text{for} \quad t\geq a \quad (i=0,1,\dots,l).$$

Then

$$\rho_i'(t)=\rho_{i-1}(t) \quad \text{for} \quad t\geq a \quad (i=1,\dots,l).$$

Since $\rho_0(t)=-(t-a)u_0^{(l+1)}(t)\geq 0$ for $t\geq a$ and $\rho_i(a)=0$ $(i=1,\dots,l)$, we have

$$\rho_i(t)\geq 0 \quad \text{for} \quad t\geq a \quad (i=0,1,\dots,l).$$

This implies

$$u_0(t) \geq \frac{(t-a)^{l-1}}{l!} u_0^{(l-1)}(t). \tag{10}$$

From (9) and (10) we obtain

$$1 \geq \frac{(t-a)}{l!(n-l)!} \int_t^\beta (s-a)^{n-2} |p(s)| ds \quad \text{for} \quad t \geq a,$$

which contradicts (5). The case $\gamma^0 = +\infty$ is thus eliminated.

Now consider the case when $\gamma^0 < +\infty$. As we have already noted, $\gamma^0 > b$, $u_{\gamma^0}(t) > 0$ for $t \in]a, \gamma^0[$ and

$$\begin{aligned} u_{\gamma^0}^{(i)}(a) &= 0 \qquad (i = 0, \dots, l-1), \\ u_{\gamma^0}^{(i)}(\gamma^0) &= 0 \qquad (i = 0, \dots, n-l-1). \end{aligned} \tag{11}$$

Hence

$$u_{\gamma^0}(t) = \int_a^{\gamma^0} G(t,s) p(s) u_{\gamma^0}(s) ds$$

where G is the Green's function of the boundary value problem (11) for the equation $u^{(n)} = 0$.

Let $t_0 \in]\alpha, \beta[$ be such that

$$\frac{u_{\gamma^0}(t)}{(t-a)^{l-1}(\gamma^0 - t)^{n-l-1}} \geq \frac{u_{\gamma^0}(t_0)}{(t_0 - a)^{l-1}(\gamma^0 - t_0)^{n-l-1}} \quad \text{for } t \in [\alpha, \beta]. \tag{12}$$

Then from the Lemma and the inequality (12) it follows that

$$\begin{aligned} u_{\gamma^0}(t_0) &\geq (\gamma^0 - \beta)(\alpha - a) \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \times \\ &\times \int_\alpha^\beta \frac{(s-a)^{n-2}(\gamma^0 - s)^{n-2}}{(\gamma^0 - a)^{n-1}} |p(s)| ds\, u_{\gamma^0}(t_0) > \\ &> \frac{(b-\beta)(\alpha - a)}{(b-a)^{n-1}} \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \times \\ &\times \int_\alpha^\beta (s-a)^{n-2}(b-s)^{n-2} |p(s)| ds\, u_{\gamma^0}(t_0). \end{aligned} \tag{13}$$

Since $\sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \geq \frac{1}{l!(n-l)!}$ the inequality (13) contradicts (5). ■

Denote

$$\mu_n^- = \min\{l!(n-l)! : l \in \{1,\dots,n-1\},\ n-l \ \ odd\},$$
$$\mu_n^+ = \min\{l!(n-l)! : l \in \{1,\dots,n-1\},\ n-l \ \ even\}.$$

It is clear that

$$\mu_n^- = \begin{cases} (\frac{n}{2}-1)!(\frac{n}{2}+1)! & \text{for } n \equiv 0 \pmod 4, \\ \left[(\frac{n}{2})!\right]^2 & \text{for } n \equiv 2 \pmod 4, \\ (\frac{n-1}{2})!(\frac{n+1}{2})! & \text{for } n \equiv 1 \pmod 2, \end{cases}$$

$$\mu_n^+ = \begin{cases} \left[(\frac{n}{2})!\right]^2 & \text{for } n \equiv 0 \pmod 4, \\ (\frac{n}{2}-1)!(\frac{n}{2}+1)! & \text{for } n \equiv 2 \pmod 4, \\ (\frac{n-1}{2})!(\frac{n+1}{2})! & \text{for } n \equiv 1 \pmod 2. \end{cases}$$

Corollary 1. *Let either* $n \ge 2$, $\mu_n = \mu_n^-$ *and* $p(t) \le 0$ *for* $t \in [a,b]$ *or* $n \ge 3$, $\mu_n = \mu_n^+$ *and* $p(t) \ge 0$ *for* $t \in [a,b]$. *Let, moreover,* α, $\beta \in]a,b[$ *exist such that* $a < \alpha < \beta < b$ *and*

$$\int_\alpha^\beta |p(t)|dt \ge \mu_n \Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}.$$

Then equation (1) *is conjugate in* $[a,b]$.

Note that

$$\max\{l!(n-l)! : l \in \{1,\dots,n-1\}\} = (n-1)!.$$

Thus from Theorem 1 easily follows

Corollary 2. *Let* $l \in \{1,\dots,n-1\}$, *the conditions* (4) *be fulfilled and let* α,
$\beta \in]a,b[$ *exist such that* $a < \alpha < \beta < b$ *and*

$$\int_\alpha^\beta |p(t)|dt \ge (n-1)!\Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}. \tag{14}$$

Then equation (1) *is* $(l, n-l)$ *conjugate in* $[a,b]$.

Note that in the inequality (14) the factor $(n-1)!$ cannot be replaced by $(n-1)!-\varepsilon$ with $\varepsilon \in]0,1[$. This is shown by the following

Example. Let $\varepsilon \in]0,1[$ be given beforehand and we choose $\alpha \in]a,b[$ and $\beta \in]\alpha,b[$ such that

$$(n-1)!\left(\frac{\alpha-a}{\beta-a}\right)^{n-1}\left(\frac{b-\beta}{b-a}\right)^{n-1} > (n-1)!-\varepsilon. \tag{15}$$

Put

$$v(t)=\begin{cases} t-a & \text{for } t\in[a,\alpha],\\ \dfrac{\alpha+\beta}{2}-a-\dfrac{1}{2(\beta-\alpha)}(t-\beta)^2 & \text{for } t\in]\alpha,\beta[,\\ \dfrac{\alpha+\beta}{2}-a & \text{for } t\in[\beta,b]. \end{cases},$$

$$u_0(t)=\frac{1}{(n-3)!}\int_a^t (t-s)^{n-3}v(s)ds \quad \text{for } t\in[a,b],\ n\geq 3,$$

$$u_0(t)=v(t) \quad \text{for } t\in[a,b],\ n=2,$$

and

$$p(t)=\frac{v''(t)}{u_0(t)} \quad \text{for } a<t<b.$$

Then the function u_0 is non-decreasing for $t\in[a,\beta]$, and the following inequality

$$u_0(t)\leq u_0(\beta)\leq \frac{1}{(n-3)!}\int_a^\beta (\beta-s)^{n-3}(s-a)ds=\frac{(\beta-a)^{n-1}}{(n-1)!}$$

is valid. Taking into account the inequality (15), we obtain

$$\int_\alpha^\beta p(t)dt=\frac{1}{\beta-\alpha}\int_\alpha^\beta \frac{dt}{u_0(t)} > \frac{(n-1)!}{(\beta-a)^{n-1}} >$$
$$> ((n-1)!-\varepsilon)\left(\frac{b-a}{(b-\beta)(\alpha-a)}\right)^{n-1}.$$

On the other hand, in the case considered, equation (1) is not conjugate in $[a,b]$ because it has the solution u_0 satisfying the following conditions

$$u_0^{(i)}(a)=0\ (i=0,\dots,n-2),\ u_0^{(n-1)}(a)=1,\ u_0(t)>0 \ \text{for } a<t\leq b.$$

This example shows that in Corollary 2 inequality (14) cannot be replaced by the inequality

$$\int_\alpha^\beta |p(t)|dt \geq ((n-1)!-\varepsilon)\left(\frac{b-a}{(b-\beta)(\alpha-a)}\right)^{n-1}$$

no matter how small $\varepsilon > 0$ is.

Now consider equation (1) on the whole axis $\mathbb{R}$ with $p \in L_{loc}(\mathbb{R})$. From Corollary 2 easily follows

Corollary 3. *Let $l \in \{1, \dots, n-1\}$, p is not zero on the set of the positive measure and*

$$n \geq 2 + \frac{1 + (-1)^{n-l}}{2}, \quad (-1)^{n-l} p(t) \geq 0 \quad \textit{for} \quad t \in \mathbb{R}.$$

Then equation (1) *is* $(l, n-l)$ *conjugate in* $\mathbb{R}$.

References

1. I.T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

2. V.A. Kondratyev, On oscillations of solutions of equation $y^{(n)} = p(x)y$. (Russian) *Trudy Moskov. Mat. Obshch.* **10**(1961), 419-436.

3. N.L. Korshikova, On zeroes of solutions of linear equation of high orders. (Russian) *Differential Equations and their Applications (Russian)*, 143-148, *Moscow University Press, Moscow*, 1984.

4. A.Yu. Levin, Non-oscillation of solutions of equation $x^{(n)} + p_1(t)x^{(n-1)} + \dots$ $+ p_n(t)x = 0$. (Russian) *Uspekhi Mat. Nauk.* **24**(1969), No. 2, 43-96.

5. A.G. Lomtatidze, On oscillatory properties of solutions of linear differential equations of second order. (Russian) *Reports of the seminar of I.N. Vekua Institute of Applied Mathematics*, **19**(1989), 39-54.

6. T.A. Chanturia, Sturm type theorems of comparison for differential equations of high orders. (Russian) *Bull. Acad. Sci. Georgian SSR*, **99**(1980), No. 2, 289-291.

7. ——, On oscillations of solutions of linear differential equations of high orders. (Russian) *Reports of the seminar of I.N. Vekua Institute of Applied Mathematics*, **16**(1982), 3-72.

8. F. Hartman, Ordinary differential equations. (Russian) *"Mir", Moscow*, 1970; *English original, Wiley, New York*, 1964.

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CRITERIA OF GENERAL WEAK TYPE INEQUALITIES FOR INTEGRAL TRANSFORMS WITH POSITIVE KERNELS

I.GENEBASHVILI, A.GOGATISHVILI, V.KOKILASHVILI

ABSTRACT. The necessary and sufficient conditions are derived in order that an inequality of the form

$$\varphi(\lambda)\theta(\beta\{(x,t)\in X\times[0,\infty):\mathcal{K}(fd\nu)(x,t)>\lambda\})\leq$$
$$\leq c\int_X \psi\left(\frac{f(x)}{\eta(\lambda)}\right)\sigma(x)d\nu(x)$$

be fulfilled for some positive c independent of λ and a ν-measurable nonnegative function $f:X\to\mathbf{R}^1$, where

$$\mathcal{K}(fd\nu)(x,t)=\int_X f(y)k(x,y,t)d\nu(y),\ t\geq 0,$$

$k:X\times X\times[0,\infty)\to\mathbf{R}^1$ is a nonnegative measurable kernel, (X,d,μ) is a homogeneous type space, $\varphi\eta$ and ψ are quasiconvex functions, $\psi\in\Delta_2$, and $t^{-\alpha}\theta(t)$ is a decreasing function for some α, $0<\alpha<1$.

A similar problem was solved in Lorentz spaces with weights.

რეზიუმე. ერთგვაროვანი ტიპის სივრცეებზე განსაზღვრული ინტეგრალური გარდაქმნებისათვის დადებითი გულებით დადგენილია ზოგადი სახის სუსტი ტიპის წონიანი უტოლობების კრიტერიუმები ორლიჩის და ლორენცის სივრცეებში.

1. INTRODUCTION

This paper presents the characterization of those weight functions and kernels for which we have general weight weak type inequalities for integral transforms of the form

$$\mathcal{K}(fd\nu)(x,t)=\int_X f(y)k(x,y,t)d\nu(y), \tag{1.1}$$

where X is a homogeneous type space, and $k:X\times X\times[0,\infty)\to\mathbf{R}^1$ a nonnegative measurable kernel.

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The homogeneous type space (X, d, μ) is a space with measure μ such that the class of compactly supported continuous functions is dense in the space $L^1(X, \mu)$. Besides, it is also assumed that there is a nonnegative real-valued function $d : X \times X \to \mathbf{R}^1$ satisfying the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$;

(ii) $d(x, y) > 0$ for all $x \neq y$ in X;

(iii) There is a constant a_0 such that $d(x, y) \leq a_0 d(y, x)$ for all x, y in X ;

(iv) There is a constant a_1 such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for all x, y, z in X ;

(v) For each neighbourhood V of x in X there is an $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in V ;

(vi) The balls $B(x, r)$ are measurable for all x and $r > 0$;

(vii) There is a constant b such that

$$\mu B(x, 2r) \leq b \mu B(x, r)$$

for all $x \in X$ and $r > 0$ (see [1], p. 2).

In the sequel $\hat{B}(x, r)$ will denote the set $B(x, r) \times [0, 2r)$ for $r > 0$ and the one-point set $\{x\}$ for $r = 0$. The set $B(x, 0)$ will be assumed to be empty. β will be a measure defined on the product of σ-algebras generated by balls in X and by intervals from $[0, \infty)$.

Let φ, ψ, and η are nonnegative nondecreasing functions on $[0, \infty)$. For our further discussion we will also need the following basic definitions of quasiconvex functions. We call ω a Young function if it is a nonnegative increasing convex function on $[0, \infty)$ with $\omega(0) = 0, \omega(\infty) = \infty$ and not identically zero or ∞ on $(0, \infty)$; it may have a jump up to ∞ at some point $t > 0$, but in that case it should be left continuous at t (see [2]).

The function ψ is called quasiconvex if there exist a Young function ω and a constant $c > 1$ such that

$$\omega(t) \leq \psi(t) \leq \omega(ct), \quad t \geq 0. \tag{1.2}$$

Clearly, $\psi(0) = 0$, and for $s \leq t$ we have $\psi(s) \leq \psi(ct)$. To the quasiconvex function ψ we can put into correspondence its complementary function $\tilde{\psi}$ defined by

$$\tilde{\psi}(t) = \sup_{s \geq 0}(st - \psi(s)).$$

The subadditivity of supremum easily implies that $\tilde{\psi}$ is always a Young function and $(\tilde{\psi})\tilde{} \leq \psi$. The equality holds if ψ is itself a Young function. If $\psi_1 \le \psi_2$, then $\tilde{\psi}_2 \le \tilde{\psi}_1$, and if

$$\psi_1(t) = a\psi(bt),$$

then

$$\tilde{\psi}_1(t) = a\tilde{\psi}\left(\frac{t}{ab}\right).$$

Hence and from (1.2) we have

$$\tilde{\omega}\left(\frac{t}{c}\right) \le \tilde{\psi}(t) \le \tilde{\omega}(t). \tag{1.3}$$

Now from the definition of $\tilde{\psi}$ we obtain Young's inequality

$$st \le \psi(s) + \tilde{\psi}(t), \quad s, t \ge 0. \tag{1.4}$$

It should be noted that unlike ψ the function $\tilde{\psi}$ may jump to ∞ at some point $t > 0$. For example, if $\psi(t) = t$, then $\tilde{\psi}(t) = \infty \cdot \chi_{(1,\infty)}(t)$. Throughout the paper we take $0 \cdot \infty$ to be zero.

We use the convention that c denotes the absolute constant which may change line to line.

The function ψ satisfies the (global) Δ_2 condition ($\psi \in \Delta_2$) if there exists $c > 0$ such that

$$\psi(2t) \le c\psi(t), \ t > 0.$$

Some properties of quasiconvex functions and also of functions satisfying the Δ_2 condition will be presented in Section 2.

Now we are ready to formulate the main results of this paper.

Ih the sequel θ will always be a positive nondecreasing function.

Theorem 1.1. *Let $k : X \times X \times [0, \infty) \to \mathbf{R}^1$ be any measurable nonnegative kernel, $\psi \in \Delta_2$, and the function $t^{-\alpha}\theta(t)$ decrease for some $\alpha \in (0, 1)$. Let, further, ν be a finite measure on X , $\sigma : X \to \mathbf{R}^1$ be an alsmost everywhere positive function which is locally summable with respect to measure ν.*

Assume that there exist positive constants ε and c_1 such that

$$\int\limits_{X \setminus B(a,r)} \tilde{\psi}\left(\varepsilon \frac{\varphi(s)\eta(s)}{s} \frac{\theta(\beta \hat{B}(a, N_0(2r + t)))}{\sigma(y)} k(a, y, t)\right) \sigma(y) d\nu(y) \le$$

$$\le c_1 \varphi(s) \theta(\beta \hat{B}(a, N_0(2r + t))) \tag{1.5}$$

for any $s > 0$, $r \ge 0$, $a \in X$ and $t \ge 0$, where $N_0 = a_1(1 + 2a_0)$.

Then there exists a positive constant c_2 such that for any $\lambda > 0$ and any nonnegative ν-measurable function $f : X \to \mathbf{R}^1$ the inequality

$$\varphi(\lambda)\theta(\beta\{(x,t) \in X \times [0,\infty) : \mathcal{K}(fd\nu)(x,t) > \lambda\}) \leq$$
$$\leq c_2 \int_X \psi\left(\frac{f(x)}{\eta(\lambda)}\right) \sigma(x)d\nu(x) \tag{1.6}$$

Assume now that the nonnegative measurable kernel k satisfies the following additional condition: there exist numbers $N \geq N_0, N_0 = a_1(1 + 2a_0)$ and c' such that

$$k(a,y,t) \leq c'k(x,y,\tau) \tag{1.7}$$

when $y \in X \setminus B(a,r)$, $(x,\tau) \in \hat{B}(a, N(r+t))$ for any $a \in X$, $r \geq 0$, $t \geq 0$.

Theorem 1.2. *Let $\varphi\eta$ and ψ be quasiconvex functions, $\psi \in \Delta_2$, the function $t^{-\alpha}\theta(t)$ decrease for some $\alpha \in (0,1)$ and k satisfy the condition* (1.7).

Then the inequality (1.6) *is equivalent to anyone of the following conditions:*

(i) *there exist positive constants ε and c_3 such that*

$$\int\limits_{X\setminus B(a,r)} \tilde{\psi}\left(\varepsilon \frac{\varphi(s)\eta(s)}{s} \frac{\theta(\beta\hat{B}(a,r+t))}{\sigma(y)} k(a,y,t)\right) \sigma(y)d\nu(y) \leq$$
$$\leq c_3\varphi(s)\theta(\beta\hat{B}(a,r+t)) \tag{1.8}$$

for arbitrary $s > 0$, $r \geq 0$, $a \in X$ and $t \geq 0$;

(ii) *there exist positive constants ε and c_4 such that*

$$\int_X \tilde{\psi}\left(\varepsilon \frac{\varphi(s)\eta(s)}{s} \frac{\theta(\beta\hat{B}(a,t))}{\sigma(y)} k(a,y,t)\right) \sigma(y)d\nu(y) \leq$$
$$\leq c_4\varphi(s)\theta(\beta\hat{B}(a,t)) \tag{1.9}$$

for any $s > 0$, $a \in X$ and $t \geq 0$;

(iii) *there exists a positive constant c_5 such that for any $a \in X$, $r \geq 0$, $t \geq 0$ and for any nonnegative ν-measurable functions $F : X \to \mathbf{R}^1$, $suppF \subset X \setminus B(a,r)$ we have*

$$\varphi\left(\mathcal{K}(Fd\nu)(a,t)\right)\theta(\beta\hat{B}(a,r+t)) \leq$$
$$\leq c_5 \int_X \psi\left(\frac{F(x)}{\eta(\mathcal{K}(Fd\nu)(a,t))}\right) \sigma(y)d\nu(y). \tag{1.10}$$

Let $k : X \times X \to \mathbf{R}^1$ be a nonnegative measurable function satisfying the following condition: there exist number $N \geq N_0$ and $c' > 0$ such that

$$k(a,y) \leq c' k(x,y)$$

for any $a \in X$, $y \in X \setminus B(a,r)$, and $x \in B(a,Nr)$.

For any positive, locally summable with respect to measure ν, function $\varrho: X \to \mathbf{R}^1$ it will be assumed below that

$$\varrho E = \int_E \varrho(x) d\nu(x)$$

for any ν-measurable set $E \subset X$.

We have

Theorem 1.3. *Let the functions φ, η , and ψ satisfy the conditions of Theorem 1.2. Then the following statements are equivalent:*

(i) *there exists a positive constant c_6 such that for any $\lambda > 0$ and for any measurable nonnegative function f*

$$\varphi(\lambda)\theta\left(\varrho\{x \in X : \int_X k(x,y)f(y)d\nu(y) > \lambda\}\right) \leq$$
$$\leq c_6 \int_X \psi\left(\frac{f(y)}{\eta(\lambda)}\right)\sigma(y)d\nu(y)$$

for any $\lambda > 0$ and for any measurable nonnegative function $f : X \to \mathbf{R}^1$;

(ii) *there exist positive numbers ε and c_7 such that*

$$\int\limits_{X \setminus B(a,r)} \tilde{\psi}\left(\varepsilon \frac{\varphi(s)\eta(s)}{s} \frac{\theta(\varrho B(a,r))}{\sigma(y)} k(a,y)\right)\sigma(y)d\nu(y) \leq$$
$$\leq c_7 \varphi(s)\theta(\varrho B(a,r))$$

for any $s > 0$, $a \in X$, and $r \geq 0$.

The above-formulated results contain the solutions of problems of description of a set of weights ensuring in Orlicz spaces the validity of both weak and extra-weak weighted inequalities for transform (1.1) which are natural analogies of inequalities of the weak type (p,q). Indeed, for $\varphi = \psi$, $\eta \equiv 1$ (1.6) becomes a weak type weighted inequality, while for $\varphi \equiv 1$, $\eta(\lambda) = \lambda$ we obtain an extra-weak type weighted inequality. It is understood that an inequality of the weak type (φ, φ) is essentially stronger than an inequality of the extra-weak type (φ, φ).

The solutions of similar problems in Lorentz spaces are derived in Section 3. Section 4 contains a discussion of the interesting corollaries of Theorems 1.2 and 1.3 for integral operators such as potentials and their generalizations, Poisson integrals and their generalizations, the Hardy operators and others. Here we would like to give a very brief survey only of the results preceding this paper.

The solution of a weak type two weight problem for Riesz potentials in Lebesgue spaces was obtained in [3], [4], the criterion found in [4] being more easily verifiable. The latter result was extended to the intergrals on homogeneous type spaces in [5]. A similar problem was treated in [6] (see also [7], Theorems 6.1.1 and 6.1.2) in Lorentz spaces over $\mathbf{R}^n$ for integral transforms

$$Kf(x) = \int_{\mathbf{R}^n} k(x,y)f(y)dy.$$

Subsequently in [8] generalizations were obtained for transforms of type (1.1) when $X = \mathbf{R}^n, d\nu(y) = dy$. More particular cases of generalized potentials and Poisson integrals were considered in [8] and [9], respectively. The latter deals with Lorentz spaces and the former with Lebesgue spaces. In Orlicz classes the problem of description of a set of weight ensuring the validity of weak type weighted inequalities was previously studied mainly for maximal functions [10], [11], [12] and the Hardy operator [13], [14].

2. Proof of the Main Theorems

In this section use will be made of some properties of quasiconvex functions satisfying the Δ_2 condition, also of the covering lemma in homogeneous type spaces.

Lemma 2.1 ([11] , p. 4). *The following statements are equaivalent:*
(i) φ *is a quasiconvex function;*
(ii) *there exists* $c > 1$ *such that*

$$\frac{\varphi(s)}{s} \leq \frac{\varphi(ct)}{t} \tag{2.1}$$

for $s < t$.

Hence for quasiconvex functions φ we immediately obtain the estimates

$$\delta\varphi(t) \leq \varphi(c\delta t), \quad t \geq 0, \quad \delta > 1, \tag{2.2}$$

$$\varphi(\delta t) \leq \delta\varphi(ct), \quad t \geq 0, \quad \delta < 1. \tag{2.3}$$

For convex functions the inequalities to be given above are valid when $c = 1$.

Lemma 2.2. *If ω is a Young function, then*

$$\omega\left(\frac{\tilde{\omega}(t)}{t}\right) \leq \tilde{\omega}(t), \quad t \geq 0. \tag{2.4}$$

Proof. By virtue of the equality $(\tilde{\omega})\tilde{} = \omega$ we have

$$\omega\left(\frac{\tilde{\omega}(t)}{t}\right) = \sup_{s\geq 0} s\left(\frac{\tilde{\omega}(t)}{t} - \frac{\tilde{\omega}(s)}{s}\right) \leq$$

$$\leq \sup_{0\leq s<t} s\frac{\tilde{\omega}(t)}{t} \leq \tilde{\omega}(t).$$

since the expression in the brackets is negative when $t < s$. ∎

Lemma 2.3 ([11], p.17). *Let ψ satisfy the Δ_2-condition. Then there exist $p > 1$ and $c > 1$ such that*

$$s^{-p}\psi(s) \leq ct^{-p}\psi(t) \tag{2.5}$$

for $0 \leq t \leq s$.

Lemma 2.4. *Let E be a bounded set in X and for each point $x \in E$ a ball $B_x = B(x, r_x)$ be given such that*

$$\sup_{x\in E} \operatorname{rad} B_x < \infty.$$

Then from the family $\{B_x\}_{x\in E}$ we can choose a (finite or infinite) sequence of pairwise disjoint balls $(B_j)_j$ for which $E \subset \cup_{j\geq 1} N_0 B_j$, $N_0 = a_1(1 + 2a_0)$, and for each $B_x \in \{B_x\}_{x\in E}$ there exists a ball B_{j_0} such that $x \in N_0 B_{j_0}$ and $\operatorname{rad} B_x \leq 2 \operatorname{rad} B_{j_0}$.

Proof. Set

$$R_1 = \sup_{x\in E} \operatorname{rad} B_x.$$

There obviously exists a ball $B_1 = B_{x_1}$ from the family $\{B_x\}_{x\in E}$ provided that $\operatorname{rad} B_1 > 2^{-1} R_1$. If $x \notin N_0 B_1 \cap E$, then $B_x \cap B_1 = \varnothing$. Indeed, making the opposite assumption that there exists a point $y \in B_x \cap B_1$, we will have

$$d(x_1, x) \leq a_1(d(x_1, y) + d(y, x)) < a_1(\operatorname{rad} B_{x_1} + a_0 d(x, y)) <$$
$$< a_1(\operatorname{rad} B_1 + a_0 \operatorname{rad} B_x) < a_1(1 + 2a_0) \operatorname{rad} B_1 = N_0 \operatorname{rad} B_1,$$

which leads to the contradiction.

Obviously, $\operatorname{rad} B_x \leq 2 \operatorname{rad} B_1$ for an arbitrary point $x \in N_0 B_1 \cap E$. Assuming now that

$$R_2 = \sup_{x\in E\setminus N_0 B_1} \operatorname{rad} B_x,$$

we can find a ball $B_2 = B_{x_2}$ from the family $\{B_x\}_{x\in E\setminus N_0B_1}$ provided that $B_2 \cap B_1 = \varnothing$, $\operatorname{rad} B_2 > 2^{-1}R_2$ and $\operatorname{rad} B_x \leq 2 \operatorname{rad} B_2$ for each point $x \in (N_0B_2 \cap E)\setminus N_0B_1$. Proceeding in this way, we arrive at the sequence $\{B_j\}_{j\geq 1}$ of nonintersecting balls. If this sequence is finite, then it will be the one we wanted to obtain.

Let the sequence be infinite. If we show that for each point $x \in E$ there exists a ball B_j for which $x \in N_0B_j$, then setting j_0 to be equal to the minimal value among similar j's, we obtain the desired covering.

Assume the opposite. Let in E there exists a point $x_0 \in E$ such that $x_0 \notin N_0B_j$ for every j. Then we will have

$$B_{x_0} \in \{B_x\}_{x\in E\setminus \cup_{j=1}^{n} N_0B_j},$$

for any natural number n and hence

$$\operatorname{rad} B_{x_0} \leq R_n < 2 \operatorname{rad} B_n$$

for each n.

On the other hand, it is obvious that $\cup_{x\in E} B_x$ is a bounded set, i.e. it is contained within some ball B_0. It therefore turns out that $(B_j)_{j\geq 1}$ is an infinite sequence of nonintersecting balls contained in B_0. Therefore $\operatorname{rad} B_n \to 0$ (see, for example, [17], p. 68). The latter result leads to the contradiction $\operatorname{rad} B_{x_0} = 0$.

The lemma is proved. ∎

Proof of Theorem 1.1. Fix the function $f \geq 0$ and $\lambda > 0$. Without loss of generality it can be assumed that

$$\frac{1}{c_1\theta(\beta(X \times [0,\infty)))\varphi(\lambda)} \int_X \psi\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu < 1. \quad (2.6)$$

Otherwise we would have

$$\varphi(\lambda)\theta(\beta\{(x,t) : \mathcal{K}(fd\nu)(x,t) > \lambda\}) \leq \varphi(\lambda)\theta(\beta(X \times [0,\infty))) \leq$$
$$\leq \frac{1}{c_1}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu$$

and, since $\psi \in \Delta_2$, the proof would be completed.

Assume $(x,t) \in E_\lambda$, where $E_\lambda = \{(x',t') \in X \times [0,\infty) : \mathcal{K}(fd\nu)(x',t') > \lambda\}$. By virtue of (2.6) for (x,t) there exists a finite $r \geq 0$ such that

$$\frac{1}{c_1\theta(\beta\widehat{B}(x, N_0(2r+t)))\varphi(\lambda)} \int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu < 1.$$

If the greatest lower bound of r is positive, then there exists a positive number $r_0 = r_0(x,t)$ such that the inequalities

$$\frac{1}{c_1\theta(\beta\widehat{B}(x,N_0(r_0+t)))\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu \geq 1, \quad (2.7)$$

$$\frac{1}{c_1\theta(\beta\widehat{B}(x,N_0(2r_0+t)))\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu < 1 \quad (2.8)$$

are simultaneously fulfilled.

