Iv.Javakhishvili Tbilisi State University

Magda Tsintsadze

Probabilistic Model of Canonically Conjugated Fuzzy Subsets

01.01.08 – Math.Cybernetics

Presented Dissertation for Scientific Degree “Candidate of Phys.-Math. Sciences”

Tbilisi, 2006

C o n t e n t

I n t r o d u c t i o n

In many cases of intellectual activity of human there exists virtually unlimited number of ways of inter-action of a subject with the object. As a result of this, the controlled inter-action is almost always incomplete. It is based on limited (generally small) number of attributes (color) of the object which corresponds to the interests of the subjects and which he/she can recognize. Sometimes these colors are not available for the direct observation but are available only in terms of their abstract modes (or quantity models), being the results of the direct perception or some specific measuring procedures. In this case the information loses the definition, univocacy and there appears the uncertainty.

Sometimes these abstract color modes, identifiable on the set of objects, are called variables. When the set of variables are defined as a result of our inter-action with the interesting object, we say that there is defined the system on the object with the given structure of uncertainty. Term “system” is observed as abstract or as a mode of set of some colors of the object not as a real thing. In other words, the system is the way to look at the object.

Today two sciences systematically study uncertainty [10]:

1. Fundamental science physics is the leading in study of the material (physical) world. There are two types of uncertainties connected with this science: probabilistic (objective) connected with deficiency in empiric information, received by the observation and virtual (intrinsic), quantum, placed in object directly by nature or is the result of the deficiency in means of description language.
2. Fundamental science of informatics [11] is the leading in study of the non-material (informational) world, one of the demonstrations of which is the uncertainty connected with the ambiguity and fuzziness. Uncertainty is intrinsic to the expert estimation and to natural language, as the means of description informational model.

Data in informatics – it is the set of so-called informational units. Each of informational units is the four: (object, sign, value, plausibility) [1][15][16]. It's important to differ the notion of inaccuracy and uncertainty. Inaccuracy belongs to information content (corresponding to the component “value”), and uncertainty – to its verity, understandable in terms of compatibility with reality (component “plausibility”). For the given various information there exists the opposition between inaccuracy of expression content and its uncertainty[24]-[28], expressed in that with the increase of expression accuracy, its uncertainty rises as well and vice versa, uncertain character of information leads to some inaccuracy of the final conclusions, received from this information. We see that from one side these notions in a certain way in contradiction, and from another side – complete each other upon the data presence.

We offer to model this situation by means of new concept of optimal pare of fuzzy subset and its canonically conjugated one [2]. Generally, fuzzy subset is constructed on the basis of expert estimation of one of the commutate component. From this point of view, fuzzy subset, constructed in this way, characterizes informational unit incompletely. We offer the method of construction of the informational unit membership function taking into account the both canonically conjugated components simultaneously and hence describing this unit in the most complete and optimal way. In the frames of optimal model the fuzzy logics and generalized information theory is constructed, corresponding to the canonically conjugated subsets. Theory of fuzzy canonically conjugated number and appropriate arithmetic, color operators theory, Zadeh operators [3] and so on.

Model of canonically complementary (canonically conjugated) subsets will be used in such method of decision-making as discrimination analysis, method of fuzzy probabilities, method of experts, fuzzy differential equations. New approach to this analysis method of fuzzy information allows us control uncertainty, organically inherent to informational units.

Now we shall shortly consider those general reasons which are laying in the basis of our model construction. In most cases there is uncountable (it is possible to tell unlimited) amount of assets of interaction of the expert with object [29][30][31].

Mostly this interaction is not full and is connected with small enough number of attributes (colors), identification of which expert can and which corresponds to his interests. Other colors are unattainable for direct supervision and thus make “uncontrollable" influences on object [17][18].

One of our main assumptions is that these uncontrollable influences are quantitatively characterized by random parameters [19][20], values which are not measured directly, occur estimations of these values on the basis of subjective decision.

On the basis of direct measurement and estimations is created an abstract image of objects, which we call variables [21]. The expert will establish corresponding values of variables on the basis of interaction with objects [22], after this we say that the system is determined on object, so the system is an abstract image of the real object and it is characterized by the pair of canonically conjugate fuzzy subsets.

From the above mentioned it’s clear that there is possibility to describe object using two approaches: if object is described in the basis of value uncertainty, expert is giving directly the membership function [15], but if we’ll base on plausibility's uncertainty - first is constructed focal distribution, that gives us possibility to construct fuzzy measure [10][23], thus the membership function.

We have to mention here that these two possibilities are “complementary” of each other:

We are offering such description of appropriate uncertainty of the object, where some characteristic of first and second type joint uncertainties would be minimal; preciously we are offering to construct such membership function of fuzzy subset, which will provide minimization of above mentioned joint uncertainty [4].

Notion of color is the base of dissertation, thus without the preliminary consideration of color theory, presentation of it is impossible. Below we offer exactly such consideration.

In the chapter I, there is presented the total and detailed consideration of color theory.

Let's given set Ω(universal set) and defined property $\wp$ in it. Lets note by $\wp_1(\Omega) and \wp_2(\Omega)$ [7] subset of Ω, defined by elements $\omega \in \Omega$ for which the expression $\wp[\omega]$ (ω possesses color $\wp$) is true or false appropriately. Further let's $\wp_0(\Omega) \subseteq \wp_{\neq 1}(\Omega)$. We can consider color $\neg \wp$ defined in Ω:

$$\neg \wp[\omega] \Leftrightarrow \omega \in \wp_0(\Omega) \qquad (1)$$

if $\overline{\wp}$ defines color complementary to $\wp$ then in Ω following relation has place:

$$\neg \wp[\omega] \Rightarrow \overline{\wp}[\omega] \qquad (2)$$

Opposite implication is fair only on set: $A(\Omega) = \wp_1(\Omega) \cup \wp_0(\Omega)$. With the help of $\wp_0$, it's possible to define such various $\neg \wp$ that if $\neg \wp$ is true, then $\wp$ is false. But opposite implication has place only on set $A(\Omega) \subseteq \Omega$.

Let's check how it's possible to construct set $\wp_0(\Omega) \subseteq \wp_{\neq 1}(\Omega)$. With this aim, let's assume that each elements of Ω can possess color $\wp$ in different amount. Further let's consider that we are able to attach its compatibility measure with color $\wp$ to each element $\omega \in \Omega$. Formally there is given such a reflection:

$$\mu_{\wp} : \Omega \to [0,1] \qquad (3)$$

that:

$$\wp[\omega] \Leftrightarrow (\mu_{\wp}(\omega) = 1) \qquad (4)$$

For each $\omega \in \Omega$, $\mu_{\wp}(\omega)$ is called the value of membership function of ω with $\wp$ or membership measure of ω to $\wp_1(\Omega)$. If $\mu_{\wp}(\omega) = 1$ we will say that ω possesses color $\wp$. If $\mu_{\wp}(\omega) = 0$ then ω does not possess color $\wp$. Further $\wp_0(\Omega)$ identify with the subset of $\wp_{\neq 1}(\Omega)$ elements, not possessing color $\wp$. Color $\wp$ in Ω satisfying the described above-mentioned conditions we will call "measurable" in Ω. If additionally assume that $\wp_1(\Omega)$ is not empty, $\wp$ we will call "completely measurable".

Let's assume that color $\wp$ is characterized by numerical parameter ξ. (analogue notion "red apple", $\wp$ is defined on physical scale of frequency where to the given color corresponds the defined frequency interval).

Main Assumption. ξ numerical characteristic of color is the random quantity. In the referent system Ω is hidden parameter.

Let's distribution of probabilistic values $\xi_{\wp}(x_\omega)$ ($\in \Re$) is characterized by density $\rho_{\wp}(x_\omega)$.($\int_{\Re} \rho_{\wp}(x_\omega) dx = 1$). Quantity

$$x_\omega^* = M\xi_{\wp} = \int_{\Re} x \rho_{\wp}(x_\omega) dx \qquad (5)$$

call calculated value of color $\wp$ in $\omega \in \Omega$.

Note, that formula (5) satisfies relation between set of calculated values X^* and universal set Ω, that is why the following definitions are clear:

(set x_ω^*, where $x_\omega^* \in X^*$ and $\omega \in \wp(\Omega)$) $\equiv$ $\wp(\Re)$,

(set x_ω^*, where $x_\omega^* \in X^*$ and $\omega \in \wp_1(\Omega)$) $\equiv$ $\wp_1(\Re)$, (6)

(set x^*_ω, where $x^*_\omega \in X^*$ and $\omega \in \wp_0(\Omega)$) $\equiv$ $\wp_0(\Re)$.

We transferred uncertainty structure (system) Ω in $\Re$.

If $\wp_1(\Omega)$ is non-empty set, exist such ω, that $\int\limits_{\wp_1(\Re)} \rho_\wp(x_\omega)dx = 1$.

Except of $M\xi$, presence of color to ω is characterized by dispersion also:

$$\sigma^2_\wp(\omega) = \int\limits_\Re (x - x^*_\omega)^2 \rho_\wp(x;\omega)dx \qquad (7)$$

In our model exactly $\sigma^2_\wp(\omega)$ is connected with definition of presence $\wp$ color to ω. If $\sigma^2_\wp(\omega) \to 0$, we'll say $\wp$ has quite define value ω. The more $\sigma^2_\wp(\omega)$ is, the uncertain $\wp$ in ω. If $\sigma^2_\wp(\omega) \to \infty$ it means ω has no $\wp$ color. Thus, if $\mu_\wp(\omega) = 1$, we will say that x^*_ω possesses color $\wp$, if $\mu_\wp(\omega) = 0$, than x^*_ω does not possess color $\wp$ [5].

$\wp_0(\Re)$ is identify with whole "colorless"(not painted in color $\wp$) elements x^*_ω. Elements of $\Re$, which do not belong to $\wp_1(\Re) \cup \wp_0(\Re)$, possessing color $\wp$ in some amount are characterized by number $\mu_\wp(\omega)$ from (0,1). Thus model of color might be transferred in $\Re$. Below we will consider universal set as numerical set $\Re : \mu_\wp(\omega) = \mu_\wp(x^*_\omega, \sigma^2_\wp)$.

Notion computability in $\Re$ corresponds to notion "measurable" in Ω and notion "completely computability" in $\Re$ - to the notion of "complete measurable" in Ω.

Assumption 2.

In $\Re(\Omega)$ are defined only $\wp$ and $\neg\wp$. Which means along with $\wp_1(\Re)$ there exists the only $\wp_0(\Re)$ but elements of $\Re(\Omega)$, not belonging to these two subsets possess color "intervening" between $\wp$ and $\neg\wp$. This accusation is expressed with the help of the following relation:

$$\mu_{\neg\wp}\left(x_\omega^*, \sigma_\wp^*\right) = 1 - \mu_\wp\left(x_\omega^*, \sigma_\wp^*\right) \qquad (8)$$

Colors $\wp$ and $\neg\wp$ are not complementary in common meaning [8], they are such only in appropriate $A(\Re)$ which means the conditions $\wp[x_\omega^*] \vee \neg\wp[x_\omega^*] = T$ and $\wp[x_\omega^*] \wedge \neg\wp[x_\omega^*] = \varnothing$ generally speaking are not fulfilled. Here T is always true expression, $\vee$ is sign of disjunction and $\wedge$ is sign of conjunction. These conditions are substituted with the conditions (8).

Definition 1. For $\forall\omega \in \Omega$ let's introduce some interval of $\wp$ values with the help of relation :

$$\mu_\wp(\omega) = 1 - \int_{\amalg_\wp(\omega)} \rho(x;\omega)dx = 1 - \int_\Re \chi_{\amalg_\wp(\omega)}(x)\rho(x;\omega)dx$$

where $\mu_\wp(\omega)$ is defined by expert, χ common characteristic function of interval $\amalg_\wp(\omega)$. Let's call interval defined by (9) as the characteristic interval of color $\wp$.

We have to mention also important paragraph 2: "Theory of informational function representation".

In theory of presents the main role plays the notion of informational function.

Definition 2. informational function of color $\wp$ let's call the following expression:

$$\sqrt{\rho_\wp(x_\omega)}e^{i\varphi} \equiv \left|x, x_\omega^*;\wp\right\rangle \qquad (10)$$

where φ is random phase and is a real quantity. We took the advantage of Dirac [6][9] nomenclature. We will use this function for the presentation of the information (uncertainty) contained in color $\wp$. Informational function module square defines membership function (precisely the appropriate density):

$$\rho_{\wp}(x,\omega) = \left| x, x_{\omega}^{*}; \wp \right\rangle^{+} \left| x, x_{\omega}^{*}; \wp \right\rangle \quad (11)$$

Any fuzzy subsets of $\tilde{\wp}$ ($\tilde{\wp}$ is $\wp$ color appropriate fuzzy subset) can be described independently from hidden parameters (ξ) type by some quantity which we will call ket-vector [7] (by Dirac nomenclature) and will note by $|\wp\rangle$. Let's $|\wp\rangle \in L^2(\Re)$ (Hilbert space). Let's consider the Fourier transformation of this ket-vector:

$$\hat{F}\left| x; x_{\omega}^{*}; \wp \right\rangle = \frac{1}{\sqrt{2\pi c}} \int_{\Re} \left| x; x_{\omega}^{*}; \wp \right\rangle e^{-\frac{i}{c} x x_c} dx \quad (12)$$

Where C is constant.

Expression (12) is identified with informational function in x_c presence:

$$\hat{F}\left| x; x_{\omega}^{*}; \wp \right\rangle = \left| x_c; x_{\omega c}^{*}; \wp_c \right\rangle \quad (13)$$

Where $\wp_c$ is canonically conjugated in relation to $\wp$ color.

$\tilde{\wp}_c$ fuzzy subset – canonically conjugated in relation to $\tilde{\wp}$ is appropriate to this color[13], membership function of which is defined by formula (9):

$$\chi_{\wp c}(\omega_c) = \int_{I_{\wp c}(\omega_c)} \left| x_c; x_{\omega c}^{*}; \wp_c \right\rangle^{+} \left| x_c; x_{\omega c}^{*}; \wp_c \right\rangle dx_c =$$

(14)

$$= \int_{\Re} I_{I_{\wp c}(\omega_c)}(x_c) \left| x_c; x^*_{\omega c}; \wp_c \right\rangle^+ \left| x_c; x^*_{\omega c}; \wp_c \right\rangle dx_c$$

In space of information function $\left| x; x^*_{\omega}; \wp \right\rangle$, operator of color $\hat{\wp}$ is appropriate to color $\wp$. If information about color is precious, then

$$\hat{\wp}\left| x; x^*_{\omega}; \wp \right\rangle = x\left| x; x^*_{\omega}; \wp \right\rangle \qquad (15)$$

Analogically

$$\hat{\wp}_c \left| x_c; x^*_{\omega c}; \wp_c \right\rangle = x_c \left| x_c; x^*_{\omega c}; \wp_c \right\rangle \qquad (16)$$

Operators $\hat{\wp}$ and $\hat{\wp}_c$ are connected with the following commutation:

$$\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c \hat{\wp} = ic\hat{E} \qquad (17)$$

Where $\hat{E}$ is operator of identity presence. This relation should define the quantity connection between canonically conjugated colors. This connection is studied in details. Hence the meaning of this study is necessary to control the uncertainty decision-making systems. Formula (17) bounds the simultaneous calculation of canonically conjugated colors. The uncertainty principle, analogical to the Heizenberg's principle, which allows introduction of definitely optimal fuzzy subset $\tilde{\wp}$, for $\wp$ and $\wp_c$ is studied. By using our theory with the set of real numbers we constructed arithmetic of optimal real numbers.

Theory of its common form is connected with the vector properties and to appropriate operators in Hilbert space: each informational state corresponds to definite estimation of membership function, and the color – to operator. Though the various formulations are possible in frames of which the informational functions in the phase space (Cartesian product of universal set on canonically conjugated) can be connected as with the informational state, so with the observable (estimated by expert) colors. As the sample of such formalism is theory of phase functions of Vigner and transformation of Veill. The dissertation presents construction of analogical formalism: colors proper vectors $|\wp\rangle$ and $|\wp_c\rangle$ or appropriate operators $\hat{\wp}$ and $\hat{\wp}_c$ are satisfying equation for proper values (15) and (16). The complete system of proper vectors satisfies the completeness condition:

$$\int dx_{\omega}|\wp\rangle\langle\wp| = \hat{I} \ , \quad \int dx_{\omega c}|\wp_c\rangle\langle\wp_c| = \hat{I} \qquad (18)$$

Where $\hat{I}$ is unique operator in Hilbert's space.