For such r_0 we would have by virtue of inequality (1.4) and condition (1.5)

$$\int\limits_{X\setminus B(x,r_0)} f(y)k(x,y,t)d\nu = \frac{\lambda}{4c_1\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_0+t)))}\times$$

$$\times \int\limits_{X\setminus B(x,r_0)} \frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(x,N_0(2r_0+t)))}{\sigma(y)}k(x,y,t)\sigma(y)d\nu \leq$$

$$\leq \frac{\lambda}{4c_1\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_0+t)))}\int\limits_{X\setminus B(x,r_0)} \psi\Big(\frac{4c_1 f(x)}{\varepsilon\eta(\lambda)}\Big)\sigma(x)d\nu +$$

$$+\frac{\lambda}{4c_1\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_0+t)))}\times$$

$$\times \int\limits_{X\setminus B(x,r_0)} \tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(x,N_0(2r_0+t)))}{\sigma(y)}k(x,y,t)\Big)\sigma(y)d\nu \leq$$

$$\leq \frac{\lambda}{4}+\frac{\lambda}{4}=\frac{\lambda}{2}.$$

But since $(x,t)\in E_\lambda$, the latter estimate implies

$$\int\limits_{B(x,r_0)} f(y)k(x,y,t)d\nu > \frac{\lambda}{2}. \quad (2.9)$$

When the measure ν is concentrated at the point x, the above-mentioned greatest lower bound may turn out to be equal to zero. Then instead of (2.9) we have

$$k(x,x,t)f(x)\nu\{x\} > \frac{\lambda}{2}. \quad (2.10)$$

Therefore due to (1.4) and (1.5)

$$\varphi(\lambda)\theta(\beta\hat{B}(x,N_0t))) \leq \frac{1}{2c_1}\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)\nu\{x\} +$$
$$+\frac{1}{2c_1}\tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\hat{B}(x,N_0t)))}{\sigma(x)}k(x,x,t)\Big)\sigma(x)\nu\{x\} \leq$$
$$\leq \frac{1}{2c_1}\int\limits_{B(x,t)}\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(y)d\nu(y) + \frac{1}{2}\varphi(\lambda)\theta(\beta\hat{B}(x,N_0t)).$$

Hence

$$\varphi(\lambda)\theta(\beta\hat{B}(x,N_0t)) \leq \frac{1}{4c_1}\int\limits_{B(x,t)}\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu(y). \quad (2.11)$$

Let us now consider the case when ν is not concentrated at the point x. Let n be the greatest nonnegative integer for which

$$b \stackrel{\text{def}}{=} \lim_{r\to 0}\beta\hat{B}(r+t)) < 2^{-n}\beta\hat{B}(x,N_0(r_0+t)).$$

n may be equal to ∞ if $b=0$. For each k, $0<k\leq n$ we set

$$r_k = \sup\{r : \beta\hat{B}(x,N_0(r+t)) \leq 2^{-k}\beta\hat{B}(x,N_0(r_0+t))\}.$$

Then $(r_k)_k$ is a decreasing (finite or infinite) sequence and

$$\beta\hat{B}(x,N_0(r_k+t)) \leq 2^{-k}\beta\hat{B}(x,N_0(r_0+t)) \leq$$
$$\leq \beta\hat{B}(x,N_0(2r_k+t)). \quad (2.12)$$

Let $B_k = B(x,r_k)$, $0\leq k\leq n$ and $B_{n+1}=\{x\}$. Since by the condition of the theorem $u^{-\alpha}\theta(u)$ decreases, we have

$$a_k^p = \frac{\beta\hat{B}(x,N_0(r_k+t))}{\theta(\beta\hat{B}(x,N_0(r_k+t)))}\cdot\frac{\theta(\beta\hat{B}(x,N_0(r_0+t)))}{\beta\hat{B}(x,N_0(r_0+t))} =$$
$$= \frac{\beta\hat{B}(x,N_0(r_k+t))}{\theta(\beta\hat{B}(x,N_0(r_k+t)))}\frac{\theta(\beta\hat{B}(x,N_0(r_0+t)))}{(\beta\hat{B}(x,N_0(r_0+t)))^{1-\alpha}(\beta\hat{B}(x,N_0(r_0+t)))^{\alpha}} \leq$$
$$\leq c\Big(\frac{\beta\hat{B}(x,N_0(r_k+t))}{\beta\hat{B}(x,N_0(r_0+t))}\Big)^{1-\alpha} \leq c\Big(\frac{1}{2^k}\Big)^{1-\alpha}.$$

Because of this

$$\sum_{k=0}^{n} a_k \leq c^{\frac{1}{p}}\sum_{k=0}^{\infty}\Big(\frac{1}{2^k}\Big)^{\frac{1-\alpha}{p}} = a < \infty.$$

(2.9) can now be rewritten as

$$\frac{\lambda}{2}\sum_{k=0}^{n}\frac{a_k}{a} \le \sum_{k=0}^{n}\int\limits_{B_k\setminus B_{k+1}} k(x,y,t)f(y)d\nu$$

whence it follows there exists k_0, $0 \le k_0 \le n$, such that

$$\frac{\lambda}{2}\frac{a_{k_0}}{a} < \int\limits_{B_{k_0}\setminus B_{k_0+1}} k(x,y,t)f(y)d\nu.$$

Therefore

$$\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_{k_0+1}+t))) \le$$

$$\le \frac{1}{2c_1}\int\limits_{B_{k_0}\setminus B_{k_0+1}} \frac{4ac_1}{\varepsilon a_{k_0}}\frac{f(y)}{\eta(\lambda)}\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(x,N_0(2r_{k_0+1}+t)))}{\sigma(y)}\times$$

$$\times k(x,y,t)\sigma(y)d\nu(y) \le \frac{1}{2c_1}\int\limits_{B_{k_0}} \psi\Big(\frac{4ac_1}{\varepsilon a_{k_0}}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu +$$

$$+\frac{1}{2c_1}\int\limits_{B_{k_0}\setminus B_{k_0+1}} \widetilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(x,N_0(2r_{k_0+1}+t)))}{\sigma(y)}\times$$

$$\times k(x,y,t)\Big)\sigma(y)d\nu \le \frac{1}{2c_1}\int\limits_{B_{k_0}} \psi\Big(\frac{4ac_1}{\varepsilon a_{k_0}}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu +$$

$$+\frac{1}{2}\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_{k_0+1}+t)))$$

and as a result we have

$$\varphi(\lambda)\theta(\beta\widehat{B}(x,N_0(2r_{k_0+1}+t))) \le ca_{k_0}^{-p}\int\limits_{B(x,r_{k_0})} \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu.$$

Next, taking into account (2.12), we obtain the estimate

$$\varphi(\lambda)\theta(\beta\widehat{B}(x,N(r_{k_0}+t)) \le ca_{k_0}^{-p}\int\limits_{B(x,r_{k_0}+t)} \psi\Big(\frac{4c_1 f(y)}{\varepsilon\eta(\lambda)}\Big)\sigma(y)d\nu. \tag{2.13}$$

which by the definition of the number a_{k_0} takes the form

$$\varphi(\lambda)\beta\widehat{B}(x,N_0(r_{k_0}+t)) \le$$

$$\le c\frac{\beta\widehat{B}(x,N_0(r_0+t))}{\theta(\beta\widehat{B}(x,N_0(r_0+t)))}\int\limits_{B(x,r_{k_0}+t)} \psi\Big(\frac{4c_1 f(y)}{\varepsilon\eta(\lambda)}\Big)\sigma(y)d\nu. \tag{2.14}$$

Rewrite now (2.7) as

$$\theta(\beta\widehat{B}(x,N_0(r_0+t))) \le \frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu. \quad (2.15)$$

If θ^{-1} is defined by

$$\theta^{-1}(u) = \sup\{\tau : \theta(\tau) \le u\},$$

then

$$\theta(\theta^{-1}(u)) \le u \quad \text{and} \quad \theta(2\theta^{-1}(u)) \ge u,$$

i.e.,

$$\frac{u}{2} \le \theta(\theta^{-1}(u)) \le u.$$

Moreover,

$$\theta^{-1}(\theta(u)) = \sup\{\tau : \theta(\tau) \le \theta(u)\} \ge u.$$

Therefore (2.15) will yield the estimate

$$\beta\widehat{B}(x,N_0(r_0+t)) \le \theta^{-1}(\theta(\beta\widehat{B}(x,N_0(r_0+t)))) \le$$
$$\le \theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu\Big).$$

Thus

$$\theta\Big(\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1 f(x)}{\varepsilon\eta(\lambda)}\Big)\sigma(x)d\nu\Big)\Big) \le$$
$$\le \frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1 f(x)}{\varepsilon\eta(\lambda)}\Big)\sigma(x)d\nu$$

which by virtue of the fact that $\frac{\theta(u)}{u}$ decreases yields the estimate

$$\frac{\beta\widehat{B}(x,N_0(r_0+t))}{\theta(\beta\widehat{B}(x,N_0(r_0+t)))} \le$$
$$\le c\frac{\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu\Big)}{\theta\Big(\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu\Big)\Big)} \le$$
$$\le 2c\frac{\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu\Big)}{\frac{1}{c_1\varphi(\lambda)}\int_X \psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu} \stackrel{\text{def}}{=} l(f,\lambda).$$

After all foregoing manipulations condition (2.14) can be formulated as follows: for each $(x,t)\in E_\lambda$ there exists a ball $B_{x,t}$ such that x is its centre, $t\leq \operatorname{rad} B_{x,t}$, and

$$\varphi(\lambda)\beta(N_0\widehat{B}_{x,t})\leq cl(f,\lambda)\int\limits_{B_{x,t}}\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu. \qquad (2.16)$$

Now fix a ball B_0 and consider the sets

$$\widehat{B}_0\cap E_\lambda \quad\text{and}\quad B_0\cap\{x:\mathcal{K}(fd\nu)(x,0)>\lambda\}\equiv B_0\cap E^0_\lambda.$$

It is obvious that latter set is contained in the former. For each $x\in E^0_\lambda$ we set

$$d(x)=\sup\{t:(x,t)\in\widehat{B}_0\cap E_\lambda\}.$$

It is easy to verify that $d(x)<2\operatorname{rad}B_0$. For each $x\in B_0\cap E^0_\lambda$ there exist $t_x\geq\frac{2d(x)}{N_0}$ such that $(x,t_x)\in\widehat{B}_0\cap E_\lambda(N_0>2)$, consequently for (x,t_x) (2.16) is valid.

As a result we have the following situation: for each $x\in B_0\cap E^0_\lambda$ there exists a ball B_x with centre at the point x such that $\operatorname{rad}B_x>\frac{d(x)}{N_0}$ and (2.16) is fulfilled for $B_x=B_{x,t}$.

If

$$\sup_{x\in B_0\cap E^0_\lambda}\operatorname{rad}B_x=\infty,$$

then, clearly, there exists a ball $B_1\in\{B_x\}_{x\in E^0_\lambda\cap B_0}$ such that $E_\lambda\cap\widehat{B}_0\subset\widehat{B}_1$.

If $\sup\operatorname{rad}B_x<\infty$, then, due to Lemma 2.4, from the family $\{B_x\}_{x\in E^0_\lambda}$ $\cap B_0$ covering the bounded set $E^0_\lambda\cap B_0$ we can choose a sequence (B_j) of nonintersecting balls for which $\cup_{j\geq1}N_0B_j\supset E^0_\lambda\cap B_0$ and (2.16) holds.

It will be shown that $(\widehat{N_0B_j})_{j\geq1}$ covers the set $E_\lambda\cap B_0$. To this end we prove that each $(x,d(x))\in\cup_{j\geq1}\widehat{N_0B_j}$. Indeed, if x is centre of some ball B_j, then there is nothing to prove. Let x not be centre of B_j; then by Lemma 2.4 for B_x there exists a ball B_j such that $x\in N_0B_j$ and

$$\operatorname{rad}B_x\leq 2\operatorname{rad}B_j.$$

Therefore

$$d(x)<N_0\operatorname{rad}B_x\leq 2N_0\operatorname{rad}B_j$$

or, which is the same, $(x,d(x))\in\widehat{N_0B_j}$.

On account of the foregoing reasoning we can derive estimates

$$\varphi(\lambda)\beta\{(x,t)\in\widehat{B_0} : \mathcal{K}(fd\nu)(x,t) > \lambda\} \le \sum_{j=1}^{\infty}\varphi(\lambda)\beta\widehat{N_0B_j} \le$$
$$\le cl(f,\lambda)\sum_{j=1}^{\infty}\int_{B_j}\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu \le$$
$$\le cl(f,\lambda)\int_X\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu \le$$
$$\le c\varphi(\lambda)\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(y)}{\eta(\lambda)}\Big)\sigma(y)d\nu\Big)$$

which yield

$$\theta(\beta\{(x,t)\in\widehat{B_0} : \mathcal{K}(fd\nu)(x,t) > \lambda\}) \le$$
$$\le \theta\Big(c\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu\Big)\Big).$$

Taking into account $\psi\in\Delta_2$ and $u^{-\alpha}\psi(u)\downarrow$, from the latter estimate we obtain the inequality

$$\theta(\beta\{(x,t)\in\widehat{B_0} : \mathcal{K}(fd\nu)(x,t) > \lambda\}) \le$$
$$\le c^{\alpha}\theta\Big(\theta^{-1}\Big(\frac{1}{c_1\varphi(\lambda)}\int_X\psi\Big(\frac{4c_1}{\varepsilon}\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu\Big)\Big) \le$$
$$\le \frac{c}{\varphi(\lambda)}\int_X\psi\Big(\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu.$$

If we now assume that rad B_0 tends to infinity, we obtain (1.6).

Theorem 1.1 is proved. ■

Consider the case $d\beta = \varrho d\nu\otimes\delta_0$ where δ_0, is the Dirac measure supported at the origin and

$$k(x,y,t) = \begin{cases} k(x,y), & t=0,\\ 0, & t>0.\end{cases}$$

In that case due to Theorem 1.1 we have

Corollary 2.1. *Let the functions φ, η and ψ satisfy the conditions of Theorem 1.1. It is further assumed that there exist positive ε and*

c_1 such that

$$\int\limits_{X\setminus B(a,r)} \tilde{\psi}\Big(\varepsilon \frac{\varphi(s)\eta(s)}{s}\frac{\theta(\varrho B(a,2N_0 r))}{\sigma(y)} k(a,y)\Big)\sigma(y)d\nu \le c_1\varphi(s)\theta(\varrho B(a,2N_0 r))$$

for any $s>0$, $r\ge 0$ and $a\in X$.

In that case there exists $c_2>0$ such that the inequality

$$\varphi(\lambda)\theta(\varrho\{x\in X: \int_X k(x,y)f(y)d\nu(y)>\lambda\}) \le$$
$$\le c_2\int_X \psi\Big(\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu$$

holds for any $\lambda>0$ and any nonnegative measurable function $f: X\to \mathbf{R}^1$.

It is time to make some remarks. Taking a closer look at the proof of Theorem 1.1, we readily find that if $\beta\hat{B}(x,r)$ is continuous with respect to r for each $x\in X$, the factor 2 in condition (1.5) can be omitted.

Moreover, if the space (X,d,μ) possesses the Besicovitch property (consisting in that for every bounded set E any family $\{B(y,r(y))\}_{y\in E}$ of balls contains a countable (or finite) subfamily $\{B_n\}=\{B(y_n,r(y_n))\}$, $n\in N$, such that $E\subset \cup B_n$ and $\sum\chi_{B_n}\le c$, where χ_{B_n} is the characteristic function of the set B_n, then in Corollary 2.1 we can set $N_0=1$.

Finally we remark that for $\varphi(\lambda)=\psi(\lambda)=\lambda^p$, $\eta\equiv 1$, $\theta(u)=u^{\frac{p}{q}}$, $X=\mathbf{R}^n$ Corollary 2.1 becomes the particular case of theorem 6.1.1 from [7], p. 171.

The proof of Theorem 1.2 rests on a number of lemmas.

Lemma 2.5. *Let θ be any increasing function and the kernel k satisfy condition (1.7). If condition (1.6) is fulfilled, then there exists a constant $c>0$ such that for any $a\in X$, $r\ge 0$, $t\ge 0$ and any nonnegative measurable function $F: X\to\mathbf{R}^1$, $\operatorname{supp}F\subset X\setminus B(a,r)$ we have the inequality*

$$\varphi(\mathcal{K}(Fd\nu)(a,t))\theta(\beta\hat{B}(a,N(r+t)))\le$$
$$\le c\int_X \psi\Big(\frac{c'F(x)}{\eta(\mathcal{K}(Fd\nu)(a,t))}\Big)\sigma(x)d\nu, \tag{2.17}$$

where N and c' are the constants from condition (1.7).

Proof. Fix $a \in X$, $r \geq 0$, $t \geq 0$, and $F : X \to \mathbf{R}^1$ assuming that $suppF \subset X \backslash B(a,r)$. Inequality (2.17) is obtained if in (1.6) we set $f = c'F$ and $\lambda = \mathcal{K}(Fd\nu)(a,t)$.

It is sufficient only to note that the inclusion

$$\widehat{B}(a, N(r+t)) \subset \{(x,\tau) : \mathcal{K}(c'Fd\nu)(x,\tau) \geq \mathcal{K}(Fd\nu)(a,t)\}$$

holds by virtue of condition (1.7).

The lemma is proved. ■

Lemma 2.6. *Let $\varphi\eta$ and ψ are quasiconvex functions and k is a nonnegative kernel. Then condition* (2.17) *with the constants c, c', and N implies the existence of ε and c_1 such that the inequality*

$$\int_{X\backslash B(a,r)} \widetilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(a,N(r+t)))}{\sigma(y)}k(a,y,t)\Big)\sigma(y)d\nu(y) \leq \tag{2.18}$$

$$\leq c_1\varphi(\lambda)\theta(\beta\widehat{B}(a,N(r+t))) \tag{2.19}$$

holds for any $\lambda > 0$, $r \geq 0$, $t \geq 0$, and $a \in X$.

Proof. It is obvious that (2.17) is fulfilled for $\sigma_1 = \sigma + \delta$, too, if $\delta > 0$. Let $a \in X$, $r \geq 0$, $t \geq 0$, $\lambda > 0$ be the fixed constants. Due to (1.8) it can be assumed without loss of generality that the function ψ is a convex. For $M > 0$ we define the value

$$I = \int_D \widetilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(a,N(r+t))}{\sigma_1(y)}k(a,y,t)\Big)\sigma_1(y)d\nu,$$

where

$$D = B(a,R)\backslash B(a,r) \cap \{y \in X : k(a,y,t) < M\},$$

$R > r$, while the constant ε will be appropriately chosen later.

We introduce the notation

$$T_\delta(y) = \varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\widehat{B}(a,N(r+t)))}{\sigma_1(y)}k(a,y,t),$$

$$g(y) = \frac{\widetilde{\psi}(T_\sigma(y))}{T_\sigma(y)}\varepsilon\eta(\lambda)\chi_D(y)$$

allowing us to write

$$I = \varphi(\lambda)\theta(\beta\widehat{B}(a,N(r+t)))\frac{\mathcal{K}(gd\nu)(a,t)}{\lambda}. \tag{2.20}$$

Our next step is to show that for sufficiently small ε 's the value I is finite.

If $\lim_{t\to\infty}\frac{\psi(t)}{t}=\infty$, then $\tilde{\psi}$ is finite everywhere and thus

$$I\le\tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\hat{B}(a,N(r+t)))}{\delta}M\Big)\int\limits_{B(a,R)}\sigma_1(x)d\nu<\infty$$

for any $\varepsilon>0$.

Let now $\psi(t)\le At$, $A>0$. Then from the condition (2.17) we obtain

$$\varphi(\mathcal{K}(Fd\nu)(a,t))\eta(\mathcal{K}(Fd\nu)(a,t))\theta(\beta\hat{B}(a,N(r+t)))\le$$
$$\le c\int_X F(x)\sigma(x)d\nu. \tag{2.21}$$

Set

$$l=\left\|\chi_{X\setminus B(a,r)}\frac{k(a,\cdot,t)}{\sigma_1}\right\|_{L^\infty}.$$

From the definition of the norm in L^∞ it follows that there exists a measurable set $E\subset X\setminus B(a,r)$, $\nu E>0$ such that

$$\frac{k(a,y,t)}{\sigma_1(y)}>\frac{l}{2}$$

for $y\in E$.

Set in (2.20)

$$F(y)=\frac{\lambda}{\nu E k(a,y,t)}\chi_E(y).$$

Recall that a and t are fixed. Obviously,

$$\mathcal{K}(Fd\nu)(a,t)=\lambda$$

and hence by virtue of (2.20) we obtain the estimate

$$\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\theta(\beta\hat{B}(a,N(r+t)))\le\frac{c}{\nu E}\int_E\frac{\sigma(y)}{k(a,y,t)}d\nu\le\frac{2c}{l}.$$

which yields

$$\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\theta(\beta\hat{B}(a,N(r+t)))\chi_{X\setminus B(a,r)}\frac{k(a,y,t)}{\sigma_1(y)}<c,$$

where the constant does not depend on λ, r, t and a.

Thus we conclude that

$$I\le\tilde{\psi}(\varepsilon c)\int\limits_{B(a,r)}\sigma_1(y)d\nu.$$

If now ε is so small that $\tilde{\psi}(c\varepsilon) < \infty$, then the value I will be finite for the respective ε.

Now it will be shown that

$$I \leq b\varphi(\lambda)\theta(\beta\hat{B}(a, N(r+t))) + cb\int_X \psi\Big(\frac{c'g(y)}{\eta(\lambda)}\Big)\sigma_1 d\nu, \quad (2.22)$$

where the constants b, c and c' do not depend on λ, r and t.

Let $a \in X$ and $t \geq 0$ be such that

$$\mathcal{K}(gd\nu)(a,t) < b\lambda,$$

where a constant b is such that

$$\frac{\varphi(s)\eta(s)}{s} \leq b\frac{\varphi(u)\eta(u)}{u} \quad (2.23)$$

for $bs < u$ (see Lemma 2.1).

Then evidently (2.19) will yield

$$I \leq b\varphi(\lambda)\theta(\beta\hat{B}(a, N(r+t))).$$

Let now

$$\mathcal{K}(gd\nu)(a,t) > b\lambda.$$

Using (2.22) and condition (2.17), from (2.19) we obtain the estimate

$$I = \varphi(\lambda)\theta(\beta\hat{B}(a, N(r+t)))\frac{\mathcal{K}(gd\nu)(a,t)}{\lambda} \leq$$

$$\leq b\frac{\varphi(\mathcal{K}(gd\nu)(a,t))\eta(\mathcal{K}(gd\nu)(a,t))}{\eta(\lambda)}\theta(\beta\hat{B}(a, N(r+t))) \leq$$

$$\leq cb\frac{\eta(\mathcal{K}(gd\nu)(a,t))}{\eta(\lambda)}\int_X \psi\Big(\frac{c'g(x)}{\eta(\mathcal{K}(gd\nu)(a,t))}\Big)\sigma_1(x)d\nu.$$

Since the function ψ is convex, estimating the right-hand part of the latter inequality by means of (2.2) we conclude that

$$I \leq cb\int_X \psi\Big(\frac{c'g(x)}{\eta(\lambda)}\Big)\sigma_1(x)d\nu.$$

Thus we have shown that inequality (2.21) is valid.

Rewrite (2.21) as

$$I \leq b\varphi(\lambda)\theta(\beta\hat{B}(a, N(r+t))) +$$

$$+cb\int_X \psi\Big(c'\varepsilon\frac{\tilde{\psi}(T_\delta(y))}{T_\delta(y)}\Big)\sigma_1(y)d\nu. \quad (2.24)$$

Let ε be chosen so small that $c'\varepsilon < 1$. Then, by virtue of the assumption that the function ψ is convex and taking into account (2.3) and (2.4), from (2.23) we have

$$\int_D \tilde{\psi}\big(T_\delta(y)\big)\sigma(y)d\nu \leq b\varphi(\lambda)\theta(\beta\hat{B}(a,N(r+t))) +$$
$$+cc'b\varepsilon \int_D \tilde{\psi}(T_\delta(y))\sigma_1(y)d\nu.$$

If ε is so small that

$$cc'b\varepsilon < 1,$$

then the latter inequality implies

$$\int_D \tilde{\psi}\big(T_\delta(y)\big)\sigma_1(y)d\nu \leq c\varphi(\lambda)\theta(\beta\hat{B}(a,N(r+t))).$$

Passing here to the limit when $R \to \infty$, $M \to \infty$, and $\delta \to 0$, we obtain the desired inequality (2.18).

The lemma is proved. ■

Lemma 2.7. *Let the kernel k satisfy condition* (1.7) *and inequality* (1.9) *be fulfilled. Then* (1.5) *is valid.*

Proof. Replace t by $N_0(2r+t)$ in condition (1.9) and take into consideration that by virtue of condition (1.7) we have

$$k(a,y,t) \leq c'k(a,y,N_0(2r+t)).$$

for any $y \in X \setminus B(a,r)$.

The lemma is proved. ■

Proof of Theorem 2.2. The proof is accomplished by the diagram

$$\begin{array}{ccc} (1.6) & \Rightarrow & (1.10) \\ \Uparrow & & \Downarrow \\ (1.9) & \Leftarrow & (1.8) \end{array}$$

By Lemma 2.5 (1.6) $\Rightarrow$ (1.10). Then by Lemma 2.6 (1.10) $\Rightarrow$ (1.8). When $r = 0$ condition (1.8) yields (1.9). Next, by Lemma 2.7 we obtain (1.5). Finally, using Theorem 1.1, we ascertain that the implication (1.5) $\Rightarrow$ (1.6) is valid.

The theorem is proved. ■

We would like to make some remarks connected with the proof of Theorem 1.3. If $k(x,y,t) = k(x,y)$, $d\beta = \varrho d\mu \otimes \delta_0$ Lemmas 2.5 and 2.6 can be reformulated in the respective. Further, proceeding from Corollary 2.1 and following the proof of Theorem 2.2, we ascertain that Theorem 1.3 is valid.

3. Criteria of General Weak Type Weighted Inequalities in Lorentz Spaces

Let (Y, ν) be a space with a positive σ-additive measure ν. When $1 \leq p \leq \infty$, $1 \leq s \leq \infty$, the Lorentz space L_ν^{ps} is a space of all ν-measurable functions f for which $\|f\|_{L^{ps}(Y,\nu)} < \infty$, where

$$\|f\|_{L^{ps}(Y,\nu)} = \left(s\int_0^\infty (\nu\{y \in Y : |f(y)| > \tau\})^{\frac{s}{p}}\tau^{s-1}d\tau\right)^{\frac{1}{s}}$$
$$\text{if } 1 \leq p < \infty, \quad 1 \leq s < \infty,$$

and

$$\|f\|_{L^{ps}(Y,\nu)} = \sup \tau(\{y \in Y : |f(y)| > \tau\})^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty, \quad s = \infty.$$

If $1 < p < \infty$ and $1 \leq s \leq \infty$, or $p = s = 1$, or $p = s = \infty$, then $L^{ps}(Y, \nu)$ is the Banach space with a norm equivalent to $\|\cdot\|_{L^{ps}(Y,\nu)}$.

In the sequel X will denote a homogeneous type space, β – a positive measure given on the product of σ-algebras generated by balls from X and by intervals from $[0, \infty)$, ν – a finite positive measure on X.

Theorem 3.1. *Let $1 \leq s \leq p < q < \infty$ and $k : X \times X \times [0, \infty) \to \mathbf{R}^1$ be an arbitrarily chosen nonnegative kernel. In that case if there exists a number $c_1 > 0$ such that the inequality*

$$\left(\beta\widehat{B}(a,\ N_0(2r+t))\right)^{\frac{1}{q}}\|\chi_{X\setminus B(a,r)}\frac{k(a,\cdot,t)}{\sigma}\|_{L^{p's'}(X,\sigma d\nu)} \leq c_1, \quad (3.1)$$

holds for any $a \in X$, $r \geq 0$, $t \geq 0$, then there exists a positive constant c_2 such that we have

$$\beta\{(x,t) \in X \times [0,\infty) : \mathcal{K}(fd\nu)(x,t) > \lambda\} \leq$$
$$\leq c_2\lambda^{-q}\|f\|^q_{L^{ps}(X,\nu)}. \quad (3.2)$$

for any measurable nonnegative $f : X \to \mathbf{R}^1$ and $\lambda > 0$.

Theorem 3.2. *Let $1 \leq s \leq p < q < \infty$ and the kernel k satisfies condition (1.7). Then the folowing statements are equivalent:*

(i) *(3.2) is fulfilled;*

(ii) *there exists a positive constant c such that*

$$\left(\beta\widehat{B}(a,\ (2r+t))\right)^{\frac{1}{q}}\|\chi_{X\setminus B(a,r)}\frac{k(a,\cdot,t)}{\sigma}\|_{L^{p's'}(X,\sigma d\nu)} \leq c$$

for any $a \in X$, $r \geq 0$, $t \geq 0$;

(iii) *there exists a number $c_1 > 0$ such that*

$$\left(\beta\widehat{B}(a,\ t)\right)^{\frac{1}{q}}\|\frac{k(a,\cdot,t)}{\sigma}\|_{L^{p's'}(x,\sigma d\nu)} \leq c_1.$$

The proofs of these theorems are accomplished in the manner described in section 2 using the technique from [7], Chapter 6, and we therefore leave them out. Note that the solution of the two weight problem in the sense of [3] was previously derived in [19] in Lebesgue spaces for a fractional integral over a homogenous type space.

4. General Weak Type Inequalities for Classical Operators

In this section we are going to discuss some specific examples for which the results of the previous sections are valid.

Consider the kernel

$$k(x,y,t) = (\mu B(x,d(x,y)+t))^{-\delta}, \quad \delta > 0.$$

It is easy to verify that it satisfies condition (1.7). Let $y \in X \backslash B(a,r)$ and $(x,\tau) \in$
$\in \widehat{B}(a,N(r+t))$, where N is an arbitrary positive number. It sufficient to show the inclusion

$$B(x,d(x,y)+\tau) \subset B(a,c(d(a,y)+t)).$$

Indeed, assuming that $z \in B(x,d(x,y)+\tau)$, we obtain a chain of inequalities

$$\begin{aligned} d(a,z) &\le a_1(d(a,x)+d(x,z)) \le a_1(N(r+t)+d(x,z)) \le \\ &\le a_1N(r+t)+a_1d(x,y)+a_1\tau \le 3a_1N(r+t)+ \\ &+a_1^2(d(x,a)+d(a,y)) \le 3a_1N(r+t)+a_1^2a_0d(a,x)+ \\ &+a_1^2d(a,y) \le (3a_1+a_1^2a_0)N(r+t)+a_1^2d(a,y) \le \\ &\le (3a_1+a_1^2a_0)Nt+((3a_1+a_1^2a_0)N+a_1^2)d(a,y) \le \\ &\le ((3a_1+a_1^2a_0)N+a_1^2)(d(a,y)+t). \end{aligned}$$

Thus condition (1.7) is fulfilled. For such kernels we have Theorems 2.2 and 3.2 and hence we obtain the solution of the general weak type weight problem in Orlicz and Lorentz spaces for classical operators such as Riesz potentials, Poisson integrals and others.

Let $X = R^n$, d be a Euclidean distance, μ a Lebesgue measure and

$$T_\gamma f(x,t) = \int_{R^n} \frac{f(y)}{(|x-y|+t)^{n-\gamma}} d\nu, \quad 0 < \gamma < n.$$

a generalized potential. Theorem 2.2 yields a solution of the general weak-type weight problem for T_γ in Orlicz spaces. It was previously solved in Lorentz spaces in [6] (see also [7], Theorem 6.5.1).

Now consider the Poisson integral in the upper half-space

$$\mathbb{P}f(x,y)=\int_{R^n} f(y)\mathcal{P}(x-y,t)dy,$$

where $\mathcal{P}(x,t)=c_n t(t^2+|x|^2)^{-\frac{n+1}{2}}$ is the Poisson kernel for R_+^{n+1}. The criterion of a two weight inequality of the weak type (p,q) was established in [10]. From Theorem 2.2 we obtain

Corollary 4.1. *Let $\varphi\eta$ and ψ be quasiconvex functions, $\psi\in\Delta_2$ and the function $t^{-\alpha}\theta(t)$ decrease for some $\alpha\in(0,1)$. Then the following statements are equivalent:*

(i) *there exists a constant $c_1>0$ such that*

$$\varphi(\lambda)\theta(\beta\{(x,t)\in R^n\times[0,\infty):\frac{1}{t}\mathbb{P}f(x,t)>\lambda\})\le$$
$$c_1\int_{R^n}\psi\Big(\frac{f(x)}{\eta(\lambda)}\Big)\sigma(x)d\nu$$

for any $\lambda>0$ and any nonnegative measurable function $f:R^n\to\mathbf{R}^1$;

(ii) *there exist positive constants ε and c_2 such that*

$$\int\limits_{R^n\setminus B(a,r)}\tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\hat{B}(a,r+t))}{\sigma(y)}\frac{1}{t}\mathcal{P}(a-y,t)\Big)\sigma(y)d\nu\le$$
$$\le c_2\varphi(\lambda)\theta(\beta\hat{B}(a,r+t))$$

for any $\lambda>0$, $a\in R^n$, $r\ge0$, $t\ge0$;

(iii) *there exists positive constants ε and c_3 such that*

$$\int_{R^n}\tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\beta\hat{B}(a,t))}{\sigma(y)}\frac{1}{t}\mathcal{P}(a-y,t)\Big)\sigma(y)d\nu\le$$
$$\le c_3\varphi(\lambda)\theta(\beta\hat{B}(a,t))$$

for any $\lambda>0$, $a\in R^n$ and $t\ge0$.

Let now $X=[0,\infty)$, d be a Euclidean distance, μ a Lebesgue measure and

$$k(x,y)=\begin{cases}1 & \text{for } x>y,\\ 0 & \text{for } x\le y.\end{cases}$$

Then for the Hardy transform $f\mapsto\int_0^x f(y)d\nu$ Theorem 2.3 yields the criterion of validity of the weak-type inequality figuring in this theorem. This criterion is written in the form

$$\int_0^x\tilde{\psi}\Big(\varepsilon\frac{\varphi(\lambda)\eta(\lambda)}{\lambda}\frac{\theta(\varrho(x,\infty))}{\sigma(y)}\Big)\sigma(y)d\nu\le c\varphi(\lambda)\theta(\varrho(x,\infty)).$$

References

1. J.O. Strömberg and A. Torchinski, Weighted Hardy spaces. *Lecture Notes in Math.* **1381**, *Springer-Verlag, New York*, 1989.

2. J.O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces. *Indiana University Math. J.* **28**(1979), 511-544.

3. E.T. Sawyer, A two-weight weak-type inequality for fractional integrals. *Trans. Amer. Math. Soc.* **281**(1984), 339-345.

4. M. Gabidzashvili, Weighted inequalities for anisotropic potentials. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* bf 89(1986), 25-36.

5. I. Genebashvili. Carleson measures and potentials defined on spaces of the homogeneous type. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **135**(1989), No.3, 505-508.

6. V. Kokilashvili and M. Gabidzashvili, Two-weight weak-type inequalities for fractional-type integrals. *Preprint,* No. 45, *Math. Inst. Czech. Acad. Sci., Prague,* 1989.

7. V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces. *World Scientific, Singapore, New Jersey, London, Hong Kong,* 1991.

8. V.M. Kokilashvili, On the weight problem for integrals with positive kernels. *Soobshch. Akad. nauk Gruzin. SSR* **140**(1990), No.3, 469-471.

9. M. Gabidzashvili, I. Genebashvili, V. Kokilashvili, Two-weight inequalities for generalized potentials. (Russian) *Trudy Mat. Inst. Steklov.* **192**(1991).

10. E. Sawyer, R.Wheeden, Carleson conditions for the Poisson integral. *Indiana University Math. J.* **40**(1991), No.2, 639-676.

11. A. Carbery, S.Y. Chang and J. Garnett, Weights and $L \log L$. *Pacific J. Math.* **120**(1985), No.1, 33-45.

12. R.J. Bagby, Weak bounds for the maximal function in weighted Orlicz spaces. *Studia Math.* **95**(1990), 195-204.

13. D. Gallardo, Weighted weak type integral inequalities for the Hardy-Littlewood maximal operator. *Izrael J. Math.* **67**(1989), No.1, 95-108.

14. L. Pick, Two weight weak type maximal inequalities in Orlicz classes. *Studia Math.* **100**(1991), No.3, 206-218.

15. A. Gogatishvili, L. Pick, Weak and extra-weak type inequalities for the maximal operator and Hilbert transform. To appear in *Chechoslov. Math. J.*

16. A. Gogatishvili, General weak-type inequalities for the maximal operators and singular integrals. *Proc. A. Razmadze Math. Inst. Georgian Acad. Sci.* **101**(1992), 47-63.