In conclusion the consideration of conjugated colors theory application is given.

Chapter I

Color Representation in Fuzzy Probabilistic Model

§ 1.1 Notion and Properties of Color

Suppose, $\tilde{A}$ fuzzy subset of Ω Universal set corresponds to A concept and suppose this concept is characterized by numerical parameter ξ.

Consider ξ is quantitatively characterizing some property of $\tilde{A}$ - let's call it "color" $\wp$.

Main definition: The numerical characteristic of color $\xi\,\wp_{\tilde{A}}[\omega]$ is a random quantity. Define appropriate distribution density of probabilities by $\rho_{\wp}(x;\omega)$.

Denote

$$x_{\omega}^{*} = M\xi\left(\wp_{\tilde{A}}[\omega]\right) = \int_{\Re} x\rho_{\wp}(x;\omega)dx \qquad (1)$$

as Calculated value of $\tilde{A}$ fuzzy subset membership function's modal value.

Note, that formula (1) satisfies relation between set of calculated values X^* and universal set Ω, that is why the following definitions are clear:

(set x_ω^*, where $x_\omega^* \in X^* and \omega \in \wp(\Omega)$) $\equiv$ $\wp(\Re)$,

(set x_ω^*, where $x_\omega^* \in X^*$) and $\omega \in \wp_1(\Omega) \equiv \wp_1(\Re)$, (*)

(set x_ω^*, where $x_\omega^* \in X^*$) and $\omega \in \wp_0(\Omega) \equiv \wp_0(\Re)$.

We transferred uncertainty structure (system) Ω in $\Re$.

Except of $M\xi$, presence of color to ω is characterized by dispersion also:

$$\sigma_\wp^2(\omega) = \int_\Re \left(x - x_\omega^*\right)^2 \rho_\wp(x;\omega)dx \qquad (2)$$

In our model exactly $\sigma_\wp^2(\omega)$ is connected with definition of presence $\wp$ color to ω. If $\sigma_\wp^2(\omega) \to 0$, we'll say $\wp$ has quite define value ω. The more $\sigma_\wp^2(\omega)$ is, the uncertain $\wp$ in ω. If $\sigma_\wp^2(\omega) \to \infty$ it means ω has no $\wp$ color. Thus, if $\mu_\wp(\omega) = 1$, we will say that x_ω^* possesses color $\wp$, if $\mu_\wp(\omega) = 0$, than x_ω^* does not possess color $\wp$.

$\wp_0(\Re)$ is identify with whole "colorless"(not painted in color $\wp$) elements x_ω^*. Elements of $\Re$, which do not belong to $\wp_1(\Re) \cup \wp_0(\Re)$, possessing color $\wp$ in some amount are characterized by number $\mu_\wp(\omega)$ from (0,1). Thus model of color might be transferred in $\Re$. Below we will consider universal set as numerical set $\Re : \mu_\wp(\omega) = \mu_\wp\left(x_\omega^*, \sigma_\wp^2\right)$.

Notion computability in $\Re$ corresponds to notion "measurable" in Ω and notion "completely computability" in $\Re$ - to the notion of "complete measurable" in Ω.

suppose:

$$
\begin{aligned}
&1)\ \wp_1(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) = 0, \forall \omega \in \wp_1(\Omega) \\
&2)\ \wp_{\neq 1}(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) \neq 0, \forall \omega \in \wp_{\neq 1}(\Omega) \qquad (3) \\
&3)\ \wp_0(\Omega) \subseteq \Omega : \sigma^2_{\wp}(\omega) = +\infty, \forall \omega \in \wp_0(\Omega)
\end{aligned}
$$

It means that:

1) for $\forall \omega \in \wp_1(\Omega)$, expression " ω has $\wp$ color" , $\wp[\omega]$ - is true;

2) for $\forall \omega \in \wp_{\neq 1}(\Omega)$, we say $\overline{\wp}[\omega]$ is true if $\wp[\omega]$ is false.

3) if for $\forall \omega \in \wp_0(\Omega)$, expression $\wp[\omega]$ is false, we say that in this case $\neg \wp[\omega]$ - is true.

The following is valid:

$$\neg \wp[\omega] \Rightarrow \overline{\wp}[\omega] \qquad (4)$$

But the reverse implication is valid only on the following subset:

$$A(\Omega) \equiv \wp_1(\Omega) \cup \wp_0(\Omega)$$

if $\wp_0(\Omega)$ is proper subset of $\wp_{\neq 1}(\Omega)$, than in Ω exist ω , that :

$$0 < \sigma^2_{\wp}(\omega) < +\infty$$

Now we may indicate easy way to separate $\wp_0(\Omega)$ subset from $\wp_{\neq 1}(\Omega)$. Assume Ω universal subset's every element is characterized $\wp$ color specified quantity. Formally it means that expert may give the reflection directly:

$$\mu_{\wp} : \Omega \to [0,1] \tag{5}$$

Which has the property:

$$\wp[\omega] \Leftrightarrow (\mu_{\wp}(\omega) > 0) \tag{6}$$

$\mu_{\wp}(\omega)$ is considered as measure (membership function) of $\wp$ color presence to ω. If $\mu_{\wp}(\omega)=1$, it's said: ω has $\wp$ color, but if $\mu_{\wp}(\omega)=0$, then ω has no $\wp$ color. $\wp_0(\Omega)$ is the set of such ω, which "are not colored in $\wp$ color": $\wp_0(\Omega)=\{\omega : \mu_{\wp}(\omega)=0, \omega \in \Omega\}$.

The elements of $\Omega \notin \wp_1(\Omega) \cup \wp_0(\Omega)$, which have $\wp$ color in some amount, are characterized by numbers from (0,1).

Proposition 1. There are defined just $\wp$ and $\neg\wp$ in Ω, so we have just one $\wp_0(\Omega)$ with $\wp_1(\Omega)$, as for other elements from Ω, which are not belonging this two subset , we are saying they have color "passing through" $\wp$ and $\neg\wp$:

$$\mu_{\neg\wp}(\omega)=1-\mu_{\wp}(\omega), \forall \omega \in \Omega \tag{7}$$

$\wp$ and $\neg\wp$ colors are not really complementary of each other ($\neg\wp \neq \overline{\wp}$), they are such just on appropriate $A(\Omega)$, so conditions $\wp[x_{\omega}^{*}] \vee \neg\wp[x_{\omega}^{*}] = T$ **and** $\wp[x_{\omega}^{*}] \wedge \neg\wp[x_{\omega}^{*}] = \varnothing$ **generally speaking are not fulfilled. Here T is always true expression,** $\vee$ **is sign of disjunction and** $\wedge$ **is sign of conjunction. These conditions are substituted with the conditions (7).**

Let's consider Ω universal set, some $\wp$ color and it's compatibility with points of Ω . It is clear that we can use normal indicator $\mathrm{I}_{\wp_1(\Omega)}(\omega)$ with values in $\{0,1\}$ as characteristic of $\wp$ color compatibility with points $\wp_1(\Omega) \cup \wp_0(\Omega)$, as for points $\wp_{\neq 1}(\Omega) \setminus \wp_0(\Omega)$, for them we'll use generalized indicator $\mu_{\wp}(\omega)$ with values from (0,1).

Proposition 2 . $\mu_{\wp}(\omega)$ is equal of fuzzy subset of $\tilde{\Omega}$, where

$$\sup p\tilde{\Omega} = \left(\wp_{\neq 1}(\Omega) \setminus \wp_0(\Omega)\right) \cup \wp_1(\Omega)$$

Note 1. $\mu_{\wp}(\omega)$ and $\left(\wp_{\neq 1}(\Omega) \setminus \wp_0(\Omega)\right) \cup \wp_1(\Omega)$ are defining $\tilde{A}$ fuzzy subset, where $\sup p\tilde{A} = \left(\wp_{\neq 1}(\Omega) \setminus \wp_0(\Omega)\right) \cup \wp_1(\Omega)$.

Definition 3. $\forall \omega \in \Omega$ introduce some $\amalg_{\wp} \subseteq \Re$ interval on scale of $\wp$ color values (i.e. on $\Re$) by

$$\mu_{\wp}(\omega) = 1 - \int_{\amalg_{\wp}(\omega)} \rho(x;\omega)dx = 1 - \int_{\Re} \chi_{\amalg_{\wp}(\omega)}(x)\rho(x;\omega)dx$$

(8)

where $\mu_{\wp}(\omega)$ is defined by expert, χ common characteristic function of interval $\amalg_{\wp}(\omega)$.

Main Definition: (1), (2), (7), (8) equations define the following set:

$$\widetilde{\Omega} = \left\{ \widetilde{\omega} \equiv \left(\omega; \mu_{\wp}(\omega) \right) : \omega \in \Omega \right\} \tag{9}$$

Call Ω universal set the probabilistic model of $\widetilde{\Omega}$ fuzzy subset.

§ 1.2 Information Functions Representation [13]

Suppose, $\chi_{\widetilde{A}}(\omega), \omega \in \Omega$ denotes $\widetilde{A}$ fuzzy subset's appropriate membership function.

Assumption 2. Lets call expression

$$\sqrt{\rho_{\widetilde{A}}(x;\omega)} \equiv \left\langle x, x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \tag{10}$$

Information function.

Here we are using Dirac's notation. We need this function to represent the information in $\widetilde{A}$ concept. Information function's magnitude square determines membership function (precisely the appropriate density)[12][13][14]:

$$\rho_{\widetilde{A}}(x;\omega) = \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \tag{11}$$

Any $\widetilde{A}$ fuzzy subset might be defined separate from some value's hidden parameters. Call such values (by Dicar nomenclature) ket-vector and denote as $\left| \widetilde{A} \right\rangle$.

We may sum ket-vectors, also product ket-vectors as on scalar also on complex values –and receive ket-vectors again.

Suppose, $\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \in L^{2}(\Re)$ (Hilbert space), consider Fourier transformation of this function:

$$F\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle = \frac{1}{2\pi c} \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} dx \qquad (12)$$

where c is const. (12) expression is equal of information function at x_{c} presentation:

$$\hat{F}\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle = \left\langle x_{c}, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle \qquad (13)$$

where $\tilde{A}^{c}$ is canonically conjugate fuzzy subset:

$$\chi_{\wp c}(\omega) = \int_{I_{\wp c}(\omega)} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle^{+} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle dx_{c} =$$

$$(14)$$

$$= \int_{\Re} I_{I_{\wp c}(\omega)}(x_{x}) \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle^{+} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle dx_{c}$$

At information function space $\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle$, $\hat{\wp}$ operator is appropriate of $\wp$ color. If information about color is precise, than

$$\hat{\wp}\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle = x\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \qquad (15)$$

analogically

$$\hat{\wp}_c\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle = x_c\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle \qquad (16)$$

Theorem 1. Suppose $\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle$ and $\frac{d}{dx}\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle \in L^2(\Re)$, $\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle = \hat{F}\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle$, than the following expression is valid for $\hat{\wp}$ and $\hat{\wp}^c$ operators:

$$\hat{\wp}_c\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle = -ic\frac{d}{dx}\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle \qquad (17)$$

And analogically

$$\hat{\wp}\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle = ic\frac{d}{dx_c}\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle \qquad (18)$$

Note that by (15),(16) and (11),(1):

$$x^*_\omega = \int_\Re \left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle^+ \hat{\wp}_c\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle dx \qquad (19)$$

$$x^*_{c\omega} = \int_\Re \left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle^+ \hat{\wp}\left\langle x_c;x^*_{c\omega}\middle|\tilde{A}^c\right\rangle dx_c \qquad (20)$$

Consider (20) , let us show that the equality is true:

$$x^*_{c\omega} = \int_\Re \left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle^+ \left(-ic\frac{d}{dx}\right)\left\langle x;x^*_\omega\middle|\tilde{A}\right\rangle dx_c =$$

$$= \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \hat{\wp}_{c} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx_{c}$$

(21)

We have:

$$x_{c\omega}^{*} = \int_{\Re} dx_{c} \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} x} \right] \hat{\wp}_{c} \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} \right] =$$

$$= \int_{\Re} dx_{c} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle x_{c} \frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} =$$

$$= \frac{1}{2\pi c} \int_{\Re} \int_{\Re} \int_{\Re} dx_{c} dx dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+'} \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} x} \left(ic \frac{d}{dx'} e^{-\frac{i}{c} x_{c} x'} \right) =$$

$$= \frac{1}{2\pi c} \int_{\Re} \int_{\Re} \int_{\Re} dx_{c} dx dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle ic \frac{d}{dx'} e^{\frac{i}{c} x_{c} (x - x')} =$$

$$= \frac{i}{2\pi} \int_{\Re} dx_{c} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left[\left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} (x - x')} \Big|_{+\infty}^{-\infty} - \right.$$

$$\left. - \int_{\Re} dx' \frac{d}{dx'} \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} (x - x')} \right] =$$

$$= -\frac{i}{2\pi} \int_{\Re} dx \int_{\Re} dx' \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \int_{\Re} dx_{c} e^{\frac{i}{c} x_{c} (x - x')} =$$

$$= \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle^{+} \left(-ic \frac{d}{dx} \right) \left\langle x; x_{\omega}^{*} \middle| \widetilde{A} \right\rangle$$

Note 2. The followings are valid:

$$x_{\omega}^{*} = \left(\left\langle x, x_{\omega}^{*} \middle| \widetilde{A} \right\rangle, \hat{\wp} \left\langle x, x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \right) =$$

$$= \left(\left\langle x_{C}, x_{C\omega}^{*} \middle| \widetilde{A}^{C} \right\rangle, \hat{\wp} \left\langle x_{C}, x_{C\omega}^{*} \middle| \widetilde{A}^{C} \right\rangle \right) \qquad (22)$$

$$x_{C\omega}^{*} = \left(\left\langle x_{C}, x_{C\omega}^{*} \middle| \widetilde{A}^{C} \right\rangle, \hat{\wp}_{C} \left\langle x_{C}, x_{C\omega}^{*} \middle| \widetilde{A}^{C} \right\rangle \right) =$$

$$= \left(\left\langle x, x_{\omega}^{*} \middle| \widetilde{A} \right\rangle, \hat{\wp}_{C} \left\langle x, x_{\omega}^{*} \middle| \widetilde{A} \right\rangle \right) \qquad (23)$$

The proves of these equalities can be done by the same way, so we aren't considering them here.

Theorem 2. Operators $\hat{\wp}$ and $\hat{\wp}_{C}$ are satisfying the following condition:

$$\hat{\wp}\,\hat{\wp}_{C} - \hat{\wp}_{C}\hat{\wp} = ic\hat{E} \qquad (24)$$

where $\hat{E}$ - identity operator.

Proof. : Suppose $\hat{\wp} f(x) = xf(x)$, $f(x)$, $xf(x)$ and $f'(x) \in L^{2}(\Re)$, then

$$\left(\hat{\wp}\,\hat{\wp}_{C} - \hat{\wp}_{C}\hat{\wp} \right) f(x) = \hat{\wp}\left(\hat{\wp}_{C} f(x) \right) - \hat{\wp}_{C}\left(\hat{\wp} f(x) \right) =$$

$$= -icx\frac{df(x)}{dx} + ic\frac{d}{dx}\left(xf(x)\right) = ic\hat{E}f(x)$$

Here we have used the following facts:

$$\hat{\wp}\left(\hat{\wp}_C f\right) = x\left(\hat{\wp}_C f\right) \quad \text{and} \quad \hat{\wp}_C\left(\hat{\wp} f\right) = -ic\frac{d}{dx}\left(\hat{\wp} f\right), \quad x \in \Re$$

according to: $\left(f, \hat{\wp}\,\hat{\wp}_C f\right) = \left(\hat{\wp} f, \hat{\wp}_C f\right) = \left(xf, \hat{\wp}_C f\right) = \left(f, x\,\hat{\wp}_C f\right)$

§ 1.3 Connection between canonically conjugated colors

Connection between canonically conjugated colors is given with the following theorem:

Theorem 3 [13]. if $\wp$ and $\wp_C$ are canonically conjugated colors, then

$$\sigma^2_{\wp}\left(x^*_{\omega}\right)\sigma^2_{\wp_C}\left(x^*_{c\omega}\right) \geq \frac{c^2}{4} \tag{25}$$

Proof. Let's enter designations

$$\hat{\alpha} \equiv \hat{\wp} - x^*_{\omega}\hat{E} \quad , \quad \hat{\beta} \equiv \hat{\wp}_C - x^* c\omega\hat{E} \tag{26}$$

appropriately

$$\sigma^2_{\wp}\left(x^*_{\omega}\right) = \left\langle \hat{\alpha}^2 \right\rangle = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\alpha}^2 \left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right)$$

(27)

$$\sigma^2_{\wp_C}\left(x^*_{c\omega}\right) = \left\langle \hat{\beta}^2 \right\rangle = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\beta}^2 \left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right)$$

We have

$$\sigma_{\wp}^{2}\left(x_{\omega}^{*}\right)\sigma_{\wp_{C}}^{2}\left(x_{C\omega}^{*}\right)=$$

$$=\int_{\Re}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\hat{\alpha}^{2}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx\int_{\Re}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\hat{\beta}^{2}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx=$$

$$=\int_{\Re}\hat{\alpha}^{+}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\hat{\alpha}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx\int_{\Re}\hat{\beta}^{+}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\hat{\beta}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx$$

(28)

Using Cauchy-Buniakovski inequality:

$$\int\left|f(x)\right|^{2}dx\int\left|g(x)\right|^{2}dx\geq\left|\int f(x)g(x)dx\right|^{2} \qquad (29)$$

And suppose that:

$$\hat{\alpha}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle\equiv f(x) \quad \text{and} \quad \hat{\beta}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle\equiv g(x) \quad ,$$

We will have:

$$\sigma_{\wp}^{2}\left(x_{\omega}^{*}\right)\sigma_{\wp_{C}}^{2}\left(x_{C\omega}^{*}\right)\geq\left|\int_{\Re}\hat{\alpha}^{+}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\hat{\beta}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx\right|^{2}=$$

$$=\left|\int_{\Re}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle\hat{\alpha}\hat{\beta}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx\right|^{2}=$$

$$=\left|\int_{\Re}\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle^{+}\left[\frac{1}{2}\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)+\frac{1}{2}\left(\hat{\alpha}\hat{\beta}-\hat{\beta}\hat{\alpha}\right)\right]\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle dx\right|^{2}=$$

(30)

$$= \frac{1}{4}\left| \int_{\Re} \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} \right) \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx \right|^{2} + \frac{1}{4}\left| \int_{\Re} \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} \right) \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx \right|^{2}$$

Missed member is equal to 0, because $\hat{\alpha}^{+} = \hat{\alpha}$ and $\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} = ic\hat{E}$.

So using (26):

$$\left(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} \right) \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle = -ic\left[x \frac{d}{dx} \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle - \frac{d}{dx}\left(\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \right) \right] = ic\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle$$

(31)

so, if at right side of equality (30) we'll ignore second summary (which ≥ 0) finally receive (25) .

Chapter II

Joint Representation of Characteristic Functions

§ 2.1 Characteristic functions of canonically conjugated colors [13]

Let us consider the operators:

$$\hat{M}(\alpha) = \exp\left(i\alpha \hat{\wp} \right) \quad (32)$$

$$\hat{M}^{c}(\beta) = \exp\left(i\beta \hat{\wp}_{c} \right) \quad (33)$$

Scalar product:

$$M(\alpha) = \left(\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle, \hat{M}(\alpha) \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \right) \quad (34)$$

$$M^{c}(\beta) = \left(\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle, \hat{M}^{c}(\beta) \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \right) \quad (35)$$

call appropriately characteristic functions of canonically conjugated colors $\hat{\wp}$ and $\hat{\wp}_c$.

Theorem1. Characteristic functions $M(\alpha)$ and $M^c(\beta)$ are caliber-invariant.

Proof. It is clear that (22) and (23) might be generalized:

$$\left(\left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle, \hat{\wp}^n \left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle\right) = \left(\left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle, \hat{\wp}^n \left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle\right) \qquad (36)$$

$$\left(\left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle, \hat{\wp}_c{}^n \left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle\right) = \left(\left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle, \hat{\wp}_c{}^n \left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle\right) \qquad (37)$$

because of :

$$\exp(i\alpha\hat{\wp}) = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} \hat{\wp}^k$$

and

$$\exp(i\beta\hat{\wp}_c) = \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!} \hat{\wp}_c{}^k$$

so, according to (36) and (37), the invariance of $M(\alpha)$ and $M^c(\beta)$ is clear:

$$M(\alpha) = \left(\left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle, \hat{M}(\alpha) \left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle\right) = \left(\left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle, \hat{M}(\alpha) \left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle\right)$$

(38)

$$M^c(\beta) = \left(\left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle, \hat{M}^c(\beta) \left\langle x, x_\omega^* \middle| \widetilde{A}\right\rangle\right) = \left(\left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle, \hat{M}^c(\beta) \left\langle x_c, x_{c\omega}^* \middle| \widetilde{A}^c\right\rangle\right)$$

(39)

note: Operators $\hat{\wp}^n$ and $\hat{\wp}_c{}^n$ might be confront with some colors , denoted as $\wp^n$ and $\wp_c{}^n$ by us.

According to (15)-(18) we have:

$$\hat{M}(\alpha)\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle = e^{i\alpha c}\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle$$

$$\hat{M}(\alpha)\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle = \left\langle x_{c} - \alpha c, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle$$

(40)

$$\hat{M}^{c}(\beta)\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle = \left\langle x + \beta c, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle$$

$$\hat{M}^{c}(\beta)\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle = e^{i\beta x}\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle$$

First and last equality of (40) directly comes from (15) and (16), as for third and fourth equalities – according to (17)-(18), we have:

$$\sum_{k=0}^{\infty}\frac{(i\alpha)^{k}}{k!}\hat{\wp}^{k}\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle = \sum_{k=0}^{\infty}\frac{(i\alpha)^{k}}{k!}(i\alpha)^{k}\frac{d^{k}}{dx^{k}}\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle =$$

$$= \sum_{k=0}^{\infty}(-1)^{k}\frac{(\alpha c)^{k}}{k!}\frac{d^{k}}{dx^{k}}\left\langle x_{c}, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle =$$

$$= \left\langle x_{c} - \alpha c, x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle$$

$$\sum_{k=0}^{\infty}\frac{(i\beta)^{k}}{k!}\hat{\wp}_{c}^{\,k}\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle = \sum_{k=0}^{\infty}\frac{(i\beta)^{k}}{k!}(-ic)^{k}\frac{d^{k}}{dx^{k}}\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle =$$

$$= \sum_{k=0}^{\infty}\frac{(\beta c)^{k}}{k!}\frac{d^{k}}{dx^{k}}\left\langle x, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle =$$

$$= \left\langle x + \beta c, x_{\omega}^{*}\middle|\widetilde{A}\right\rangle$$

Theorem 2. If $\hat{M}(\alpha)$ and $\hat{M}^{c}(\beta)$ are defined by formulas (32)-(33), $\left\langle x; x_{\omega}^{*}\middle|\widetilde{A}\right\rangle \in L^{2}(\Re)$ and $\left\langle x_{c}; x_{c\omega}^{*}\middle|\widetilde{A}^{c}\right\rangle = \hat{F}\left\langle x; x_{\omega}^{*}\middle|\widetilde{A}\right\rangle$ then

$$\rho_{\wp}\left(x,x^{*}\right)=\frac{1}{2\pi}\int_{\Re}\left(\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle,\hat{M}(\alpha)\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle\right)e^{-i\alpha x}d\alpha \qquad (41)$$

$$\chi_{\wp_C}\left(x,x_C{}^{*}\right)=\frac{1}{2\pi}\int_{\Re}\left(\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle,\hat{M}^{C}(\beta)\left\langle x,x_{\omega}^{*}\middle|\tilde{A}\right\rangle\right)e^{-i\beta x}d\beta \qquad (42)$$

note. According to the fact of caliber-invariance (see theorem 1.) probabilities distribution densities of canonically conjugated colors $\hat{\wp}$ and $\hat{\wp}_c$ allowing following presentations:

$$\rho_{\wp}\left(x,x^{*}\right)=\frac{1}{2\pi}\int_{\Re}\left(\left\langle x_C;x_{C\omega}^{*}\middle|\tilde{A}^{C}\right\rangle,\hat{M}(\alpha)\left\langle x_C;x_{C\omega}^{*}\middle|\tilde{A}^{C}\right\rangle\right)e^{-i\alpha x}d\alpha \qquad (41')$$

$$\chi_{\wp_C}\left(x,x_C{}^{*}\right)=\frac{1}{2\pi}\int_{\Re}\left(\left\langle x_C;x_{C\omega}^{*}\middle|\tilde{A}^{C}\right\rangle,\hat{M}^{C}(\beta)\left\langle x_C;x_{C\omega}^{*}\middle|\tilde{A}^{C}\right\rangle\right)e^{-i\beta x}d\beta$$

(42')

§ 2. 2 Canonically conjugated colors joint distribution

Suppose $\hat{\wp}$ and $\hat{\wp}_C$ are canonically conjugated operators. Denote x and x_C their possible values. It is clear to count $e^{i\left(\tau\hat{\wp}+\theta\hat{\wp}_C\right)}$ [5],[12],[34],[35] to define phase distribution (joint distribution), the average value of which will be denoted as membership function

$$M(\tau,\theta)=\left\langle \tilde{A}\middle|e^{i\left(\tau\hat{\wp}+\theta\hat{\wp}_C\right)}\middle|\tilde{A}\right\rangle \qquad (43)$$

Using well known opposite formula of Fourier transaction for discrete proper values we have:

$$F\left(x_i, x_{ck}\right) = \lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{+T} \int_{-T}^{+T} \left\langle \tilde{A} \left| e^{i\left(\tau\hat{\wp} + \theta\hat{\wp}_c\right)} \right| \tilde{A} \right\rangle e^{-i\left(\tau x_i + \theta x_{ck}\right)} d\theta d\tau$$

(44)

And for continuous proper values

$$F\left(x_i, x_c\right) = \frac{1}{4T^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\langle \tilde{A} \left| e^{i\left(\tau\hat{\wp} + \theta\hat{\wp}_c\right)} \right| \tilde{A} \right\rangle e^{-i\left(\tau x_i + \theta x_{ck}\right)} d\theta d\tau$$

(45)

When $\hat{\wp}$ and $\hat{\wp}_c$ are canonically conjugated: $\left(\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c\hat{\wp}\right) = -ic\hat{E}$, so the view of characteristic function would be very simple:

$$\hat{M}(\theta, \tau) = e^{-\frac{1}{2}i\tau\hat{\wp}_c} e^{i\theta\hat{\wp}} e^{+\frac{1}{2}i\tau\hat{\wp}_c} \tag{46}$$

If we consider expert functions as $\hat{\wp}$ operator proper value, then

$$M(\theta, \tau) = \left\langle \hat{M}(\theta, \tau) \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\langle \tilde{A} \middle| x^* - \frac{1}{2}c\tau \right\rangle e^{i\theta x} \left\langle x^* + \frac{1}{2}c\tau \middle| \tilde{A} \right\rangle d\tau$$

(47)

So,

$$F\left(x, x_c\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\langle \tilde{A} \middle| x^* - \frac{1}{2}c\tau \right\rangle e^{-ix_c\tau} \left\langle x^* + \frac{1}{2}c\tau \middle| \tilde{A} \right\rangle d\tau =$$

$$= \frac{1}{\sqrt{c}} e^{-\frac{1}{2}ic\frac{\partial^2}{\partial x \partial x_c}} \left\langle \tilde{A} \middle| x^* \right\rangle \left\langle \tilde{A}^c \middle| x_c^* \right\rangle e^{icxx_c}$$

(48)

Let's consider average value of complementary variable common function $G(x, x_c)$ to the phase distribution Function $F(x, x_c)$

$$\langle G \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, x_c) F(x, x_c) dx dx_c = \qquad (49)$$

Hereinafter we'll consider $\hat{G}$ as "energetic function"

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, x_c) M(\tau, \theta) e^{-i\left(\tau x + \theta x_c\right)} dx dx_c d\tau d\theta =$$

$$= \left\langle \tilde{A} \left\| \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \gamma(\tau, \theta) e^{i\left(\tau \hat{\wp} + \theta \hat{\wp}_c\right)} d\tau d\theta \right) \right\| \tilde{A} \right\rangle$$

(50)

where

$$\gamma(\tau, \theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, x_c) e^{-i\left(\tau x + \theta x_c\right)} dx dx_c \qquad (51)$$

Thus, the appropriate operator of canonically conjugated variables common function might be presented by the following way:

$$\hat{G} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \gamma(\tau, \theta) e^{i\left(\tau \hat{\wp} + \theta \hat{\wp}_c\right)} d\tau d\theta = e^{-\frac{1}{2} ic \frac{\partial^2}{\partial x \partial x_c}} \hat{G}_0\left(\hat{\wp}, \hat{\wp}_c\right)$$

(52)

where $\hat{G}_0(\hat{\wp},\hat{\wp}_c)$ is received by $G(p,q)$ function in case of changing x and x_c with beforehand right order of appropriate operators $\hat{\wp}$ and $\hat{\wp}_c$. We should save this ordering in case of $e^{-\frac{1}{2}ic\frac{\partial^2}{\partial x \partial x_c}}$ operator's operating on G_0. We'll use "Energetic functions" of color in time derivation consideration.

If connection of information functions on time is presented by exponential phase items ($e^{i\alpha(x,x_c)t}$), what means that information about $\tilde{A}$ fuzzy subset is not depending on time, then

$$\frac{d\hat{\wp}}{dt} = -\frac{i}{c}\left(\hat{\wp}\hat{G} - \hat{G}\hat{\wp}\right) \tag{53}$$

We will consider case, when

$$\hat{G} = \alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2} \tag{54}$$

Now is easy to calculate following:

$$\left(\frac{d\xi}{dt}\right)^* = -ic\left(\int_{-\infty}^{+\infty}\left\langle\tilde{A}\middle|x^*\right\rangle\hat{\wp}\hat{G}\left\langle x^*\middle|\tilde{A}\right\rangle dx - \int_{-\infty}^{+\infty}\left\langle\tilde{A}\middle|x^*\right\rangle\hat{G}\hat{\wp}\left\langle x^*\middle|\tilde{A}\right\rangle dx\right)$$

Using (11) and (12) we receive:

$$\frac{d\hat{\wp}}{dt} = -\frac{i}{c}\left(\hat{\wp}\left(\alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2}\right) - \left(\alpha\hat{\wp}^2 + \beta\hat{\wp}_c^{\,2}\right)\hat{\wp}\right) =$$

$$= -\frac{i\beta}{c}\left(\hat{\wp}\,\hat{\wp}_c^{\,2} - \hat{\wp}_c^2\hat{\wp}\right) = 2\beta\hat{\wp}_c \tag{55}$$

Thus,

$$\sigma\left(\frac{d\hat{\wp}}{dt}\right)=2\beta\sigma\left(\hat{\wp}_C\right) \qquad (56)$$

and

$$\mu_{\hat{\wp}\times\hat{\wp}_C}\left(x^*, x_C{}^*\right)=1-\iint\limits_{I\left(x^*\right)\times I_C\left(x_C{}^*\right)} F\left(x, x_C\right)dx\,dx_C$$

Chapter III
Optimal F Real numbers

§ 3.1 Main Definitions

Definition: Numbers, corresponding to the minimal value of $\sigma_{\wp}^2\cdot\sigma_{\wp_C}^2$ product, call Optimal.