17. R.R. Coifman, G. Weiss, Analyse harmonique noncommutative sur certains espaces homogenes. *Lecture Notes in Math.* **242**, *Springer Verlag, Berlin and New York,* 1971.

18. Pan Wenjie. Fractional integrals on spaces of homogeneous type. *Approx. Theory and its Appl.* **8**(1992), No.1, 1-15.

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ON THE TWO-POINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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ABSTRACT. The sufficient conditions of solvability and unique solvability of the two-point boundary value problems of Vallée-Poussin and Cauchy-Niccoletti have been found for a system of ordinary differential equations of the form

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}),$$

where the vector function $f :]a, b[\times \mathbb{R}^{nl} \to \mathbb{R}^l$ has nonintegrable singularities with respect to the first argument at the points a and b.

რეზიუმე. დადგენილია ვალე-პუსენისა და კოში-ნიკოლეტის ორ-წერტილოვანი სასაზღვრო ამოცანების ამოხსნადობისა და ცალსახად ამოხსნადობის საკმარისი პირობები

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)})$$

სახის ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისათვის, სადაც $f :]a, b[\times \mathbb{R}^{nl} \to \mathbb{R}^l$ ვექტორულ ფუნქციას პირველი არგუმენტის მიმართ გააჩნია არაინტეგრებადი განსაკუთრებულებანი a და b წერტილებში.

§ 1. STATEMENT OF THE MAIN RESULTS

In this paper for an l-dimensional system of differential equations

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1.1}$$

we consider the boundary value problem of Vallée-Poussin

$$\begin{aligned} u(a+) = \cdots = u^{(m-1)}(a+) = 0, \\ u(b-) = \cdots = u^{(n-m-1)}(b-) = 0 \end{aligned} \tag{1.2}$$

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and that of Cauchy-Niccoletti

$$\begin{aligned} u(a+) = \cdots = u^{(m-1)}(a+) = 0, \\ u^{(m)}(b-) = \cdots = u^{(n-1)}(b-) = 0, \end{aligned} \tag{1.3}$$

where $l \geq 1$, $n \geq 2$, m is an integer part of the number $\frac{n}{2}$, $-\infty < a < b < +\infty$, and the vector function $f :]a,b[\times\mathbb{R}^{nl} \to \mathbb{R}^l$ satisfies the Caratheodori conditions on each compact contained in $]a,b[\times\mathbb{R}^{nl}$. We are interested mainly in the singular case when f is nonintegrable with respect to the first argument on $[a,b]$, having singularities at the ends of this interval. The above problems were investigated for $l = 1$ in [2-6].

The following notations will be used:

$$I_n(a,b) = \begin{cases}]a,b[& \text{for } n = 2m \\ ]a,b] & \text{for } n = 2m+1 \end{cases};$$

$$\mu_n = \begin{cases} 1 & \text{for } n = 2m \\ \frac{n}{2} & \text{for } n = 2m+1 \end{cases};$$

$$\lambda_{im}(a,b;t) = \frac{\min\{(t-a)^{2m-i}, (b-t)^{2m-i}\}}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}}$$
$$(i = 1, \dots, m);$$

$\mathbb{R}$ is a set of real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$\xi = (\xi_j)_{j=1}^l \in \mathbb{R}^l$ and $A = (a_{kj})_{k,j=1}^l \in \mathbb{R}^{l\times l}$ are respectively an l-dimensional column vector and an $l \times l$ matrix with real components ξ_j $(j = 1, \dots, l)$ and a_{kj} $(k,j = 1, \dots, l)$,

$$|\xi| = (|\xi_j|)_{j=1}^l, \quad \|\xi\| = \sum_{j=1}^l |\xi_j|, \quad \|A\| = \sum_{k,j=1}^l |a_{kj}|,$$

$$S(\xi) = \begin{pmatrix} \operatorname{sign}\xi_1 & 0 & \dots & 0 \\ 0 & \operatorname{sign}\xi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \operatorname{sign}\xi_l \end{pmatrix};$$

$r(A)$ is the spectral radius of the matrix A;

$\mathbb{R}_+^l$ and $\mathbb{R}_+^{l\times l}$ are sets of l-dimensional vectors and $l \times l$ matrices with nonnegative components;

the inequalities $\xi \leq \bar{\xi}$ and $A \leq \bar{A}$, where ξ and $\bar{\xi} \in \mathbb{R}^l$ and A and $\bar{A} \in \mathbb{R}^{l\times l}$, imply respectively $\bar{\xi} - \xi \in \mathbb{R}_+^l$ and $\bar{A} - A \in \mathbb{R}_+^{l\times l}$;

$L_{loc}(I;\mathbb{R}_+)$, where $I \subset \mathbb{R}$ is an interval, is a set of functions $x : I \to \mathbb{R}_+$ which are Lebesgue integrable on each segment contained in I;

$K_{loc}(I \times \mathbb{R}^p;\mathbb{R}^l)$, where p is a natural number, is a set of vector functions mapping $I \times \mathbb{R}^p$ into $\mathbb{R}^l$ and satisfying the Caratheodori conditions on each compact contained in $I \times \mathbb{R}^p$;

$\widetilde{C}^p_{loc}(I;\mathbb{R}^l)$ is a set of vector functions $u : I \to \mathbb{R}^l$ which are absolutely continuous together with all their derivatives up to order p inclusive on each segment contained in I;

$\widetilde{C}^{n-1,m}(I;\mathbb{R}^l)$ is a set of vector functions $u \in \widetilde{C}^{n-1}_{loc}(I;\mathbb{R}^l)$ satisfying the condition

$$\int_I \|u^{(m)}(\tau)\|^2 d\tau < +\infty.$$

As mentioned above, throughout this paper it is assumed that

$$f \in K_{loc}(]a,b[\times\mathbb{R}^{nl};\mathbb{R}^l).$$

Theorem 1.1. *Let the following inequalities be fulfilled on* $]a,b[\times\mathbb{R}^{nl}$*:*

$$(-1)^{n-m-1}S(x_1)f(t,x_1,\dots,x_n) \geq -\sum_{i=1}^{m} H_i(t)|x_i| - h(t) \tag{1.4}$$

and

$$\|f(t,x_1,\dots,x_n)\| \leq q(t,x_1,\dots,x_m) \sum_{i=m+1}^{n} (1+\|x_i\|)^{\frac{2n-2m-1}{2i-2m-1}}, \tag{1.5}$$

where

$$q \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml};\mathbb{R}_+), \tag{1.6}$$

and $H_i :]a,b[\to \mathbb{R}^{l\times l}_+$ $(i=1,\dots,m)$ *and* $h :]a,b[\to \mathbb{R}^l_+$ *are respectively measurable matrix and vector functions satisfying the conditions*

$$\int_a^b (\tau-a)^{n-m-\frac{1}{2}}(b-\tau)^{m-\frac{1}{2}}\|h(\tau)\|d\tau < +\infty, \tag{1.7}$$

$$\int_a^b (\tau-a)^{n-i}(b-\tau)^{2m-i}\|H_i(\tau)\|d\tau < +\infty \quad (i=1,\dots,m), \tag{1.8}$$

$$r\Big(\sum_{i=1}^{m} \int_a^b (\tau-a)^{n-2m}\lambda_{im}(a,b;\tau)H_i(\tau)d\tau\Big) < \mu_n. \tag{1.9}$$

Then the problem (1.1),(1.2) *is solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Theorem 1.2. *Let on* $]a,b[\times\mathbb{R}^{nl}$ *the inequalities* (1.4) *and* (1.5) *be fulfilled, where*

$$q \in K_{loc}(]a,b] \times \mathbb{R}^{ml};\mathbb{R}_+),$$

and $H_i :]a,b] \to \mathbb{R}_+^{l\times l}$ $(i=1,\dots,m)$ *and* $h :]a,b] \to \mathbb{R}_+^l$ *are respectively measurable matrix and vector functions satisfying the conditions*

$$\int_a^b (\tau-a)^{n-m-\frac{1}{2}}\|h(\tau)\|d\tau < +\infty, \tag{1.10}$$

$$\int_a^b (\tau-a)^{n-i}\|H_i(\tau)\|d\tau < +\infty \quad (i=1,\dots,m), \tag{1.11}$$

$$r\bigg(\sum_{i=1}^m \frac{1}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}} \times$$

$$\times \int_a^b (\tau-a)^{n-i}H_i(\tau)d\tau\bigg) < \mu_n. \tag{1.12}$$

Then the problem (1.1), (1.3) *is solvable in the class* $\widetilde{C}^{n-1,m}(]a,b];\mathbb{R}^l)$.

For a differential system

$$u^{(n)} = f(t,u,u',\dots,u^{(m-1)}), \tag{1.1$'$}$$

not containing intermediate derivatives of order higher than $(m-1)$, Theorems 1.1 and 1.2 can be formulated as follows:

Theorem 1.1$'$. *Let*

$$f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml};\mathbb{R}^l)$$

and on $]a,b[\times\mathbb{R}^{ml}$

$$(-1)^{n-m-1}S(x_1)f(t,x_1,\dots,x_m) \ge -\sum_{i=1}^m H_i(t)|x_i| - h(t), \tag{1.4$'$}$$

where $H_i :]a,b[\to \mathbb{R}_+^{l\times l}$ $(i=1,\dots,m)$ *and* $h :]a,b[\to \mathbb{R}_+^l$ *are measurable matrix and vector functions satisfying the conditions* (1.7)-(1.9). *Then the problem* (1.1$'$), (1.2) *is solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Theorem 1.2$'$. *Let*

$$f \in K_{loc}(]a,b] \times \mathbb{R}^{ml};\mathbb{R}^l)$$

and on $]a,b[\times\mathbb{R}^{ml}$ *the inequality (1.4$'$) be fulfilled, where* $H_i :]a,b] \to \mathbb{R}_+^{l\times l}$ $(i=1,\dots,m)$ *and* $h :]a,b] \to \mathbb{R}_+^l$ *are measurable matrix and vector functions satisfying the conditions* (1.10)-(1.12). *Then the problem* (1.1$'$), (1.3) *is solvable in the class* $\widetilde{C}^{n-1,m}(]a,b];\mathbb{R}^l)$.

Theorem 1.3. *Let*

$$f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml}; \mathbb{R}^l),$$
$$\int_a^b (\tau - a)^{n-m-\frac{1}{2}} (b-\tau)^{m-\frac{1}{2}} \|f(\tau, 0, \dots, 0)\| d\tau < +\infty \tag{1.13}$$

and on $]a,b[\times \mathbb{R}^{ml}$

$$(-1)^{n-m-1} S(x_1 - y_1)[f(t, x_1, \dots, x_m) - f(t, y_1, \dots, y_m)] \geq$$
$$\geq -\sum_{i=1}^{m} H_i(t)|x_i - y_i|, \tag{1.14}$$

where $H_i :]a,b[\to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *are measurable matrix functions satisfying the conditions* (1.8) *and* (1.9). *Then the problem* (1.1′), (1.2) *is uniquely solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b); \mathbb{R}^l)$.

Theorem 1.4. *Let*

$$f \in K_{loc}(]a,b] \times \mathbb{R}^{ml}; \mathbb{R}^l), \quad \int_a^b (\tau - a)^{n-m-\frac{1}{2}} \|f(\tau, 0, \dots, 0)\| d\tau < +\infty$$

and on $]a,b[\times \mathbb{R}^{ml}$ *the inequality* (1.14) *be fulfilled, where* $H_i :]a,b] \to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *are measurable matrix functions satisfying the conditions* (1.11) *and* (1.12). *Then the problem* (1.1′), (1.3) *is uniquely solvable in the class* $\widetilde{C}^{n-1,m}(]a,b]; \mathbb{R}^l)$.

§ 2. Auxiliary propositions

Lemma 2.1. *Let* $I \subset \mathbb{R}$ *be some interval,* k *be a natural number,* $\rho_0 \in]0, +\infty[$ *and*

$$\varphi \in L_{loc}(I; \mathbb{R}_+). \tag{2.1}$$

Then there exists a continuous function $\rho : I \to \mathbb{R}_+$ *such that for any vector function* $v \in \widetilde{C}^k_{loc}(I; \mathbb{R}^l)$ *satisfying almost everywhere on* I *the differential inequality*

$$\|v^{(k+1)}(t)\| \leq \varphi(t) \left[1 + \sum_{i=0}^{k} \|v^{(i)}(t)\|^{\frac{2k+1}{2i+1}}\right] \tag{2.2}$$

and the condition

$$\int_I \|v(\tau)\|^2 d\tau \leq \rho_0^2, \tag{2.3}$$

the estimates

$$\|v^{(i)}(t)\| < \rho(t) \quad \text{for } t \in I \ \ (i = 0, \dots, k) \tag{2.4}$$

hold.

Proof. In the case $I = [a, b]$ it is not difficult to verify by Lemma 2.2 from [6] that there exists a positive constant $\tilde{\rho}$ such that the estimates

$$\|v^{(i)}(t)\| < \tilde{\rho} \quad \text{for } a \le t \le b \ (i = 0, \dots, k)$$

hold for any vector function $v \in \tilde{C}^k_{loc}(I; \mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3); in other words, we have (2.4), where $\rho(t) \equiv \tilde{\rho}$.

Now consider the case $I =]a, b]$. Choose any decreasing sequence $a_j \in]a, b]$ $(j = 0, 1, 2, \dots)$ such that $a_0 = b$ and

$$\lim_{j \to +\infty} a_j = a.$$

Then, by virtue of the above reasoning, for any natural number j there exists a positive constant ρ_j such that any vector function $v \in \tilde{C}^k_{loc}(I; \mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3) admits the estimates

$$\|v^{(i)}(t)\| < \rho_j \quad \text{for } a_j \le t \le b \ (i = 0, \dots, k). \tag{2.5}$$

Without loss of generality the sequence $(\rho_j)_{j=1}^{+\infty}$ can be assumed to be nondecreasing. Then (2.5) yields the estimates (2.4), where

$$\rho(t) = \rho_j + \frac{t - a_{j-1}}{a_j - a_{j-1}} (\rho_{j+1} - \rho_j) \quad \text{for } a_j < t \le a_{j-1} \ (j = 1, 2, \dots)$$

with $\rho : I \to \mathbb{R}_+$ being continuous and independent of v.

The cases $I = [a, b[$ and $I =]a, b[$ are considered similarly. ■

Lemma 2.2. *Let* $H_i :]a, b[\to \mathbb{R}_+^{l \times l}$ $(i = 1, \dots, m)$ *and* $h :]a, b[\to \mathbb{R}_+^l$ *be measurable matrix and vector functions satisfying the conditions* (1.7)-(1.9) *and*

$$H = \sum_{i=1}^{m} \int_a^b (\tau - a)^{n-2m} \lambda_{im}(a, b; \tau) H_i(\tau) \, d\tau. \tag{2.6}$$

Then for any vector function $u \in \tilde{C}^{n-1,m}(]a, b[; \mathbb{R}^l)$ *satisfying a system of differential inequalities*

$$(-1)^{n-m-1} S(u(t)) u^{(n)}(t) \ge -\sum_{i=1}^{m} H_i(t) |u^{(i-1)}(t)| - h(t) \tag{2.7}$$

$$\text{for } a < t < b$$

and the boundary conditions (1.2) *we have the estimates*

$$\int_a^b \|u^{(m)}(\tau)\|^2 \, d\tau \le \rho_0^2 \tag{2.8}$$

and

$$\|u^{(i-1)}(t)\| \le \rho_0 \sigma_{im}(a,b;t) \quad for \ a < t < b \ (i = 1,\dots,m), \tag{2.9}$$

where

$$\sigma_{im}(a,b;t) = \frac{\min\{(t-a)^{m-i+\frac{1}{2}}, (b-t)^{m-i+\frac{1}{2}}\}}{(m-i)!\sqrt{2m-2i+1}},$$
$$\rho_0 = \sqrt{l}\|(\mu_n E - H)^{-1}\| \times$$
$$\times \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a,b;\tau)\|h(\tau)\|\, d\tau \tag{2.10}$$

and E is the unit $l \times l$ matrix.

To prove this lemma we need

Lemma 2.3. *Let*

$$w(t) = \sum_{i=1}^{n-m} \sum_{k=i}^{n-m} c_{ik}(t) v^{(n-k)}(t) v^{(i-1)}(t),$$

where

$$v \in \widetilde{C}^{n-1,m}(]a,b[;\mathbb{R}),$$
$$v^{(i-1)}(a+) = 0 \ (i = 1,\dots,m), \quad v^{(j-1)}(b-) = 0 \tag{2.11}$$
$$(j = 1,\dots,n-m)$$

and each $c_{ik} : [a,b] \to \mathbb{R}$ is a $(n-k-i+1)$-times continuously differentiable function; in that case there exists a positive constant c_0 such that

$$|c_{ii}(t)| \le c_0 (t-a)^{n-2m} \quad for \ a \le t \le b \tag{2.12}$$
$$(i = 1,\dots,n-m).$$

Then

$$\liminf_{t\to a+} |w(t)| = 0, \quad \liminf_{t\to b-} |w(t)| = 0.$$

Proof. In the first place it will be shown that

$$\liminf_{t\to a+} |w(t)| = 0. \tag{2.13}$$

Let the opposite be true. Then without loss of generality one may assume that the inequality

$$w(t) \ge \delta \quad \text{for } a < t \le a + 2\varepsilon_0$$

is fulfilled for some $\delta \in]0,+\infty[$ and $\varepsilon_0 \in \left]0, \frac{b-a}{4}\right[\cap]0,1[$.

Therefore

$$\sum_{i=1}^{n-m}\sum_{k=i}^{n-m} q_{ik}(t;\varepsilon)v^{(n-k)}(t)v^{(i-1)}(t) \geq \delta(t-a-\varepsilon)^n(a+2\varepsilon-t)^n \tag{2.14}$$

$$\text{for} \quad a+\varepsilon \leq t \leq a+2\varepsilon, \quad 0<\varepsilon\leq\varepsilon_0,$$

where

$$q_{ik}(t;\varepsilon) = (t-a-\varepsilon)^n(a+2\varepsilon-t)^n c_{ik}(t).$$

After integrating the latter inequality from $a+\varepsilon$ to $a+2\varepsilon$ according to Lemma 4.1 from [7], we obtain

$$\sum_{i=1}^{n-m}\sum_{k=i}^{n-m}\sum_{j=0}^{m_{ik}} \nu_{ikj}\int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2 d\tau \geq$$
$$\geq \delta\int_{a+\varepsilon}^{a+2\varepsilon}(\tau-a-\varepsilon)^n(a+2\varepsilon-\tau)^n\,d\tau, \tag{2.15}$$

where m_{ik} is the integer part of the number $\frac{1}{2}(n-k-i+1)$ and ν_{ikj} $(i=1,\dots,n-m;\ k=i,\dots,n-m;\ j=0,\dots,m_{ik})$ are the positive constants intependent of a, ε and v.

If $k\in\{i+1,\dots,n-m\}$, then we have

$$i+j-1\leq m-1, \quad 2n-(n-k-i-2j+1)\geq 2i+2j+n$$

for any $j\in\{0,\dots,m_{ik}\}$.

Therefore, taking into account (2.11) and (2.14), we find

$$[v^{(i+j-1)}(t)]^2 = \left[\frac{1}{(m-i-j)!}\int_a^t (t-\tau)^{m-i-j}v^{(m)}(\tau)\,d\tau\right]^2 \leq$$
$$\leq \alpha(\varepsilon)\varepsilon^{2m-2i-2j+1} \quad \text{for } a<t\leq a+2\varepsilon \tag{2.16}$$

and

$$|q_{ik}^{(n-k-i-2j+1)}(t;\varepsilon)| \leq \alpha_1\varepsilon^{2i+2j+n} \quad \text{for } a\leq t\leq a+2\varepsilon,$$

where

$$\alpha(\varepsilon) = 2^{2m-1}\int_a^{a+2\varepsilon}[v^{(m)}(\tau)]^2\,d\tau \to 0 \quad \text{for } \varepsilon\to 0 \tag{2.17}$$

and α_1 is a positive constant independent of ε. Therefore

$$\left|\int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2\,d\tau\right| \leq \alpha_1\alpha(\varepsilon)\varepsilon^{2m+2+n} \leq$$
$$\leq \alpha_1\alpha(\varepsilon)\varepsilon^{2n+1}.$$

Consider now the case $k = i$. By virtue of (2.12) and (2.14) we have

$$|q_{ik}^{(n-k-i-2j+1)}(t;\varepsilon)| = |q_{ii}^{(n-2i-2j+1)}(t;\varepsilon)| \leq \alpha_2 \varepsilon^{2n-2m+2i+2j-1}$$
$$\text{for } a \leq t \leq a + 2\varepsilon,$$

where α_2 is a positive constant independent of ε. Therefore if $i+j-1 = m$, then

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}^{(n-2i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2 \, d\tau \right| =$$
$$= \left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}(\tau;\varepsilon)[v^{(m)}(\tau)]^2 \, d\tau \right| \leq \alpha_2 \alpha(\varepsilon) \varepsilon^{2n+1},$$

if however $i + j - 1 < m$, then, taking into account (2.16), we obtain

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}^{(n-2i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2 \, d\tau \right| \leq \alpha_2 \alpha(\varepsilon) \varepsilon^{2n+1}.$$

Thus

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2 \, d\tau \right| \leq \alpha_0 \alpha(\varepsilon) \varepsilon^{2n+1}, \quad (2.18)$$
$$(i = 1, \dots, n - m; \quad k = i, \dots, n - m, \quad j = 0, \dots, m_{ik}),$$

where $\alpha_0 = \max\{\alpha_1, \alpha_2\}$.

On the other hand,

$$\int_{a+\varepsilon}^{a+2\varepsilon} (\tau - a - \varepsilon)^n (a + 2\varepsilon - \tau)^n \, d\tau \geq \frac{\varepsilon^n}{2^n} \int_{a+\varepsilon}^{a+\frac{3\varepsilon}{2}} (\tau - a - \varepsilon)^n d\tau =$$
$$= \frac{1}{2^{2n+1}(n+1)} \varepsilon^{2n+1}.$$

Due to (2.18) and the latter inequality we find from (2.15) that

$$\alpha(\varepsilon) \geq \delta_0 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0,$$

where δ_0 is a positive constant independent of ε. But the latter inequality contradicts the condition (2.17). This contradiction proves that (2.13) holds.

The equality

$$\liminf_{t \to b-} |w(t)| = 0$$

is proved similarly, the only difference being that for $n = 2m + 1$ instead of (2.12) the condition

$$v^{(m)}(b-) = 0$$

is used. ■

Proof of Lemma 2.2. For each component u_j $(j = 1, \dots, l)$ of the solution u of the problem (2.7), (1.2) we have

$$|u_j^{(i-1)}(t)| = \left| \frac{1}{(m-i)!} \int_a^t (t-\tau)^{m-i} u_j^{(m)}(\tau)\, d\tau \right| \leq$$

$$\leq \frac{1}{(m-i)!\sqrt{2m-2i+1}} (t-a)^{m-i+\frac{1}{2}} \left(\int_a^b [u_j^{(m)}(\tau)]^2\, d\tau \right)^{\frac{1}{2}}$$

$$\text{for } a < t < b \quad (i = 1, \dots, m)$$

and

$$|u_j^{(i-1)}(t)| = \left| \frac{1}{(m-i)!} \int_t^b (\tau-t)^{m-i} u_j^{(m)}(\tau)\, d\tau \right| \leq$$

$$\leq \frac{1}{(m-i)!\sqrt{2m-2i+1}} (b-t)^{m-i+\frac{1}{2}} \left(\int_a^b [u_j^{(m)}(\tau)]^2\, d\tau \right)^{\frac{1}{2}}$$

$$\text{for } a < t < b \quad (i = 1, \dots, m).$$

Therefore

$$|u_j^{(i-1)}(t)| \leq \sigma_{im}(a, b; t)\rho_j \quad \text{for } a < t < b \ (i = 1, \dots, m), \tag{2.19}$$

where

$$\rho_j = \left(\int_a^b [u_j^{(m)}(\tau)]^2\, d\tau \right)^{\frac{1}{2}}.$$

Let

$$H_i(t) = \big(h_{ijk}(t)\big)_{j,k=1}^{l} \quad (i = 1, \dots, m), \quad h(t) = (h_j(t))_{j=1}^{l}.$$

Rewrite (2.7) in terms of components as

$$(-1)^{n-m-1} u_j^{(n)}(t) \operatorname{sign} u_j(t) \geq$$

$$\geq -\sum_{i=1}^{m} \sum_{k=1}^{l} h_{ijk}(t) |u_k^{(i-1)}(t)| - h_j(t) \ (j = 1, \dots, l). \tag{2.7'}$$

After multiplying both sides of (2.7′) by $(t-a)^{n-2m}|u_j(t)|$ and integrating from s to t, we obtain

$$(-1)^{n-m} \int_s^t (\tau-a)^{n-2m} u_j^{(n)}(\tau) u_j(\tau)\, d\tau \leq$$

$$\leq \sum_{i=1}^{m} \sum_{k=1}^{l} \int_s^t (\tau-a)^{n-2m} h_{ijk}(\tau) |u_k^{(i-1)}(\tau)| |u_j(\tau)|\, d\tau +$$

$$+ \int_s^t (\tau-a)^{n-2m} h_j(\tau) |u_j(\tau)|\, d\tau \quad \text{for } a < s \leq t < b. \tag{2.20}$$

By virtue of (2.19)

$$\sum_{k=1}^{l}\int_s^t(\tau-a)^{n-2m}h_{ijk}(\tau)|u_k^{(i-1)}(\tau)||u_j(\tau)|\,d\tau\le$$
$$\le\rho_j\sum_{k=1}^{l}\rho_k\int_s^t(\tau-a)^{n-2m}\sigma_{1m}(a,b;\tau)\sigma_{im}(a,b;\tau)h_{ijk}(\tau)\,d\tau=$$
$$=\rho_j\sum_{k=1}^{l}\rho_k\int_s^t(\tau-a)^{n-2m}\lambda_{im}(a,b;\tau)h_{ijk}(\tau)\,d\tau \tag{2.21}$$
$$(i=1,\dots,m),$$
$$\int_s^t(\tau-a)^{n-2m}h_i(\tau)|u_j(\tau)|\,d\tau\le$$
$$\le\rho_j\int_s^t(\tau-a)^{n-2m}\sigma_{1m}(a,b;\tau)h_j(\tau)\,d\tau. \tag{2.22}$$

On the other hand, by Lemma 4.1 from [7]

$$\int_s^t(\tau-a)^{n-2m}u_j^{(n)}(\tau)u_j(\tau)\,d\tau=$$
$$=w_j(t)-w_j(s)+(-1)^{n-m}\mu_n\int_s^t[u_j^{(m)}(\tau)]^2\,d\tau, \tag{2.23}$$

where

$$w_j(t)=\begin{cases}\sum\limits_{p=1}^{n-m}(-1)^{p-1}u_j^{(n-p)}(t)u_j^{(p-1)}(t) & \text{for } n=2m,\\ \sum\limits_{p=1}^{n-m-1}(-1)^{p-1}[(t-a)u_j^{(n-p)}(t)- & \\ -pu_j^{(n-p-1)}(t)]u_j^{(p-1)}(t)+(-1)^m\frac{t-a}{2}[u_j^{(m)}(t)]^2 & \text{for } n=2m+1.\end{cases}$$

As one may readily verify, the functions w_j $(j=1,\dots,l)$ satisfy the conditions of Lemma 2.3 and therefore

$$\liminf_{s\to a+}|w_j(s)|=0,\quad \liminf_{t\to b-}|w_j(t)|=0\quad (j=1,\dots,l).$$

Taking into account latter equalities and conditions (1.7) and (1.8) from (2.20)-(2.23) we obtain

$$\mu_n\rho_j^2\le\rho_j\sum_{i=1}^{m}\sum_{k=1}^{l}\rho_k\int_a^b(\tau-a)^{n-2m}\lambda_{im}(a,b;\tau)h_{ijk}(\tau)d\tau+$$
$$+\rho_j\int_a^b(\tau-a)^{n-2m}\sigma_{1m}(a,b;\tau)h_j(\tau)d\tau\quad (j=1,\dots,l).$$

Hence by virtue of (2.6) we have

$$\mu_n \rho \leq H\rho + \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) h(\tau) d\tau,$$

where $\rho = (\rho_j)_{j=1}^l$. In view of (1.9) and the notation (2.10) from the latter inequality we find

$$\rho \leq (\mu_n E - H)^{-1} \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) h(\tau) d\tau$$

and

$$\|\rho\| \leq \|(\mu_n E - H)^{-1}\| \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) \|h(\tau)\| d\tau = l^{-\frac{1}{2}} \rho_0.$$

Hence

$$\int_a^b \|u^{(m)}(\tau)\|^2 d\tau \leq l\|\rho\|^2 \leq \rho_0^2.$$

On the other hand, in view of (2.19)

$$\|u^{(i-1)}(t)\| \leq \sigma_{im}(a, b; t)\|\rho\| \leq \rho_0 \sigma_{im}(a, b; t) \quad \text{for} \quad a < t < b$$
$$(i = 1, \dots, m).$$

Therefore, the estimates (2.8) and (2.9) hold. ■

We prove quite similarly

Lemma 2.4. *Let $H_i :]a, b] \to \mathbb{R}_+^{l \times l}$ $(i = 1, \dots, m)$ and $h :]a, b] \to \mathbb{R}_+^l$ be measurable matrix and vector functions satisfying the conditions (1.10)-(1.12). Then for any solution $u \in \widetilde{C}^{n-1,m}(]a, b]; \mathbb{R}^l)$ of the problem (2.7), (1.3) we have the estimates*

$$\int_a^b \|u^{(m)}(\tau)\|^2 d\tau \leq \rho_0^2$$

and

$$\|u^{(i-1)}(t)\| \leq \rho_0 (t - a)^{m-i+\frac{1}{2}} \quad \text{for} \quad a < t < b \ (i = 1, \dots, m),$$

where

$$\rho_0 = \frac{\sqrt{l}}{(m-1)!\sqrt{2m-1}} \|(\mu_n E - H)^{-1}\| \int_a^b (\tau - a)^{n-m-\frac{1}{2}} \|h(\tau)\| \, d\tau,$$

$$H = \sum_{i=1}^m \frac{1}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}} \int_a^b (\tau - a)^{n-i} H_i(\tau) d\tau,$$

and E is the unit $l \times l$ matrix.

§ 3. Proof of the main results

Proof of Theorem 1.1. Let ρ_0 and $\sigma_{im}(a,b;t)$ $(i=1,\dots,m)$ be respectively the number and functions from Lemma 2.2 and

$$\varphi(t)=4^n\sup\{q(t,x_1,\dots,x_m)\,:\,\|x_i\|\le\rho_0\sigma_{im}(a,b;t)\ (i=1,\dots,m)\}. \tag{3.1}$$

Than due to (1.6), (2.1) holds with $I=I_n(a,b)$.

For $k=n-m-1$, ρ_0 and φ by virtue of Lemma 2.1 there exists a continuous function $\rho:I_n(a,b)\to\mathbb{R}_+$ such that estimates (2.4) are valid for any vector function $v\in\widetilde{C}^k_{loc}(I_n(a,b);\mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3).