It's not hard to establish view of vector $\left|x_\omega; x_\omega^*\right\rangle$, which is minimizing functional $\sigma_{\wp}^2\cdot\sigma_{\wp_C}^2$, i.e. providing equality in the following ratio:

$$\sigma_{\wp}^2\cdot\sigma_{\wp_C}^2\geq\frac{c^2}{4}$$

It's known that in Cauchy-Buniakovsky [] inequation equality has place when $f=\gamma g$, where γ is some (commonly complex) number. It follows that minimal value of dispersion product is reach only in case of following conditions fulfillment:

$$\hat{\alpha}\left|x_\omega; x_\omega^*\right\rangle=\gamma\hat{\beta}\left|x_\omega; x_\omega^*\right\rangle \qquad (57)$$

$$\int_{\Re}\langle x_\omega; x_\omega^*|\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)|x_\omega; x_\omega^*\rangle dx = 0 \qquad (58)$$

Expression (30) and condition (57) leads us to the following differential equation:

$$\frac{d|x_\omega; x_\omega^*\rangle}{dx} = \left[\frac{i}{\gamma c}\left(x - x^*\right) + \frac{i x_c^*}{c}\right]|x_\omega; x_\omega^*\rangle \qquad (59)$$

Actually in accordance of (30), we have:

$$\left(\hat{\wp} - x^*\hat{E}\right)|x_\omega; x_\omega^*\rangle = \gamma\left(\hat{\wp}^c - x_c{}^*\hat{E}\right)|x_\omega; x_\omega^*\rangle,$$

$$x|x_\omega; x_\omega^*\rangle - x^*|x_\omega; x_\omega^*\rangle = -ic\gamma\frac{d|x_\omega; x_\omega^*\rangle}{dx} - \gamma x_c^*|x_\omega; x_\omega^*\rangle,$$

From here (59) is true. This equation could be directly integrated:

$$\frac{d|x_\omega; x_\omega^*\rangle}{|x_\omega; x_\omega^*\rangle} = \frac{i}{c}\left[\frac{\left(x - x^*\right)}{\gamma} + x_c^*\right]dx \;;$$

If random constant define by lnN, we’ll receive:

$$\begin{gathered}\ln|x_\omega; x_\omega^*\rangle = \frac{i}{2\gamma c}\left(x - x^*\right)^2 + \frac{i}{c}x_c^* x + \ln N \\ |x_\omega; x_\omega^*\rangle = N\cdot\exp\left[\frac{i}{2\gamma c}\left(x - x^*\right)^2 + \frac{i}{c}x_c^* x\right]\end{gathered} \qquad (60)$$

N is defined from normality condition.

Condition (58) with (57) is giving:

$$\left(\frac{1}{\gamma}+\frac{1}{\gamma^*}\right)\int_{\Re}\left\langle x_\omega; x_\omega^*\right|\hat{\alpha}^2\left|x_\omega; x_\omega^*\right\rangle dx = 0 \qquad (61)$$

It is clear that from above mentioned γ should be imaginary. Further because of

$\left|x_\omega, x_\omega^*\right\rangle \in L^2(\Re)$, γ should has the view $\gamma = -i|\gamma|$

So,

$$\left|x_\omega, x_\omega^*\right\rangle = N \cdot \exp\left[-\frac{\left(x-x^*\right)^2}{2|\gamma|c}+\frac{i}{c}\Delta_c^* x\right] \qquad (62)$$

As we already mentioned, N is defined from normality condition, and $|\gamma|$ from

$$\int_{\Re}\left(x-x^*\right)^2\left\|x_\omega, x_\omega^*\right\rangle\Big|^2 dx = \sigma_{\wp}^2(x^*) \qquad (63)$$

Let's write normality condition:

$$1=|N|^2\int_{\Re}\exp\left[-\frac{\left(x-x^*\right)^2}{c|\gamma|}\right]dx\ ; \quad 1=|N|^2\sqrt{\pi|\gamma|c} \qquad (64)$$

Condition (63) is giving the following:

$$\sigma_{\wp}^2\left(x^*\right)=|N|^2\int_{\Re}\left(x-x^*\right)^2\exp\left[-\frac{\left(x-x^*\right)^2}{c|\gamma|}\right]\ ; \quad \sigma_{\wp}^2\left(x^*\right)=\frac{1}{2}|N|^2 c|\gamma|\sqrt{c|\gamma|\pi}$$

(65)

From last two correlations we have:

$$|N| = \frac{1}{\sqrt[4]{\pi c|\gamma|}} \ ; \ \sigma_{\wp}^{2}\left(x^{*}\right) = \frac{1}{2}c|\gamma|\sqrt{c|\gamma|\pi} \cdot \frac{1}{\sqrt{\pi c|\gamma|}} \ ; \ c|\gamma| = 2\sigma_{\wp}^{2}\left(x^{*}\right) \tag{66}$$

Note the important feature: in optimal case $\sigma_{\wp}^{2}\left(x^{*}\right)$ is not depended on x^{*}.

So, for normalized vector $\left|x_{\omega}, x_{\omega}^{*}\right\rangle$ we have received the following expression, which provides equality in (29):

$$\left|x_{\omega}, x_{\omega}^{*}\right\rangle = \frac{1}{\sqrt[4]{2\pi\sigma_{\wp}^{2}}} \exp\left[-\frac{\left(x - x^{*}\right)^{2}}{4\sigma_{\wp}^{2}} + \frac{i}{c} x_{c}^{*} x \right] \tag{67}$$

Because of it, the membership function of optimal fuzzy real number will have a view:

$$\rho_{\wp}\left(x; x^{*}\right) = \frac{1}{\sqrt{2\pi\sigma_{\wp}^{2}}} \exp\left[-\frac{\left(x - x^{*}\right)^{2}}{2\sigma_{\wp}^{2}} \right] \tag{68}$$

We see that measure of $\wp$ calibration in optimal case is Gaussian.

Absolutely analogical argumentation could be given in case of canonically conjugated calibration.

Theorem1. If calibration $\wp$ scale is optimal, than the canonically conjugated calibration would be also optimal, with parameter: $\sigma_{\wp c} = \frac{c^2}{4\sigma_{\wp}^2}$

Proof: we have:

$$\left|x_{c\omega}, x_{c\omega}^*\right\rangle = \frac{1}{\sqrt{2\pi}} \int_{\Re} \left|y, x_{\omega}^*\right\rangle e^{-\frac{i}{c}xy} dy =$$

$$= \frac{1}{\sqrt[4]{(2\pi)^3 \sigma_{\wp}^2}} \int_{\Re} \exp\left[-\frac{\left(y - x^*\right)^2}{4\sigma_{\wp}^2} + \frac{i}{c}\left(x_c^* - x\right) y\right] dy =$$

$$= \frac{1}{\sqrt[4]{(2\pi)^3 \sigma_{\wp}^2}} \exp\left[\frac{i}{c}\left(x_c^* - x\right) x^*\right] \int_{\Re} \exp\left[-\frac{t^2}{4\sigma_{\wp}^2} + \frac{i}{c}\left(x_c^* - x\right) t\right] dt =$$

$$= \sqrt[4]{\frac{2\sigma_{\wp}^2}{\pi c^2}} \exp\left[-\frac{\sigma_{\wp}{}^2}{c^2}\left(x_c^* - x\right)^2 + \frac{i}{c} x^* \left(x_c^* - x\right)\right]$$

(69)

If here suppose that $\sigma_{\wp}^2 = \frac{c^2}{4\sigma_{\wp_c}^2}$ then:

$$\left|x_{c\omega}, x_{c\omega}^*\right\rangle = \frac{1}{\sqrt[4]{2\pi\sigma_{\wp_c}^2}} \exp\left[-\frac{\left(x_c^* - x\right)^2}{4\sigma_{\wp_c}^2} + \frac{i}{c}\left(x_c^* - x\right)\right] \qquad (70)$$

$$\chi_{\wp c}(y;y_c^*)=\frac{1}{\sqrt{2\pi\sigma_{\wp c}^2}}\exp\left[-\frac{\left(y-y_c^*\right)^2}{2\sigma_{\wp c}^2}\right] \tag{71}$$

Note: The optimal case should be considered as such model of R, when closest calculations are realizable at given values of canonically conjugated color dispersions.

For Intervals I and I_c pick up the special view:

$$\mathrm{I}[\omega]=\left[x_\omega^*-\alpha_\omega\sigma_\wp;x_\omega^*+\alpha_\omega\sigma_\wp\right]$$
$$\mathrm{I}_c[\omega]=\left[x_{c\omega}^*-\alpha_{c\omega}\sigma_{\wp c};x_{c\omega}^*+\alpha_{c\omega}\sigma_{\wp c}\right]$$

(72)

Where α and α_c depends appropriately from x^* and x_c^*, the value of $\alpha\sigma$ defines width of interval on scale of "hidden" parameters.

By experienced way, with help of inquiry, expert can define the interval (72). The same inquiry will help to estimate frequencies of individual values i.e. let to estimate x_c^* and $\sigma_\wp^2$, which , for its part will help to estimate distribution $\rho_\wp\left(x;x^*\right)$.

It's clear that for any $\mu_\wp(\omega)$ $\left(\mu_{\wp c}(\omega)\right)$, selection of such α $\left(\alpha_c\right)$ to satisfy terms (8) and (18) is always possible.

For the considered optimal model, the membership functions appropriate to canonically conjugated calibrations would be presented by the following view:

$$\mu_{\wp}(\omega)=\frac{1}{\sqrt{2\pi\sigma_{\wp}^{2}}}\int_{x_{\omega}^{*}-\alpha\sigma_{\wp}}^{x_{\omega}^{*}+\alpha\sigma_{\wp}}\exp\left[-\frac{x-x_{\omega}^{*}}{2\sigma_{\wp}^{2}}\right]dx=$$

$$=\frac{1}{2}\left[\Phi\left(\frac{\alpha}{\sqrt{2}}\right)-\Phi\left(-\frac{\alpha}{\sqrt{2}}\right)\right]=\Phi\left(\frac{\alpha\omega}{\sqrt{2}}\right)$$

(73)

Analogically ,

$$\mu_{\wp_C}(\omega)=\Phi\left(\frac{\alpha_C\omega}{\sqrt{2}}\right) \qquad (74)$$

In such expressions $\Phi(z)$ is Probability Integral .

For x>0, $\Phi(x)$ is increasing :

$$\lim_{x\to\infty}\Phi(x)=1$$

Where probabilistic integral :

$$\Phi(x)\equiv erfx=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^2}dt$$

If physical notion of color is characterizing by objective data x^* and σ^2, then for all $\omega\in\Omega$ they are the same. Membership function is defined within increasing mapping.

§ 3.2 Interpretation of the Basic fuzzy Set Theoretic Operations for Optimal Model

Because of (72), (73) and (74):

$$\left(\mu_{\wp 1}(\omega) > \mu_{\wp 2}(\omega)\right) \Leftrightarrow \left(\mathrm{I}_1(\omega) \supset \mathrm{I}_2(\omega)\right) \tag{75}$$

We have to mention that interval width occurs depended only on α :

$$\alpha_1 > \alpha_2 \Rightarrow \left(\mathrm{I}_1(\omega) \supseteq \mathrm{I}_2(\omega)\right)$$

So:

$$1)\quad \left(\tilde{\Re}_1 \supset \tilde{\Re}_2\right) \Leftrightarrow \left(\mathrm{I}_1(x^*) \supset \mathrm{I}_2(x^*); x^* \in \Re\right)$$

$$2)\quad \left(\tilde{\Re}_1 \cup \tilde{\Re}_2\right) \Leftrightarrow \max\left(\mu_{\wp 1}, \mu_{\wp 2}\right) \Leftrightarrow \left(\mathrm{I}_1(x^*) \cup \mathrm{I}_2(x^*); x^* \in \Re\right)$$

$$3)\quad \left(\tilde{\Re}_1 \cap \tilde{\Re}_2\right) \Leftrightarrow \min\left(\mu_{\wp 1}, \mu_{\wp 2}\right) \Leftrightarrow \left(\mathrm{I}_1(x^*) \cap \mathrm{I}_2(x^*); x^* \in \Re\right)$$

$$4)\quad \left(\neg \tilde{\Re}\right) \Leftrightarrow \left(\mu_{\neg \wp} = 1 - \mu_{\wp}\right) \Leftrightarrow \left(\Re \setminus \mathrm{I}(x^*); x^* \in \Re\right)$$

The same correlation results $\tilde{\Re}^c$.

We see that common operations performed on appropriate intervals are appropriate of fuzzy set-theoretic operations.

First three ratios show that in optimal model, a fuzzy set-theoretic operation finally reduce to appropriate common operations, but in case of fourth ratio situation is more complicated.

As in the previous cases $\mu_{\neg \wp}(x^*)$ Membership function value calculation is reduced to the operation of common supplement taking:

$$\mu_{\neg\wp}(x^*) = 1 - \mu_{\wp}(x^*) = \int_{-\infty}^{x^* - \alpha\sigma_{\wp}} \rho_{\wp}\left(x, x^*\right)dx + \int_{x^* + \alpha\sigma_{\wp}}^{+\infty} \rho_{\wp}\left(x, x^*\right)dx =$$

$$= \int_{\Re \backslash \mathrm{I}\left(x^*\right)} \rho_{\wp}\left(x, x^*\right)dx \qquad (76)$$

Hence, to $\neg\wp$ color is appropriate its interval

$$\mathrm{I}_{\neg}\left(x^*\right) = \left[x^* - \alpha_{\neg}\sigma^2_{\neg\wp}; x^* + \alpha_{\neg}\sigma^2_{\neg\wp}\right]$$

which is not a supplement of $\mathrm{I}\left(x^*\right)$ till R, but as we've already mentioned before, is fully defined by this supplement. For $\alpha_{\neg}$ definition we have equation:

$$\Phi\left(\frac{\alpha_{\neg}}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{\alpha}{\sqrt{2}}\right) \qquad (77)$$

All above mentioned for color $\neg\wp$, is true also for canonically conjugated color $\neg\wp_C$.

The following operations are pertinent only to F sets with only and same $\wp$ color and thus might be reduced to operations on intervals:

5) $$\left(\tilde{\Re}_1 \cdot \tilde{\Re}_2\right) \Leftrightarrow \left(\mu_{\wp 1} \cdot \mu_{\wp 2}\right) \Leftrightarrow \left(\Phi\left(\frac{\alpha\sigma_{\wp}}{\sqrt{2}}\right) = \Phi\left(\frac{\alpha_1\sigma_{\wp 1}}{\sqrt{2}}\right) \cdot \Phi\left(\frac{\alpha_2\sigma_{\wp 2}}{\sqrt{2}}\right)\right)$$

6) $$\left(CON\tilde{\Re} = \tilde{\Re} \cdot \tilde{\Re} \equiv \tilde{\Re}^2\right) \Leftrightarrow \left(\mu^2_{\wp}\right) \Leftrightarrow \left(\Phi\left(\frac{\alpha_{CON}}{\sqrt{2}}\right) = \left[\Phi\left(\frac{\alpha\sigma_{\wp}}{\sqrt{2}}\right)\right]^2\right)$$

7) $$\left(DIL\tilde{\Re} \equiv \tilde{\Re}^{\frac{1}{2}}\right) \Leftrightarrow \left(\mu_{\wp}^{\frac{1}{2}}\right) \Leftrightarrow \left(\Phi\left(\frac{\alpha_{DIL}\sigma_{\wp}}{\sqrt{2}}\right) = \left[\Phi\left(\frac{\alpha\sigma_{\wp}}{\sqrt{2}}\right)\right]^{\frac{1}{2}}\right)$$

In much the same way we'll act for canonically conjugated σ_c color.

It is important to underline that considered operations would not take us outside of optimal model.

§3.3 Cartesian product of Colors

Construction of probabilistic model for F sets Cartesian product provokes interest as here given definition is different from the one offered by Zadeh and the main interest represents the fact that special(complementary) character of connection between canonically conjugated $\hat{\wp}$ and $\hat{\wp}_C$ colors is most fully reflected in this model.

During calculation of membership function of two canonically conjugated F sets, we will be based on appropriate characteristic function.

Let be given two fuzzy sets $\tilde{\Re}_1$ and $\tilde{\Re}_2$, $\wp_1$ and $\wp_2$ are colors to define different elements of two sets. Let's $\hat{\wp}_1$ and $\hat{\wp}_2$ be the appropriate operators. First we will consider case when this operators commute. Denote as $\left|x_{1\omega}, x_{2\omega}; \overset{*}{x}_{1\omega}, \overset{*}{x}_{2\omega}\right\rangle$ vector which $\in L^2(\Re \times \Re)$ and defines membership function of Cartesian product $\tilde{\Re} = \tilde{\Re}_1 \times \tilde{\Re}_2$:

$$\rho_{\wp_1 \times \wp_2}\left(x_1, x_2; \overset{*}{x}_1, \overset{*}{x}_2\right) = \left\|x_{1\omega}, x_{2\omega}; \overset{*}{x}_{1\omega}, \overset{*}{x}_{2\omega}\right\rangle\right|^2 , \qquad (78)$$

$$\mu_{\wp_1\times\wp_2}\left(x_1^*,x_2^*\right)=\iint\limits_{I\left(x_1^*,x_2^*\right)}\left\|x_{1\omega},x_{2\omega};x_{1\omega}^*,x_{2\omega}^*\right\rangle\Big|^2 dx_1dx_2 \tag{79}$$

Where

$$I\left(x_1^*,x_2^*\right)=I_1\left(x_1^*\right)\times I_2\left(x_2^*\right) \tag{80}$$

Definition. Two F numbers $\tilde{x}_1^*$ and $\tilde{x}_2^*$ called "non-interacting", if $\left|x_{1\omega},x_{2\omega};x_{1\omega}^*,x_{2\omega}^*\right\rangle=\left|x_{1\omega},x_{1\omega}^*\right\rangle\times\left|x_{2\omega},x_{2\omega}^*\right\rangle$, otherwise they called interacting.

Let's consider operator

$$\hat{M}(\alpha_1,\alpha_2)=\exp\left[i\left(\alpha_1\hat{\wp}_1+\alpha_2\hat{\wp}_2\right)\right] \tag{81}$$

If $\hat{\wp}_1\hat{\wp}_2-\hat{\wp}_2\hat{\wp}_1=0$, it has quite unique mining.

Definition. Call characteristic function of $\wp_1\times\wp_2$ color, the following scalar product:

$$M(\alpha_1,\alpha_2)=\left(\left\langle x_{1\omega},x_{2\omega};x_{1\omega}^*,x_{2\omega}^*\right|,\hat{M}(\alpha_1,\alpha_2)\left|x_{1\omega},x_{2\omega};x_{1\omega}^*,x_{2\omega}^*\right\rangle\right)$$

(82)

Theorem 1. Density $\rho_{\wp_1\times\wp_2}$ is calculated by formula:

$$\rho_{\wp_1 \times \wp_2}\left(x_1, x_2; x_1^*, x_2^*\right) = \frac{1}{4\pi^2} \iint_{\Re\times\Re} M(\alpha_1, \alpha_2) e^{-i(\alpha_1 x_1 + \alpha_2 x_2)} d\alpha_1 d\alpha_2$$

(83)

Proof. Put (82) in (83): $\rho_{\wp_1 \times \wp_2}\left(x_1, x_2; x_1^*, x_2^*\right) =$

$$= \frac{1}{4\pi^2} \iint_{\Re\times\Re} d\alpha_1 d\alpha_2 e^{-i(\alpha_1 x_1 + \alpha_2 x_2)} \iint_{\Re\times\Re} dx_1^{'} dx_2^{'} \left\langle x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right|$$

$$\exp\left[i\left(\alpha_1 \hat{\wp}_1 + \alpha_2 \hat{\wp}_2\right)\right] \left| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle$$

When x_1^* and x_2^* are non-interacting F-numbers, it is clear that:

$$\hat{\wp}_i \left| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle = x_i \left| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle \quad \text{i=1, 2} \qquad (84)$$

It is natural to suppose that the same relation has place in case of interacting F-numbers, so it is possible to write:

$$\rho_{\wp_1 \times \wp_2}\left(x_1, x_2; x_1^*, x_2^*\right) =$$

$$= \frac{1}{4\pi^2} \iint_{\Re\times\Re} d\alpha_1 d\alpha_2 e^{-i(\alpha_1 x_1 + \alpha_2 x_2)} \iint_{\Re\times\Re} dx_1^{'} dx_2^{'} \exp\left[i\left(\alpha_1 x_1^{'} + \right.\right.$$

$$\left.\left. + \alpha_2 x_2^{'}\right)\right] \left\| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle \Big|^2 =$$

$$= \iint_{\Re\times\Re} dx_1^{'} dx_2^{'} \left\| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle \Big|^2 \frac{1}{2\pi} \int_{\Re} e^{i\alpha_1\left(x_1^{'} - x_1\right)} d\alpha_1 \frac{1}{2\pi} \int_{\Re} e^{i\alpha_2\left(x_2^{'} - x_2\right)} d\alpha_2 =$$

$$= \iint_{\Re\times\Re} dx_1^{'} dx_2^{'} \left\| x_{1\omega}, x_{2\omega}; x_{1\omega}^*, x_{2\omega}^* \right\rangle \Big|^2 \delta\left(x_1^{'} - x_1\right) \delta\left(x_2^{'} - x_2\right) =$$

$$= \left\| x_{1\omega}, x_{2\omega}; x^*_{1\omega}, x^*_{2\omega} \right\rangle \Big|^2$$

With help of density $\rho_{\wp_1 \times \wp_2}$, it is possible to restore its components densities:

$$\rho_{\wp_1}\left(x; x^*_1\right) = \int_{\Re} \rho_{\wp_1 \times \wp_2}\left(x, x'; x^*_1, x^*_2\right) dx' \quad (85)$$

$$\rho_{\wp_2}\left(x; x^*_2\right) = \int_{\Re} \rho_{\wp_1 \times \wp_2}\left(x', x; x^*_1, x^*_2\right) dx' \quad (86)$$

Relations (81)-(86) are easily generalized on case of Cartesian product of $\forall$ finite number of efficient with "commute" colors:

$$\hat{M}\left(\alpha_1, ..., \alpha_n\right) = \exp\left[i \sum_{k=1}^{n} \alpha_k \hat{\wp}^k \right] \quad (87)$$

$$M\left(\alpha_1, ..., \alpha_n\right) = \left(\left\langle x_{1\omega} ... x_{n\omega}; x^*_{1\omega} ... x^*_{n\omega} \right|, \hat{M}\left(\alpha_1, ..., \alpha_n\right) \left| x_{1\omega} ... x_{n\omega}; x^*_{1\omega} ... x^*_{n\omega} \right\rangle \right)$$

(88)

$$\rho_{\wp_1 \times .. \times \wp_n}\left(x_1, ..., x_n; x^*_1, ..., x^*_n \right) =$$

$$= \frac{1}{(2\pi)^n} \int_{\Re^n} M\left(\alpha_1, ..., \alpha_n\right) e^{-i \sum_{k=1}^{n} \alpha_k x^k} \prod_{k=1}^{n} d\alpha_k$$

(89)

It is possible to receive densities appropriate to the smaller number of efficient by **integration on** defined variables with help of this density.

Now we are proceeding to consideration of Cartesian product of two F-sets with "non-interacting" colors.

At considered case the view of characteristic function depends on appropriate commutation between $\hat{\wp}_1$ and $\hat{\wp}_2$.

Especially simple view assumes the (81) for canonically conjugated colors ($\hat{\wp}\,\hat{\wp}_C - \hat{\wp}_C\hat{\wp} = -ic\hat{E}$):

$$\hat{M}(\alpha,\alpha_C) = \exp\left(-\frac{i}{2}\alpha_C\hat{\wp}_C\right)\exp\left(i\alpha\hat{\wp}\right)\exp\left(\frac{i}{2}\alpha_C\hat{\wp}_C\right) \qquad (90)$$

Characteristic function of $\wp\times\wp_C$ color is the following:

$$M(\alpha,\alpha_C) = \left(\left\langle x_\omega, x_\omega^*\right|, \hat{M}(\alpha,\alpha_C)\left|x_\omega, x_\omega^*\right\rangle\right) =$$

$$= \int_{\Re}\left\langle x_\omega, x_\omega^*\right| e^{-\frac{i}{2}\alpha_C\hat{\wp}_C} e^{i\alpha\hat{\wp}} e^{\frac{i}{2}\alpha_C\hat{\wp}_C}\left|x_\omega, x_\omega^*\right\rangle dx$$

According to (27),(33) and (40) we receive:

$$M(\alpha,\alpha_C) = \int_{\Re}\left\langle x_\omega - \frac{c\alpha_c}{2}, x_\omega^*\right| e^{i\alpha x}\left| x_\omega + \frac{c\alpha_c}{2}, x_\omega^*\right\rangle dx$$

(91)

The view of density appropriate of this characteristic function is:

$$\rho_{\wp\times\wp_C}\left(x, x'; x^*, x_C^*\right) = \frac{1}{4\pi^2}\iint_{\Re\times\Re} M(\alpha,\alpha_C) e^{-i\left(\alpha x + \alpha_C x'\right)} d\alpha d\alpha_C =$$

$$= \frac{1}{4\pi^2}\iint_{\Re\times\Re} d\alpha d\alpha_C e^{-i\left(\alpha x + \alpha_C x'\right)} \int_{\Re}\left\langle y - \frac{c\alpha_c}{2}, x_\omega^*\right| e^{i\alpha y}\left| y + \frac{c\alpha_c}{2}, x_\omega^*\right\rangle dy =$$

$$= \frac{1}{2\pi}\int_{\Re} d\alpha_C e^{-i\alpha_C x'} \int_{\Re} dy \left\langle y - \frac{c\alpha_c}{2}; x_\omega^*\right|\left. y + \frac{c\alpha_c}{2}; x_\omega^*\right\rangle\left[\frac{1}{2\pi}\int_{\Re} e^{i\alpha(y-x)} d\alpha\right] =$$

$$= \frac{1}{2\pi} \int_{\Re} d\alpha_c e^{-i\alpha_c x'} \left\langle x_\omega - \frac{c\alpha_c}{2}; x_\omega^* \middle| x_\omega + \frac{c\alpha_c}{2}; x_\omega^* \right\rangle$$

So, for Cartesian product of canonically conjugated fuzzy sets $\tilde{\Re} \times \tilde{\Re}^c$ we have:

Density is expressing by Vigner Formula:

$$\rho_{\wp \times \wp_c}\left(x, x'; x^*, x_c^*\right) = \frac{1}{2\pi} \int_{\Re} \left\langle x_\omega - \frac{c\alpha_c}{2}; x_\omega^* \middle| x_\omega + \frac{c\alpha_c}{2}; x_\omega^* \right\rangle e^{-i\alpha_c x'} d\alpha_c$$

(92)

And the membership function is:

$$\mu_{\wp \times \wp_c}\left(x^*, x_c^*\right) = \iint_{I\left(x^*\right) \times I_c\left(x_c^*\right)} \rho_{\wp \times \wp_c}\left(x, x'; x^*, x_c^*\right) dx dx'$$

(93)

Note: Instead of (92) and (93) receipt of other expressions are possible by replacing Veil formula with some other expression.

Note: Using caliber-invariance of scalar product, instead of (92) (based on the same Veil formula (90)) it is possible to receive:

$$\rho_{\wp \times \wp_c}\left(x, x'; x^*, x_c^*\right) = \frac{1}{2\pi} \int_{\Re} \left\langle x_{c\omega} - \frac{c\alpha_c}{2}; x_{c\omega}^* \middle| x_{c\omega} + \frac{c\alpha_c}{2}; x_{c\omega}^* \right\rangle e^{-i\alpha x} d\alpha$$

(94)

§3.4 Color Value Calculation Condition

Color value calculation condition is conditional moment, calculated with help of density (92). For example $\wp_c^{\ n}$ color (appropriate operator $\hat{\wp}_c^{\ n}$) value condition is defined by formula:

$$\left(x_c^M\right)_x^* \rho_\wp\left(x,x^*\right) = \int_\Re x'^n \rho_{\wp\times\wp_c}\left(x,x';x^*,x_c^*\right)dx' \qquad (95)$$

Insert in consideration conditional characteristic function:

$$M(\alpha|x) = \frac{1}{\rho_\wp\left(x,x^*\right)}\int_\Re \rho_{\wp\times\wp_c}\left(x,x';x^*,x_c^*\right)e^{i\alpha_c x'}dx' =$$

$$= \frac{1}{\rho_\wp\left(x,x^*\right)}\int_\Re dx' e^{i\alpha_c x'}\frac{1}{2\pi}\int_\Re \left\langle x_\omega - \frac{c\alpha'_c}{2};x_\omega^*\middle| x_\omega + \frac{c\alpha'_c}{2};x_\omega^*\right\rangle e^{-i\alpha'_c x'}d\alpha'_c =$$

$$= \frac{1}{\rho_\wp\left(x,x^*\right)}\int_\Re d\alpha'_c \left\langle x_\omega - \frac{c\alpha'_c}{2};x_\omega^*\middle| x_\omega + \frac{c\alpha'_c}{2};x_\omega^*\right\rangle \int_\Re e^{-i\left(\alpha_c - \alpha'_c\right)x'}dx' =$$

$$= \frac{1}{\rho_\wp\left(x,x^*\right)}\left\langle x_\omega - \frac{c\alpha'_c}{2};x_\omega^*\middle| x_\omega + \frac{c\alpha'_c}{2};x_\omega^*\right\rangle \qquad (96)$$

Supposing that:

$$\left|x_\omega;x_\omega^*\right\rangle = \rho_\wp^{1/2}\left(x,x^*\right)e^{\frac{i}{c}S(x)} \qquad (97)$$

We may write logarithm $M\left(\alpha_c|x\right)$, or conjugated function, in the following view:

$$K(\alpha_c|x) = \ln M(\alpha_c|x) = \frac{1}{2}\ln\rho_\wp\left(x+\frac{c\alpha_c}{2};x\right)+$$

$$+\frac{1}{2}\ln\rho_\wp\left(x-\frac{c\alpha_c}{2};x\right) - \ln\rho_\wp\left(x;x^*\right)+$$

$$+\frac{i}{c}\left[S\left(x+\frac{c\alpha_c}{2}\right)-S\left(x-\frac{c\alpha_c}{2}\right)\right] \quad , \qquad (98)$$

and so, for cumulants $\overline{\aleph}_n(x)$ of given distribution (coefficients under $(i\alpha_c)^n/n!$ in Taylor expansion K), the simple expressions are received:

$$\overline{\aleph}_{2n+1}(x)=\left(\frac{c}{2i}\right)^{2n}\frac{d^{2n+1}}{dx^{2n+1}}S(x) \qquad (99)$$

$$\overline{\aleph}_{2n}(x)=\left(\frac{c}{2i}\right)^{2n}\frac{d^{2n}}{dx^{2n}}\ln\rho_{\wp}\left(x;x^*\right) \quad . \quad (91)$$

Quantities $\overline{\aleph}_n$ simply connected with $(x_c')_x^*$ counted values. Particularly for n=1 we have:

$$\overline{\aleph}_1(x)=(x_c')_x^*=\frac{dS}{dx} \qquad (92)$$

It gives possibility of $\psi_{x^*}(x)$ comulate function argument interpretation as $S(x)$

potential conditional calculated (conditional average) values $(x_c')_x^*$.

Conditional dispersion of $\wp_c$ color is:

$$\overline{\aleph}_2(x)=\sigma^2_{\wp_c}\left(x_c^*|x\right)=\left(x_c'^{\,2}\right)_x^*-\left[(x_c')_x^*\right]^2=-\frac{c^2}{4}\frac{d^2}{dx^2}\ln\rho_{\wp}\left(x;x^*\right) \qquad (93)$$

Note: Distribution skewness purely defined by its odd cumulants, hence canonically conjugated color numerical value conditional distribution skewness depends only on $S(x)$.

Now we will show that from two F-sets Cartesian product probabilistic model relation (25) is flowing. This fact surely counts in favor of offered Cartesian product model.