Let

$$\rho_i(t)=\begin{cases}\rho_0\sigma_{im}(a,b;t) & \text{for}\quad i\in\{1,\dots,m\}\\ \rho(t) & \text{for}\quad i\in\{m+1,\dots,n\}\end{cases} \tag{3.2}$$

and

$$f^*(t)=\sup\{\|f(t,x_1,\dots,x_n)\|\,:\,\|x_i\|\le\rho_i(t)\ (i=1,\dots,n)\}.$$

For any $i\in\{1,\dots,n\}$ and $\xi=(\xi_p)_{p=1}^l$ we set

$$\chi_{ip}(t,\xi)=\begin{cases}\xi_p & \text{for}\quad |\xi_p|\le\rho_i(t)\\ \rho_i(t)\operatorname{sign}\xi_p & \text{for}\quad |\xi_p|>\rho_i(t)\end{cases},\qquad \chi_i(t,\xi)=\big(\chi_{ip}(t,\xi)\big)_{p=1}^l. \tag{3.3}$$

Let j be an arbitrary natural number,

$$I_{nj}(a,b)=\begin{cases}[a+\frac{b-a}{3j},\,b-\frac{b-a}{3j}] & \text{for}\quad n=2m\\ [a+\frac{b-a}{3j},\,b] & \text{for}\quad n=2m+1\end{cases},$$

$$f_j(t,x_1,\dots,x_n)=$$
$$=\begin{cases}f(t,\chi_1(t,x_1),\dots,\chi_n(t,x_n)) & \text{for}\quad t\in I_{nj}(a,b)\\ 0 & \text{for}\quad t\in[a,b]\setminus I_{nj}(a,b)\end{cases}, \tag{3.4}$$

$$f_j^*(t)=\begin{cases}f^*(t) & \text{for}\quad t\in I_{nj}(a,b)\\ 0 & \text{for}\quad t\in[a,b]\setminus I_{nj}(a,b)\end{cases}.$$

Clearly, that $f_j^*:[a,b]\to\mathbb{R}_+$ is the Lebesgue integrable function and on the $[a,b]\times\mathbb{R}^{nl}$ the inequality

$$\|f_j(t,x_1,\dots,x_n)\|\le f_j^*(t)$$

holds. On the other hand the homogeneous differential system

$$u^{(n)} = 0$$

by boundary conditions (1.2) has only the trivial solution. Therefore by virtue of the Conti theorem [1] [1] the differential system

$$u^{(n)} = f_j(t, u, \dots, u^{(n-1)})$$

has a solution $u_j \in \widetilde{C}^{n-1}_{loc}([a,b];\mathbb{R}^l)$ satisfying the boundary conditions (1.2). It is obvious that $u_j \in \widetilde{C}^{n-1,m}([a,b];\mathbb{R}^l)$. Simultaneously, from (1.4), (3.3) and (3.4) follows that u_j is the solution of the system of the differential inequalities (2.7). Therefore by virtue of Lemma 2.2

$$\int_a^b \|u_j^{(m)}(\tau)\|^2 \, d\tau \le \rho_0^2 \tag{3.5}$$

and

$$\|u_j^{(i-1)}(t)\| \le \rho_0 \sigma_{im}(a,b;t) \quad \text{for} \quad a < t < b \quad (i = 1, \dots, m). \tag{3.6}$$

From conditions (1.5) and (3.1)-(3.6) is clear that the vector function $v_j(t) = u_j^{(m)}(t)$ satifies the inequalities (2.2) and (2.3). Hence by Lemma 2.1

$$\|u_j^{(i-1)}(t)\| < \rho(t) \quad \text{for} \quad t \in I_n(a,b) \quad (i = m+1, \dots, n).$$

Therefore

$$\|u_j^{(i-1)}(t)\| < \rho_i(t) \quad \text{for} \quad a < t < b \quad (i = 1, \dots, n) \tag{3.7}$$

and

$$\|u_j^{(n)}(t)\| \le f^*(t) \quad \text{for} \quad a < t < b. \tag{3.8}$$

Moreover, in view of (3.3),(3.4) and (3.7) it is clear that

$$u_j^{(n)}(t) = f(t, u_j(t), \dots, u_j^{(n-1)}(t)) \quad \text{for } t \in I_{nj}(a,b). \tag{3.9}$$

Since $f^* \in L_{loc}(I_n(a,b);\mathbb{R}_+)$, the estimates (3.7) and (3.8) imply that the sequences $(u_j^{(i-1)})_{j=1}^{+\infty}$ $(i = 1, \dots, n)$ are uniformly bounded and equicontinuous on each segment contained in $I_n(a,b)$. Therefore, by virtue of the Arcela-Ascoli lemma these sequences can be regarded without loss of generality as uniformly converging on each segment from $I_n(a,b)$.

[1] also see [8], corollary 2.1

If we set

$$\lim_{j\to+\infty} u_j(t) = u(t) \quad \text{for } t \in I_n(a,b),$$

then

$$\lim_{j\to+\infty} u_j^{(i-1)}(t) = u^{(i-1)}(t) \quad \text{for } t \in I_n(a,b) \ \ (i = 1,\dots,n) \tag{3.10}$$

uniformly on each segment contained in $I_n(a,b)$. Therefore from (3.5) and (3.6) we obtain

$$\int_a^b \|u^{(m)}(\tau)\|^2\,d\tau \le \rho_0^2, \tag{3.11}$$

$$\|u^{(i-1)}(t)\| \le \rho_0\sigma_{im}(a,b;t) \quad \text{for} \quad t \in I_n(a,b) \tag{3.12}$$
$$(i = 1,\dots,m).$$

In view of (3.9) for arbitrary fixed s and $t \in I_n(a,b)$ there exists a natural number j_0 such that

$$u_j^{(n-1)}(t) - u_j^{(n-1)}(s) = \int_s^t f(\tau, u_j(\tau),\dots,u_j^{(n-1)}(\tau))d\tau$$
$$(j = j_0, j_0+1,\dots)$$

and

$$s,t \in I_{nj}(a,b) \quad \text{for} \quad j \ge j_0.$$

Passing to the limit in the latter equality by $j \to +\infty$, we obtain

$$u^{(n-1)}(t) - u^{(n-1)}(s) = \int_s^t f(\tau, u(\tau),\dots,u^{(n-1)}(\tau))d\tau.$$

Therefore u is the solution of the system (1.1). Simultaneously, (3.10)-(3.12) imply that $u \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$ and satisfies the boundary conditions (1.2). ■

Theorem 1.1′ immediately follows from Theorem 1.1, since in the case when

$$f(t,x_1,\dots,x_n) \equiv f(t,x_1,\dots,x_m)$$

and $f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml};\mathbb{R}^l)$, the inequality (1.5) is fulfilled automatically and the function $q(t,x_1,\dots,x_m) \equiv \|f(t,x_1,\dots,x_m)\|$ satisfies the condition (1.6).

Proof of Theorem 1.3. (1.13) and (1.14) yield the conditions (1.4′) and (1.7), where

$$h(t) = |f(t,0,\dots,0)|.$$

Therefore by virtue of Theorem 1.1′ the problem (1.1′), (1.2) is solvable in the class $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

To complete the proof of the theorem it remains for us to verify that the problem under consideration has at most one solution in the class $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Let $u, \bar{u} \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$ be two arbitrary solutions of the problem (1.1′), (1.2). We set

$$v(t) = u(t) - \bar{u}(t) \quad \text{for} \quad t \in I_n(a,b).$$

It is clear that

$$v \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$$

and

$$v(a+) = \cdots = v^{(m-1)}(a+) = 0, \quad v(b-) = \cdots = v^{(n-m-1)}(b-) = 0.$$

On the other hand, by the condition (1.14) from the equality

$$v^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)) - f(t, \bar{u}(t), \ldots, \bar{u}^{(n-1)}(t))$$

we have

$$(-1)^{n-m-1} S(v(t)) v^{(n)}(t) \geq -\sum_{i=1}^{m} H_i(t) |v^{(i-1)}(t)|.$$

Therefore due to Lemma 2.2

$$v(t) \equiv 0,$$

i.e.,

$$u(t) \equiv \bar{u}(t). \quad \blacksquare$$

Theorems 1.2 and 1.2′ are proved similarly to Theorems 1.1 and 1.1′, while Theorem 1.4 similarly to Theorem 1.3 with the only difference consisting in that Lemma 2.4 is used instead of Lemma 2.2.

References

1. R. Conti, Equazioni differenziali oridinarie quasilineari con condizioni lineari. *Ann. mat. pura ed appl.* **57**(1962), 49-61.

2. I.T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi,* 1975.

3. ——, On some singular boundary value problems for ordinary differential equations. *Equadiff, 5. Proc. Czech. Conf. Diff. Equations and Applications*, 174-178, *Teubner Verlag, Leipzig*, 1982.

4. I.T. Kiguradze, On solvability of boundary value problem of de la Vallèe Poussin. (Russian) *Differentsial'nie Uravneniya* **21**(1985), No. 3, 391-398.

5. ——, On boundary value problems for high order ordinary differential equations with singularities. (Russian) *Uspekhi Mat. Nauk.* **41**(1986), No. 4, 166-167.

6. ——, On two-point singular boundary value problems for nonlinear ordinary differential equations. (Russian) *Proceedings of All-Union symposium on current problems of mathematical phisics, vol. 1 (Russian),* 276-279, *Tbilisi University Press, Tbilisi*, 1987.

7. I.T. Kiguradze and T.A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. (Russian) *"Nauka", Moscow*, 1990.

8. I.T. Kiguradze, The boundary value problems for systems of ordinary differential equations. (Russian) *Current problems in mathematics. Newest results, vol. 30 (Russian),* 3-103, *Itogi nauki i tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Techn. Inform., Moscow,* 1987.

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MEASURES OF CONTROLLABILITY

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ABSTRACT. We introduce here a new notion, the measure of *controllability* aimed at expressing that one system is "more controllable" than another one. First estimates are given.

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1. INTRODUCTION

Let Ω be an open set in $\mathbb{R}^n$, bounded or not, with boundary Γ, smooth or not.

In the domain Ω and for $t > 0$, we consider the system whose state $y : y(x,t) = y(x,t;v)$ is given as follows:

$$\frac{\partial y}{\partial t} + Ay = v(x,t)\chi_{\mathcal{O}} \quad \text{in} \quad \Omega \times \{t > 0\}, \tag{1.1}$$

where A = second order elliptic operator in Ω (its coefficients are not necessarily smooth and they may depend on t),

$\mathcal{O}$ = open set $\subset \Omega$,

$\chi_{\mathcal{O}}$ = characteristic function of $\mathcal{O}$,

$v = v(x,t)$ = control function.

We add to (1.1) the *initial* and boundary *conditions* respectively given by

$$y(x,0) = y^0(x) \quad \text{in} \quad \Omega, \quad y^0 \text{ given in } L^2(\Omega), \tag{1.2}$$

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and

$$y = 0 \quad \text{on} \quad \Gamma \times \{t > 0\}. \tag{1.3}$$

Under reasonable conditions on the coefficients of A (cf.for instance J.L.Lions [3]), and assuming that

$$v \in L^2(\mathcal{O} \times (0,T)), \tag{1.4}$$

equations (1.1),(1.2),(1.3) admit a *unique solution* y, which is such that

$$y, \frac{\partial y}{\partial x_i} \in L^2(\Omega \times (0,T)). \tag{1.5}$$

This defines the state of the system, with distributed control with support in $\mathcal{O}$. ■

Remark 1.1. Boundary condition (1.3) is taken here *to fix ideas*. What follows readily applies to other situations corresponding to other boundary conditions. ■

Remark 1.2. All what follows readily extends to higher order parabolic equations, to systems of parabolic equations and actually to *all evolution equations*, provided they are *linear*. This will be reported elsewhere. Cf. also the Remarks of the last section of this paper. ■

Remark 1.3. One knows that (J.L.Lions [3]) after a possible change on a set of 0 measure, the function $t \to y(t) = y(\cdot,t)$ is continuous from $[0,T] \to L^2(\Omega)$. ■

Approximate controllability is defined as follows (cf. for instance J.L.Lions [4]). We are given T and $y^1 \in L^2(\Omega)$. Let B denote the unit ball in $L^2(\Omega)$ and let β be a positive number arbitrarily small.

It is known (J.L.Lions [5]) that, when v spans $L^2(\mathcal{O} \times (0,T))$, the functions $y(\cdot,T;v)$ describe an affine space in $L^2(\Omega)$ which is dense in $L^2(\Omega)$. Therefore one can always find functions v (controls) such that

$$y(T;v) \in y^1 + \beta B \tag{1.6}$$

and there are *infinitely many v 's* such that (1.6) takes place. One says that the system is *approximately controllable*. It is natural to look for the (actually unique) element v such that

$$\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt = \min \tag{1.7}$$

where v is restricted to those elements such that (1.6) takes place.

The question we want to address here is the following: *when can we say that a system is more controllable than another one?*

In this question we assume that Ω and that $\mathcal{O}$ do not change. Then the min in (1.7) is a quantity which depends on A, y^0, y^1 and β and T. We write

$$\begin{aligned} &\inf_v \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt = M(A, y^0, y^1, \beta, T), \\ &y(T; v) \in y^1 + \beta B. \end{aligned} \tag{1.8}$$

We have to introduce a quantity which is *independent* of y^0 and of y^1 but which only depends on the sets described by y^0 and by y^1.

We shall assume

$$y^0 \in \alpha_0 B, \quad y^1 \in \alpha_1 B \tag{1.9}$$

and we introduce as a "measure of controllability" the quantity

$$M(A, \alpha_0, \alpha_1, \beta, T) = \sup_{\substack{y^0 \in \alpha_0 B \\ y^1 \in \alpha_1 B}} M(A, y^0, y^1, \beta, T). \tag{1.10}$$

Remark 1.4. This quantity seems to be introduced here for the first time. The study of the function

$$A \to M(A, \alpha_0, \alpha_1, \beta, T) \tag{1.11}$$

leads to many seemingly interesting open questions. We shall return to these questions in other occasions. ■

Remark 1.5. It is not obvious that the quantity introduced in (1.9) is always finite. Indeed this quantity is finite iff $\beta > \alpha_1$. ■

Remark 1.6. We shall give below a number of simple formulas reducing the number of variables $\alpha_0, \alpha_1, \beta$ to actually one variable. ■

We are now going to give a formula for $M(A, \alpha_0, \alpha_1, \beta, T)$ which is based on *duality arguments*.

2. Duality Formula for the Measure of Controllability

We introduce the decomposition

$$y(x, t; v) = y(v) = y_0 + z(v) \tag{2.1}$$

where

$$\begin{aligned} &\frac{\partial y_0}{\partial t} + A y_0 = 0, \\ &y_0(0) = y^0, \quad y_0 = 0 \quad \text{on} \quad \Gamma \times (0, T) \end{aligned} \tag{2.2}$$

and

$$\frac{\partial z}{\partial t} + Az = v\chi_{\mathcal{O}}, \qquad z(0) = 0, \quad z = 0 \quad \text{on} \quad \Gamma \times (0,T). \tag{2.3}$$

Then

$$M(A, y^0, y^1, \beta, T) = \inf \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt, \qquad z(T; v) \in y^1 - y_0(T) + \beta B. \tag{2.4}$$

We introduce the convex functions defined by

$$F_0(v) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt, \quad v \in L^2(\mathcal{O} \times (0,T)), \tag{2.5}$$

$$F_1(f) = \begin{cases} 0 & \text{if } f \in y^1 - y_0(T) + \beta B, \\ +\infty & \text{otherwise in } \ L^2(\Omega). \end{cases} \tag{2.6}$$

We define the linear operator L by

$$Lv = z(T; v). \tag{2.7}$$

One has

$$L \in \mathcal{L}(L^2(\mathcal{O} \times (0,T)); L^2(\Omega)). \tag{2.8}$$

With those notations (this is only a matter of definition)

$$M(A, y^0, y^1, \beta, T) = \inf_{v \in L^2(\mathcal{O}\times(0,T))} F_0(v) + F_1(Lv). \ \blacksquare \tag{2.9}$$

The next step is to use Fenchel-Rockafellar duality (cf. T.R.Rockafellar [6] and the presentation made in I.Ekeland and R.Temam [1]).

In general, the conjugate function F_i^* of F_i is defined by

$$F_i^*(f) = \sup_{\hat{f}} [(f, \hat{f}) - F_i(\hat{f})].$$

With these definitions, one has

$$F_0^*(v) = F_0(v), \qquad F_1^*(f) = (f, y^1 - y_0(T)) + \beta \|f\|, \qquad \text{where } \|f\| \in \left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}. \tag{2.10}$$

Let L^* denote the adjoint of L. Then (T.R.Rockafellar, loc.cit.)

$$\begin{aligned} &\inf_{v\in L^2(\mathcal{O}\times(0,T))} F_0(v)+F_1(Lv)= \\ &-\inf_{f\in L^2(\Omega)} F_0^*(L^*f)+F_1^*(-f). \ \blacksquare \end{aligned} \tag{2.11}$$

The operator L^* is given as follows. If f is given in $L^2(\Omega)$, we solve

$$\begin{aligned} &-\frac{\partial\psi}{\partial t}+A^*\psi=0, \quad t<T, \\ &\psi(x,T)=f(x) \quad \text{in} \quad \Omega, \\ &\psi=0 \quad \text{on} \quad \Gamma\times\{t<T\}, \end{aligned} \tag{2.12}$$

where A^* = adjoint of A.

This problem admits a unique solution $\psi(x,t)=\psi(x,t;f)=\psi(f)$. Then one easily verifies that

$$L^*f=\psi\chi_{\mathcal{O}}. \tag{2.13}$$

Using this result, (2.11), and (2.10), we obtain

$$\begin{aligned} M(A,y^0,y^1,\beta,T) = -&\inf_{f\in L^2(\Omega)} \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \psi^2 dx\,dt - \\ &-(f,y^1-y_0(T))+\beta\|f\|. \end{aligned} \tag{2.14}$$

If we multiply (2.12) by y_0, we obtain after integration by parts

$$-(f,y_0(T))+(\psi(0),y^0)=0 \tag{2.15}$$

so that (2.14) can be written

$$\begin{aligned} &M(A,y^0,y^1,\beta,T)= \\ &=-\inf_{f\in L^2(\Omega)} \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} v^2 dx\,dt - \\ &-(f,y^1)+(\psi(0),y^0)+\|f\|. \ \blacksquare \end{aligned} \tag{2.16}$$

By definition

$$\begin{aligned} &M(A,\alpha_0,\alpha_1,\beta,T)= \\ &=\sup_{y^0\in\alpha_0 B,\, y^1\in\alpha_1 B} M(A,y^0,y^1,\beta,T) = (\text{using } (2.16)) = \\ &=-\inf_{y^0\in\alpha_0 B,\, y^1\in\alpha_1 B,\, f\in L^2(\Omega)} \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \psi^2 dx\,dt - \\ &-(f,y^1)+(\psi(0),y^0)+\beta\|f\|, \end{aligned} \tag{2.17}$$

i.e.

$$M(A,\alpha_0,\alpha_1,\beta,T) = \\ = \inf_f \Big[\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \psi^2 dx\,dt + \\ + (\beta-\alpha_1)\|f\| - \alpha_0\|\psi(0)\|\Big]. \tag{2.18}$$

In summary:

the measure of controllability is given by formula (2.18), *where* $\psi = \psi(f)$ *is given by* (2.12). ■ (2.19)

Remark 2.1. One can show that the $\inf_f$ in (2.18) is finite iff $\beta > \alpha_1$. ■

One has

$$M(A,\alpha_0,\alpha_1,\beta,T) = M(A,\alpha_0,0,\beta-\alpha_1,T), \quad \beta > \alpha_1. \tag{2.20}$$

Therefore it suffices to consider the following situation:

$$\sup_{y^0\in\alpha\beta} \inf \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} v^2 dx\,dt = M_0(A,\alpha,\beta,T), \\ y(T;v) \in \beta B \tag{2.21}$$

(Then $M(A,\alpha_0,\alpha_1,\beta,T) = M_0(A,\alpha_0,\beta-\alpha_1,T)$).

One verifies directly that

$$M_0(A,\alpha,\beta,T) = \alpha^2 M_0(A,1,\frac{\beta}{\alpha},T), \tag{2.22}$$

$$M_0(A,\alpha,\beta,T) = \begin{cases} 0 & \text{for } \beta \text{ large enough}, \\ \text{increases to } +\infty \text{ as } \beta \text{ decreases to } 0. \end{cases} \tag{2.23}$$

Remark 2.2. Formula (2.18) is constructive. One can deduce from it numerical algorithms for the approximation of M. Cf. R.Glowinski and J.L.Lions [2]. ■

REFERENCES

1. I.Ekeland and R.Temam, Analyse Convexe et problèmes variationnels. *Dunod, Gauthier Villars, Paris-Bruxeles-Montréal,* 1974.
2. R.Glowinski and J.L.Lions, To appear in *Acta Numerica,* 1993.
3. J.L.Lions, Equations Differentielles Opérationnelles et Problèmes aux limites. *Springer-Verlag, Berlin-New York,* 1961.

4. J.L.Lions, Exact Controllability for distributed systems. Some trends and some problems. *Applied and Industrial Mathematics, R.Spigler (ed.)*, 59-84, *Kluwer*, 1991.

5. ——, Controle Optimal de systèmes gouvernés par les équations aux dérivées partielles. *Dunod, Gauthier-Villars, Paris,* 1968.

6. T.R.Rockafellar, Duality and stability in extremum problems involving convex functions. *Pac. J. Math.* **21**(1967), 167-187.

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ON SOME ENTIRE MODULAR FORMS OF WEIGHTS 5 AND 6 FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

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Abstract. Two entire modular forms of weight 5 and two of weight 6 for the congruence subgroup $\Gamma_0(4N)$ are constructed, which will be useful for revealing the arithmetical sense of additional terms in the formulae for the number of representations of positive integers by the quadratic forms in 10 and 12 variables.

რეზიუმე. აგებულია $\Gamma_0(4N)$ შედარებათა ქვეჯგუფის მიმართ 5 და 6 წონის ორ-ორი მთელი მოდულური ფორმა, რომლებიც სასარგებლო იქნებიან შესაბამისად 10 და 12 ცვლადიანი კვადრატული ფორმებით ნატურალური რიცხვის წარმოდგენათა რაოდენობის გამომსახველი ფორმულების დამატებითი წევრების არითმეტიკული აზრის გამოსავლენად.

In this paper N, a, k, n, r, s, t denote positive integers; u are odd positive integers; H, c, g, h, j, m, α, β, γ, δ, ξ, η are integers; A, B, C, D are complex numbers and z, τ ($\operatorname{Im}\tau > 0$) complex variables. Further, $(\frac{h}{u})$ is the generalized Jacobi symbol, $\binom{n}{t}$ a binomial coefficient, $\varphi(k)$ Euler's function, $e(z) = \exp 2\pi i z$, $\eta(\gamma) = 1$ if $\gamma \geq 0$ and $\eta(\gamma) = -1$ if $\gamma < 0$.

Let

$$\Gamma = \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \middle| \alpha\delta - \beta\gamma = 1 \right\},$$

$$\Gamma_0(4N) = \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \in \Gamma \middle| \gamma \equiv 0 \pmod{4N} \right\}.$$

Definition. We shall say that a function F defined on $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ is an entire modular form of weight r and character $\chi(\delta)$ for the congruence subgroup $\Gamma_0(4N)$ if

1) F is regular on $\mathcal{H}$,

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2) for all substitutions from $\Gamma_0(4N)$ and all $\tau \in \mathcal{H}$

$$F\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \chi(\delta)(\gamma\tau+\delta)^r F(\tau),$$

3) in the neighbourhood of the point $\tau = i\infty$

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau),$$

4) for all substitutions from Γ, in the neighbourhood of each rational point $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,\ (\gamma,\delta)=1)$

$$(\gamma\tau+\delta)^r F(\tau) = \sum_{m=0}^{\infty} A'_m e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big).$$

For $\eta \neq 0$, ξ, g, h, N with $\xi g + \eta h + \xi\eta N \equiv 0 \pmod 2$, put

$$S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix};c,N\right) = \sum_{\substack{m \bmod N|\eta| \\ m\equiv c \pmod N}} (-1)^{h(m-c)/N} e\Big(\frac{\xi}{2N}\Big(m+\frac{g}{c}\Big)^2\Big).$$

It is known ([2] p. 323, formulae (2.4)-(2.6)) that

$$\begin{aligned} S_{g+2j,h}\left(\begin{matrix}\xi\\ \eta\end{matrix};c,N\right) &= S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix};c+j,N\right),\\ S_{g,h+2j}\left(\begin{matrix}\xi\\ \eta\end{matrix};c,N\right) &= S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix};c,N\right), \qquad (1)\\ S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix};c+Nj,N\right) &= (-1)^{hj} S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix};c,N\right). \end{aligned}$$

Let

$$\vartheta_{gh}(z|\tau;c,N) = \sum_{m\equiv c \pmod N} (-1)^{h(m-c)/N} e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big) e\Big(\Big(m+\frac{g}{2}\Big)z\Big), \qquad (2)$$

hence

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau;c,N) = (\pi i)^n \sum_{m\equiv c \pmod N} (-1)^{h(m-c)/N} (2m+g)^n \times$$
$$\times e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big) e\Big(\Big(m+\frac{g}{2}\Big)z\Big). \qquad (3)$$

Put

$$\vartheta^{(n)}_{gh}(\tau;c,N)=\frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau;c,N)\Big|_{z=0}, \qquad (4)$$
$$\vartheta^{(0)}_{gh}(\tau;c,N)=\vartheta_{gh}(\tau;c,N)=\vartheta_{gh}(0|\tau;c,N).$$

It is known ([2], p.318, form.(1.3); p. 321, form.(1.12); p.327, formulae (3.9), (3.5), (3.3), (3.7); p.324, form.(2.16); p.327, form.(3.10), (3.11)) that

$$\vartheta_{g,h+2j}(z|\tau;c,N)=\vartheta_{gh}(z|\tau;c,N); \qquad (5)$$

$$\vartheta_{gh}\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau};c,N\Big)=\Big(-\frac{i\tau}{N}\Big)^{1/2}e\Big(\frac{Nz^2}{2\tau}\Big)\times$$
$$\times\sum_{H \bmod N} e\Big(-\frac{1}{N}\Big(c+\frac{g}{2}\Big)\Big(H+\frac{h}{2}\Big)\Big)\vartheta_{gh}(z|\tau;H,N); \qquad (6)$$

$$\vartheta_{gh}\Big(\frac{z}{\gamma\tau+\delta}\Big|\frac{\alpha\tau+\beta}{\gamma\tau+\delta};c,N\Big)=\Big(\frac{-i(\gamma\tau+\delta)\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{\frac{1}{2}}e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\times$$
$$\times\sum_{H \bmod N}\varphi_{g'gh}(c,H;N)\vartheta_{g'h'}(z|\tau;H,N)\quad(\gamma\neq 0), \qquad (7)$$

where

$$g'=\alpha g+\gamma h+\alpha\gamma N,\quad h'=\beta g+\delta h+\beta\delta N, \qquad (8)$$

$$\varphi_{g'gh}(c,H;N)=e\Big(-\frac{\beta\delta}{2N}\Big(H+\frac{g'}{2}\Big)^2\Big)e\Big(\frac{\beta}{N}\Big(c+\frac{g}{2}\Big)\Big(H+\frac{g'}{2}\Big)\Big)\times$$
$$\times S_{g-\delta g',h+\beta g'}\Big(\begin{matrix}\alpha\\ \gamma\end{matrix};c-\delta H,N\Big); \qquad (9)$$

$$\vartheta_{gh}(z|\tau+\beta;c,N)=e\Big(\frac{\beta}{2N}\Big(c+\frac{g}{2}\Big)^2\Big)\vartheta_{g,h+\beta g+\beta N}(z|\tau;c,N),$$
$$\vartheta_{gh}(-z|\tau-\beta;c,N)= \qquad (10)$$
$$=e\Big(-\frac{\beta}{2N}\Big(c+\frac{g}{2}\Big)^2\Big)\vartheta_{-g,-h+\beta g-\beta N}(z|\tau;-c,N).$$

From (5) and (10), according to the notations (4), it follows that

$$\vartheta_{g,h+2j}(\tau;c,N)=\vartheta_{gh}(\tau;c,N), \qquad (11)$$
$$\vartheta^{(n)}_{g,h+2j}(\tau;c,N)=\vartheta^{(n)}_{gh}(\tau;c,N);$$

$$\vartheta^{(n)}_{gh}(\tau+\beta;c,N)=e\Big(\frac{\beta}{2N}\Big(c+\frac{g}{2}\Big)^2\Big)\vartheta^{(n)}_{g,h+\beta g+\beta N}(\tau;c,N),$$
$$\vartheta^{(n)}_{gh}(\tau-\beta;c,N)= \tag{12}$$
$$=(-1)^n e\Big(-\frac{\beta}{2N}\Big(c+\frac{g}{2}\Big)^2\Big)\vartheta^{(n)}_{-g,-h+\beta g-\beta N}(\tau;-c,N).$$

From (2) and (3), according to the notations (4), in particular, it follows that

$$\vartheta_{gh}(\tau;0,N)=\sum_{m=-\infty}^{\infty}(-1)^{hm}e\Big(\frac{1}{2N}\Big(Nm+\frac{g}{2}\Big)^2\tau\Big),$$
$$\vartheta^{(n)}_{gh}(\tau;0,N)= \tag{13}$$
$$=(\pi i)^n\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)^n e\Big(\frac{1}{2N}\Big(Nm+\frac{g}{2}\Big)^2\tau\Big).$$

1.

Lemma 1. *For* $n\geq 0$

$$\vartheta^{(n)}_{gh}\Big(-\frac{1}{\tau};c,N\Big)=(Ni)^n\Big(-\frac{i\tau}{N}\Big)^{(2n+1)/2}\times$$
$$\times\sum_{H\bmod N}e\Big(-\frac{1}{N}\Big(c+\frac{g}{2}\Big)\Big(H+\frac{h}{2}\Big)\Big)\times$$
$$\times\Big\{\vartheta^{(n)}_{hg}(\tau;H,N)+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{hg}(\tau;H,N)\Big\},$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\big(\frac{N\pi i}{\tau}\big)^k & \text{if } t=2k\\ 0 & \text{if } t\neq 2k\end{cases}$$
$$(t=1,2,\dots,n;\ k=1,2,\dots,t). \tag{1.1}$$

Proof. From (6), by Leibnitz's formula, we obtain

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau};c,N\Big)=\tau^n\Big(\frac{-i\tau}{N}\Big)^{1/2}\frac{\partial^n}{\partial z^n}\Big\{e\Big(\frac{Nz^2}{2\tau}\Big)\times$$
$$\times\sum_{H\bmod N}e\Big(-\frac{1}{N}\Big(c+\frac{g}{2}\Big)\Big(H+\frac{h}{2}\Big)\Big)\vartheta_{hg}(z|\tau;H,N)\Big\}=$$
$$=(Ni)^n\Big(\frac{-i\tau}{N}\Big)^{(2n+1)/2}\times$$

$$\times \sum_{H \bmod N} e\Big(-\frac{1}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{h}{2}\big)\Big)\Big\{e\Big(\frac{Nz^2}{2\tau}\Big)\frac{\partial^n}{\partial z^n}\vartheta_{hg}(z|\tau;H,N)\Big\}+$$

$$+\sum_{t=1}^{n}\binom{n}{t}\frac{\partial^n}{\partial z^n}e\Big(\frac{Nz^2}{2\tau}\Big)\frac{\partial^{n-t}}{\partial z^{n-t}}\vartheta_{hg}(z|\tau;H,N). \qquad (1.2)$$

According to formulae (a) and (b) of [1], p. 37[1]

$$\frac{\partial^t}{\partial z^t}e\Big(\frac{Nz^2}{2\tau}\Big) = A_{t1}e\Big(\frac{Nz^2}{2\tau}\Big)+\frac{A_{t2}}{2!}e\Big(\frac{Nz^2}{2\tau}\Big)+\cdots+\frac{A_{tt}}{t!}e\Big(\frac{Nz^2}{2\tau}\Big)$$
$$(t=1,2,\dots,n),$$

where

$$A_{tk} = \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^k - k\Big(\frac{N\pi i z^2}{\tau}\Big)\frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^{k-1}+$$
$$+\frac{k(k-1)}{2!}\Big(\frac{N\pi i z^2}{\tau}\Big)^2\frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^{k-2}+$$
$$+\cdots+(-1)^{k-1}k\Big(\frac{N\pi i z^2}{\tau}\Big)^{k-1}\frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big) \quad (k=1,2,\dots,t),$$

hence

$$\frac{\partial^t}{\partial z^t}e\Big(\frac{Nz^2}{2\tau}\Big)\Big|_{z=0} = \sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0} \quad (t=1,2,\dots,n) \qquad (1.3)$$

and

$$A_{tk}\Big|_{z=0} = \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^k\Big|_{z=0} \quad (t=1,2,\dots,n). \qquad (1.4)$$

Thus, in view of notations (4), the lemma follows from (1.2)-(1.4). ■

Lemma 2. *If $\gamma \neq 0$, then for $n \geq 0$*

$$\vartheta_{gh}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};c,N\Big) = (N|\gamma| i\operatorname{sgn}\gamma)^n\Big(-i(\gamma\tau+\delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{(2n+1)/2}\times$$
$$\times\sum_{H \bmod N}\varphi_{g'gh}(c,H;N)\Big\{\vartheta_{g'h'}^{(n)}(\tau;H,N)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{g'h'}^{(n-t)}(\tau;H,N)\Big\},$$

[1]P. 75 in the Russian version of [1] published in 1933

where $g'\,h'$ *and* $\varphi_{g'gh}(c,H;N)$ *are defined by the formulas* (8) *and* (9),

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\left(\frac{N\gamma\pi i}{\gamma\tau+\delta}\right)^k & \text{if } t=2k\\ 0 & \text{if } t\neq 2k\end{cases}$$

$$(t=1,2,\ldots,n;\ k=1,2,\ldots,t).$$

Proof. From (7), according to Leibnitz's formula, we obtain

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}\Big(\frac{z}{\gamma\tau+\delta}\Big|\frac{\alpha\tau+\beta}{\gamma\tau+\delta};c,N\Big)=(\gamma\tau+\delta)^n\Big(-i(\gamma\tau+\delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{1/2}\times$$

$$\times\frac{\partial^n}{\partial z^n}\Big\{e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\sum_{H\bmod N}\varphi_{g'gh}(c,H;N)\vartheta_{g'h'}(z|\tau;H,N)\Big\}=$$

$$=(N|\gamma|i\operatorname{sgn}\gamma)^n\Big(-i(\gamma\tau+\delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{(2n+1)/2}\times$$

$$\times\sum_{H\bmod N}\varphi_{g'gh}(c,H;N)\Big\{e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\frac{\partial^n}{\partial z^n}\vartheta_{g'h'}(z|\tau;H,N)+$$

$$+\sum_{t=1}^{n}\binom{n}{t}\frac{\partial^t}{\partial z^t}e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\frac{\partial^{n-t}}{\partial z^{n-t}}\vartheta_{g'h'}(z|\tau;H,N)\Big\}.\qquad(1.5)$$

As in Lemma 1, but with $e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)$ instead of $e\Big(\frac{Nz^2}{2\tau}\Big)$, we have

$$\frac{\partial^t}{\partial z^t}e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\Big|_{z=0}=\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\quad(t=1,2,\ldots,n)\qquad(1.6)$$

and

$$A_{tk}\Big|_{z=0}=\frac{\partial^t}{\partial z^t}\Big(\frac{N\gamma\pi iz}{\gamma\tau+\delta}\Big)^{2k}\Big|_{z=0}\quad(k=1,2,\ldots,t).\qquad(1.7)$$

Thus, according to the notations (4), the lemma follows from (1.5)-(1.7). ■

Lemma 3. *If* g *is even, then for* $n\geq 0$ *and all substitutions from* $\Gamma_0(4N)$ *we have*

$$\vartheta_{gh}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big)=$$

$$=(\operatorname{sgn}\delta)^n\,i^{(2n+1)\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\,i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{(2n+1)/2}e\Big(-\frac{\alpha\gamma\delta^2h^2}{16N}\Big)e\Big(\frac{\beta\delta g^2}{4}\frac{dl^{2\varphi(2N)-2}}{4N}\Big)\vartheta_{\alpha g,h}^{(n)}(\tau;0,2N).$$

Proof.