Let us denote by $\hat{\alpha}$ and $\hat{\beta}$ operators with average zero values:

$$\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\alpha}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)=\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\beta}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)=0$$

Well known Schwarz theorem read as follows:

$$\left|\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\alpha}\hat{\beta}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)\right|\leq\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\alpha}^{2}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)^{1/2}\cdot\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\beta}^{2}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)^{1/2} \quad (94)$$

It is also known that it is possible to put appropriate $\alpha(x)$ and $\beta(x)$ random quantities to $\hat{\alpha}$ and $\hat{\beta}$ operators for fulfillment of the following relations:

$$\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\alpha}\hat{\beta}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)=\int_{\Re}\alpha(x)\beta(x)\rho_{\wp}\left(x;x^{*}\right)dx\equiv\overline{\alpha\beta} \quad (95)$$

$$\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\alpha}^{2}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)=\int_{\Re}\alpha^{2}(x)\rho_{\wp}\left(x;x^{*}\right)dx\equiv\overline{\alpha^{2}}=\sigma_{\alpha}^{2} \quad (96)$$

$$\left(\left\langle x_{\omega};x_{\omega}^{*}\right|,\hat{\beta}^{2}\left|x_{\omega};x_{\omega}^{*}\right\rangle\right)=\int_{\Re}\beta^{2}(x)\rho_{\wp}\left(x;x^{*}\right)dx\equiv\overline{\beta^{2}}=\sigma_{\beta}^{2}$$

Therefore if put $\alpha(x)=(x_{C}')_{X}^{*}-\left[(x_{C}')_{X}^{*}\right]^{*}$ and $\beta(x)=x-x^{*}$, where

$$\left[(x_{C}')_{X}^{*}\right]^{*}=\int_{\Re}(x_{C}')_{X}^{*}\rho_{\wp}\left(x;x^{*}\right)dx \quad (97)$$

then

$$\int_{\Re}\left((x_{C}')_{X}^{*}-\left[(x_{C}')_{X}^{*}\right]^{*}\right)\left(x-x^{*}\right)\rho_{\wp}\left(x;x^{*}\right)dx\leq\sigma_{\wp}\sigma_{\alpha} \quad (98)$$

Further, consider the random quantity:

$$\alpha'(x)=\frac{d}{dx}\ln\rho_{\wp}\left(x;x^{*}\right)=\rho_{\wp}^{-1}\left(x;x^{*}\right)\frac{d}{dx}\rho_{\wp}\left(x;x^{*}\right) \quad (99)$$

it is clear that

$$\overline{\alpha'} = \int_{\Re}\left[\frac{d}{dx}\ln\rho_{\wp}\left(x;x^{*}\right)\right]\rho_{\wp}\left(x;x^{*}\right)dx = \int_{\Re}\rho'_{\wp}\left(x;x^{*}\right)dx = \rho_{\wp}\left(x;x^{*}\right)\Big|_{-\infty}^{+\infty} = 0$$

$$\overline{\alpha'^{2}} = \int_{\Re}\left[\frac{d}{dx}\ln\rho_{\wp}\left(x;x^{*}\right)\right]^{2}\rho_{\wp}\left(x;x^{*}\right)dx =$$

$$= \int_{\Re}\left[\frac{1}{\rho_{\wp}\left(x;x^{*}\right)}\right]^{2}\left[\frac{d}{dx}\rho_{\wp}\left(x;x^{*}\right)\right]^{2}\rho_{\wp}\left(x;x^{*}\right)dx$$

(100)

Suppose $\psi'_{x^{*}}(x)\in L^{2}(\Re)$, in this case:

$$\int_{\Re}\left[\frac{d^{2}}{dx^{2}}\ln\rho_{\wp}\left(x;x^{*}\right)\right]\rho_{\wp}\left(x;x^{*}\right)dx =$$

$$= \int_{\Re}\frac{d}{dx}\left[\frac{\rho'_{\wp}\left(x;x^{*}\right)}{\rho_{\wp}\left(x;x^{*}\right)}\right]\rho_{\wp}\left(x;x^{*}\right)dx =$$

$$= \int_{\Re}\rho''_{\wp}\left(x;x^{*}\right)dx - \int_{\Re}\left[\frac{\rho'_{\wp}\left(x;x^{*}\right)}{\rho_{\wp}\left(x;x^{*}\right)}\right]^{2}\rho_{\wp}\left(x;x^{*}\right)dx =$$

$$= \rho'_{\wp}\left(x;x^{*}\right)\Big|_{-\infty}^{+\infty} - \int_{\Re}\left[\frac{\rho'_{\wp}\left(x;x^{*}\right)}{\rho_{\wp}\left(x;x^{*}\right)}\right]^{2}\rho_{\wp}\left(x;x^{*}\right)dx =$$

$$= - \int_{\Re} \left[\frac{\rho'_{\wp}\left(x; x^*\right)}{\rho_{\wp}\left(x; x^*\right)} \right]^2 \rho_{\wp}\left(x; x^*\right) dx$$

thus,

$$\overline{\alpha'^2} = - \int_{\Re} \left[\frac{d^2}{dx^2} \ln \rho_{\wp}\left(x; x^*\right) \right] \rho_{\wp}\left(x; x^*\right) dx$$

By taking into account formula (92), we receive:

$$\overline{\alpha'^2} = \frac{4}{c^2} \int_{\Re} \sigma^2_{\wp_C}\left(x^*_C | x\right) \rho_{\wp}\left(x; x^*\right) dx \qquad (101)$$

$$\overline{\alpha'\left(x - x^*\right)} = \int_{\Re} \left(x - x^*\right) \left[\frac{d}{dx} \ln \rho_{\wp}\left(x; x^*\right) \right] \rho_{\wp}\left(x; x^*\right) dx =$$

$$= \int_{\Re} \left(x - x^*\right) \frac{d}{dx} \rho_{\wp}\left(x; x^*\right) dx =$$

$$= \left(x - x^*\right) \rho_{\wp}\left(x; x^*\right) \Big|_{-\infty}^{+\infty} - \int_{\Re} \rho_{\wp}\left(x; x^*\right) dx = -1 \qquad (102)$$

While comparing (92) and (101) and defining $\alpha(x)$, we see that it is possible to put $\alpha'(x) = \alpha(x)$, according to it formula (98) gives:

$$\sigma^2_{\wp} \int_{\Re} \sigma^2_{\wp_C}\left(x^*_C | x\right) \rho_{\wp}\left(x; x^*\right) dx \geq \frac{c^2}{4} \qquad (103)$$

Since integral in this inequality is $\sigma^2_{\wp_C}$, we are immediately receiving (25).

§3.5 Conditional Color Fuzzy Subsets

Let us consider conditional density:

$$\rho_{\wp_2}\left(x'\middle|x;(x')_X^*\right)=\frac{\rho_{\wp_1\times\wp_2}\left(x,x';x^*,x'^*\right)}{\rho_{\wp_1}\left(x;x^*\right)} \qquad (104)$$

Definition:

$$\tilde{\Re}_{2X}=\left\{(x')_X^*;\mu_{\wp_2}\left((x')_X^*\right):(x')_X^*\in\Re;0\le\mu_{\wp_2}\left((x')_X^*\right)\le 1\right\}, \qquad (105)$$

fuzzy subset of $\Re$,where $(x')_X^*$ is defined by formula (95), when $n=1$, and

$$\mu_{\wp_2}\left((x')_X^*\right)=\int_{I_2\left((x')_X^*\right)}\rho_{\wp_2}\left(x'\middle|x;(x')_X^*\right)dx' \qquad (106)$$

also according to (82):

$$I_2\left((x')_X^*\right)=\left[(x')_X^*-\alpha_2\sigma_{\wp_2}\left((x')_X^*\right);(x')_X^*+\alpha_2\sigma_{\wp_2}\left((x')_X^*\right)\right] \qquad (107)$$

call conditional $\wp_1$ color fuzzy subset.

Color definitions of theoretical-linguistic operations are valid also for conditional color fuzzy subset.

Note: It is clear that for non-interacting fuzzy numbers conditioned color fuzzy subsets comes to the common “unconditional” (absolute?).

Bellow we will consider two important examples:

Example1: Let appropriate density of $\wp\times\wp_C$ has the view:

$$\rho_{\wp_1 \times \wp_2}\left(x, x'; x^*, x_c^*\right) = \frac{1}{\pi} e^{-\frac{\left(x - x^*\right)^2}{2\sigma_\wp^2} - \frac{2\sigma_\wp^2\left(x' - x_c^*\right)^2}{c^2}} \quad (108)$$

This density corresponds to the optimal model of considering Cartesian product as $\sigma_\wp^2 \cdot \sigma_{\wp_c}^2 = \frac{c^2}{4}$, besides it is obvious that it describes non-interacting optimal fuzzy numbers: $\rho_{\wp \times \wp_c} = \rho_\wp \cdot \rho_{\wp_c}$.

Therefore in this example conditional and non-conditional fuzzy subsets coincide.

Example2. Consider the density:

$$\rho_{\wp_1 \times \wp_2}\left(x, x'; x^*, x_c^*\right) = \frac{1}{\pi c}\left[\frac{2}{c}\left(x^2 + x'^2\right) - 1\right] e^{-\frac{1}{c}\left(x^2 + x'^2\right)} \quad (109)$$

It is clear that $x^* = x_c^* = 0$, further :

$$\rho_\wp(x;0) = \int_{\Re} \rho_{\wp \times \wp c}\left(x, x'; x^*, x_c^*\right) dx' = \frac{2}{\sqrt{\pi c^3}} x^2 e^{-\frac{1}{c}x^2} \quad (110)$$

Since (109) symmetry by x and x', it is possible to write

$$\rho_{\wp_c}(x;0) = \rho_\wp(x;0) \quad \text{and} \quad \sigma_\wp^2(0) = \sigma_{\wp_c}^2(0) = \frac{3}{2}c \quad (111)$$

We see that density (109) is appropriate of interacting fuzzy numbers: $\rho_{\wp \times \wp_c} \neq \rho_\wp \cdot \rho_{\wp_c}$ and also the model is not optimal while

$$\sigma_\wp^2 \cdot \sigma_{\wp_c}^2 = \frac{9c^2}{4} > \frac{c^2}{4}$$

Conditional density is:

$$\rho_{\wp_c}\left(x'\middle|x;(x')^*_x\right)=\frac{\sqrt{c}}{2}\frac{1}{x^2}\left[\frac{2}{c}\left(x^2+x'^2\right)-1\right]e^{-\frac{1}{c}x'^2} \qquad (112)$$

$\wp_c$ color conditional value $(x')^*_x=0$ and

$$\sigma^2_{\wp_c}(0)=\int_{\Re}\rho_{\wp_c}\left(x'\middle|x;0\right)x'^2dx'=\frac{\sqrt{\pi c}}{2}\left(1+\frac{3c-2}{x^2}\right) \qquad (113)$$

Formulas (106), (107), (112) and (113) are defining reflection $\Re$ in set of fuzzy subsets $\left\{\tilde{\Re}^c_x\right\}$: $x\to\tilde{\Re}^c_x$.

Chapter IV
Arithmetic Operations on Fuzzy Numbers in Probabilistic Model

In consideration of arithmetic operations, we won't go beyond optimal F numbers [12][13].

Definition.

$$\Delta x=\left|x-x^*\right| \qquad (114)$$

Math. Expectation of this quantity call "error" of counted value:

$$\Delta x^*=\int_{\Re}\Delta x\rho_{\wp}\left(x;x^*\right)dx \qquad (115)$$

For optimal model we have:

$$\Delta x^*=\frac{1}{\sqrt{2\pi\sigma^2_{\wp}}}\int_{\Re}\left|x-x^*\right|\exp\left[-\frac{\left(x-x^*\right)^2}{2\sigma^2_{\wp}}\right]dx=\frac{2}{\sqrt{2\pi\sigma^2_{\wp}}}\int_0^{\infty}ye^{-\frac{y^2}{2\sigma^2_{\wp}}}dy=$$

$$= \frac{1}{\sqrt{2\pi\sigma_{\wp}^{2}}} \int_{0}^{\infty} e^{-\frac{t}{2\sigma_{\wp}^{2}}} dt = -\frac{1}{\sqrt{2\pi\sigma_{\wp}^{2}}} e^{-\frac{t}{2\sigma_{\wp}^{2}}} \Bigg|_{0}^{\infty} 2\sigma_{\wp}^{2} = \sqrt{\frac{2}{\pi}} \sigma_{\wp} \qquad (116)$$

As known arithmetic operations result error is defined by formulas:

$$\Delta\left(x_1^* + x_2^*\right) = \Delta x_1^* + \Delta x_2^*$$

$$\Delta\left(x_1^* \bullet x_2^*\right) = \left| x_1^* \Delta x_2^* + x_2^* \Delta x_1^* \right| \qquad (117)$$

$$\Delta\left(\frac{x_1^*}{x_2^*}\right) = \left| \frac{x_2^* \Delta x_1^* - x_1^* \Delta x_2^*}{x_2^{*2}} \right|$$

Because of (116) for arithmetic operations on optimal F numbers we have following regulations (1^*) (see p. 101):

$$\sigma_{\wp}\left(x_1^* + x_2^*\right) = \sigma_{\wp}\left(x_1^*\right) + \sigma_{\wp}\left(x_2^*\right) \qquad (118)$$

$$\sigma_{\wp}\left(x_1^* \bullet x_2^*\right) = \left| x_1^* \sigma_{\wp}\left(x_2^*\right) + x_2^* \sigma_{\wp}\left(x_1^*\right) \right| \qquad (119)$$

$$\sigma_{\wp}\left(\frac{x_1^*}{x_2^*}\right) = \left| \frac{x_2^* \sigma_{\wp}\left(x_1^*\right) - x_1^* \sigma_{\wp}\left(x_2^*\right)}{x_2^{*2}} \right| \qquad (120)$$

Connection between canonically conjugated colors appears in case of arithmetic operations. Because of (29), $\tilde{x}^*_{c1} \oplus \tilde{x}^*_{c2}$ operation for canonically conjugated numbers, characterizing by dispersion:

$$\sigma_{\wp c}\left(x^*_{c1} \oplus x^*_{c2}\right) = \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{\sigma_{\wp c}\left(x^*_{c1}\right) + \sigma_{\wp c}\left(x^*_{c2}\right)} \qquad (121)$$

is appropriate of addition operation for two fuzzy numbers $\tilde{x}^*_1 + \tilde{x}^*_{c2}$.

For multiplication $\tilde{x}^*_1 \times \tilde{x}^*_2 \rightarrow \tilde{x}^*_{c1} \otimes \tilde{x}^*_{c2}$ and

$$\sigma_{\wp c}\left(x^*_{c1} \otimes x^*_{c2}\right) = \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_1 \sigma_{\wp c}\left(x^*_{c2}\right) + x^*_2 \sigma_{\wp c}\left(x^*_{c1}\right)} \qquad (122)$$

For division $\frac{\tilde{x}^*_1}{\tilde{x}^*_2} \rightarrow \tilde{x}^*_{c1} \div \tilde{x}^*_{c2}$ and

$$\sigma_{\wp c}\left(x^*_{c1} \div x^*_{c2}\right) = \frac{{x^*_2}^2 \sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_2 \sigma_{\wp c}\left(x^*_{c2}\right) - x^*_1 \sigma_{\wp c}\left(x^*_{c1}\right)} \qquad (123)$$

Note: Rules (118-120) and (121-123) are pertinent only to F numbers of one color.
It is possible to say that these rules are appropriate to calculations in $\wp$ calibration. In case of $\wp_c$ calibration, we have to change $\sigma_\wp$ by $\sigma_{\wp c}$, also x^* by $x_c{}^*$.

Arithmetic operations for two F numbers, appropriate to canonically conjugated colors are defined.

Generally speaking, rules (121-123) are establishing equivalence connection in F numbers set of $\tilde{\Re}^c$, because , by (61) density $\rho_{\wp_c}\left(x;x_c^*\right)$ depend not only on dispersion, but also on calculated value of x_c^*. Calibration invariance of this value allows to switch from equivalence classes consideration in $\tilde{\Re}^c$ to equivalence classes consideration in $\tilde{\Re}$, this procedure simplifies consideration of this issue. True, because of (27) we can write:

$$\overset{*}{x}_{c1}\overset{\circ}{\circ}\overset{*}{x}_{c2}=\left(\left\langle x_{1\omega}\circ x_{2\omega};\overset{*}{x}_{1\omega}\circ\overset{*}{x}_{2\omega}\right|,\hat{\wp}^c\left|x_{1\omega}\circ x_{2\omega};\overset{*}{x}_{1\omega}\circ\overset{*}{x}_{2\omega}\right\rangle\right)$$

(124)

Where “o” indicates common operations: “+”,”$\cdot$””:” ,and “$\overset{\circ}{\circ}$” is appropriately $\oplus,\otimes,\div$. As we see from (67), in ratio for $\left|x_{1\omega}\circ x_{2\omega};\overset{*}{x}_{1\omega}\circ\overset{*}{x}_{2\omega}\right\rangle$, figures number x_c - appropriate to number $\overset{*}{x}_1\circ\overset{*}{x}_2$, so it’s clear that

$$\overset{*}{x}_{c1}\overset{\circ}{\circ}\overset{*}{x}_{c2}=x_c \qquad (125)$$

Because of above mentioned, x_c type numbers in $\tilde{\Re}$, which is corresponded to the same number $\sigma_{\wp c}\left(\overset{*}{x}_{c1}\overset{\circ}{\circ}\overset{*}{x}_{c2}\right)$ are equivalent.