1) Let $\gamma \neq 0$. In [4] (p.18, form.(5.1)) it is shown that

$$S_{g0}\left(\begin{smallmatrix}\beta\\ \delta\end{smallmatrix};0,2N\right)=$$

$$=e\left(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2n)-2}}{4N}\right)i^{(1-|\delta|)/2}\left(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\right)|\delta|^{1/2}. \qquad (1.8)$$

Replacing α, β, γ, δ, τ, c, N by β, $-\alpha$, δ, $-\gamma$, τ', 0, $2N$ in Lemma 2, we obtain

$$\vartheta_{gh}^{(n)}\left(\frac{\beta\tau'-\alpha}{\delta\tau'-\gamma};0,2N\right)=(2N|\delta|i\operatorname{sgn}\delta)^n\left(-i(\delta\tau'-\gamma)\frac{\operatorname{sgn}\delta}{2N|\delta|}\right)^{(2n+1)/2}\times$$

$$\times\sum_{H\bmod 2N}\varphi_{h'gh}(0,H;2N)\Big\{\vartheta_{h',\alpha g}^{(n)}(\tau';H,2N)+$$

$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\bigg|_{z=0}\cdot\vartheta_{h',\alpha g}^{(n-t)}(\tau';H,2N)\Big\}, \qquad (1.9)$$

where by (8),(9),(11) and (1)

$$h'=\beta g+\delta h+2\beta\delta N, \qquad (1.10)$$

$$\varphi_{h'gh}(0,H;2N)=$$

$$=e\left(-\frac{\alpha\gamma h'^2}{16N}\right)e\left(-\frac{\alpha g}{4N}\left(H+\frac{h'}{2}\right)\right)S_{g0}\left(\begin{smallmatrix}\beta\\ \delta\end{smallmatrix};0,2N\right), \qquad (1.11)$$

$$A_{tk}\bigg|_{z=0}=\begin{cases}(2k)!\left(\frac{2N\delta\pi i}{\delta\tau'-\gamma}\right)^k & \text{if } t=2k\\ 0 & \text{if } t\neq 2k\end{cases}$$

$$(k=1,2,\dots,t). \qquad (1.12)$$

Writing $-\frac{1}{\tau}$ instead of τ' in (1.9) and (1.12), according to (1.11), we obtain

$$\vartheta_{gh}^{(n)}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\right)=(2Ni)^n(|\delta|\operatorname{sgn}\delta)^n\left(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\right)^{(2n+1)/2}\times$$

$$\times e\left(-\frac{\alpha\gamma h'^2}{16N}\right)e\left(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2N)-2}}{4N}\right)i^{(1-|\delta|)/2}\left(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\right)|\delta|^{1/2}\times$$

$$\times\sum_{H\bmod 2N}e\left(-\frac{\alpha g}{4N}\left(H+\frac{h'}{2}\right)\right)\Big\{\vartheta_{h',\alpha g}^{(n)}\left(-\frac{1}{\tau};H,2N\right)+$$

$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\bigg|_{z=0}\cdot\vartheta_{h',\alpha g}^{(n-t)}\left(-\frac{1}{\tau};H,2N\right)\Big\}, \qquad (1.13)$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\left(\frac{2N\delta\pi i\tau}{\gamma\tau+\delta}\right)^k & \text{if } t=2k\\ 0 \quad \text{if } t\neq 2k\end{cases}\qquad (k=1,2,\dots,t).$$

Writing αg, h', $-\frac{1}{\tau}$, 0, $2N$ instead of g, h, τ, c, N in Lemma 1, we obtain

$$\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N)=(2Ni)^n\Big(\frac{i}{2N\tau}\Big)^{(2n+1)/2}\sum_{H\bmod 2N}e\Big(-\frac{\alpha g}{4N}\Big(H+\frac{h'}{2}\Big)\Big)\times$$
$$\times\Big\{\vartheta^{(n)}_{h',\alpha g}\Big(-\frac{1}{\tau};H,2N\Big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{h',\alpha g}\Big(-\frac{1}{\tau};H,2N\Big)\Big\},\qquad (1.14)$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!(-2N\pi i\tau)^k & \text{if } t=2k\\ 0 \quad \text{if } t\neq 2k\end{cases}\qquad (k=1,2,\dots,t).$$

From (1.14) it follows that

$$\sum_{H\bmod 2N}e\Big(-\frac{\alpha g}{4N}\Big(H+\frac{h'}{2}\Big)\Big)\Big\{\vartheta^{(n)}_{h',\alpha g}\Big(-\frac{1}{\tau};H,2N\Big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{h',\alpha g}\Big(-\frac{1}{\tau};H,2N\Big)\Big\}=$$
$$=(2\pi i)^{-n}(-2Ni\tau)^{(2n+1)/2}\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N).\qquad (1.15)$$

From (1.13) and (1.15) we obtain

$$\vartheta^{(n)}_{gh}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big)=(-2Ni\tau)^{(2n+1)/2}(|\delta|\operatorname{sgn}\delta)^n\times$$
$$\times\Big(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\Big)^{(2n+1)/2}\times i^{(1-|\delta|)/2}e\Big(-\frac{\alpha\gamma h'^2}{16N}\Big)\times$$
$$\times e\Big(\frac{\beta\delta g^2}{4}\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}\times\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N).\ (1.16)$$

In [2] (p.85, form.(6.16) with $2N$ instead of N) it is shown that

$$\Big(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\Big)^{1/2}(-2Ni\tau)^{1/2}=$$
$$=i^{\operatorname{sgn}\gamma(\operatorname{sgn}\delta-1)/2}\Big(\frac{\gamma\tau+\delta}{|\delta|}\Big)^{1/2}.\qquad (1.17)$$

From (1.16) and (1.17) it follows that

$$\vartheta_{gh}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big)=$$
$$=(|\delta|\operatorname{sgn}\delta)^n\Big(i^{\operatorname{sgn}\gamma(\operatorname{sgn}\delta-1)}\frac{\gamma\tau+\delta}{|\delta|}\Big)^{(2n+1)/2}i^{(1-|\delta|)/2}\times$$
$$\times e\Big(-\frac{\alpha\gamma h'^2}{16N}\Big)e\Big(\frac{\beta\delta g^2}{4}\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}\vartheta_{\alpha g,h'}^{(n)}(\tau;0,2N).$$

Thus, in view of (1.10) and (1.11), the lemma is proved for $\gamma\neq 0$.

2) Let $\gamma=0$. Then $\alpha=\delta=1$ or $\alpha=\delta=-1$ in (12). Putting $c=0$ in (12) and writing $2N$ instead of N, by (11), we obtain

$$\vartheta_{gh}^{(n)}(\tau+\beta;0,2N)=e\Big(\frac{\beta g^2}{16N}\Big)\vartheta_{gh}^{(n)}(\tau;0,2N),$$
$$\vartheta_{gh}^{(n)}(\tau-\beta;0,2N)=(-1)^n e\Big(-\frac{\beta g^2}{16N}\Big)\vartheta_{-g,h}^{(n)}(\tau;0,2N),$$

i.e. the lemma is also proved for $\gamma=0$. ■

Lemma 4. *If $\gamma\neq 0$, then for $n\geq 0$*

$$(\gamma\tau+\delta)^{(2n+1)/2}\vartheta_{gh}^{(n)}(\tau;0,2N)=$$
$$=e\Big((2n+1)\operatorname{sgn}\gamma/8\Big)(2N|\gamma|)^{-1/2}(-i\operatorname{sgn}\gamma)^n\times$$
$$\times\sum_{H\bmod 2N}\varphi_{g'gh}(0,H;2N)\Big\{\vartheta_{g'h'}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{g'h'}^{(n-t)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)\Big\},$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!(-2N\gamma\pi i(\gamma\tau+\delta))^k & \text{if } t=2k\\ 0 \quad \text{if } t\neq 2k\end{cases}$$
$$(t=1,2,\dots,n;\ k=1,2,\dots,t). \tag{1.18}$$

Remark. From (1.18) it follows that

$$\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}=\begin{cases}\frac{A_{2k,k}}{k!} & \text{if } 2|t\\ 0 \quad \text{if } 2\nmid t.\end{cases} \tag{1.19}$$

Proof. Replacing α, β, γ, δ, τ, c, N by δ, $-\beta$, $-\gamma$, α, τ', 0, $2N$ in Lemma 2, we obtain

$$\vartheta_{gh}^{(n)}\Big(\frac{\delta\tau'-\beta}{-\gamma\tau'+\alpha};0,2N\Big)=$$
$$=(-2N|\gamma|i\operatorname{sgn}\gamma)^n\Big(i(-\gamma\tau'+\alpha)\frac{\operatorname{sgn}\gamma}{2N|\gamma|}\Big)^{(2n+1)/2}\times$$
$$\times\sum_{H\bmod 2N}\varphi_{g'gh}(0,H;2N)\Big\{\vartheta_{g'h'}^{(n)}(\tau';H,2N)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{g'h'}^{(n-t)}(\tau';H,2N)\Big\},\tag{1.20}$$

where

$$g'=\delta g-\gamma h-2\gamma\delta N,\quad h'=-\beta g+\alpha h-2\alpha\beta N,\tag{1.21}$$
$$\varphi_{g'gh}(0,H;2N)=e\Big(-\frac{\alpha\beta}{4N}\Big(H+\frac{g'}{2}\Big)^2\Big)e\Big(-\frac{\beta g}{4N}\Big(H+\frac{g'}{2}\Big)\Big)\times$$
$$\times S_{g-\alpha g',h-\beta g'}\Big(\frac{\delta}{-\gamma};-\alpha H,2N\Big),\tag{1.22}$$

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\Big(\frac{2N\gamma\pi i}{\gamma\tau'-\alpha}\Big)^k & \text{if } t=2k,\\ 0 \quad \text{if } t\neq 2k.\end{cases}\tag{1.23}$$

In [2, p.87, form.(6.23)] (with $2N$ instead of N) it is shown that

$$\Big(\frac{i\operatorname{sgn}\gamma}{2N|\gamma|(\gamma\tau+\delta)}\Big)^{1/2}=\frac{e(\operatorname{sgn}\gamma/8)}{(2N|\gamma|)^{1/2}(\gamma\tau+\delta)^{1/2}}.\tag{1.24}$$

Taking $\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ instead of τ' in (1.20) and (1.23) and using (1.24), we complete the proof of the lemma. ■

2.

Lemma 5. *For a given N let*

$$\Psi_1(\tau)=\Psi_1(\tau;g_1,g_2,h_1,h_2,c_1,c_2,N_1,N_2)=$$
$$=\frac{1}{N_1}\vartheta'''_{g_1h_1}(\tau;c_1,2N_1)\vartheta'_{g_2h_2}(\tau;c_2,2N_2)-$$
$$-\frac{1}{N_2}\vartheta'''_{g_2h_2}(\tau;c_2,2N_2)\vartheta'_{g_1h_1}(\tau;c_1,2N_1)\tag{2.1}$$

and

$$\Psi_2(\tau) = \Psi_2(\tau; g_1, g_2, h_1, h_2, c_1, c_2, N_1, N_2) =$$
$$= \frac{1}{N_1^2}\vartheta^{(4)}_{g_1h_1}(\tau; c_1, 2N_1)\vartheta_{g_2h_2}(\tau; c_2, 2N_2) +$$
$$+ \frac{1}{N_2^2}\vartheta^{(4)}_{g_2h_2}(\tau; c_2, 2N_2)\vartheta_{g_1h_1}(\tau; c_1, 2N_1) -$$
$$- \frac{6}{N_1N_2}\vartheta''_{g_1h_1}(\tau; c_1, 2N_1)\vartheta''_{g_2h_2}(\tau; c_2, 2N_2), \tag{2.2}$$

where

$$2|g_1, 2|g_2, N_1|N, N_2|N, 4|N\Big(\frac{h_1}{N_1} + \frac{h_2}{N_2}\Big). \tag{2.3}$$

For all substitutions from Γ *in the neighbourhood of each rational point* $\tau = -\frac{\delta}{\gamma}\ (\gamma \neq 0, (\gamma, \delta) = 1)$, *we then have*

$$(\gamma\tau + \delta)^5\Psi_j(\tau; g_1, g_2, h_1, h_2, 0, 0, N_1, N_2) =$$
$$= \sum_{n=0}^{\infty} C_n^{(j)} e\Big(\frac{n}{4N}\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big) \quad (j = 1, 2). \tag{2.4}$$

Proof.

I. From Lemma 4 for $n = 3$ (with g_1, h_1, N_1, g_1', h_1', H_1 instead of g, h, N, g', h', H) and $n = 1$ (with g_2, h_2, N_2, g_2', h_2', H_2 instead of g,h, N, g', h', H), according to (1.19), it follows that

$$\frac{1}{N_1}(\gamma\tau + \delta)^5\vartheta'''_{g_1h_1}(\tau; 0, 2N_1)\vartheta'_{g_2h_2}(\tau; 0, 2N_2) =$$
$$= \frac{1}{N_1}e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1} \times$$
$$\times \sum_{H_1 \bmod 2N_1} \varphi_{g_1'g_1h_1}(0, H_1; 2N_1)\Big\{\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\Big) +$$
$$+ \frac{3\cdot 2}{2!}A_{21}\Big|_{z=0}\cdot\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\Big)\Big\} \times$$
$$\times \sum_{H_2 \bmod 2N_2} \varphi_{g_2'g_2h_2}(0, H_2; 2N_2)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\Big) =$$
$$= e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(2|\gamma|(N_1N_2)^{1/2}\Big)^{-1} \times$$
$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g_1'g_1h_1}(0, H_1; 2N_1)\varphi_{g_2'g_2h_2}(0, H_2; 2N_2)\times$$

$$\times\Big\{\frac{1}{N_1}\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$
$$\times\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{2.5}$$

Replacing N_1, g_1, h_1, H_1, g_1', h_1' by N_2, g_2, h_2, H_2, g_2', h_2' in (2.5) and vice versa, we obtain

$$\frac{1}{N_2}(\gamma\tau+\delta)^5\vartheta'''_{g_2'h_2'}(\tau;0,2N_2)\vartheta'_{g_1'h_1'}(\tau;0,2N_1)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$
$$\times\sum_{\substack{H_2 \bmod 2N_2\\ H_1 \bmod 2N_1}}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\varphi_{g_1'g_1h_1}(\tau;0,H_1;2N_1)\times$$
$$\times\Big\{\frac{1}{N_2}\vartheta'''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$
$$\times\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.6}$$

Subtracting (2.6) from (2.5), according to (2.1), we obtain

$$(\gamma\tau+\delta)^5\Psi_1(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$
$$\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$
$$\times\Psi_1\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1',g_2',h_1',h_2',H_1,H_2,N_1,N_2\Big). \tag{2.7}$$

In (2.7) let γ be even. Then, by (1.21) and (2.3), g_1' and g_2' are also even. Therefore, according to (3) and the notations (4), we have

$$\vartheta'''_{g_r'h_r'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\Big)\vartheta'_{g_s'h_s'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_s,2N_s\Big)=$$
$$=\sum_{n_1=0}^{\infty}B_{n_1}e\Big(\frac{Nn_1/N_r}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\sum_{n_2=0}^{\infty}B_{n_2}e\Big(\frac{Nn_2/N_s}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)=$$

$$= \sum_{n=0}^{\infty} B_n e\Big(\frac{n}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)$$

$$\text{for} \quad r=1,\ s=2 \ \text{and}\ r=2,\ s=1. \tag{2.8}$$

Hence, for even γ, (2.4) follows from (2.1), (2.7) and (2.8) if $j = 1$.

In (2.7) now let γ be odd. If h_1 and h_2 are both even, then by (1.21) and (2.3), g_1' and g_2' are also even, and we obtain the same result. But if h_r is odd, then by (1.21), g_r' will also be odd and in (3) we shall have

$$\Big(m+\frac{g_r'}{2}\Big)^2 = \Big(m+\frac{1}{2}(g_r'-1)\Big)^2 + \Big(m+\frac{1}{2}(g_r'-1)\Big) + \frac{1}{4},$$

hence

$$\vartheta'''_{g_r'h_r'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big) = -\pi^3 i e\Big(\frac{1}{16N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times$$

$$\times \sum_{m\equiv H_r \pmod{2N_r}} (-1)^{h_r'(m-H_r)/2N_r}(2m+g_r')^3 \times$$

$$\times e\Big\{\frac{1}{4N_r}\Big(\big(m+\frac{1}{2}(g_r'-1)\big)^2 + \big(m+\frac{1}{2}(g_r'-1)\big)\Big)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big\} =$$

$$= e\Big(\frac{h_r}{16N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\sum_{n_1=0}^{\infty} B'_{n_1} e\Big(\frac{n_1}{4N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big),$$

since by (5) we can imply that $h_r = 1$. Analogously,

$$\vartheta'_{g_s'h_s'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_s, 2N_s\Big) = e\Big(\frac{h_s}{16N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\sum_{n_2=0}^{\infty} B'_{n_2} e\Big(\frac{n_2}{4N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big),$$

where we can imply that $h_s = 1$ if h_s is odd and $h_s = 0$ if h_s is even. Thus, if among h_1 and h_2 at least one is odd, then we shall have for $r = 1$, $s = 2$ and $r = 2$, $s = 1$

$$\vartheta'''_{g_r'h_r'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big)\vartheta'_{g_s'h_s'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_s, 2N_s\Big) =$$

$$= e\Big(\frac{N/4(h_r/N_r + h_s/N_s)}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\sum_{n_1=0}^{\infty} B'_{n_1} e\Big(\frac{n_1}{4N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times$$

$$\times \sum_{n_2=0}^{\infty} B'_{n_2} e\Big(\frac{n_2}{4N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \sum_{n=0}^{\infty} B'_n e\Big(\frac{n}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big), \tag{2.9}$$

since, by (2.3), $\frac{N}{4}\big(\frac{h_r}{N_r}+\frac{h_s}{N_s}\big)$ is an integer. Hence for odd γ (2.4) follows from (2.1),(2.7) and (2.9) if $j = 1$.

II. From Lemma 4 for $n = 4$ and $n = 0$, as in I, it follows that

$$\frac{1}{N_1^2}(\gamma\tau+\delta)^5\vartheta^{(4)}_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2) =$$

$$= \frac{1}{N_1^2}e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$

$$\times\sum_{H_1 \bmod 2N_1}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\Big\{\vartheta^{(4)}_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$

$$+\frac{4\cdot 3}{2!}A_{21}\Big|_{z=0}\cdot\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$

$$+\frac{A_{42}}{2!}\Big|_{z=0}\cdot\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times$$

$$\times\sum_{H_2 \bmod 2N_2}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big) =$$

$$= e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(2|\gamma|(N_1N_2)^{1/2}\Big)^{-1}\times$$

$$\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$

$$\times\Big\{\frac{1}{N_1^2}\vartheta^{(4)}_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$

$$\times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$

$$\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-48\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$

$$\times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{2.10}$$

Replacing $N_1, g_1, h_1, H_1, g_1', h_1'$ by $N_2, g_2, H_2, g_2', h_2'$ in (2.10) and vice versa, we obtain

$$\frac{1}{N_1^2}(\gamma\tau+\delta)^5\vartheta^{(4)}_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1) =$$

$$= e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$

$$\times\sum_{\substack{H_2 \bmod 2N_2\\ H_1 \bmod 2N_1}}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times$$

$$\times\Big\{\frac{1}{N_2^2}\vartheta^{(4)}_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$

$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$

$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-48\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$

$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \qquad (2.11)$$

Analogously, from Lemma 4 for $n=2$ it follows that

$$\frac{6}{N_1N_2}(\gamma\tau+\delta)^5\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)=$$

$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$

$$\times\sum_{\substack{H_1\bmod 2N_1\\ H_2\bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$

$$\times\Big\{\frac{6}{N_1N"_2}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$

$$-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$

$$-\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$

$$-96\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$

$$\times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \qquad (2.12)$$

Subtracting (2.12) from the sum of (2.10) and (2.11), according to (2.2), we obtain

$$(\gamma\tau+\delta)^5\Psi_2(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)=$$

$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\Big(2|\gamma|(N_1N_2)^{1/2}\Big)^{-1}\times$$

$$\times\sum_{\substack{H_1\bmod 2N_1\\ H_2\bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$

$$\times\Psi_2\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1',g_2',h_1',h_2',H_1,H_2,N_1,N_2\Big). \qquad (2.13)$$

Further, reasoning as in I, from (2.2) and (2.13) we obtain (2.4) if $j = 2$. ■

Theorem 1. *For a given N the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ are entire modular forms of weight 5 and character $\chi(\delta) = sgn\,\delta(\frac{-\Delta}{|\delta|})$ (Δ is the determinant of an arbitrary positive quadratic form in 5 variables) for the congruence group $\Gamma_0(4N)$ if the following conditions hold:*

$$1)\quad 2|g_1, 2|g_2, N_1|N, N_2|N, \tag{2.14}$$

$$2)\quad 4\Big|N\Big(\frac{h_1^2}{N_1}+\frac{h_2^2}{N_2}\Big), 4\Big|\frac{g_1^2}{4N_1}+\frac{g_2^2}{4N_2}, \tag{2.15}$$

3) *for all α and δ with $\alpha \equiv \delta \equiv 1 \pmod{4N}$*

$$\Big(\frac{N_1N_2}{|\delta|}\Big)\Psi_j(\tau;\alpha g_1,\alpha g_2,h_1,h_2,0,0,N_1,N_2) = \\ = \Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)\ (j=1,2). \tag{2.16}$$

Proof.

I. It is well-known that the theta-series (2)-(3) are regular on $\mathcal{H}$, hence the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ satisfy the condition 1) of the definition.

II. From (2.15), since $\delta \equiv 1 \pmod 4$, it follows that

$$4\Big|N\delta^2\Big(\frac{h_1^2}{N_1}+\frac{h_2^2}{N_2}\Big), 4\Big|\frac{g_1^2}{4N_1}\delta^{2\varphi(2N_1)-2}+\frac{g_2^2}{4N_2}\delta^{2\varphi(2N_2-2)}. \tag{2.17}$$

By Lemma 3 for $n = 3$ and $n = 1$ (with g_r, h_r, N_r and g_s, h_s, N_s instead of g, h, N), according to (2.14) and (2.17), for each substitution from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta'_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big) = i^{\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\times \\ \times(\gamma\tau+\delta)^5\Big(\frac{N_rN_s}{|\delta|}\Big)\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) = \\ = \operatorname{sgn}\delta\Big(\frac{-N_rN_s}{|\delta|}\Big)(\gamma\tau+\delta)^5\times \\ \times\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) \tag{2.18}$$

for $r = 1,\ s = 2$ and $r = 2,\ s = 1$.

Analogously, by Lemma 3, we have:

1) for $n = 4$ and $n = 0$

$$\vartheta^{(4)}_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_s h_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big) = i^{\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\times$$
$$\times(\gamma\tau+\delta)^5\Big(\frac{N_r N_s}{|\delta|}\Big)\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_s,h_s}(\tau;0,2N_s) =$$
$$= \operatorname{sgn}\delta\Big(\frac{-N_r N_s}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$
$$\times\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) \tag{2.19}$$

if $r=1,\ s=2$ and $r=2,\ s=1$;

2) for $n=2$

$$\vartheta''_{g_1 h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)\vartheta''_{g_2 h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\Big) =$$
$$= \operatorname{sgn}\delta\Big(\frac{-N_1 N_2}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$
$$\times\vartheta''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2). \tag{2.20}$$

Hence, according to (2.16), for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$, we have

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,g_2,h_1,h_2,0,0,N_1,N_2\Big) =$$
$$= \operatorname{sgn}\delta\Big(\frac{-N_1N_2}{|\delta|}\Big)(\gamma\tau+\delta)^5\Psi_j(\tau;\alpha g_1,\alpha g_2,h_1,h_2,0,0,N_1,N_2) =$$
$$= \operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^5\Psi_j(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)$$

for $j=1$, by (2.1) and (2.16), and for $j=2$, by (2.2),(2.19) and (2.20). Thus the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ satisfy the condition 2) of the definition.

III. From (13) it follows for $r=1,\ s=2$ and $r=2,\ s=1$

$$\vartheta'''_{g_r h_r}(\tau;0,2N_r)\vartheta'_{g_s h_s}(\tau;0,2N_s) = \pi^4\sum_{m_r,m_s=-\infty}^{\infty}(-1)^{h_r m_r+h_s m_s}\times$$
$$\times(4N_r m_r+g_r)^3(4N_s m_s+g_s)e(\Lambda\tau),$$
$$\vartheta^{(4)}_{g_r h_r}(\tau;0,2N_r)\vartheta_{g_s h_s}(\tau;0,2N_s) = \pi^4\sum_{m_r,m_s=-\infty}^{\infty}(-1)^{h_r m_r+h_s m_s}\times$$
$$\times(4N_r m_r+g_r)^4 e(\Lambda\tau) \tag{2.21}$$

and also

$$\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2) = \pi^4 \sum_{m_1,m_2=-\infty}^{\infty} (-1)^{h_1m_1+h_2m_2} \times$$
$$\times(4N_1m_1+g_1)^2(4N_2m_2+g_2)^2 e(\Lambda\tau),$$

where

$$\Lambda = \sum_{k=1}^{2} \frac{1}{4N_k}(2N_km_k+g_k/2)^2 = \sum_{k=1}^{2}(N_km_k^2+m_kg_k/2)+\frac{1}{4}\sum_{k=1}^{2} g_k^2/4N_k,$$

by (2.14) and (2.15), is an integer. Thus, the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ satisfy the condition 3) of the definition.

IV. By Lemma 5 the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ also satisfy the condition 4) of the definition. ■

3.

Lemma 6. *For a given N let*

$$\Phi_1(\tau) = \Phi_1(\tau;g_1,\ldots,g_4,h_1,\ldots,h_4,0,\ldots,0,N_1,\ldots,N_4) =$$
$$= \frac{1}{N_1}\vartheta'''_{g_1h_1}(\tau;c_1,2N_1)\vartheta'_{g_2h_2}(\tau;c_2,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;c_k,2N_k) -$$
$$-\frac{1}{N_2}\vartheta'''_{g_2h_2}(\tau;c_2,2N_2))\vartheta'_{g_1h_1}(\tau;c_1,2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;c_k,2N_k) \quad (3.1)$$

and

$$\Phi_2(\tau) = \Phi_2(\tau;g_1,\ldots,g_4,h_1,\ldots,h_4,0,\ldots,0,N_1,\ldots,N_4) =$$
$$= \frac{1}{N_1^2}\vartheta^{(4)}_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;c_k,2N_k) +$$
$$+\frac{1}{N_2^2}\vartheta^{(4)}_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;c_k,2N_k) -$$
$$-\frac{6}{N_1N_2}\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;c_k,2N_k), \quad (3.2)$$

where

$$2|g_k,\ N_k|N\ (k=1,2,3,4),\ \ 4\Big|N\sum_{k=1}^{4}\frac{h_k}{N_k}. \quad (3.3)$$

For all substitutions from Γ *in the neighbourhood of each rational point* $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,\ (\gamma, \delta) = 1)$, *we then have*

$$(\gamma\tau + \delta)^6 \Phi_j(\tau; g_1, \ldots, g_4, h_1, \ldots, h_4, 0, \ldots, 0, N_1, \ldots, N_4) = \\ = \sum_{n=0}^{\infty} D_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 1, 2). \tag{3.4}$$

Proof. From Lemma 4 for $n = 3$, $n = 1$ and $n = 0$, as in the proof of Lemma 5, it follows that

$$\frac{1}{N_1}(\gamma\tau + \delta)^6 \vartheta'''_{g_1h_1}(\tau; 0, 2N_1)\vartheta'_{g_2h_2}(\tau; 0, 2N_2) \prod_{k=3}^{4} \vartheta_{g_kh_k}(\tau; 0, 2N_k) = \\ = e\left(\frac{3}{2}\operatorname{sgn}\gamma\right)\left(4\gamma^2\left(\prod_{k=1}^{4} N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_kg_kh_k}(0, H_k; 2N_k) \times \\ \times\left\{\left(\frac{1}{N_1}\vartheta'''_{g'_1h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) - \right.\right. \\ \left. -12\gamma\pi i(\gamma\tau + \delta)\vartheta'_{g'_1h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right)\right) \times \\ \left. \times\vartheta'_{g'_2h'_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) \prod_{k=3}^{4} \vartheta_{g'_kh'_k}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right)\right\} \tag{3.5}$$

and

$$\frac{1}{N_2}(\gamma\tau + \delta)^6 \vartheta'''_{g_2h_2}(\tau; 0, 2N_2)\vartheta'_{g_1h_1}(\tau; 0, 2N_1) \prod_{k=3}^{4} \vartheta_{g_kh_k}(\tau; 0, 2N_k) = \\ = e\left(\frac{3}{2}\operatorname{sgn}\gamma\right)\left(4\gamma^2\left(\prod_{k=1}^{4} N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_kg_kh_k}(0, H_k; 2N_k) \times \\ \times\left\{\left(\frac{1}{N_2}\vartheta'''_{g'_2h'_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) - \right.\right. \\ \left. -12\gamma\pi i(\gamma\tau + \delta)\vartheta'_{g'_2h'_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right)\right) \times \\ \left. \times\vartheta'_{g'_1h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) \prod_{k=3}^{4} \vartheta_{g'_kh'_k}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right)\right\}. \tag{3.6}$$

As in the proof of Lemma 5, from Lemma 4 it follows that

1) for $n=4$ and $n=0$

$$\frac{1}{N_1^2}(\gamma\tau+\delta)^6\vartheta^{(4)}_{g_1h_1}(\tau;0,2N_1)\prod_{k=2}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$

$$=e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\prod_{k=1}^{4}\varphi_{g'_kg_kh_k}\big(0,H_k;2N_k\big)\times$$

$$\times\Big\{\Big(\frac{1}{N_1^2}\vartheta^{(4)}_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$

$$-\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$

$$-48\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big)\times$$

$$\times\prod_{k=2}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big\}\tag{3.7}$$

and

$$\frac{1}{N_2^2}(\gamma\tau+\delta)^6\vartheta^{(4)}_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$

$$=e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\prod_{k=1}^{4}\varphi_{g'_kg_kh_k}\big(0,H_k;2N_k\big)\times$$

$$\times\Big\{\Big(\frac{1}{N_2^2}\vartheta^{(4)}_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$

$$-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta''_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big)\times$$

$$\times\vartheta_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\prod_{k=3}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)-$$

$$-48\gamma^2\pi^2(\gamma\tau+\delta)^2\prod_{k=1}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big\},\tag{3.8}$$

2) for $n=2$ and $n=0$

$$\frac{6}{N_1N_2}(\gamma\tau+\delta)^6\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$

$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\times$$

$$\times\Big\{\Big(\frac{6}{N_1N_2}\vartheta''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$

$$-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big)\times$$

$$\times\vartheta''_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\prod_{k=3}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)-$$

$$-\Big(\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$

$$+96\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big)\times$$

$$\times\prod_{k=2}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big\}. \qquad (3.9)$$

Subtracting (3.6) from (3.5), and (3.9) from the sum of (3.7) and (3.8), according to (3.1) and (3.2) respectively, we obtain

$$(\gamma\tau+\delta)^6\Phi_j(\tau;g_1,\dots,g_4,h_1,\dots,h_4,0,\dots,N_1,\dots,N_4)=$$

$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\times$$

$$\times\Phi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4,h_1,\dots,h_4,0,\dots,0,N_1,\dots,N_4\Big)$$

$$(j=1,2). \qquad (3.10)$$

Further, reasoning just as in Lemma 5, from (3.10) we obtain (3.4). ■

Theorem 2. *For a given N the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ are entire modular forms of weight 6 and character $\chi(\delta)=\big(\frac{\Delta}{|\delta|}\big)$ (Δ is the determinant of an arbitrary positive quadratic form in 6 variables) for* *the congruence group* $\Gamma_0(4N)$ *if the following conditions hold:*

$$1)\quad 2|g_k, N_k|N \qquad (k=1,2,3,4), \tag{3.11}$$

$$2)\quad 4\Big|N\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k}, \tag{3.12}$$

3) *for all* α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$

$$\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\Phi_j(\tau;\alpha g_1,\ldots,\alpha g_4,h_1,\ldots,h_4,0,\ldots,0,N_1,\ldots,N_4)=$$

$$=\Big(\frac{\Delta}{|\delta|}\Big)\Phi_j(\tau;g_1,\ldots,g_4,h_1,\ldots,h_4,0,\ldots,0,N_1,\ldots,N_4)$$

$$(j=1,2). \tag{3.13}$$

Proof.