However, referring to linearity of $\hat{F}$, we can suppose that appropriate $\psi_{x^*} \to \hat{F}\psi_{x^*} = \varphi_{x^*}$ is isomorphism in case of "o" and "$\overset{\circ}{\circ}$" operations identity assumption. All these considerations need more strict basis.

If we'll nevertheless assume this point of view we'll come to a conclusion that along with the arithmetic operations on F numbers, defined by (118-120) rules, exist operations on canonically conjugated F-numbers – expressed by the following rules:

$$\sigma_{\wp c}\left(x^*_{c1} \oplus x^*_{c2}\right) = \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{\sigma_{\wp c}\left(x^*_{c1}\right) + \sigma_{\wp c}\left(x^*_{c2}\right)} \tag{121'}$$

$$\sigma_{\wp c}\left(x^*_{c1} \otimes x^*_{c2}\right) = \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_1\sigma_{\wp c}\left(x^*_{c2}\right) + x^*_2\sigma_{\wp c}\left(x^*_{c1}\right)} \tag{122'}$$

$$\sigma_{\wp c}\left(x^*_{c1} \circ x^*_{c2}\right) = \frac{x^{*2}_2\,\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_1\sigma_{\wp c}\left(x^*_{c2}\right) - x^*_2\sigma_{\wp c}\left(x^*_{c1}\right)} \tag{123'}$$

Call these rules arithmetic operations in $\wp$ calibration. It's clear that also exist another pair of rules, which we'll call arithmetic rules in $\wp_c$ calibration. These rules are received by substitution: $\left(\sigma_{\wp}, x^*\right) \overrightarrow{\leftarrow} \left(\sigma_{\wp c}, x_c{}^*\right)$. All statements on operations in $\wp$ are valid also in $\wp_c$ calibration.

Let's consider properties of introduced arithmetic operations:

1. <u>Operation "+"</u>

i. Commutation:

$$\tilde{x}_1^* + \tilde{x}_2^* = \tilde{x}_2^* + \tilde{x}_1^*$$

Proof:

$$\tilde{x}_1^* + \tilde{x}_2^* = \tilde{x}_2^* + \tilde{x}_1^*$$

and

$$\sigma_{\wp}\left(\tilde{x}_1^* + \tilde{x}_2^*\right) = \sigma_{\wp}\left(\tilde{x}_1^*\right) + \sigma_{\wp}\left(\tilde{x}_2^*\right) = \sigma_{\wp}\left(\tilde{x}_2^*\right) + \sigma_{\wp}\left(\tilde{x}_1^*\right) = \sigma_{\wp}\left(\tilde{x}_2^* + \tilde{x}_1^*\right)$$

ii. Associative property:

$$\left(\tilde{x}_1^* + \tilde{x}_2^*\right) + \tilde{x}_3^* = \tilde{x}_1^* + \left(\tilde{x}_2^* + \tilde{x}_3^*\right)$$

Proof:

$$\left(x_1^* + x_2^*\right) + x_3^* = x_1^* + \left(x_2^* + x_3^*\right)$$

and

$$\sigma_{\wp}\left(\left(x_1^* + x_2^*\right) + \sigma_{\wp}\left(x_3^*\right)\right) = \sigma_{\wp}\left(x_1^* + x_2^*\right) + \sigma_{\wp}\left(x_3^*\right) =$$

$$= \sigma_{\wp}\left(x_1^*\right) + \sigma_{\wp}\left(x_2^*\right) + \sigma_{\wp}\left(x_3^*\right) =$$

$$= \sigma_{\wp}\left(x_1^*\right) + \sigma_{\wp}\left(x_2^* + x_3^*\right) = \sigma_{\wp}\left(x_1^* + \left(x_2^* + x_3^*\right)\right)$$

2. Operation "$\bullet$ "

i. Commutation:

$$\tilde{x}_1^* \bullet \tilde{x}_2^* = \tilde{x}_2^* \bullet \tilde{x}_1^*$$

Proof:

$$x_1^* \bullet x_2^* = x_2^* \bullet x_1^*$$

and

$$\sigma_{\wp}\left(x_1^* \bullet x_2^*\right) = x_2^* \sigma_{\wp}\left(x_1^*\right) + x_1^* \sigma_{\wp}\left(x_2^*\right) =$$

$$= x_1^* \sigma_{\wp}\left(x_2^*\right) + x_2^* \sigma_{\wp}\left(x_1^*\right) = \sigma_{\wp}\left(x_2^* \bullet x_1^*\right)$$

ii. Associative property:

$$\left(\tilde{x}_1^* \bullet \tilde{x}_2^*\right) \bullet \tilde{x}_3^* = \tilde{x}_1^* \bullet \left(\tilde{x}_2^* \bullet \tilde{x}_3^*\right)$$

Proof:

$$\left(x_1^* \bullet x_2^*\right) \bullet x_3^* = x_1^* \bullet \left(x_2^* \bullet x_3^*\right)$$

and

$$\sigma_{\wp}\left(\left(x_1^* \bullet x_2^*\right) \bullet \sigma_{\wp}\left(x_3^*\right)\right) = x_3^* \sigma_{\wp}\left(x_1^* \bullet x_2^*\right) + x_1^* x_2^* \sigma_{\wp}\left(x_3^*\right) =$$

$$= x_3^*\left(x_2^*\sigma_\wp\left(x_1^*\right) + x_1^*\sigma_\wp\left(x_2^*\right)\right) + x_1^*x_2^*\sigma_\wp\left(x_3^*\right) =$$

$$= x_2^*x_3^*\sigma_\wp\left(x_1^*\right) + x_1^*x_3^*\sigma_\wp\left(x_2^*\right) + x_1^*x_2^*\sigma_\wp\left(x_3^*\right) =$$

$$= x_2^*x_3^*\sigma_\wp\left(x_1^*\right) + x_1^*\left(x_3^*\sigma_\wp\left(x_2^*\right) + x_2^*\sigma_\wp\left(x_3^*\right)\right) =$$

$$= x_2^*x_3^*\sigma_\wp\left(x_1^*\right) + x_1^*\sigma_\wp\left(x_2^* \bullet x_3^*\right) = \sigma_\wp\left(x_1^* \bullet \left(x_2^* \bullet x_3^*\right)\right)$$

3. <u>Distributive rule:</u>

$$\tilde{x}_1^* \bullet \left(\tilde{x}_2^* + \tilde{x}_3^*\right) = \tilde{x}_1^* \bullet \tilde{x}_2^* + \tilde{x}_1^* \bullet \tilde{x}_3^*$$

Proof: $$x_1^* \bullet \left(x_2^* + x_3^*\right) = x_1^* \bullet x_2^* + x_1^* \bullet x_3^*$$

and

$$\sigma_\wp\left(x_1^* \bullet \left(x_2^* + x_3^*\right)\right) = \left(x_2^* + x_3^*\right)\sigma_\wp\left(x_1^*\right) + x_1^*\sigma_\wp\left(x_2^* + x_3^*\right) =$$

$$= x_2^*\sigma_\wp\left(x_1^*\right) + x_3^*\sigma_\wp\left(x_1^*\right) + x_1^*\sigma_\wp\left(x_2^*\right) +$$

$$+x_1^*\sigma_{\wp}\left(x_3^*\right)=\left(x_2^*\sigma_{\wp}\left(x_1^*\right)+x_1^*\sigma_{\wp}\left(x_2^*\right)\right)+\left(x_3^*\sigma_{\wp}\left(x_1^*\right)+x_1^*\sigma_{\wp}\left(x_3^*\right)\right)=$$

$$=\sigma_{\wp}\left(x_1^*\bullet x_2^*\right)+\sigma_{\wp}\left(x_1^*\bullet x_3^*\right)=$$

$$=\sigma_{\wp}\left(x_1^*\bullet x_2^*+x_1^*\bullet x_3^*\right)$$

1_c . Operation "$\oplus$":

i. Commutation:

$$\tilde{x}_{c1}^*\oplus\tilde{x}_{c2}^*=\tilde{x}_{c2}^*\oplus\tilde{x}_{c1}^*$$

Proof: $$x_{c1}^*\oplus x_{c2}^*=x_{c1}^*+x_{c2}^*=x_{c2}^*+x_{c1}^*=x_{c2}^*\oplus x_{c1}^*$$

and

$$\sigma_{\wp c}\left(x_{c1}^*\oplus x_{c2}^*\right)=\frac{\sigma_{\wp c}\left(x_{c1}^*\right)\sigma_{\wp c}\left(x_{c2}^*\right)}{\sigma_{\wp c}\left(x_{c1}^*\right)+\sigma_{\wp c}\left(x_{c2}^*\right)}=$$

$$=\frac{\sigma_{\wp c}\left(x_{c2}^*\right)\sigma_{\wp c}\left(x_{c1}^*\right)}{\sigma_{\wp c}\left(x_{c2}^*\right)+\sigma_{\wp c}\left(x_{c1}^*\right)}=\sigma_{\wp c}\left(x_{c2}^*\oplus x_{c1}^*\right)$$

ii. Associative property:

$$\left(\tilde{x}_{c1}^{*} \oplus \tilde{x}_{c2}^{*}\right) \oplus \tilde{x}_{c3}^{*} = \tilde{x}_{c1}^{*} \oplus \left(\tilde{x}_{c2}^{*} \oplus \tilde{x}_{c3}^{*}\right)$$

Proof:

$$\left(x_{c1}^{*} \oplus x_{c2}^{*}\right) \oplus x_{c3}^{*} = \left(x_{c1}^{*} + x_{c2}^{*}\right) + x_{c3}^{*} =$$

$$= x_{c1}^{*} + \left(x_{c2}^{*} + x_{c3}^{*}\right) = x_{c1}^{*} \oplus \left(x_{c2}^{*} \oplus x_{c3}^{*}\right)$$

and

$$\sigma_{\wp c}\left(\left(x_{c1}^{*} \oplus x_{c2}^{*}\right) \oplus x_{c3}^{*}\right) =$$

$$= \frac{\sigma_{\wp c}\left(x_{c1}^{*} \oplus x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*} \oplus x_{c2}^{*}\right) + \sigma_{\wp c}\left(x_{c3}^{*}\right)} =$$

$$= \frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right) + \sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right) + \sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)} =$$

$$= \frac{\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*}\right) + \sigma_{\wp c}\left(x_{c2}^{*}\right)}\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*}\right) + \sigma_{\wp c}\left(x_{c2}^{*}\right)} + \sigma_{\wp c}\left(x_{c3}^{*}\right)} =$$

$$= \frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\frac{\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c2}^{*}\right)+\sigma_{\wp c}\left(x_{c3}^{*}\right)}}{\sigma_{\wp c}\left(x_{c1}^{*}\right)+\frac{\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c2}^{*}\right)+\sigma_{\wp c}\left(x_{c3}^{*}\right)}} =$$

$$= \frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\oplus x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*}\right)+\sigma_{\wp c}\left(x_{c2}^{*}\oplus x_{c3}^{*}\right)} = \sigma_{\wp c}\left(x_{c1}^{*}\oplus\left(x_{c2}^{*}\oplus x_{c3}^{*}\right)\right)$$

2_c . <u>Operation</u> "$\otimes$":

i. Commutation:

$$\tilde{x}_{c1}^{*}\otimes\tilde{x}_{c2}^{*} = \tilde{x}_{c2}^{*}\otimes\tilde{x}_{c1}^{*}$$

Proof: $$x_{c1}^{*}\otimes x_{c2}^{*} = x_{c1}^{*}\bullet x_{c2}^{*} = x_{c2}^{*}\bullet x_{c1}^{*} = x_{c2}^{*}\otimes x_{c1}^{*}$$

and

$$\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c2}^{*}\right) = \left|\frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)}{x_{1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+x_{2}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)}\right| =$$

$$= \left|\frac{\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c1}^{*}\right)}{x_{2}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)+x_{1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)}\right| = \sigma_{\wp c}\left(x_{c2}^{*}\otimes x_{c1}^{*}\right)$$

ii. Associative property:

$$\left(\tilde{x}^*_{c1} \otimes \tilde{x}^*_{c2}\right) \otimes \tilde{x}^*_{c3} = \tilde{x}^*_{c1} \otimes \left(\tilde{x}^*_{c2} \otimes \tilde{x}^*_{c3}\right)$$

Proof:

$$\left(x^*_{c1} \otimes x^*_{c2}\right) \otimes x^*_{c3} = \left(x^*_{c1} \bullet x^*_{c2}\right) \bullet x^*_{c3} =$$

$$= x^*_{c1} \bullet \left(x^*_{c2} \bullet x^*_{c3}\right) = x^*_{c1} \otimes \left(x^*_{c2} \otimes x^*_{c3}\right)$$

and

$$\sigma_{\wp c}\left(\left(x^*_{c1} \otimes x^*_{c2}\right) \otimes x^*_{c3}\right) =$$

$$= \frac{\sigma_{\wp c}\left(x^*_{c1} \otimes x^*_{c2}\right)\sigma_{\wp c}\left(x^*_{c3}\right)}{x^*_{c1}x^*_{c2}\sigma_{\wp c}\left(x^*_{c1} \otimes x^*_{c2}\right) + x^*_{c3}\sigma_{\wp c}\left(x^*_{c3}\right)} =$$

$$= \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)\sigma_{\wp c}\left(x^*_{c3}\right)}{x^*_{c1}x^*_{c2}\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right) + x^*_{c1}x^*_{c3}\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c3}\right) + x^*_{c2}x^*_{c3}\sigma_{\wp c}\left(x^*_{c2}\right)\sigma_{\wp c}\left(x^*_{c3}\right)} =$$

$$= \frac{\dfrac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_{c1}\sigma_{\wp c}\left(x^*_{c1}\right) + x^*_{c2}\sigma_{\wp c}\left(x^*_{c2}\right)}\sigma_{\wp c}\left(x^*_{c3}\right)}{\dfrac{x^*_{c1}x^*_{c2}\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)}{x^*_{c1}\sigma_{\wp c}\left(x^*_{c1}\right) + x^*_{c2}\sigma_{\wp c}\left(x^*_{c2}\right)} + x^*_{c3}\sigma_{\wp c}\left(x^*_{c3}\right)} =$$

$$= \frac{\dfrac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\right)\sigma_{\wp c}\left(x^*_{c3}\right)}{x^*_{c2}\sigma_{\wp c}\left(x^*_{c2}\right)+x^*_{c3}\sigma_{\wp c}\left(x^*_{c3}\right)}}{x^*_{c1}\sigma_{\wp c}\left(x^*_{c1}\right)+\dfrac{x^*_{c2}x^*_{c3}\sigma_{\wp c}\left(x^*_{c2}\right)\sigma_{\wp c}\left(x^*_{c3}\right)}{x^*_{c2}\sigma_{\wp c}\left(x^*_{c2}\right)+x^*_{c3}\sigma_{\wp c}\left(x^*_{c3}\right)}} =$$

$$= \frac{\sigma_{\wp c}\left(x^*_{c1}\right)\sigma_{\wp c}\left(x^*_{c2}\otimes x^*_{c3}\right)}{x^*_{c1}\sigma_{\wp c}\left(x^*_{c1}\right)+x^*_{c2}x^*_{c3}\sigma_{\wp c}\left(x^*_{c2}\otimes x^*_{c3}\right)} = \sigma_{\wp c}\left(x^*_{c1}\otimes\left(x^*_{c2}\otimes x^*_{c3}\right)\right)$$

3_c. Distributive rule:

$$\tilde{x}^*_{c1}\otimes\left(\tilde{x}^*_{c2}\oplus\tilde{x}c^*_3\right)=\tilde{x}^*_{c1}\otimes\tilde{x}^*_{c2}\oplus\tilde{x}^*_{c1}\otimes\tilde{x}^*_{c3}$$

Proof:

$$x^*_{c1}\otimes\left(x^*_{c2}\oplus x^*_{c3}\right)=x^*_{c1}\bullet\left(x^*_{c2}+x^*_{c3}\right)=$$

$$=x^*_{c1}\bullet x^*_{c2}+x^*_{c1}\bullet x^*_{c3}=x^*_{c1}\otimes x^*_{c2}\oplus x^*_{c1}\otimes x^*_{c3}$$

and

$$\sigma_{\wp c}\left(x^*_{c1}\otimes\left(x^*_{c2}\oplus x^*_{c3}\right)\right)=$$

$$=\frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\oplus x_{c3}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+\left(x_{c2}^{*}+x_{c3}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\oplus x_{c3}^{*}\right)}=$$

$$=\frac{\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c2}^{*}\right)+\sigma_{\wp c}\left(x_{c3}^{*}\right)}}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+\dfrac{\left(x_{c2}^{*}+x_{c3}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c2}^{*}\right)+\sigma_{\wp c}\left(x_{c3}^{*}\right)}}=$$

$$=\frac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)+x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)+x_{c2}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)+x_{c3}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}=$$

$$=\frac{\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+x_{c2}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)}\bullet\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+x_{c3}^{*}\sigma_{\wp c}\left(x_{c3}^{*}\right)}}{\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c2}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+x_{c2}^{*}\sigma_{\wp c}\left(x_{c2}^{*}\right)}+\dfrac{\sigma_{\wp c}\left(x_{c1}^{*}\right)\sigma_{\wp c}\left(x_{c3}^{*}\right)}{x_{c1}^{*}\sigma_{\wp c}\left(x_{c1}^{*}\right)+x_{c3}^{*}\sigma_{\wp c}\left(x_{c3}^{*}\right)}}=$$

$$=\frac{\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c2}^{*}\right)\bullet\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c3}^{*}\right)}{\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c2}^{*}\right)+\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c3}^{*}\right)}=$$

$$=\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c2}^{*}\oplus x_{c1}^{*}\otimes x_{c3}^{*}\right)=\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c2}^{*}\right)\oplus\sigma_{\wp c}\left(x_{c1}^{*}\otimes x_{c3}^{*}\right)$$

Note: Assumption on identity of “o” and “$\overset{\circ}{\circ}$” operations is not fundamental, but this assumption is simplifying consideration.