I. As in the case of Theorem 1, the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ satisfy the condition 1) and, by Lemma 6, also the condition 4) of the definition.

II. From (3.12), since $\delta\equiv 1 \pmod 4$, it follows that

$$4\Big|N\delta^2\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k}\delta^{2\varphi(2N_k)-2}. \tag{3.14}$$

By Lemma 3, for $n=3$, $n=1$ and $n=0$, according to (3.11) and (3.14), for each substitution from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta'_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)\times$$

$$\times\prod_{k=3}^{4}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$

$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s)\times$$

$$\times\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{3.15}$$

for $r=1$, $s=2$ and $r=2$, $s=1$.

Analogously, by Lemma 3, we have:

1) for $n = 4$ and $n = 0$

$$\vartheta^{(4)}_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_s h_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_s,h_s}(\tau;0,2N_s)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{3.16}$$

for $r = 1$, $s = 2$ and $r = 2$, $s = 1$,

2) for $n = 2$ and $n = 0$

$$\prod_{k=1}^{2}\vartheta''_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)\prod_{k=3}^{4}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\prod_{k=1}^{2}\vartheta''_{\alpha g_k,h_k}(\tau;0,2N_k)\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k).$$

Hence, according to (3.13), for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$, we have

$$\Phi_j(\tau;g_1,\ldots,g_4,h_1,\ldots,h_4,0,\ldots,N_1,\ldots,N_4)=$$
$$=\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)(\gamma\tau+\delta)^6\times$$
$$\times\Phi_j(\tau;\alpha g_1,\ldots,\alpha g_4,h_1,\ldots,h_4,0,\ldots,0,N_1,\ldots,N_4)=$$
$$=\Big(\frac{\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^6\Phi_j(\tau;g_1,\ldots,g_4,h_1,\ldots,h_4,0,\ldots,N_1,\ldots,N_4)$$

for $j = 1$, by (3.1) and (3.15), and for $j = 2$, by (3.2),(3.16) and (3.17). Thus the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ satisfy the conditions 2) of the definition.

III. From (13) it follows for $r=1,\ s=2$ and $r=2,\ s=1$ that

$$\vartheta'''_{g_r h_r}(\tau;0,2N_r)\vartheta'_{g_s h_s}(\tau;0,2N_s)\prod_{k=3}^{4}\vartheta_{g_k h_k}(\tau;0,2N_k)=$$

$$=\pi^4\sum_{m_r,m_s,m_3,m_4=-\infty}^{\infty}(-1)^{m_r+m_s+m_3+m_4}(4N_r m_r+g_r)^3(4N_s m_s+g_s)e(\Lambda\tau),$$

$$\vartheta^{(4)}_{g_r h_r}(\tau;0,2N_r)\vartheta_{g_s h_s}(\tau;0,2N_s)\prod_{k=3}^{4}\vartheta_{g_k h_k}(\tau;0,2N_k)=$$

$$=\pi^4\sum_{m_r,m_s,m_3,m_4=-\infty}^{\infty}(-1)^{m_r+m_s+m_3+m_4}(4N_r m_r+g_r)^4 e(\Lambda\tau),$$

and also

$$\vartheta''_{g_1 h_1}(\tau;0,2N_1)\vartheta''_{g_2 h_2}(\tau;0,2N_2)\prod_{k=3}^{4}\vartheta_{g_k h_k}(\tau;0,2N_k)=$$

$$=\pi^4\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{\sum_{k=1}^{4}h_k m_k}(4N_1 m_1+g_1)^2(4N_2 m_2+g_2)^2 e(\Lambda\tau),$$

where

$$\Lambda=\sum_{k=1}^{4}\frac{1}{4N_k}\Big(2N_k m_k+\frac{g_k}{2}\Big)^2=\sum_{k=1}^{4}\Big(N_k m_k^2+\frac{1}{2}m_k g_k\Big)+\frac{1}{4}\sum_{k=1}^{4}\frac{g_k^2}{4N_k},$$

by (3.11) and (3.12), is an integer. Thus the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ satisfy the condition 3) of the definition. ■

REFERENCES

1. É. Goursat, Cours d'analyse mathematique. T. I. *Gauthier-Villars, Paris,* 1902.

2. H.D. Kloosterman, The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I. *Ann. of Math.* **47**(1946), No.3, 317-375.

3. G.A.Lomadze, On the representation of numbers by sums of generalized polygonal numbers. I. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **22**(1956), 77-102.

4. G.A.Lomadze, On the representation of numbers by certain quadratic forms in six variables. I. (Russian) *Trudy Tbiliss. Univ. Mat. Mech. Astron.* **117**(1966), 7-43.

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WEIGHTED ESTIMATES FOR THE HILBERT TRANSFORM OF ODD FUNCTIONS

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ABSTRACT. The aim of the present paper is to characterize the clases of weights which ensure the validity of one weighted strong, weak or extra-weak type estimates in Orlicz clases for the integral operator

$$H_0 f(x) = \frac{2}{\pi} \int_0^\infty \frac{y f(y)}{x^2 - y^2} dy, \quad x \in (0, \infty).$$

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$$H_0 f(x) = \frac{2}{\pi} \int_0^\infty \frac{y f(y)}{x^2 - y^2} dy, \quad x \in (0, \infty).$$

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1. Introduction. The Hilbert transform is given for any function f satisfying

$$\int_{-\infty}^{\infty} |f(x)| \, (1 + |x|)^{-1} \, dx < \infty$$

by the Cauchy principal value integral

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\mathbf{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(y)}{x - y} dy.$$

If f is an odd function, then Hf is even, and $Hf(x) = H_o f(|x|)$, where

$$H_o f(x) = \frac{2}{\pi} \int_0^\infty \frac{y f(y)}{x^2 - y^2} dy, \qquad x \in (0, \infty).$$

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The Hilbert transform is closely related to the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy.$$

If ϱ is a weight (measurable and nonnegative function) and $1 \le p < \infty$, strong type inequalities

$$\int_\Omega |Tf(x)|^p \varrho(x)\, dx \le C \int_\Omega |f(x)|^p\, \varrho(x)\, dx, \qquad (1.1)$$

as well as weak type inequalities

$$\varrho(\{|Tf| > \lambda\}) \le C\, \lambda^{-p} \int_\Omega |f(x)|^p\, \varrho(x)\, dx, \qquad (1.2)$$

have been widely studied by many authors. The pioneering result of Muckenhoupt [13] stated that (1.1) holds with $\Omega = \mathbf{R}$, $T = M$ and $p > 1$ if and only if $\varrho \in A_p$, that is,

$$\sup_I \varrho_I \cdot ((\varrho^{1-p'})_I)^{p-1} \le C,$$

and (1.2) holds with $\Omega = \mathbf{R}, T = M$ and $p \ge 1$ if and only if $\varrho \in A_p$, where $\varrho \in A_1$ means $\varrho_I \le C\text{ess inf}_I \varrho$. Hunt, Muckenhoupt and Wheeden [10] proved the same result for $\Omega = \mathbf{R}$ and $T = H$. The class of good weights for (1.1) or (1.2) with $\Omega = (0, \infty)$ and $T = H_o$ appears to be strictly larger than A_p. This result is due to Andersen who showed that (1.1) with $\Omega = (0, \infty)$, $p > 1$, and $T = H_o$ holds if and only if $\varrho \in A_p^o$, that is,

$$\varrho(a,b)\Big(\int_a^b \varrho^{1-p'}(x) x^{p'}\, dx\Big)^{p-1} \le C(b^2 - a^2)^p, \qquad (a,b) \subset \mathbf{R}, \qquad (1.3)$$

and (1.2) with $\Omega = (0, \infty)$, $p \ge 1$, and $T = H_o$ holds if and only if $\varrho \in A_p^o$, where $\varrho \in A_1^o$ means

$$\frac{\varrho(a,b)}{b^2 - a^2} \le C\text{ess inf}_{(a,b)} \frac{\varrho(x)}{x}.$$

Our aim is to study analogous inequalities where the power function t^p is replaced by a general convex function $\Phi(t)$. More precisely, we shall study the inequalities

$$\int_\Omega \Phi(|Tf(x)|)\,\varrho(x)\,dx \le C \int_\Omega \Phi(C|f(x)|)\,\varrho(x)\,dx, \tag{1.4}$$

$$\varrho(\{|Tf| > \lambda\}) \cdot \Phi(\lambda) \le C \int_\Omega \Phi(C|f(x)|)\,\varrho(x)\,dx\,, \tag{1.5}$$

and

$$\varrho(\{|Tf| > \lambda\}) \le C \int_\Omega \Phi(C\lambda^{-1}|f(x)|)\,\varrho(x)\,dx. \tag{1.6}$$

We call (1.4) strong type inequality, (1.5) weak type inequality, and (1.6) extra–weak type inequality. While (1.4) is an analogue of (1.1), (1.5) and (1.6) are two different analogues of (1.2). It is always true that (1.4)$\Rightarrow$(1.5)$\Rightarrow$(1.6), and none of these implications is reversible in general. The interest in these types of inequalities stems from their use in various problems of Fourier analysis. For example, extra–weak type inequalities have interesting interpolation applications (see [2]).

We throughout assume that Φ is a convex nondecreasing function on $(0,\infty)$, $\Phi(0) = 0$. In fact, it is not hard to prove that for all the above operators the inequalities (1.4) or (1.5) always imply at least quasiconvexity of Φ. For more discussion we refer to [9].

Weak and extra–weak type inequalities together were apparently firstly studied in [14] for $T = M$ and $\Omega = \mathbf{R}^n$. In [9] the following results were obtained (for definitions see Section 2 below):

Theorem A. *The inequality* (1.5) *holds with* $T = H$ *and* $\Omega = \mathbf{R}$ *if and only if* $\Phi \in \Delta_2$ *and* $\varrho \in A_\Phi$.

Theorem B. *Let* $\Phi \in \Delta_2^0$. *Then* (1.6) *holds with* $T = H$ *and* $\Omega = \mathbf{R}$ *if and only if* $\varrho \in E_\Phi$.

The main aim of the present paper is to characterize the classes of weights for which the inequalities (1.4–6) hold with $\Omega = (0,\infty)$ and $T = H_o$ (Theorems 3–5 in Section 4). Moreover, we get a characterization for the strong type inequality (1.4) with $\Omega = \mathbf{R}$ and $T = H$. This is given in Section 3 (Theorem 2) as well as the similar assertion for $T = M$ (Theorem 1). However, in the case $T = M$ we do not obtain a full characterization but we are left with a small but significant gap between the necessary and the sufficient condition.

It should be mentioned that Andersen obtained in [1] L^p–results also for the operator

$$H_e f(x) = \int_0^\infty \frac{x f(y)}{x^2 - y^2}\, dy, \tag{1.7}$$

the Hilbert transform for even functions. However, our methods do not provide analogous results for H_e with t^p replaced by $\Phi(t)$.

Let us finally mention that the result of Andersen was generalized to the case of multiple Hilbert transform in [17]. For other related results we refer also to [15], [16]. Some of the results of this paper were taken over to the comprehensive monograph [18].

2. Preliminaries. Let Φ be a convex nondecreasing function on $[0,\infty)$, $\Phi(0) = 0$, which does not vanish identically on $[0,\infty)$, but it is allowed that $\Phi \equiv 0$ on $[0,a]$ and/or $\Phi \equiv \infty$ on (a,∞) for some $a > 0$ provided that $\Phi(a-)$ is finite. The *complementary function to* Φ, $\tilde{\Phi}(t) = \sup_{s>0}(st - \Phi(s))$, has the same properties as Φ (for example, convexity of $\tilde{\Phi}$ follows easily from the subadditivity of supremum). Moreover, the Young inequality

$$st \le \Phi(s) + \tilde{\Phi}(t) \tag{2.1}$$

holds for all s,t positive. Both Φ and $\tilde{\Phi}$ are invertible on $(0,\infty)$ and it follows immediately from (2.1) that

$$\Phi^{-1}(t) \cdot \tilde{\Phi}^{-1}(t) \le 2t, \qquad t > 0. \tag{2.2}$$

We say that Φ satisfies *the* Δ_2 *condition*, ($\Phi \in \Delta_2$), if $\Phi(2t) \le C\Phi(t)$. If this estimate holds merely near 0 (near ∞), we write $\Phi \in \Delta_2^0$ ($\Phi \in \Delta_2^\infty$). We recall that $\Phi \in \Delta_2$ is equivalent to $2\Phi^{-1}(t) \le \Phi^{-1}(Ct)$.

The functions

$$R_\Phi(t) = \Phi(t)/t, \qquad S_\Phi(t) = \tilde{\Phi}(t)/t,$$

will play crucial role in the sequel. Clearly, R_Φ and S_Φ are nondecreasing on $[0,\infty)$. It is known [14], [9] that

$$\Phi(S_\Phi(t)) \le C\tilde{\Phi}(t), \qquad t \ge 0, \tag{2.3}$$

and, by convexity,

$$\Phi(\lambda S_\Phi(t)) \le C\lambda\tilde{\Phi}(t), \qquad t \ge 0, \lambda \in (0,1). \tag{2.4}$$

We say that Φ is *of bounded type near zero (near infinity)*, and write $\Phi \in B_0$ ($\Phi \in B_\infty$) if $R_\Phi(t) \ge a > 0$ (or $R_\Phi(t) \le a < \infty$) for all $t > 0$.

This classification was introduced in [9]. It was proved in [9] that

$$
\begin{aligned}
R_\Phi(t) \geq a, \quad & t > 0 & \Leftrightarrow \quad & \tilde{\Phi}(t) \equiv 0, & t \in [0,a], \\
R_\Phi(t) \leq a, \quad & t > 0 & \Leftrightarrow \quad & \tilde{\Phi}(t) \equiv \infty, & t \in (a,\infty), \\
\Phi(t) \equiv 0, \quad & t \in [0,a] & \Leftrightarrow \quad & S_\Phi(t) \geq a, & t > 0, \\
\Phi(t) \equiv \infty, \quad & t \in (a,\infty) & \Leftrightarrow \quad & S_\Phi(t) \leq a, & t > 0.
\end{aligned}
$$

The functions R_Φ and S_Φ need not be injective. However, thanks to convexity of Φ, they can be constant on intervals only in a few special cases (this is the main difference between R_Φ and Φ'), namely, if R_Φ is equal to a constant on an interval (a,b), then it must be $a = 0$ (b may be ∞). On the rest of its domain R_Φ is strictly increasing and thus invertible. Of course, the same holds for S_Φ.

It follows easily from (2.3) that

$$R_\Phi(t) \leq S_\Phi^{-1}(Ct) \tag{2.5}$$

holds for admissible t (that is, for t such that Ct belongs to the range of the invertible part of S_Φ). We shall also make use of the (converse) estimate

$$S_\Phi^{-1}(t) \leq 2R_\Phi(2t), \qquad \text{admissible } t. \tag{2.6}$$

To prove (2.6) substitute in (2.2) $t \to \tilde{\Phi}(t)$ to get $\tilde{\Phi}(t) \leq \Phi(2S_\Phi(t))$. The complementary version of the last inequality reads as $\Phi(t) \leq \tilde{\Phi}(2R_\Phi(t))$, which yields $t \leq 2S_\Phi(2R_\Phi(t))$. Putting now $t \to 2t$ and assuming that $2t$ is admissible we get (2.6).

Let us introduce a notion of index of a nondecreasing function.

Putting $h(\lambda) = \sup_{t>0} \Phi(\lambda t)/\Phi(t)$, $\lambda \geq 0$, we define the *lower index of* Φ as $i(\Phi) = \lim\limits_{\lambda \to 0+} \log h(\lambda)/\log\lambda$ and the *upper index of* Φ as $I(\Phi) = \lim\limits_{\lambda\to\infty} \log h(\lambda)/\log\lambda$.

It follows easily from the definitions that for every $\varepsilon > 0$ there exists $C_\varepsilon \geq 1$ such that

$$\Phi(\lambda t) \leq C_\varepsilon \max\left\{\lambda^{i(\Phi)-\varepsilon}, \lambda^{I(\Phi)+\varepsilon}\right\} \Phi(t), \quad t \geq 0, \quad \lambda \geq 0, \tag{2.7}$$

and

$$\min\left\{\mu^{i(\Phi)-\varepsilon}, \mu^{I(\Phi)+\varepsilon}\right\} \Phi(t) \leq C_\varepsilon \Phi(\mu t), \quad t \geq 0, \quad \mu \geq 0. \tag{2.8}$$

Let us recall that $\Phi \in \Delta_2$ is equivalent to $I(\Phi) < \infty$, and $\tilde{\Phi} \in \Delta_2$ is equivalent to $i(\Phi) > 1$.

We define the *weighted modular* by $m_\varrho(f,\Phi) = \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\,dx$, then the *weighted Orlicz space* $L_{\Phi,\varrho}$ is the set of all functions f for which $m_\varrho(f/\lambda,\Phi)$ is finite for some $\lambda > 0$. This space can be equipped with *the Orlicz norm*

$$|||f|||_{\Phi,\varrho} = \sup\left\{\int_{-\infty}^{\infty} fg\varrho,\ m_\varrho(g,\tilde{\Phi}) \le 1\right\},$$

and also with *the Luxemburg norm*

$$\|f\|_{\Phi,\varrho} = \inf\{\lambda > 0, m_\varrho(f/\lambda,\Phi) \le 1\}.$$

The norms are equivalent, and the unit ball in $L_{\Phi,\varrho}$ with respect to the Luxemburg norm coincides with the set of all functions f such that $m_\varrho(f,\Phi) \le 1$. The Hölder inequality

$$\int_{-\infty}^{\infty} fg\varrho\,dx \le \|f\|_{\Phi,\varrho} \cdot |||g|||_{\tilde{\Phi},\varrho}$$

holds, and is saturated in the sense that

$$\|f\|_{\Phi,\varrho} = \sup\left\{\int_{-\infty}^{\infty} fg\varrho\,dx, |||g|||_{\tilde{\Phi},\varrho} \le 1\right\},$$

and

$$|||f|||_{\Phi,\varrho} = \sup\left\{\int_{-\infty}^{\infty} fg\varrho\,dx, \|g\|_{\tilde{\Phi},\varrho} \le 1\right\}.$$

The norm topology is stronger than the modular one, whence the modular inequality $\int \Phi(Tf)\varrho \le C\int \Phi(C|f|)\varrho$ implies its norm counterpart $\|Tf\|_{\Phi,\varrho} \le C\|f\|_{\Phi,\varrho}$, where T is any positive homogeneous operator.

As usual, given measurable functions h, g and a measurable set E, $|E|$ means $\int_E dx$, $h(E)$ means $\int_E h$, h_E means $|E|^{-1}h(E)$, and $h(\{g > \lambda\})$ means $\int_{\{x\in\mathbf{R}, g(x)>\lambda\}} h(t)dt$.

The letter I will always denote an interval in $\mathbf{R}$, and if $I = (a,b)$, we put $I' = (b, 2b-a)$, $I^* = (a, 2b-a)$, and αI, $\alpha > 0$, is the interval concentric with I and α times as long.

If $\varrho(2I) \le C\varrho(I)$ for all I, we say that ϱ is a *doubling weight*.

We say that $\varrho \in A_\Phi$ if either $\Phi \notin B_0 \cup B_\infty$ and there exist C, ε such that

$$\sup_{\alpha>0}\sup_I \alpha\varrho_I R_\Phi\left(\frac{\varepsilon}{|I|}\int_I S_\Phi\left(\frac{1}{\alpha\varrho(x)}\right)dx\right) \le C,$$

or $\Phi \in B_0 \cup B_\infty$ and $\varrho \in A_1$.

We say that $\varrho \in E_\Phi$ if there exist $C, \varepsilon > 0$ such that

$$\sup_I \frac{1}{|I|} \int_I S_\Phi\Big(\varepsilon \frac{\varrho_I}{\varrho(x)}\Big)\, dx \le C.$$

3. Strong type inequalities for the maximal operator and the Hilbert transform. We start by considering the strong type inequality for the operator M. As known [6], the non-weighted inequality

$$\int \Phi(Mf) \le C \int \Phi(|f|)$$

holds if and only if $\tilde\Phi \in \Delta_2$. Kerman and Torchinsky [11] proved that under the assumption that both Φ and $\tilde\Phi$ satisfy the Δ_2 condition the weighted inequality

$$\int \Phi(Mf)\varrho \le C \int \Phi(|f|)\,\varrho$$

is equivalent to the condition

$$\sup_{\alpha, I} \left(\frac{\alpha}{|I|} \int_I \varrho(x)\, dx\right) \phi \left(\frac{1}{|I|} \int_I \phi^{-1}\left(\frac{1}{\alpha \varrho(x)}\right) dx\right) \le C,$$

where $\phi = \Phi'$.

As we shall see, the assumption $\Phi \in \Delta_2$ can be removed. On the other hand, the assumption $\tilde\Phi \in \Delta_2$, at least near infinity, is necessary.

Theorem 1. *Assume that ϱ and Φ are such that $\varrho \in A_\Phi$ and $\tilde\Phi \in \Delta_2$. Then there exists C so that for every f the inequality*

$$\int_{-\infty}^{\infty} \Phi(Mf(x))\varrho(x)\, dx \le C \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\, dx \tag{3.1}$$

holds.

Conversely, if (3.1) holds with C independent of f, then $\varrho \in A_\Phi$ and $\tilde\Phi \in \Delta_2^\infty$.

We shall need the following two observations:

Lemma 1. *If*

$$|||Mf|||_{\Phi,\varrho} \le C\, |||f|||_{\Phi,\varrho}, \tag{3.2}$$

then $\tilde\Phi \in \Delta_2^\infty$.

Lemma 2. *If $\tilde\Phi \in \Delta_2$ and $\varrho \in A_\Phi$, then there exists a function Φ_0 such that $\varrho \in A_{\Phi_0}$ and $i(\Phi_0) < i(\Phi)$.*

Proof of Theorem 1. Necessity. As mentioned above, the modular inequality (3.1) implies (3.2). Necessity of $\Phi \in \Delta_2^\infty$ thus follows from Lemma 1. As proved in [14], $\varrho \in A_\Phi$ is necessary even for the weak type inequality, the more so for (3.1).

Sufficiency. Let Φ_0 be the function from Lemma 2. Put $p = i(\Phi_0)$ and $F_p(t) = \Phi(t^{1/p})$. Then $i(F_p) = \frac{1}{p} i(\Phi) > 1$, whence the weighted maximal operator $M_\varrho f = \sup \varrho(I)^{-1} \int_I |f| \varrho$ is bounded on $L_{F_p,\varrho}$ [6]. Moreover, $\varrho \in A_{\Phi_0}$ implies $\varrho \in A_p$, and $(Mf)^p \le C M_\varrho(f^p)$ [11]. Thus,

$$\int \Phi(Mf)\varrho = \int F_p((Mf)^p)\,\varrho \le \int F_p(M_\varrho(C|f|^p))\,\varrho$$
$$\le C \int F_p(C|f|^p)\,\varrho = C \int \Phi(C|f|)\varrho. \quad \blacksquare$$

Proof of Lemma 1. Fix a $K > 0$ such that the set $E = \{K^{-1} \le \varrho(x) \le K\}$ has positive measure. Let x be a density point of E, with no loss of generality let $x = 0$. Fix a_0 such that $|E \cap [0, a)| \ge \frac{3}{4} a$ for all $a \le a_0$. Then, for such a,

$$|E \cap (4^{-1}a, a)| \ge \frac{1}{2} a. \tag{3.3}$$

Indeed, it is

$$|E \cap (4^{-1}a, a)| = |E \cap (0, a)| - |E \cap (0, 4^{-1}a)| \ge \frac{3}{4} a - \frac{1}{4} a.$$

From this we obtain the following observation to be used below: Since $\frac{1}{x}$ is a decreasing function, we have for every $a \in (0, a_0)$

$$\int_{E \cap (4^{-1}a, a)} \frac{dx}{x} \ge \int_{2^{-1}a}^{a} \frac{dx}{x} = \log 2. \tag{3.4}$$

Moreover, for every $a \in (0, a_0)$,

$$|E \cap (0, a)| \le a = 4^m |(0, 4^{-m}a)| \le 3^{-1} 4^{m+1} |E \cap (0, 4^{-m}a)|,$$

and so, by the definition of E,

$$\varrho(E \cap (0, a)) \le K|E \cap (0, a)| \le K 3^{-1} 4^{m+1} |E \cap (0, 4^{-m}a)|$$
$$\le K^2 3^{-1} 4^{m+1} \varrho(E \cap (0, 4^{-m}a)). \tag{3.5}$$

For $m \in \mathbf{N}$ and a fixed $b \in (0, a_0)$ put $f_m(x) = \chi_{E \cap (0, 4^{-m}b)}(x)$. Then, by (3.5),

$$|||f_m|||_{\Phi,\varrho} = \varrho(E \cap (0,4^{-m}b)) \cdot \tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,4^{-m}b))}\right)$$

$$\leq K|E \cap (0,4^{-m}b)| \cdot \tilde{\Phi}^{-1}\Big(\frac{K^2 3^{-1} 4^{m+1}}{\varrho(E \cap (0,b))}\Big). \qquad (3.6)$$

Moreover, for $x \in (4^{-m}b, b)$, $Mf_m(x) \geq x^{-1}|E \cap (0,4^{-m}b)|$. Therefore, setting

$$g(x) = \tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \cdot \chi_{E\cap(0,b)}(x),$$

we get $\int \tilde{\Phi}(g)\varrho = 1$, and thus

$$|||Mf_m|||_{\Phi,\varrho} \geq \int_{-\infty}^{\infty} Mf_m(x)g(x)\varrho(x)\,dx$$

$$\geq |E \cap (0,4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \int_{E\cap(4^{-m}b,b)} \frac{\varrho(x)}{x}\,dx$$

$$\geq |E \cap (0,4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) K^{-1} \sum_{n=1}^{m} \int_{E\cap(4^{-n}b,4^{-n+1}b)} \frac{dx}{x}$$

$\geq$ by (3.4)

$$\geq |E \cap (0,4^{-m}b)|\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) K^{-1} m \log 2. \qquad (3.7)$$

Combining (3.2), (3.6) and (3.7), we arrive at

$$\tilde{\Phi}^{-1}\left(\frac{1}{\varrho(E \cap (0,b))}\right) \cdot m \leq \frac{CK^2}{\log 2} \cdot \tilde{\Phi}^{-1}\left(\frac{K^2 3^{-1} 4^{m+1}}{\varrho(E \cap (0,b))}\right).$$

Choose $m = 2CK^2/\log 2$. Since m does not depend on b, the last inequality can be rewritten as

$$2\tilde{\Phi}^{-1}(t) \leq \tilde{\Phi}^{-1}(C_0 t), \qquad t \geq t_0,$$

with $C_0 = 3^{-1}4^{m+1}K^2$ and $t_0 = (\varrho(E \cap (0,a_0)))^{-1}$. In other words, $\tilde{\Phi} \in \Delta_2^{\infty}$. ■

Proof of Lemma 2. Fix $\alpha > 0$ and I and define $v = S_\Phi(1/\alpha\varrho)$. We claim that $v \in A_\infty$, that is, there exist α and β independent of I such that the set

$$E_\beta = \{x \in I; v(x) > \beta v_I\}$$

satisfies $|E_\beta| > \alpha|I|$.

We have to distinguish several cases. First assume that $i(\Phi) = \infty$. Since $\varrho \in A_\Phi$ always implies $\varrho \in A_\infty$ [9], and $\varrho \in A_\infty$ always implies $\varrho \in A_p$ for certain $p < \infty$, in this case the assertion of the lemma is easily satisfied.

Suppose $i(\Phi) < \infty$. Since $\tilde{\Phi} \in \Delta_2$, Φ cannot be of any bounded type. However, this is not true for $\tilde{\Phi}$, so it can be either

(i) $S_\Phi(0,\infty) = (0,\infty)$;
(ii) $S_\Phi(0,\infty) = (0,a)$;
(iii) $S_\Phi(0,\infty) = (a,\infty)$; or
(iv) $S_\Phi(0,\infty) = [a,\infty)$;

with some positive a. Note that in the case (ii) S_Φ is invertible on $(0,a)$, in the cases (iii) and (iv) S_Φ is invertible on (a,∞). Choose $\gamma \in (1, i(\Phi))$ arbitrarily. Then, by (2.7),

$$\Phi(\lambda t) \leq C_\gamma \cdot \lambda^\gamma \cdot \Phi(t), \qquad t \geq 0, \quad \lambda \in (0,1). \tag{3.8}$$

Let ε be the constant from A_Φ. Choose $\beta \leq \varepsilon/2$ in order that

$$C_\varrho \cdot 2^\gamma \cdot C_\gamma \cdot \left(\frac{\beta}{\varepsilon}\right)^{\gamma-1} \leq \frac{1}{2}, \tag{3.9}$$

where C_ϱ is an A_Φ constant for the weight ϱ. Given fixed I, suppose that βv_I is admissible for S_Φ^{-1}. We then may conclude from A_Φ that

$$\frac{C_\varrho}{R_\Phi(\varepsilon v_I)} \geq \frac{1}{|I|}\int_I \frac{dx}{S_\Phi^{-1}(v(x))} \geq \frac{|I \setminus E_\beta|}{|I|}\frac{1}{S_\Phi^{-1}(\beta v_I)}. \tag{3.10}$$

Hence, by (3.10), (2.6), (3.8) and (3.9),

$$\begin{aligned}\frac{|I\setminus E_\beta|}{|I|} &\leq C_\varrho \frac{S_\Phi^{-1}(\beta v_I)}{R_\Phi(\varepsilon v_I)} \leq 2C_\varrho \frac{R_\Phi(2\beta v_I)}{R_\Phi(\varepsilon v_I)} = \frac{C_\varrho \varepsilon}{\beta}\frac{\Phi(\frac{2\beta}{\varepsilon}\varepsilon v_I)}{\Phi(\varepsilon v_I)} \\ &\leq C_\varrho 2^\gamma C_\gamma \left(\frac{\beta}{\varepsilon}\right)^{\gamma-1} \leq \frac{1}{2},\end{aligned} \tag{3.11}$$

or, $|E_\beta| > \frac{1}{2}|I|$.

Now suppose that βv_I is not admissible for S_Φ^{-1}. This is possible only in the case (iii) or (iv) if $\beta v_I \leq a$. But then, of course, $E_\beta = I$, and the desired estimate is trivial. Therefore, $v \in A_\infty$.

Now, as known [5], v satisfies the reverse Hölder inequality, that is, there are positive C and δ such that

$$\left(\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} \leq \frac{C}{|I|}\int_I v(x)\,dx \tag{3.12}$$

for all I.

We define the function Φ_0 by means of its complementary function: put

$$S_{\Phi_0} = S_\Phi^{1+\delta}, \qquad \text{that is,} \qquad \tilde{\Phi}_0(t) = t \cdot [S_\Phi(t)]^{1+\delta}.$$

Then, obviously,

$$I(\tilde{\Phi}_0) = I(\tilde{\Phi}) + \delta(I(\tilde{\Phi}) - 1).$$

The case $I(\tilde{\Phi}) = 1$ (that is, $i(\Phi) = \infty$), was already excluded at the beginning. On the other hand, $I(\tilde{\Phi})$ cannot be ∞, since $\tilde{\Phi} \in \Delta_2$. Consequently, $I(\tilde{\Phi}_0) > I(\tilde{\Phi})$, which is of course equivalent to $i(\Phi_0) < i(\Phi)$.

It remains to prove $\varrho \in A_{\Phi_0}$. We start with rewriting (3.12) as

$$\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} \le \frac{\varepsilon}{2|I|}\int_I v(x)\,dx. \tag{3.13}$$

Suppose first that everything is admissible for S_Φ^{-1}. Then, as S_Φ^{-1} is nondecreasing,

$$S_\Phi^{-1}\left(\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)}\right) \le S_\Phi^{-1}\left(\frac{\varepsilon}{2|I|}\int_I v(x)\,dx\right). \tag{3.14}$$

Note that $S_{\Phi_0}^{-1}(t) = S_\Phi^{-1}(t^{1/(1+\delta)})$ for admissible t. So, (3.14) gives

$$S_{\Phi_0}^{-1}\left(\left(\frac{\varepsilon}{2C}\right)^{1+\delta}\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right) \le S_\Phi^{-1}\left(\frac{\varepsilon}{2|I|}\int_I v(x)\,dx\right), \tag{3.15}$$

which by means of (2.5) and (2.6) yields

$$R_{\Phi_0}\left(\left(\frac{\varepsilon}{2C}\right)^{1+\delta}\frac{1}{|I|}\int_I S_{\Phi_0}\left(\frac{1}{\alpha\varrho(x)}\right)dx\right) \le 2R_\Phi\left(\frac{\varepsilon}{|I|}\int_I S_\Phi\left(\frac{1}{\alpha\varrho(x)}\right)dx\right). \tag{3.16}$$

Now assume that it was not possible to apply S_{Φ}^{-1} in (3.13). This can happen only in the case (iii) or (iv) and

$$\frac{\varepsilon}{2C}\left(\frac{1}{|I|}\int_I v^{1+\delta}(x)\,dx\right)^{1/(1+\delta)} < a. \tag{3.17}$$

Note that in the cases (iii) or (iv) it is for all t $S_{\Phi}(t) \geq a$, that is, $S_{\Phi_0}(t) \geq a^{1+\delta}$, which is equivalent to $R_{\Phi_0}(t) = 0$ for $t \leq a^{1+\delta}$. Thus, in this case (3.16) trivially holds. It is clear that from A_{Φ} and (3.16) already follows A_{Φ_0}. The proof is finished. ■

The method of the proof of Lemma 2 is quite the same as that in [11], the only slight change is that we have replaced derivatives of Φ and $\tilde{\Phi}$ by R_{Φ} and S_{Φ}. Actually, our proof shows that the condition $\tilde{\Phi} \in \Delta_2$ is not necessary to be required as an assumption, and allows us to insert it as the part of the statement of the strong maximal theorem.

Let us turn our attention to the Hilbert transform.

It follows easily from the Kerman – Torchinsky theorem that if $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$, then $\varrho \in A_{\Phi}$ is necessary and sufficient for

$$\int \Phi(|Hf|)\varrho \leq C \int \Phi(C|f|)\varrho. \tag{3.18}$$

Indeed, for sufficiency we use Coifman's inequality [4]

$$\int \Phi(|Hf|)\varrho \leq C \int \Phi(Mf)\varrho$$

which is valid provided that $\Phi \in \Delta_2$ and $\varrho \in A_\infty$. However, $\Phi \in \Delta_2$ is an assumption, and $\varrho \in A_\infty$ follows from $\varrho \in A_{\Phi}$ [9].

It may be found of certain interest that both $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$ are also necessary for (3.18). We have the following characterization of the strong type inequality for the Hilbert transform.

Theorem 2. *The inequality*

$$\int_{-\infty}^{\infty} \Phi(|Hf(x)|)\varrho(x)\,dx \leq C \int_{-\infty}^{\infty} \Phi(|f(x)|)\varrho(x)\,dx \tag{3.19}$$

holds if and only if $\Phi \in \Delta_2$, $\tilde{\Phi} \in \Delta_2$, *and* $\varrho \in A_{\Phi}$.

We shall make use of the following assertion.

Lemma 3. *Let us define the operator*

$$G_\varrho f(x) = \frac{1}{\varrho(x)} \cdot H(f\varrho)(x).$$

Then the following statements are equivalent.

(i) *There is C such that*

$$\int_{-\infty}^{\infty} \Phi(|Hf(x)|)\varrho(x)\,dx \le C \int_{-\infty}^{\infty} \Phi(C|f(x)|)\varrho(x)\,dx;$$

(ii) *There is C such that*

$$\int_{-\infty}^{\infty} \tilde{\Phi}(|G_\varrho f(x)|)\varrho(x)\,dx \le C \int_{-\infty}^{\infty} \tilde{\Phi}(C|f(x)|)\varrho(x)\,dx. \tag{3.20}$$

Proof of Theorem 2. The "if part" was already established. It thus remains to show $\Phi \in \Delta_2$ and $\tilde{\Phi} \in \Delta_2$. It was proved in [9] that even the weak type inequality with Hilbert transform implies $\Phi \in \Delta_2$ and $\varrho \in A_\Phi$.

It remains to prove $\tilde{\Phi} \in \Delta_2$. By Lemma 3, (3.19) is equivalent to (3.20). Of course, (3.20) implies the weak type inequality

$$\varrho(\{|G_\varrho f| > \lambda\}) \cdot \Phi(\lambda) \le C \int_{-\infty}^{\infty} \Phi(Cf)\varrho. \tag{3.21}$$

Take K positive such that the set $E = \{K^{-1} \le \varrho(x) \le K\}$ has positive measure, and for any $\lambda > 0$ define $f = \frac{\lambda}{2C}\chi_{E_0}$, where E_0 is any bounded subset of E. Inserting f into (3.21) we get

$$\tilde{\Phi}(\lambda) \le C \frac{\varrho(E_0)}{\varrho(\{|G_\varrho f| > 2C\})} \cdot \tilde{\Phi}\left(\frac{\lambda}{2}\right),$$

in other words, $\tilde{\Phi} \in \Delta_2$. The idea is due to A. Gogatishvili [8]. ■

To prove Lemma 3 we employ the following result of D. Gallardo which was communicated to the author personally [7]. We give a sketch of the proof since as far as we know the author has not published it. When this manuscript was written, we learned that the same assertion was proved in a preprint by Bloom and Kerman ([3], Proposition 2.5).

Lemma 4. *Let T be a positively homogeneous operator. Then the modular estimate*

$$\int \Phi(|Tf(x)|)\varrho(x)\,dx \le C \int \Phi(C|f(x)|)\varrho(x)\,dx$$

is equivalent to the existence of C such that for all ε and f the norm inequality

$$\|Tf\|_{\Phi,\varepsilon\varrho} \leq C\|f\|_{\Phi,\varepsilon\varrho}$$

holds.

Proof of Lemma 4. Let the norm inequality be satisfied with C independent of ε and f. Then, by the definition of the Luxemburg norm, $\int \Phi((C\|f\|_{\Phi,\varepsilon\varrho})^{-1}|Tf(x)|)\varepsilon\varrho(x)\,dx \leq 1$. Fix f, a function with finite modular, and put $\varepsilon = (\int \Phi(C|f|)\varrho)^{-1} > 0$. Then $\|Cf\|_{\Phi,\varepsilon\varrho} = 1$, and so

$$\int \Phi(|Tf|)\varepsilon\varrho = \int \Phi\left(\frac{C|Tf(x)|}{C\|Cf\|_{\Phi,\varepsilon\varrho}}\right)\varepsilon\varrho \leq 1.$$

Inserting ε, we are done. The converse implication is evident. ■

Proof of Lemma 3. By Lemma 4, (i) is equivalent to

$$\|Hf\|_{\Phi,\varepsilon\varrho} \leq C\|f\|_{\Phi,\varepsilon\varrho}, \qquad \text{all } \varepsilon.$$

That is,

$$\begin{aligned} C &\geq \sup_{\|f\|_{\Phi,\varepsilon\varrho}\leq 1} \|Hf\|_{\Phi,\varepsilon\varrho} \\ &= \sup_{\|f\|_{\Phi,\varepsilon\varrho}\leq 1} \sup_{\|g\|_{\tilde{\Phi},\varepsilon\varrho}\leq 1} \int_{-\infty}^{\infty} |Hf(x)|g(x)\varepsilon\varrho(x)\,dx \\ &= \sup_{\|g\|_{\tilde{\Phi},\varepsilon\varrho}\leq 1} \sup_{\|f\|_{\Phi,\varepsilon\varrho}\leq 1} \int_{-\infty}^{\infty} |f(x)| \cdot \frac{1}{\varrho(x)}|H(g\varrho)(x)|\varepsilon\varrho(x)\,dx \\ &= \sup_{\|g\|_{\tilde{\Phi},\varepsilon\varrho}\leq 1} \|G_\varrho g\|_{\tilde{\Phi},\varepsilon\varrho}, \end{aligned}$$

which is, again by Lemma 4, equivalent to (ii). ■

4. The Hilbert transform for odd functions. In this section we shall make use of the measure ν defined on $(0,\infty)$ by $d\nu(x) = x\,dx$.

We say that $\varrho \in A_\Phi^o$ if either $\Phi \notin B_0 \cup B_\infty$ and there exist positive C, ε such that

$$\sup_{\alpha,I} \left(\frac{\alpha}{\nu(I)}\int_I \frac{\varrho(x)}{x}\,d\nu\right) R_\Phi\left(\frac{\varepsilon}{\nu(I)}\int_I S_\Phi\left(\varepsilon\frac{x}{\alpha\varrho(x)}\right)d\nu\right) \leq C$$

or $\Phi \in B_0 \cup B_\infty$ and $\varrho \in A_1^o$, that is,

$$\frac{\varrho(I)}{\nu(I)} \leq C\operatorname{ess\,inf}_I \frac{\varrho(x)}{x}.$$

We say that $\varrho \in E^{\circ}_{\Phi}$ if there exist positive C, ε such that

$$\sup_I \frac{1}{\nu(I)} \int_I S_\Phi \left(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)} \right) d\nu \le C.$$

Remark. Obviously, $\varrho \in A^{\circ}_1$ implies $\varrho \in A^{\circ}_{\Phi}$ for any Φ. Further, putting $\alpha = \frac{\nu(I)}{\varrho(I)}$ we obtain that $\varrho \in A^{\circ}_{\Phi}$ implies $\varrho \in E^{\circ}_{\Phi}$ for any Φ.

We shall prove the following theorems.

Theorem 3. *The strong type inequality*

$$\int_0^\infty \Phi(|H_\circ f(x)|)\varrho(x)\, dx \le C \int_0^\infty \Phi(C|f(x)|)\varrho(x)\, dx, \tag{4.1}$$

holds with C independent of f if and only if $\Phi \in \Delta_2$, $\tilde{\Phi} \in \Delta_2$, and $\varrho \in A^{\circ}_{\Phi}$.

Theorem 4. *The weak type inequality*

$$\varrho(\{|H_\circ f| > \lambda\}) \cdot \Phi(\lambda) \le C \int_0^\infty \Phi(C\,|f(x)|)\, \varrho(x)\, dx, \tag{4.2}$$

holds with C independent of f and λ if and only if $\Phi \in \Delta_2$, and $\varrho \in A^{\circ}_{\Phi}$.

Theorem 5. *Let $\Phi \in \Delta^{\circ}_2$. Then the extra–weak type inequality*

$$\varrho(\{|H_\circ f| > \lambda\}) \le C \int_0^\infty \Phi(\frac{C\,|f(x)|}{\lambda})\, \varrho(x)\, dx, \tag{4.3}$$

holds with C independent of f and λ if and only if $\varrho \in E^{\circ}_{\Phi}$.

The following auxiliary assertion is a modification of Lemma 1 from [1].

Lemma 5. *Define*

$$\sigma(x) = \frac{\varrho(\sqrt{|x|})}{2\sqrt{|x|}}, \qquad x \ne 0.$$

Then $\varrho \in A^{\circ}_{\Phi}$ if and only if $\sigma \in A_\Phi$, and $\varrho \in E^{\circ}_{\Phi}$ if and only if $\sigma \in E_\Phi$.

Proof of Lemma 5. Let $I = (a, b)$, $a > 0$, and put $J = (\sqrt{a}, \sqrt{b})$. Easily, $|I| = 2\nu(J)$ and $\sigma(I) = \varrho(J)$. Therefore,

$$\alpha\sigma_I R_\Phi\left(\frac{\gamma}{|I|}\int_I S_\Phi\left(\frac{1}{\alpha\sigma(x)}\right)dx\right)$$
$$=\frac{\alpha\varrho(J)}{2\nu(J)}R_\Phi\left(\frac{\gamma}{\nu(J)}\int_J S_\Phi\left(\frac{2y}{\alpha\varrho(y)}\right)d\nu(y)\right),$$

and analogously

$$\int_I\tilde{\Phi}\left(\varepsilon\frac{\sigma_I}{\sigma(x)}\right)\frac{\sigma(x)}{\sigma(I)}dx=\int_J\tilde{\Phi}\left(\varepsilon\frac{y\varrho(J)}{\varrho(y)\nu(J)}\right)\frac{\varrho(y)}{\varrho(J)}dy.$$

Similar argument holds for $b<0$, and, in the remaining case, we split the interval into two. ■

Proof of Theorem 4. Necessity. First we claim that if (4.3) holds, then ϱ is a doubling weight.

Let $I=(a,b)$, $0<a<b<\infty$. Assume that $\operatorname{supp} f\subset I$. Since $\nu(I^*)=2b(b-a)>(b^2-a^2)$,

$$H_\circ f(x)\ge\frac{1}{\pi\nu(I^*)}\int_I f(y)\,d\nu(y),\qquad x\in I'. \tag{4.4}$$

Now, easily $4\nu(I)>\nu(I^*)$, so inserting $f=\chi_I$ in (4.4) yields $H_\circ\chi_I(x)>(4\pi)^{-1}$ for all $x\in I'$. This together with (4.3) leads to $\varrho(I')\le C\varrho(I)$. By symmetry,

$$C^{-1}\varrho(I)\le\varrho(I')\le C\varrho(I), \tag{4.5}$$

and the doubling condition follows.

Now we shall show, using again the idea from [8], that $\Phi\in\Delta_2$. Given λ, set $f=(2C)^{-1}\lambda\chi_I$, where C is from (4.2) and I is an appropriate interval. It then follows from (4.2) that

$$\Phi(\lambda)\le C\frac{\varrho(I)}{\varrho(\{|H_\circ\chi_I|>2C\})}\Phi(\lambda/2),\lambda>0,$$

that is, $\Phi\in\Delta_2$.

It remains to show that $\varrho\in A^\circ_\Phi$. Given $\alpha>0$, and $I=(a,b)$, put

$$f=C^{-1}S_\Phi\left(\frac{\gamma}{\alpha}\frac{x}{\varrho(x)}\right)\chi_I(x),$$

where C is from (4.2), and

$$\lambda=\frac{1}{2\pi\nu(I^*)}\int_I f\,d\nu.$$

Then by (4.4), (2.1), (4.2) and (2.3)

$$\varrho(I') \cdot \Phi(\lambda) \le C \int_I \Phi S_\Phi \left(\frac{\gamma}{\alpha} \frac{x}{\varrho(x)} \right) \varrho(x)\, dx \le C\gamma\alpha^{-1} \int_I f\, d\nu,$$

or, by (4.5),

$$\frac{\alpha\varrho(I)}{\nu(I)} \Phi \left(\frac{1}{8\pi C \nu(I)} \int_I S_\Phi \left(\frac{\gamma}{\alpha} \frac{x}{\varrho(x)} \right) d\nu(x) \right)$$
$$\le \frac{C\gamma}{\nu(I)} \int_I S_\Phi \left(\frac{\gamma}{\alpha} \frac{x}{\varrho(x)} \right) d\nu(x) \,. \tag{4.6}$$

Denote $A = \int_I f\, d\nu$. Obviously, $A > 0$. Assume that $A = \infty$. Then $\int_I \tilde{\Phi}(\frac{\gamma}{\alpha} \frac{x}{\varrho(x)}) \varrho(x) dx = \infty$, and there must exist a function $g \in L_{\Phi,\varrho}(I)$ such that

$$\infty = \int_I \frac{\gamma}{\alpha} \frac{x}{\varrho(x)} g(x)\, \varrho(x)\, dx = \frac{\gamma}{\alpha} \int_I g\, d\nu.$$

This and (4.4) would give $H_o(\varepsilon g)(x) = \infty$ for all $x \in I'$, and $\varepsilon > 0$ and, by (4.2),

$$\varrho(I')\Phi(\lambda) \le C \int_I \Phi(\varepsilon C g(x)) \varrho(x)\, dx, \qquad \lambda, \varepsilon > 0.$$

Since $g \in L_{\Phi,\varrho}$, there must be ε such that the last integral is finite, and so it follows that $\varrho(I') = 0$. However, since ϱ is doubling and nontrivial, this is impossible. Hence $0 < A < \infty$ and we can divide both sides of (4.6) by $\nu(I)^{-1} A$ to get $\varrho \in A_\Phi^o$. ■

Sufficiency. By Lemma 5, $\varrho \in A_\Phi^o$ implies $\sigma \in A_\Phi$. We thus have from Theorem A

$$\sigma(\{Hg > \lambda\}) \cdot \Phi(\lambda) \le C \int_{-\infty}^{\infty} \Phi(C|g(x)|) \sigma(x)\, dx.$$

For given f on $(0, \infty)$ put $g(x) = f(\sqrt{x})$ for $x > 0$ and 0 elsewhere. Then $Hg(x) = (H_o f)(\sqrt{x})$ [1], and therefore

$$\varrho(\{x > 0, |H_o f(x)| > \lambda\}) \le \sigma(\{x \in \mathbf{R}, |Hg(x)| > \lambda\})$$
$$\le \frac{C}{\Phi(\lambda)} \int_0^\infty \Phi(Cf(y)) \varrho(y)\, dy. \quad ■$$

Proof of Theorem 3. Necessity. By Theorem 4, $\varrho \in A_\Phi$ and $\Phi \in \Delta_2$ are necessary even for the weak type inequality. It remains to prove $\tilde{\Phi} \in \Delta_2$. In the same way as in Lemma 3 and Lemma 4 we can prove that (4.1) implies

$$\varrho(\{\frac{1}{\varrho(x)}|H_e(f\varrho)(x)| > \lambda\})\tilde{\Phi}(\lambda) \leq C \int_0^\infty \tilde{\Phi}(C|f(x)|)\varrho(x)\,dx.$$

For the definition of H_e see (1.7). Putting $f = \frac{\lambda}{2C}\chi_{E_0}$ similarly as in the proof of Lemma 3 we get $\tilde{\Phi} \in \Delta_2$.

Sufficiency. By Lemma 5 and Lemma 2, $\varrho \in A_\Phi^\circ$ implies $\sigma \in A_{\Phi_0}$ with $i(\Phi_0) < i(\Phi)$. By Theorem A, $\sigma \in A_{\Phi_0}$ and $\Phi_0 \in \Delta_2$ imply the weak type inequality

$$\sigma(\{x \in \mathbf{R}; Hg(x) > \lambda\})\Phi_0(\lambda) \leq C \int_{-\infty}^\infty \Phi_0(Cg(x))\sigma(x)\,dx$$

for every g. Given f on $(0,\infty)$, we put $g = f(\sqrt{x}) \cdot \chi_{\{x>0\}}$, $x \in \mathbf{R}$. Change of variables then gives

$$\varrho(\{x > 0; H_o f > \lambda\})\Phi_0(\lambda) \leq C \int_0^\infty \Phi_0(Cf(x))\varrho(x)\,dx,$$

which yields the assertion by a usual interpolation argument. ∎

Proof of Theorem 5. Necessity. First assume that $\Phi \notin B_\infty$. Note that then S_Φ is finite on $(0,\infty)$. Fix $k \in \mathbf{N}$ and an interval I, put $I_k = \{x \in I, x \leq k\varrho(x)\}$, and define

$$h(x) = h_k(x) = S_\Phi\left(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\right)\chi_{I_k}(x)$$

with ε to be specified later. Put

$$\beta_I = \frac{1}{\nu(I)} \int_I h\,d\nu.$$

Now, assume that K is the biggest of the constants C from (2.4), (4.3), and (4.5). We then have from (4.5) that

$$\varrho(I) \leq K\varrho(\{|H_0 h| \geq (4\pi)^{-1}\beta_I\}).$$

Therefore, by (4.3) with $f = h$ and $\lambda < (4\pi)^{-1}\beta_I$,

$$\int_{I_k} \tilde{\Phi}\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big) \varrho(x)\, dx = \varepsilon \frac{\varrho(I)}{\nu(I)} \int_{I_k} S_\Phi\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big)\, d\nu$$
$$= \varepsilon \varrho(I) \beta_I \leq 4\pi K \varepsilon \varrho(I) + \delta_I,$$

where $\delta_I = 0$ if $\beta_I \leq 4\pi K$, and

$$\delta_I = K^2 \varepsilon \beta_I \int_{I_k} \Phi\Big(\frac{4\pi K}{\beta_I} h(x)\Big) \varrho(x)\, dx, \qquad \text{if } \beta_I > 4\pi K.$$

In any case, using (2.4) with $\lambda = 4\pi K/\beta_I$ we get

$$\int_{I_k} \tilde{\Phi}\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big) \varrho(x)\, dx$$
$$\leq 4\pi K \varepsilon \varrho(I) + 4\pi K^3 \varepsilon \int_{I_k} \tilde{\Phi}\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big) \varrho(x)\, dx.$$

Now, since S_Φ is finite, we have

$$\int_{I_k} \tilde{\Phi}\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big) \varrho(x)\, dx = \varepsilon \frac{\varrho(I)}{\nu(I)} \int_{I_k} S_\Phi\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big)\, d\nu$$
$$\leq \varepsilon \varrho(I) S_\Phi\Big(\varepsilon k \frac{\varrho(I)}{\nu(I)}\Big) < \infty,$$

and hence we can put $\varepsilon < (4\pi K^3)^{-1}$ and subtract to get

$$\int_{I_k} \tilde{\Phi}\Big(\varepsilon \frac{x}{\varrho(x)} \frac{\varrho(I)}{\nu(I)}\Big) \varrho(x)\, dx \leq \frac{4\pi K \varepsilon}{1 - 4\pi K^3 \varepsilon} \varrho(I),$$

which yields $\varrho \in E^\circ_\Phi$ as the constant on the right does not depend on k.

If $\Phi \in B_\infty$, then $\Phi(t) \leq Ct$ for all t and therefore, inserting $f = \chi_E$ and $\lambda = \nu(E)/(2\pi \nu(I^*))$ into (4.3) we obtain

$$\varrho(I) \leq C \varrho(E) \Phi\Big(C \frac{\nu(I^*)}{\nu(E)}\Big) \leq C \varrho(E) \frac{\nu(I^*)}{\nu(E)},$$

that is, $\varrho \in A^\circ_1$. Therefore, in this case $\varrho \in A^\circ_\Phi$ for any Φ (see Remark above).

Sufficiency. By Lemma 5, $\varrho \in E_\Phi^\circ$ implies $\sigma \in E_\Phi$, whence, using Theorem B, we have

$$\sigma(\{Hg > \lambda\}) \leq C \int_{-\infty}^{\infty} \Phi(\frac{C|g(x)|}{\lambda})\sigma(x)\,dx.$$

The same argument as in the proof of Theorem 4 now leads to the assertion. ■

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REFERENCES

1. K.F. Andersen, Weighted norm inequalities for Hilbert transforms and conjugate functions of even and odd functions. *Proc. Amer. Math. Soc.* **56**, (1976), No.4, 99-107.

2. R.J. Bagby, Weak bounds for the maximal function in weighted Orlicz spaces. *Studia Math.* **95**(1990), 195-204.

3. S. Bloom and R. Kerman, Weighted L_Φ integral inequalities for operators of Hardy type. *Preprint.*

4. R.R. Coifman, Distribution function inequalities for singular integrals. *Proc. Nat. Acad. Sci. USA* **69**(1972), 2838-2839.

5. R.R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51**(1974), 241-250.

6. D. Gallardo, Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded. *Publ. Math.* **32**(1988), 261-266.

7. D. Gallardo, *Personal communication.*

8. A. Gogatishvili, Riesz transforms and maximal functions in $\Phi(L)$ classes. *Bull. Acad. Sci. Georgian SSR* **137**(1990), No.3, 489-492

9. A. Gogatishvili and L. Pick, Weighted inequalities of weak and extra-weak type for the maximal operator and the Hilbert transform. *To appear in Czechoslovak Math. J.*

10. R.A. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176**(1973), 227-251.

11. R.A. Kerman and A. Torchinsky, Integral inequalities with weights for the Hardy maximal function. *Studia Math.* **71**(1982), 277-284.

12. W.A.J. Luxemburg, Banach function spaces. *Thesis, Delft*, 1955.

13. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–227.

14. L. Pick, Two weight weak type maximal inequalities in Orlicz classes. *Studia Math.* **100**(1991), No.1, 207-218.

15. V.M. Kokilashvili, Singular integral operators in weighted spaces. *Colloq. Math. Soc. János Bolyai.* **35.** *Functions, Series, Operators* (*Budapest, Hungary,* 1980), 707-714, *North-Holland, Amsterdam-New York,* 1982.

16. V.M. Kokilashvili, Singular operators in weighted Orlicz spaces. (Russian) *Trudy Tbilissk. Mat. Inst. Razmadze* **89**(1988), 42-50.

17. V.M. Kokilashvili, Weighted inequalities for some integral transforms. (Russian) *Trudy Tbilissk. Mat. Inst. Razmadze* **76**(1985), 101-106.

18. V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces. *World Sci., Singapore*, 1991.

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ORTHOGONAL RANDOM VECTORS AND THE HURWITZ-RADON-ECKMANN THEOREM

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Abstract. In several different aspects of algebra and topology the following problem is of interest: find the maximal number of unitary antisymmetric operators U_i in $H = \mathbb{R}^n$ with the property $U_iU_j = -U_jU_i$ $(i \neq j)$. The solution of this problem is given by the Hurwitz-Radon-Eckmann formula. We generalize this formula in two directions: all the operators U_i must commute with a given arbitrary self-adjoint operator and H can be infinite-dimensional. Our second main result deals with the conditions for almost sure orthogonality of two random vectors taking values in a finite or infinite-dimensional Hilbert space H. Finally, both results are used to get the formula for the maximal number of pairwise almost surely orthogonal random vectors in H with the same covariance operator and each pair having a linear support in $H \oplus H$.

The paper is based on the results obtained jointly with N.P.Kandelaki (see [1,2,3]).

რეზიუმე. ალგებრასა და ტოპოლოგიაში რამდენიმე სხვადასხვა ასპექტით აინტერესებთ $U_iU_j = -U_jU_i$ $(i \neq j)$ თვისების მქონე U_i ორთოგონალური ანტისიმეტრიული ოპერატორების მაქსიმალური რაოდენობის დადგენა $H = \mathbb{R}^n$ სივრცეში. პრობლემის ამოხსნას იძლევა ჰურვიც-რადონ-ეკმანის ფორმულა. ჩვენ ვაზოგადებთ ამ ფორმულას ორი მიმართულებით: U_i ოპერატორები გადასმადია მოცემულ ნებისმიერ თვითშეუღლებულ ოპერატორთან; H შეიძლება იყოს უსასრულოგანზომილებიანი. ჩვენი მეორე მთავარი შედეგი ეხება თითქმის ყველგან ორთოგონალობის პირობებს შემთხვევითი ვექტორებისათვის, რომლებიც მნიშვნელობებს ჰილბერტის სასრული ან უსასრულო განზომილებიან H სივრცეში იღებენ. ბოლოს, ორივე შედეგი გამოყენებულია H სივრცეში წყვილ-წყვილად თითქმის ყველგან ორთოგონალური ერთი და იგივე კოვარიაციის ოპერატორიანი ისეთი შემთხვევითი ვექტორების მაქსიმალური რაოდენობის დასადგენად, რომელთაგან შედგენილ ყოველ წყვილს წრფივი მზიდი აქვს $H \oplus H$ სივრცეში.

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1. Introduction. Two kinds of results will be given in this paper. One is of stochastic nature and deals with random vectors taking values in a finite- or infinite- dimensional real Hilbert space H. The other is algebraic or functional-analytic, and deals with unitary operators in H. Our initial problem was to find conditions for almost sure orthogonality of random vectors with values in H. Then the question arose: what is the maximal number of pairwise almost surely orthogonal random vectors in H. The analysis of this question led us to a problem which is a natural extension of an old problem in linear algebra, finally solved in 1942. It can be called the Hurwitz-Radon-Eckmann (HRE) problem in recognition of the authors who made the crucial contribution in obtaining the final solution during the different stages of the investigation.

In section 2 we give the formulation of this problem, provide its solution, and also give a brief enumeration of areas in which this problem is of primary interest. In section 3 we give the solution of the generalized HRE problem. Section 4 is for the conditions of almost sure orthogonality of two random vectors in H. In section 5 we give an analysis of these conditions. In section 6 our initial problem of determining the maximal number of pairwise orthogonal random vectors is solved under some restrictions. These restrictions simplify the problem, so that the generalized HRE formula can provide the solution. Finally, in section 7 we give the proofs of the theorems formulated in previous sections.

2. The Hurwitz-Radon-Eckmann theorem. In this section we deal only with finite-dimensional case: $H = \mathbb{R}^n$. To begin the formulation of the problem, we first recall that a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ is called unitary (or orthogonal) if $U^* = U^{-1}$ (and hence it preserves the distances).

HRE Problem. Find the maximal number of unitary operators $U_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions (I is the identity operator):

$$U_i^2 = -I, \quad U_iU_j = -U_jU_i, \quad i \neq j. \tag{1}$$

The solution of this problem is the number $\rho(n) - 1$ where $\rho(n)$ is defined as follows: represent the number n as a product of an odd number and a power of two, $n = (2a(n) + 1)2^{b(n)}$, and divide $b(n)$ by 4, $b(n) = c(n) + 4d(n)$, where $0 \leq c(n) \leq 3$. Then

$$\rho(n) = 2^{c(n)} + 8d(n). \tag{2}$$

The HRE problem is directly connected with the problem of orthogonal multiplication in vector spaces. A bilinear mapping $p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is called an orthogonal multiplication if $\|p(x,y)\| = \|x\| \cdot \|y\|$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$. An orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ exists if $k \leq \rho(n)$ and it can easily be constructed if we have $k-1$ unitary operators satisfying conditions (1). Conversely, if we have an orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ then we can easily construct $k-1$ unitary operators with the properties (1). Of course, there can be different sets of orthogonal operators satisfying (1), and correspondingly, there can be different orthogonal multiplications $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$.

Formula (2) shows that we always have $\rho(n) \leq n$. The equality $\rho(n) = n$ holds only for $n = 1,2,4,8$ and so there exists an inner multiplication in $\mathbb{R}^n$ only for those values of the dimension (and $\mathbb{R}^n$ becomes an algebra for those n). For $n = 1$, the corresponding algebra is the usual algebra of real numbers. For $n = 2,4$ and 8 we can choose the unitary operators in such a way that the corresponding algebras will, respectively, be the algebras of complex numbers, quaternions and Kelly numbers. Properties like (1) arose also in the theory of representation of Clifford algebras.

The HRE problem first appeared in the investigation of the classical problem of computing the maximal number of linearly independent vector fields on the surface S^{n-1} of the unit ball in $\mathbb{R}^n$. At first the linear vector fields have been considered, and the final result for this case given by B.Eckmann (1942) represents this number as $\rho(n) - 1$. As was later shown by J.Adams (1962) with the implementation of the K-theory, this number does not increase if we consider instead of linear vector fields general (continuous) vector fields.

The information given in this section can be found with discussions and further references among others in [4], Chapter 10.

3. The Generalized HRE Problem and its Solution. Now and in what follows H can be infinite-dimensional, and in this case we suppose that it is separable. Note that of a continuous linear operator U is unitary means that the image im $U = H$ and $U^* = U^{-1}$.

Let B be a given arbitrary continuous self-adjoint linear operator in H.

Generalized HRE Problem. Find the maximal number of unitary operators U_i in H satisfying conditions (1) along with the additional condition $U_iB = BU_i$ for all i.

Clearly, if $H = \mathbb{R}^n$ and $B = I$, this problem coincides with the

HRE problem. To formulate the solution of this problem we need the following auxiliary assertion.

Theorem on Multiplicity of the Spectrum (see [5], Theorem VII.6). *For any continuous linear self-adjoint operator B there exists a decomposition $H = H_1 \oplus H_2 \oplus \cdots \oplus H_\infty$ that satisfies the following conditions:*

a) *Each H_m $(m = 1, 2, \cdots, \infty)$ is invariant with respect to B;*

b) *The restriction of B to H_m is an operator of homogeneous multiplicity m, i.e., is unitarily equivalent to the operator of multiplication by the independent variable in the product of m copies of the space $L_2(\mu_m)$;*

c) *The measures μ_m are given on the spectrum of B, are finite, and are mutually singular for different m (in fact, it is not the measures themselves that are of importance, but the collections of corresponding sets of zero measure).*

Remark. For some m the measures μ_m can be zero; the collection of the remaining m is denoted by $\mathfrak{M}$. Now let

$$\rho(B) = \min_{m \in \mathfrak{M}} \rho(m), \tag{3}$$

where $\rho(m)$ is defined by the equality (2) for $m = 1, 2, \cdots$ and $\rho(\infty) = \infty$.

Note that if the operator B has purely point spectrum, then the relation (3) gives

$$\rho(B) = \min_j \rho(m_j),$$

where $m_1, m_2 \cdots$ are the multiplicities of eigenvalues $\lambda_1, \lambda_2, \cdots$ of B. Particularly, $\rho(I) = \rho(n)$ if $H = \mathbb{R}^n$ and $\rho(I) = \infty$ if H is infinite-dimensional.

Now we give the formulation of one of the main results of this paper.

Theorem 1 (Solution of the generalized HRE problem). *The maximal number of unitary operators in H satisfying the conditions* (1) *and also commuting with B is equal to $\rho(B) - 1$.*

Remark 1. For the case $H = \mathbb{R}^n$ and $B = I$ this theorem gives the HRE result. However, our proof of Theorem 1 is based on the HRE theorem and so, of course, we do not pretend to have a new proof of it.

Remark 2. As it was noticed, $\rho(I) = \infty$ if H is infinite-dimensional. So, Theorem 1 tells that in the infinite-dimensional case there exists an infinite set of unitary operators satisfying the condition (1).

As an easy consequence of Theorem 1 we get the following simple assertions.

Corollary 1. *No self-adjoint operator having an eigenvalue of an odd multiplicity can commute with an unitary antisymmetric operator.*

Corollary 2. *There does not exist a compact self-adjoint operator in H which commutes with infinitely many unitary operators satisfying condition* (1).

4. Orthogonality Conditions for Two Random Vectors. We begin this section with some preliminaries which are meant mostly for those readers who usually do not deal with probabilistic terminology. (This preliminary material can be found with detailed discussions and proofs in [6], Chapter 3). Let $(\Omega, \mathcal{B}, P)$ be a basic probability space, i.e., a triple where Ω is an abstract set, $\mathcal{B}$ is some σ-algebra of its subsets and P is a normed measure on $\mathcal{B}$, $P(\Omega) = 1$. Let ξ be a random vector with values in H. Because of assumed separability of H the two main definitions of measurable sets in H coincide and a random vector means nothing but a Borel measurable function $\Omega \to H$. We will assume for simplicity that ξ is centered (has zero mean):

$$E(\xi|h) = 0 \quad \text{for all} \quad h \in H,$$

where E stands for the integral over Ω with respect to the measure P and $(\cdot|\cdot)$ denotes the scalar product in H. We will consider only random vectors having weak second order:

$$E(\xi|h)^2 < +\infty \quad \text{for all} \quad h \in H.$$

This restriction is less than the demand of strong second order ($E\|\xi\|^2 < +\infty$) and coincides with it only if H is finite-dimensional.

For any random vector ξ having weak second order we can define an analogue of covariance matrix which will be a continuous linear operator $B : H \to H$ defined by the relation

$$(Bh|g) = E(\xi|h)(\xi|g), \quad h, g \in H \tag{4}$$

(we remind that ξ is assumed to be centered).

Any covariance operator is self-adjoint and positive: $(Bh|h) \geq 0$, $h \in H$.

If we have two random vectors ξ_1 and ξ_2 we can define also the cross-covariance (or mutual covariance) operator $T = T_{\xi_1\xi_2}$ as follows (we assume again, for simplicity, that ξ_1 and ξ_2 are centered):

$$(Th|g) = E(\xi_1|h)(\xi_2|g). \quad h, g \in H.$$

The cross-covariance operator T is also a continuous linear operator and satisfies the condition

$$(Th|g)^2 \le (B_1h|h)(B_2g,g), \quad h,g \in H, \tag{5}$$

where B_i is the covariance operator for $\xi_i (i = 1,2)$.

The pair (ξ_1, ξ_2) can be regarded as a random vector with values in the Hilbert space $H \oplus H$ and the usual definition, like (4), of covariance operator can be applied. Then, using the fact that the inner product in the Hilbert direct sum $H \oplus H$ is given as the sum of the inner products of the components, we easily get that the covariance operator K of the pair (ξ_1, ξ_2) is determined by 2×2 matrix with operator-valued elements:

$$K = \begin{pmatrix} B_1, & T^* \\ T, & B_2 \end{pmatrix},$$

where T^* is the operator adjoint to T (in fact it is equal to $T_{\xi_2\xi_1}$).

Now we can formulate the main result of this section, which gives sufficient conditions for almost sure (P-almost everywhere) orthogonality of random vectors ξ_1 and ξ_2. It may seem somewhat surprising that the conditions can be expressed only in terms of the covariance operator K (second moment characteristics) and more specific properties of the distribution have no effect.

Theorem 2. *If the covariance operator K satisfies the conditions*

$$T^*B_1 = -B_1T, \quad TB_2 = -B_2T^*, \quad T^2 = -B_2B_1, \tag{6}$$

then any (centered) random vector (ξ_1, ξ_2) with this covariance operator has almost surely orthogonal components, i.e., $P\{(\xi_1|\xi_2) = 0\} = 1$.

Generally speaking, condition (6) is not necessary. Here is a simple example: $\xi_1 = \epsilon_1\zeta$, $\xi_2 = \epsilon_2U\zeta$, where ϵ_1 and ϵ_2 are independent Bernoulli random variables ($P\{\epsilon_i = 1\} = P\{\epsilon_i = -1\} = 1/2$; $i = 1,2$), U is a continuous linear antisymmetric operator in $H (U^* = -U)$ and ζ is any non-degenerate random vector in H.

However, the necessity holds for a wide class of distributions, containing Gaussian ones.

Theorem 3. *If the support of random vector $\xi = (\xi_1, \xi_2)$ is a linear subspace of $H \oplus H$, then conditions* (6) *are also necessary for ξ_1 and ξ_2 to be almost surely orthogonal.*

5. Analysis of the Orthogonality Conditions. The orthogonality conditions (6) are in fact an operator equation with triples (B_1, B_2, T) as its solutions. For the special case $H = R^2$ the general solution of this equation can easily be given. For the case $H = R^n$ with $n > 2$ and, especially, for the infinite-dimensional case we cannot expect to have the same simple picture. However some basic properties of solutions can be described.

We begin with simple properties.

Theorem 4. *If conditions* (6) *hold for the operators* B_1, B_2, T, *then the following assertions are true:*

a) $\ker B_1 \subset \ker T, \quad \ker B_2 \subset \ker T^*;$ (7)

$\overline{imT^*} \subset \overline{imB_1}, \quad \overline{imT} \subset \overline{imB_2};$ (8)

b) T *commutes with* B_1 *if and only if it commutes with* B_2 *and this happens if and only if* $T^* = -T$. *In this case we have also* $B_1B_2 = B_2B_1$;

c) $T = T^*$ *only if* $T = 0$;

d) TT^* *is not necessarily equal to* T^*T *(so* T *is not necessarily a normal operator)*;

e) $TB_2B_1 = B_2B_1T, \quad T^*B_1B_2 = B_1B_2T^*$;

f) *If* $B_1 = B_2$, *then* $TB = BT$ *and* $T^* = -T$;

g) *In the finite-dimensional case* $H = R^n$ *with an odd* n *either* $\ker B_1 \neq 0$ *or* $\ker B_2 \neq 0$;

h) *In any finite-dimensional case the trace of* T *is zero;*

i) *In the two-dimensional case* $H = R^2$ *we always have* $B_1B_2 = B_2B_1$.

The main part of the next theorem shows that conditions (6) for the covariance operator of a random vector (ξ_1, ξ_2) are essentially equivalent to the existence of a linear antisymmetric connection between the components. To avoid the word "essentially", we assume that one of the covariance operators B_1 or B_2 is nonsingular. Suppose for definiteness that $\ker B_1 = 0$ $(B_1h = 0 \Rightarrow h = 0)$. This assumption implies that the inverse operator B_1^{-1} exists; in general it is unbounded, and not defined on the whole of H but only on the range imB_1. Consider the operator TB_1^{-1} on this dense linear manifold. It is easy to verify that under conditions (6) TB_1^{-1} is always closable. Denote the closure by U and its domain by $\mathcal{D}(U)$. Clearly, $imB_1 \subset \mathcal{D}(U) \subset H$. In some cases we can have $\mathcal{D}(U) = H$ and then U is continuous. Finally, denote by Γ the graph of U, i.e.,

$$\Gamma = \{(x, Ux),\ x \in \mathcal{D}(U)\}$$

and let also

$$\Gamma' = \{(Ux, x),\ x \in \mathcal{D}(U)\}.$$

Theorem 5. *Suppose that the covariance operator K of the random vector (ξ_1, ξ_2) satisfies conditions* (6) *and* $\ker B_1 = 0$. *Then the following assertions are true:*

a) $\overline{im\, K} = \Gamma$, $\ker K = \Gamma'$;

b) $\mathcal{D}(U^*) \supset \mathcal{D}(U)$ *and* $U^* = -U$ *on* $\mathcal{D}(U)$;

c) $B_2 = -UB_1U$, *and moreover,* $\mathcal{D}(U)$ *is a dense Bortel set in* H, $P\{\xi_1 \in \mathcal{D}(U)\} = 1$ *and* $P\{\xi_2 = U\xi_1\} = 1$.

Remark 1. If instead of $\ker B_1 = 0$ we assume $\ker B_2 = 0$, then we can introduce the operator V which is the closure of $T^* B_2^{-1}$. Of course, for V the theorem is again true (with natural slight alterations). If both B_1 and B_2 are nonsingular, we can introduce both U and V; they will be convertible and we will have $U^{-1} = V$.

Remark 2. Let both B_1 and B_2 be nonsingular. Then we have both U and V. Generally, neither U nor V is necessarily extended to a continuous operator in H. The example below shows that in fact all four possibilities can be realized.

Example 1. Let some basis in H be fixed and B_1 be given as a diagonal matrix with positive numbers $\lambda_1, \lambda_2, \ldots$ on the diagonal. Let B_2 be also diagonal with the positive numbers $a_1^2\lambda_2$, $a_1^2\lambda_1$, $a_2^2\lambda_4$, $a_2^2\lambda_3$, $a_3^2\lambda_6$, $a_3^2\lambda_5, \ldots$ on the diagonal. Finally let T be quasi-diagonal with the following two-dimensional blocks on the diagonal:

$$\begin{pmatrix} 0, & a_1\lambda_2 \\ -a_1\lambda_1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & a_2\lambda_4 \\ -a_2\lambda_3, & 0 \end{pmatrix}, \begin{pmatrix} 0, & a_3\lambda_6 \\ -a_3\lambda_5, & 0 \end{pmatrix}, \ldots$$

Remark 3. Let A denote the linear operator determined in $H \oplus H$ by the matrix $\|A_{ij}\|$, where $A_{11} = A_{22} = O$ and $A_{12} = A_{21} = I$, and let $d = AK$. Conditions (6) can be written as $d^2 = O$ ("differentiality" of d). According to assertion a) in Theorem 5 we have $\overline{im\, d} = \ker d$ (and get zero homology), provided B_1 or B_2 is nonsingular. If this is not the case, then the inclusion $\overline{im\, d} \subset \ker d$ (which is the consequence of $d^2 = O$) can be strict. Here is a simple example.

Example 2. $H = R^4$, B_1 and B_2 are given by diagonal matrices with the numbers $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 = \lambda_4 = 0$ and $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 > 0$ respectively, and $T = 0$.

Remark 4. According to assertion c) of the theorem, any triple (B_1, B_2, T) with nonsingular B_1, satisfying conditions (6) can be given with the pair of operators (B, U) by the following relations: $B_1 = B$, $B_2 = -UBU$, $T = UB$. In a finite-dimensional case the converse is also true: any pair of operators (B, U), where B is an arbitrary nonsingular self-adjoint positive operator and U is an arbitrary antisymmetric operator, gives the triple satisfying conditions (6). In the infinite-dimensional case, as shown by Example 1, unbounded operators can arise in the direct assertion. Therefore, if we want to obtain the general solution of the system (6) we should either confine the collection of possible pairs (B, U) or extend the triples (B_1, B_2, T) to admit unbounded operators. However, when $B_1 = B_2$ unbounded operators do not occur and the problem of the description of the general solution of (6) can be solved using only continuous operators.

Theorem 6. *If $B_1 = B_2 = B$ $(\ker B = 0)$, then $\mathcal{D}(U) = H$ and hence U is continuous. Furthermore, we have $UB = BU$, $U^2 = -I$ and so the antisymmetric operator U is also unitary $(U^* = U^{-1})$.*

6. Systems of Pairwise Orthogonal Random Vectors. Now we consider systems of $k > 2$ random vectors and look for conditions of pairwise almost sure orthogonality. A direct application of the conditions for two random vectors to each pair of the system might yield a solution, but the latter would be too complicated to be of interest. Therefore, to ease the problem we impose some restrictions on the systems under consideration. Namely, we assume that all random vectors $\xi_1, \xi_2, \dots, \xi_k$ have the same covariance operator B, and that each pair (ξ_i, ξ_j) has a linear support in $H \oplus H$. We assume also, without losing generality now, that $\ker B = 0$ and all ξ_i are centered. Such systems of random vectors we call $S(H, B)$-systems. An $S(H, B)$-system $\xi_1, \xi_2, \dots, \xi_k$ is said to be an $SO(H, B)$-system if $P\{(\xi_i|\xi_j) = 0\} = 1$ for $i, j = 1, 2, \dots, k$; $i \neq j$.

Let now $(\xi_1, \xi_2, \dots, \xi_k)$ be any $SO(H, B)$-system. Fix one of ξ_i's, say ξ_1, and consider the pairs (ξ_1, ξ_2), $(\xi_1, \xi_3), \dots, (\xi_1, \xi_k)$. Denote by T_i the cross-covariance operator $T_{\xi_1 \xi_i}$ and let $U_i = T_i B^{-1} (i = 2, 3, \dots, k)$. According to Theorem 3 and Theorem 6 we have $\xi_i = U_i \xi_1$ almost surely, and each of U_i's is unitary, commutes with B and also we have $U_i^2 = -I$. It is easy to show that orthogonality of ξ_i and ξ_j gives the condition $U_i U_j = -U_j U_i$ and if we apply Theorem 1, we get $k \leq \rho(B)$. Conversely, let now $U_2, U_3, \dots, U_{\rho(B)}$ be $\rho(B) - 1$ unitary operators from the generalized HRE problem which exist again by Theorem 1. Let also ξ_1 be any centered random vector with a linear support in H and with covariance operator B. It is easy to verify that $(\xi_1, U_2\xi_1, U_3\xi_1, \dots, U_{\rho(B)}\xi_1)$ is an $SO(H,B)$-system. Therefore, we have derived the following result.

Theorem 7. *For any covariance operator B there exists an $SO(H,B)$-system containing $\rho(B)$ random vectors and this is the maximal number of random vectors forming any $SO(H,B)$-system.*

Finally we give some corollaries of this theorem concerning Gaussian random vectors.

Corollary 1. *For any natural number k there exists an $SO(\mathbb{R}^n, B)$-system, consisting of k Gaussian random vectors (n and B should be chosen properly).*

Corollary 2. *For any natural number k there exists an $SO(H,B)$-system, consisting of k Gaussian random vectors such that H is infinite-dimensional and the Gaussian random vectors are also infinite-dimensional.*

Corollary 3. *There does not exist an infinite $SO(H,B)$-system, consisting of Gaussian random vectors.*

Remark. Corollary 3 means that an infinite system of centered Gaussian random vectors which are pairwise almost surely orthogonal does not exist if: a) all pairs of the system have linear supports in $H \oplus H$; b) all vectors of the system have the same covariance operator. In connection with this we note that such kind of system does exist if we drop either one of these two restrictions.

7. Proofs of the Results.

Proof of Theorem 1. The proof is performed in two steps. First we consider the case of the operator B having a homogeneous multiplicity.

Lemma 1. *Let B be a linear bounded operator of the homogeneous multiplicity m $(1 \le m \le \infty)$. There exist k $(0 \le k \le \infty)$ unitary operators satisfying conditions* (1) *and commuting with B if and only if $k \le \rho(m) - 1$ (we remind that $\rho(\infty)$ is defined as ∞).*

Proof. According to the condition on B there exists a linear isometry v from H onto $L_2^m(\mu)$ such that $B = v^{-1}\bar{B}v$ where $L_2^m(\mu)$ is the Hilbert direct sum of m copies of $L_2(\mu)$ with some finite Borel measure μ supported by a compact set $M \subset R^1$ and $\bar{B}$ is the operator from $L_2^m(\mu)$ to itself defined by the equality

$$(\bar{B}f)(\lambda) = \lambda f(\lambda), \quad \lambda \in M.$$

Here $f = (f_1, f_2, \dots, f_m)$ with $f_i \in L_2(\mu)$ $(i = 1, 2, \dots, m)$ for the case of finite m and $f = (f_1, f_2, \dots)$ with the additional assumption $\sum \|f_i\|^2 < \infty$ if $m = \infty$.

Consider first the case $m < \infty$. To prove the sufficiency part of the lemma we construct $\rho(m) - 1$ unitary operators $\bar{U}_i$ in $L_2^m(\mu)$ that satisfy conditions (1) and commute with $\bar{B}$; the operators $U_i = v^{-1}\bar{U}_i v$ will solve the problem in H. In virtue of classical HRE theorem there exist $\rho(m) - 1$ orthogonal operators $\tilde{U}_i$ in R^m satisfying conditions (1). Let $\|\tilde{U}_i(p,q)\|$ $(p, q = 1, 2, \dots, m)$ be the matrix of $\tilde{U}_i$ in the natural basis of R^m, and define the operator $\bar{U}_i : L_2^m(\mu) \to L_2^m(\mu)$ as $\bar{U}_i f = g$ $(i = 1, 2, \dots, \rho(m) - 1)$, where

$$g_p(\lambda) = \sum_{q=1}^{m} \tilde{U}_i(p,q) f_q(\lambda), \quad p = 1, 2, \dots, m. \tag{9}$$

It is easy to check that the operators $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{\rho(m)-1}$ have all the needed properties.

To prove the necessity part of the lemma we have to show that $k \leq \rho(m) - 1$ if $U_1, U_2, \dots, U_k$ is any system of unitary operators satisfying (1) and commuting with B. Let $\bar{U}_i = vU_iv^{-1}$ and $\bar{B} = vBv^{-1}$ be the corresponding isometric images of U_i and B. These are the operators acting from $L_2^m(\mu)$ to itself. Any linear operator $L_2^m(\mu) \to L_2^m(\mu)$ can be written in a standard way as a $m \times m$ matrix with entries that are $L_2(\mu) \to L_2(\mu)$ operators. Let $\|\bar{U}_i(p,q)\|$ be the matrix of the operator $\bar{U}_i$ $(i = 1, 2, \dots, k;\ p, q = 1, 2, \dots, m)$. Since $\bar{U}_i\bar{B} = \bar{B}\bar{U}_i$, we have $\bar{U}_i f_0(\bar{B}) = f_0(\bar{B})\bar{U}_i$ for any continuous function $f_0 : R^1 \to R^1$. Clearly, the operator $f_0(\bar{B})$ is the multiplication by the function f_0. Therefore for all $i = 1, 2, \dots, k$ and $f \in L_2^m(\mu)$ we have the following m relations $(p = 1, 2, \dots, m)$:

$$\sum_{q=1}^{m} \left[\bar{U}_i(p,q) f_0 f_q\right](\lambda) = f_0(\lambda) \sum_{q=1}^{m} \left[\bar{U}_i(p,q) f_q\right](\lambda).$$

If we take now $f = (f_1, f_2, \dots, f_m)$ with $f_q \equiv 0$ for $q \neq s$ and $f_s \equiv 1$ $(s = 1, 2, \dots, m)$, then we get

$$\left[\bar{U}_i(p,s) f_0\right](\lambda) = \bar{V}_i(p,s;\lambda) f_0(\lambda) \tag{10}$$

where

$$\bar{V}_i(p,s;\lambda) = \left[\bar{U}_i(p,s)1\right](\lambda).$$

Since the operators $\bar{U}_i(p,s)$ are bounded, the relations (10) hold not only for continuous f_0 but also for all $f_0 \in L_2(\mu)$. Now it can be shown by elementary reasonings that for almost all fixed values of λ

the $R^m \to R^m$ operators corresponding to the k matrices $\|\bar{V}_i(p,q)\|$ are unitary and satisfy conditions (1). Therefore, $k \le \rho(m) - 1$ by the classical HRE theorem.

To finish the proof of the lemma, we consider the case $m = \infty$. In this case only sufficiency part is to be proved. The existence of an infinite system of unitary operators in H satisfying conditions (1) was proved in [7]. The proof that we give here (see also [3]) is based on the same idea though the use of block matrices simplifies the technique of the proof.

Let Δ_i be the quadratic matrix of order 2^i with the second (non-principal) diagonal consisting of $+1$'s in the upper half and -1's in the lower half and all other entries equal to zero. Denote by $\tilde{U}_i$ ($i = 1, 2, \dots$) the infinite diagonal block matrix with the matrices Δ_i on the (principal) diagonal. Clearly, Δ_i are unitary and $\Delta_i^2 = -I$. Hence $\tilde{U}_i$ are unitary and $\tilde{U}_i^2 = -I$. To prove the property $\tilde{U}_i\tilde{U}_j = -\tilde{U}_j\tilde{U}_i$ ($i \neq j$), it is convenient to consider $\tilde{U}_j$ (if $j < i$) as a diagonal block matrix with the matrices (blocks) of the same order 2^i as in case of $\tilde{U}_i$. This can be achieved combining 2^{i-j} diagonal blocks of $\tilde{U}_j$ in one with zeros as other entries. Denote this matrix (block) of order 2^i by $\Delta_{j,i}$. Now $\tilde{U}_j$ is a diagonal block matrix with the diagonal blocks $\Delta_{j,i}$ of the same order 2^i and it is enough to show that $\Delta_i\Delta_{j,i} = -\Delta_{j,i}\Delta_i$. For this we remind that $\Delta_{j,i}$ is a diagonal block matrix with 2^{i-j} diagonal blocks of order 2^j each, and represent Δ_i also as a block matrix with the blocks of order 2^j each. This way we get a block matrix with the blocks that are all zero matrices except those situated on the second (non-principal) diagonal which are δ in the uppeer half and $-\delta$ in the lower one. Here δ is the matrix of order 2^j with $+1$'s on the second (non-principal) diagonal and all other entries equal to zero. Now it is quite easy to show that the needed equality $\Delta_i\Delta_{j,i} = -\Delta_{j,i}\Delta_i$ is a consequence of the elementary one: $\Delta_j\delta = -\delta\Delta_j$, and this completes the proof of $\rho(\infty) = \infty$. The proof of the lemma is also finished now: define the operators $\bar{U}_i$ ($i = 1, 2, \dots$)in $L_2^m(\mu)$, $m = \infty$, by the relations (9) with $m = \infty$; it can easily be checked that the operators $\bar{U}_i$ ($i = 1, 2, \dots$) satisfy conditions (1) and commute with $\bar{B}$.

Now we can finish the proof of Theorem 1. Clearly, $\rho(B) = \rho(m)$ if B is of homogeneous multiplicity m, and Lemma 1 coincides with Theorem 1 for this case. For the general case we will write B as the diagonal matrix with the restrictions of B to $H_1, H_2, \dots$ on the diagonal (this is possible because every H_m is invariant for B). The restriction of B to H_m, $m \in \mathfrak{M}$, is of homogeneous multiplicity m, and by Lemma 1 there exist $\rho(B)-1$ unitary operators U_i^m ($i = 1,2,\ldots,\rho(B)-1$) in each H_m ($m \in \mathfrak{M}$) that satisfy conditions (1) and commute with $B|_{H_m}$. The $\rho(B)-1$ unitary operators corresponding to diagonal matrices with the operators $U_i^1, U_i^2, \ldots$ on the diagonal satisfy conditions (1) and commute with B.

Finally, let U_i ($i = 1,2,\ldots,k$) be unitary operators in H satisfying conditions (1) and commuting with B, and H_m ($m \in \mathfrak{M}$) be the invariant subspaces corresponding to B. Because of commutativity, the subspaces H_m are invariant also for all U_i and Lemma 1 easily gives that $k \le \rho(B)-1$. ∎

Proof of Theorem 2. Let A denote the 2×2 matrix with the operator-valued elements $A_{11} = A_{22} = O$ and $A_{12} = A_{21} = \frac{1}{2}I$, where O and I denote, as before, zero and identity operators. It is obvious that $(\xi_1|\xi_2) = (A\xi|\xi)$, $\xi = (\xi_1,\xi_2)$, and hence the problem is transformed to the problem of orthogonality of ξ and $A\xi$ in the Hilbert space $H_1 \oplus H_2$. It is easily seen, using the definition, that the covariance operator of $A\xi$ is AKA. We have the equalities

$$(AKA)K = (AK)^2 = \frac{1}{4}\left[\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} B_1, & T^* \\ T, & B_2 \end{pmatrix}\right]^2 =$$

$$= \frac{1}{4}\begin{pmatrix} T^2 + B_2B_1, & TB_2 + B_2T^* \\ B_1T + T^*B_1, & B_1B_2 + T^{*2} \end{pmatrix} = O$$

and the use of the following lemma ends the proof.

Lemma 2. *Let ζ and η be centered random vectors in a Hilbert space H, with covariance operators B_ζ and B_η, respectively. If $B_\zeta B_\eta = O$, then $P\{(\zeta|\eta) = 0\} = 1$.*

Proof. Topological support S_ζ of a random vector ζ with values in a (separable) Hilbert space H is defined as the minimal closed set in H having probability 1, i.e. as the intersection of all closed sets $F \subset H$ such that $P\{\zeta \in F\} = 1$. Denote by $l(S_\zeta)$ the minimal closed subspace in H containing S_ζ. If $h \perp l(S_\zeta)$, then $h \perp im\, B_\zeta$ and $(B_\zeta h, h) = O$; hence $(B_\zeta g, h) = 0$ for all $g \in H$, and $h \perp im\, B_\zeta$. Conversely, if $h \perp im\, B_\zeta$ then $P\{\zeta \perp h\} = 1$ and so the closed subspace orthogonal to h has probability 1; hence it contains $l(S_\zeta)$ and we get $h \perp l(S_\zeta)$. Therefore we have

$$l(S_\zeta) = \overline{im\, B_\zeta}. \tag{11}$$

The condition of the lemma gives that $(B_\zeta h|B_\eta g)! =!0$ for all $h, g \in H$. So, $im\, B_\zeta \perp im\, B_\eta$ and the application of relation (11) together with the following obvious relation

$$P\{\zeta \in l(S_\zeta),\ \eta \in l(S_\eta)\} = 1$$

completes the proof. ■

Proof of Theorem 3. We have $S_\xi = \overline{im\,K}$ because of linearity of S_ξ and the relation (11) written for ξ (remaind that $B_\xi = K$). On the other hand, the condition $\xi_1 \perp \xi_2$ a.s. gives that $S_\xi \subset L$ where $L = \{(u,v),\ u,v \in H,\ (u|v) = 0\}$. Therefore, $imK \subset L$ and so, for all $x, y \in H$ we have the relation

$$(B_1 x + T^* y | Tx + B_2 y) = 0. \tag{12}$$

If we take $y = 0$ then we get the relation $(T^* B_1 x, x) = 0$ for all $x \in H$ which shows that the operator $T^* B_1$ is antisymmetric and thus proves the first equality in (6). The second one is proved in the same way by taking $x = 0$ in (12). Now the relation (12) gives that $(T^* y | Tx) + (B_1 x | B_2 y) = ((T^2 + B_2 B_1) x | y) = 0$ for all $x, y \in H$ and the third equality in (6) is also proved. ■

Remark. The necessity of conditions (6) was originally proved for the case of Gaussian random vectors. The possibility of extension to this more general case was noticed later by S.A.Chobanyan.

Proof of Theorem 4. a) Relations (7) are an easy consequence of inequality (5). Relations (8) follow from (7) because $\ker A \dotplus \overline{im\,A^*} = H$ for any linear bounded operator A.

b) It is enough to show that $TB_1 = B_1 T$ gives $T^* = -T$ (the implication $TB_2 = B_2 T \Rightarrow T^* = -T$ can be shown analogously). The condition $T^* B_1 = -B_1 T$ gives $T^* x = -Tx$ for $x \in \overline{im\,B_1}$. Let now $x \in \ker B_1$. Then, according to (7), $Tx = 0$ and it suffices to show that $T^* x = 0$ too, or $(T^* x | y) = 0$ for all $y \in H$. If $y \in \ker B_1$, this is clear; if $y \in im\,B_1$, then $y = B_1 z$ for some $z \in H$ and $(T^* x | y) = (x | TB_1 z) = (B_1 x | Tz) = 0$ since $x \in \ker B_1$. The last assertion is an easy consequence of $T^* = -T$ (which gives $T^{*2} = T^2$).

c) The last equality in (6) shows that if $T^* = T$, then $B_1 B_2 = B_2 B_1$. Therefore, $B_1 B_2$ is a positive operator and $(T^2 x, x) = -(B_2 B_1 x, x) \leq 0$. On the other hand, $(T^2 x, x) = (Tx, Tx) \geq 0)$. Consequently, $Tx = 0$ for all $x \in H$.

d) The counterexample can be given even in R^2. Let

$$B_1 = \begin{pmatrix} a, & 0 \\ 0, & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0, & 0 \\ 0, & b \end{pmatrix},\ T = \begin{pmatrix} 0, & 0 \\ t, & 0 \end{pmatrix},$$

where $a > 0,\ b > 0,\ t \in \mathbb{R}^1$.

e) These equalities immediately follow from the first and second equalities in (6).

f) If $B_1 = B_2 = B$, then T commutes with B^2 in virtue of the previous statement and hence it commutes with B too in virtue of Lemma of the square root (see [5], Theorem VI.9). It is enough now to apply statement b) of this theorem.

g) If $n = 2m+1$, then $\det(T^2) = \det(-B_2B_1) = (-1)^{2m+1}\det B_1 \times \times \det B_2 \leq 0$. On the other hand, $\det(T^2) = (\det T)^2 \geq 0$. Therefore $\det(T^2) = 0$ and hence $\det B_1 \det B_2 = 0$.

h) The first condition in (6) gives the relation $(Tx|B_1x) = 0$ which shows that $(Tf|f) = 0$ if $Bf = \lambda f$ and $\lambda \neq 0$. The last condition $\lambda \neq 0$ can be omitted due to the first relation in (7). Therefore, $(Tf|f) = 0$ for any eigen-vector of B_1 and it is enough to note that in a finite-dimensional space normed eigen-vectors of any self-adjoint operator constitute a basis and the trace does not depend on the choice of the basis.

i) It can be checked directly that any linear operator T in $\mathbb{R}^2$ with $trT = 0$ has the property $T^2 = T^{*2}$. So, this statement is an immediate consequence of statement h) and of the last equality in (6). ■

Proof of Theorem 5. According to relations (6), we have the equality $TB_1^{-1}T^* = B_2$ on $im\, B_1$, and hence if $B_1x + T^*y$ is denoted by h, then $Tx + B_2y$ will be equal to $TB^{-1}h$. Therefore the following equality in $H \oplus H$ is true

$$\{(B_1x + T^*y, Tx + B_2y) : x, y \in im\, B_1\} =$$
$$\{(h, TB^{-1}h) : h \in imB_1\},$$

which gives the first equality in a). To prove the second one, note that $\ker K$ is the collection of pairs (x, y) satisfying the system of equations: $B_1x + T^*y = 0$, $Tx + B_2y = 0$. The first equaition gives, because of the equality $B_1^{-1}T^* = -TB_1^{-1}$ on $im\, B_1$, the relation $B_1(x - U_y) = 0$ and hence $x = Uy$. It is also easy to show that the pair (Uy, y) satisfies the second equation for all $y \in \mathcal{D}(U)$ as well.

Using again the equality $B_1^{-1}T^* = -TB_1^{-1}$ on $im\, B_1$, we get for any fixed $y \in im\, B_1$ the equality

$$(TB_1^{-1}x|y) = -(x|TB_1^{-1}y) \quad \text{for all } x \in \mathcal{D}(U)$$

that just means the validity of statement b).

Finally we prove statement c). The equality $B_2 = -UB_1U$ can be verified directly. It is clear that $\mathcal{D}(U)$ is the projection of Γ on H that is continuous one-to-one mapping of the closed set $\Gamma \subset H \oplus H$

into H. Therefore $\mathcal{D}(U)$ is a Borel set (this can be shown, for example, by Kuratowski theorem, [6], p.5). Furthermore, any random vector belongs a.s. to its topological support, and the support of the random vector (ξ_1, ξ_2) is included in $\overline{im\,K}$ (see relation (11)) that is equal to Γ according to statement a). Therefore, $(\xi_1, \xi_2) \in \Gamma$ a.s. which means that $\xi_1 \in \mathcal{D}(U)$ and $\xi_2 = U\xi_1$ a.s. ∎

Proof of Theorem 6. Continuity of everywhere defined closed operators is well known. We show that $\mathcal{D}(U) = H$. Since $\ker B = 0$, $\overline{im\,B} = H$ and it is enough to show that convergence of Bz_n implies convergence of $TB^{-1}(Bz_n)$, $z_n \in H$, $n = 1,2\dots$ which is easily checked by using $T^* = -T$ (Theorem 4) and $T^2 = -B^2$ (relation 6). Finally, since $BT = TB$ (Theorem 4), $UB = TB^{-1}B = BTB^{-1} = BU$, and $-B^2 = T^2 = (UB)^2 = U^2B^2$ which gives that $U^2 = -I$. ∎

References

1. N.N.Vakhania and N.P.Kandelaki, On orthogonal random vectors in Hilbert space. (Russian) *Dokl. Akad. Nauk SSSR* **294**(1987), No.3, 528-531; *English transl. in Soviet Math. Dokl.* **35**(1987), No.3.

2. ——, A generalization of the Hurwitz-Radon-Eckmann theorem and orthogonal random vectors. (Russian) *Dokl. Akad. Nauk SSSR* **296**(1987), No.2, 265-266; *English transl. in Soviet Math. Dokl.* **36**(1988), No.2.

3. ——, Orthogonal random vectors in Hilbert space. (Russian) *Trudy Inst. Vichisl. Mat. Acad. Nauk Gruzin. SSR* **28:1**(1988), 11-37.

4. J.T.Schwartz, Differential Geometry and Topology. *Gordon and Breach, New York*, 1968.

5. M.Reed and B.Simon, Methods of modern mathematical physics, v.1. *Academic Press, New York/London*, 1972.

6. N.N.Vakhania, V.I.Tarieladze and S.A.Chobanyan, Probability distributions on Banach spaces. *D.Reidel, Dordrecht/Boston/Lancaster/Tokyo*, 1987, (*transl. from Russian, "Nauka", Moscow*, 1985).

7. N.P.Kandelaki, I.N.Kartsivadze and T.L.Chantladze, On orthogonal multiplication in Hilbert space. (Russian) *Trudy Tbiliss. Univ. Mat. Mech. Astron.* **179**(1976), 43-57.

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ავტორთა საყურადღებოდ

"საქართველოს მეცნიერებათა აკადემიის მაცნე. მათემატიკა" გამოდის 1993 წლის თებერვლიდან ორ თვეში ერთხელ. ჟურნალი აქვეყნებს შრომებს წმინდა და გამოყენებითი მათემატიკის ყველა დარგში. შრომები უნდა შეიცავდენ ახალ შედეგებს სრული დამტკიცებებით.

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CONTENTS

Vol.1, No.1 February 1993

76202