§4.2 Opposite F Number

Consideration will be held in $\wp$ calibration. Change for $\wp_C$ calibration takes place on the basis of obvious replacements.
Let’s consider ratio:

$$\tilde{a}^* \cdot \tilde{y}^* = \tilde{1}^* \quad (\tilde{1}^* \equiv 1) \tag{126}$$

Here $\tilde{y}^*$ is an unknown F number. According to (119) we have:

$$\left| y^* \sigma_{\wp}\left(a^*\right) + a^* \sigma_{\wp}\left(y^*\right) \right| = \sigma_{\wp}(1) \quad , \qquad a^* y^* = 1 \tag{127}$$

It is clear that a^* and y^* have like signs.

If $a^*, y^* > 0$ then from (127) follows:

$$\sigma_{\wp}\left(y^*\right) = \frac{\sigma_{\wp}(1) - y^* \sigma_{\wp}\left(a^*\right)}{a^*}$$

put $y^* = \dfrac{1}{a^*}$, finally we’ll receive:

$$\sigma_{\wp}\left(y^{*}\right)=\frac{a^{*}\sigma_{\wp}(1)-\sigma_{\wp}\left(a^{*}\right)}{a^{*2}} \qquad (128)$$

So, if $a^{*}\sigma_{\wp}(1)-\sigma_{\wp}\left(a^{*}\right)\geq 0$, $\frac{\sigma_{\wp}\left(a^{*}\right)}{\sigma_{\wp}(1)}\leq a^{*}$ - then solution of (126) exists,

however while comparing (128) and (120) we find that this solution might be presented in the following view:

$$\tilde{y}^{*}=\frac{\tilde{1}^{*}}{\tilde{a}^{*}} \qquad (129)$$

Call this number opposite F number regarding to the $\tilde{a}^{*}$.

Depending on $\tilde{1}^{*}$ there exist infinite number of opposites.

If $a^{*},y^{*}<0$, then $-y^{*}\sigma_{\wp}\left(a^{*}\right)-a^{*}\sigma_{\wp}\left(y^{*}\right)=\sigma_{\wp}(1)$, or

$$\sigma_{\wp}\left(y^{*}\right)=-\frac{a^{*}\sigma_{\wp}(1)+\sigma_{\wp}\left(a^{*}\right)}{a^{*2}} \quad , \frac{\sigma_{\wp}\left(a^{*}\right)}{\sigma_{\wp}(1)}\leq -a^{*} \qquad (128')$$

According to (120) solution would be written in the form of (129).

For canonically conjugated color we have:

$$\tilde{a}_{c}^{*}\otimes\tilde{y}_{c}^{*}=\tilde{1}_{c}^{*} \quad (\tilde{1}_{c}^{*}\equiv 1) \qquad (126')$$

According to (121’), we have:

$$\frac{\sigma_{\wp_C}\left(a_C^*\right)\sigma_{\wp_C}\left(y_C^*\right)}{a_C^*\sigma_{\wp_C}\left(a_C^*\right)+y_C^*\sigma_{\wp_C}\left(y_C^*\right)}=\sigma_{\wp_C}(1)$$

as $y_C^*=\frac{1}{a_C^*}$, this ratio might be rewritten

$$\frac{\left|a_C^*\right|\sigma_{\wp_C}\left(a_C^*\right)\sigma_{\wp_C}\left(y_C^*\right)}{a_C^{*2}\sigma_{\wp_C}\left(a_C^*\right)+\sigma_{\wp_C}\left(y_C^*\right)}=\sigma_{\wp_C}(1)$$

solution of (126') is presented by:

$$\tilde{y}_C^{\,*}=\frac{\tilde{1}_C^{\,*}}{\tilde{a}_C^{\,*}} \quad \text{i.e.} \quad \sigma_{\wp_C}\left(y_C^{\,*}\right)=\frac{a_C^{*2}\sigma_{\wp_C}\left(a_C^*\right)\sigma_{\wp_C}(1)}{\left|a_C^*\right|\sigma_{\wp_C}\left(a_C^*\right)-\sigma_{\wp_C}(1)}\ , \quad y_C^{\,*}=\frac{1}{a_C^{\,*}}$$

(130)

Solvability condition for the color $\wp_c$ is following:

$$\frac{\sigma_{\wp_C}(1)}{\sigma_{\wp_C}\left(a_C^*\right)}<\left|a_C^*\right| \qquad (130')$$

While comparing this condition with appropriate condition for $\wp$ color, we'll ascertain that those conditions are "complementary" of each other in determined sense.

§4.3 Mixed Fuzzy Real Numbers

Above considered Fuzzy real numbers appropriate of $\psi_{x^*}(x)$ functions call Pure F-numbers.

We will show below that there exist F-numbers to which it is impossible to confront any defined $\psi_{x^*}(x)$ function. Call such numbers mixed F-numbers.

Let us consider connection between probabilities distribution density of Cartesian product $\wp \times \wp_c$ and statistic operator of von Neumann.

According to Neumann introduce Cartesian ensemble of F-numbers (each of them is pure). Suppose every Pure F-number is characterized by some w_k ensemble appearance frequency, then it is possible to write:

$$\rho_{\wp\times\wp c}\left(x, x'; x^*, x_c{}^*\right) = \sum_k w_k \rho^k{}_{\wp\times\wp c}\left(x, x'; x^*, x_c{}^*\right) \qquad (131)$$

Consider full, orthogonal system $\left\{\psi_{x_j^*}(x)\right\}$ of functions from $L^2(\Re)$. According formula (92) for every density $\rho^k_{\wp\times\wp_c}$ from (131), there exist appropriate real F-number $\tilde{x}_k{}^*$ ($\tilde{x}_{k_c}{}^*$) or function $\psi^k_{x^*}(x)$ ($\varphi^k_{x_c^*}(x')$), which might be expanded:

$$\psi^k_{x^*}(x) = \sum_j a^k_j \psi^k_{x_j^*}(x) \qquad (132)$$

Hence,

$$\rho_{\wp\times\wp c}\left(x, x'; x^*, x_c{}^*\right) = \sum_k \sum_i \sum_j w_k a_i^{k^*} a_j^k f_{ij}(x, x') \qquad (133)$$

where

$$f_{ij}(x, x') = \frac{1}{2\pi} \int_{\Re} \psi^*_{x_i^*}\left(x - \frac{c\alpha_c}{2}\right) \psi_{x_j^*}\left(x + \frac{c\alpha_c}{2}\right) e^{-i\alpha_c x'} d\alpha_c \qquad (134)$$

Note that if $\left\{\psi_{x_j^*}(x)\right\}$ is full, orthogonal system of functions in $L^2(\Re)$, then

$\left\{(2\pi c) f_{ij}(x,x')\right\}$ is also full, orthogonal system of functions in $L^2(\Re)$ too.

Functions $f_{ij}(x,x')$ makes basis not only for densities, but covers all

space $L^2(\Re\times\Re)$.

Via calculation of matrix $\left\| 2\pi c \iint\limits_{\Re\times\Re} \rho_{\wp\times\wp c}\left(x,x';x^*,x_c{}^*\right) f^*{}_{ij}(x,x')dxdx' \right\|$ we

have:

$$\left\|\sum_k w_k a_i^{k*} a_j^k\right\| = \left\|\rho_{ij}\right\| \left(\Rightarrow \rho_{\wp\times\wp c}\left(x,x';x^*,x_c{}^*\right) = \sum_{i,j} \rho_{ij} f_{ij}(x,x')\right), \quad (135)$$

which represents the statistical operator of Von Neumann.

Consider the random quantity $g(x.x')$. Define appropriate matrix of this quantity by:

$$\left\|g_{ij}\right\| = \left\| \iint\limits_{\Re\times\Re} g\left(x,x'\right) f^*{}_{ij}(x,x')dxdx' \right\| \qquad (136)$$

Mathematical expectation of $g(x.x')$ is defining by rule of Neumann:

$$\langle\hat{g}\rangle = S_p(\hat{g}\hat{\rho}) = \sum_i\sum_j g_{ij}\rho_{ji} =$$

$$= \sum_i\sum_j \iint\limits_{\Re\times\Re} g\left(x,x'\right) f^*{}_{ij}(x,x')\left(\sum_k w_k a_i^{k*} a_j^k\right)dxdx'$$

It is not difficult to show that $f^*{}_{ij}(x,x') = f_{ji}(x,x')$, so by this relation we will

have:

$$\langle\hat{g}\rangle = S_p(\hat{g}\hat{\rho}) =$$

$$= \iint_{\Re\times\Re} g\left(x, x'\right)\left[\sum_k \sum_i \sum_j w_k a_i^{k^*} a_j^k f_{ij}(x, x')\right] dx dx' =$$

$$= \iint_{\Re\times\Re} g\left(x, x'\right) \rho_{\wp\times\wp c}\left(x, x'; x^*, x_c^*\right) dx dx' = \left\langle g\left(x, x'\right)\right\rangle \qquad (137)$$

Received result shows that $\rho_{\wp\times\wp c}$ density at informational-statistical relation is equivalent Von Neumann's statistical operator $\hat{\rho}$.

Let us introduce often-used properties of $\left\{ f_{ij} \right\}$ system:

1. $$\iint_{\Re\times\Re} f_{lk}(x, x') f^*_{l'k'}(x, x') dx dx' = \frac{1}{2\pi c} \delta_{ll'} \delta_{kk'}$$

(138_1)

2. $$\sum_k \sum_l f_{lk}(x, x') f^*_{lk}(y, y') = \frac{1}{2\pi c} \delta(x - y) \delta(x' - y')$$

(138_2)

3. $$\iint_{\Re\times\Re} f_{lk}(x, x') dx dx' = \delta_{lk}$$

(138_3)

4. $$\sum_l f_{ll}(x, x') = \frac{1}{2\pi c}$$

(138_4)

It is clear that pure F-numbers appropriate density matrix is projection operator $\left\| \rho_{ij}^k \right\| = \left\| a_i^{k^*} a_j^k \right\|$. So the mixture density matrix might be presented in view:

$$\hat{\rho} = \sum_k w_k \hat{P}^k \qquad (139)$$

The following formula has place:

$$\langle\hat{g}\rangle = S_p(\hat{\rho}\hat{g}) = S_p\left(\sum_k w_k \hat{P}^k \hat{g}\right) = \sum_k w_k S_p\left(\hat{P}^k \hat{g}\right) = \sum_k w_k \langle\hat{g}\rangle^k \quad (140)$$

Let us give formula for Cartesian product membership function. $I_1 \times I_2$ set's indicator define via $\chi_{I_1 \times I_2}(x, x')$. According of formula (136) define the appropriate matrix:

$$\hat{\chi}_{I_1 \times I_2}(x, x') = \left\| \chi^{ij}_{I_1 \times I_2} \right\| =$$

$$= \left\| \iint_{\Re \times \Re} \chi_{I_1 \times I_2}(x, x') f^*{}_{ij}(x, x') dx dx' \right\| \quad (141)$$

The following has place:

Theorem1. If I_1 and I_2 are indicators on canonically conjugated scales $\wp$ and $\wp_c$ appropriately and the indicator operator is defined by (141), then the $\tilde{\Re} \times \tilde{\Re}^c$ Cartesian product membership function formula is the following:

$$\mu_{\wp \times \wp_c}\left(x^*, x_c^*\right) = S_p\left(\hat{\chi}_{I \times I_c} \cdot \hat{\rho}\right) = \sum_k w_k S_p\left(\hat{P}^k \cdot \hat{\chi}_{I \times I_c}\right) \quad (142)$$

Proof: According definition we may write:

$$\sum_k w_k S_p\left(\hat{P}^k \cdot \hat{\chi}_{I \times I_c}\right) = \sum_k w_k \sum_{i,j} P^k_{ij} \chi^{ji}_{I \times I_c} =$$

$$= \sum_{i,j}\left(\sum_k w_k a_i^{k^*} a_j \iint_{\Re \times \Re} \chi_{I \times I_c}\left(x, x'\right) f^*_{ji}(x, x') dx dx'\right) =$$

$$= \iint_{\Re \times \Re} \chi_{I \times I_c}\left(x, x'\right) \sum_{i,j}\left(\sum_k w_k a_i^{k^*} a_j^k f_{ij}(x, x')\right) dx dx' =$$

$$= \iint_{\Re\times\Re} \chi_{I\times I_c}\left(x, x'\right)\rho_{\wp\times\wp c}\left(x, x'; x^*, x_c{}^*\right)dxdx' =$$

$$= \iint_{I\times I_c} \rho_{\wp\times\wp c}\left(x, x'; x^*, x_c{}^*\right)dxdx'$$

Which adjust with formula (93).

Note: At above introduced formulas w_k depended on x_1^* and x_2^* (in generalized case on x_1^*,…) thus, in (154) upon integration by one of variable x or x', the received density will be depended on x_1^* and x_2^*.Thus there would exist no function ψ_{x^*} appropriate to it.

A p p l i c a t i o n s

Fuzzy Linear Equation Solution

$\wp$ presentation:

$$\tilde{k}^*\tilde{x}^* + \tilde{b}^* = \tilde{0}^* \quad , \quad \left(\tilde{0}^* \equiv 0\right) \tag{143}$$

solution:

$$\tilde{x}^* = \frac{-\tilde{b}^*}{\tilde{k}^*} + \tilde{0}^* \tag{144}$$

note that sign "-" we put not in front of fraction, but in front of $\tilde{b}^*$- underlining that opposite F number "-$\tilde{b}^*$" correspondent to the given fuzzy zero.

If solution of equation (143) really might be presented in form (144) , then counted (by 118-120 formulas) values of x^* and $\sigma_\wp\left(x^*\right)$ from (143) and (144) should coincide.

For equation (143) we have:

$$k^* x^* + b^* = 0^*$$

and

$$\sigma_\wp\left(k^* x^* + b^*\right) = \sigma_\wp\left(k^* x^*\right) + \sigma_\wp\left(b^*\right) = \left|k^* \sigma_\wp\left(x^*\right) + x^* \sigma_\wp\left(k^*\right)\right| + \sigma_\wp\left(b^*\right) =$$

$$= \frac{1}{k^*} \cdot \left|k^{*2} \sigma_\wp\left(x^*\right) - b^* \sigma_\wp\left(k^*\right)\right| + \sigma_\wp\left(b^*\right)$$

It's clear that conditions of solvability have view:

$$\sigma_\wp(0) \geq \sigma_\wp\left(b^*\right) \tag{145}$$

it this case:

$$\left|k^{*2} \sigma_\wp\left(x^*\right) - b^* \sigma_\wp\left(k^*\right)\right| = \left|k^*\right|\left(\sigma_\wp(0) - \sigma_\wp\left(b^*\right)\right)$$

1) $k^{*2} \sigma_\wp\left(x^*\right) - b^* \sigma_\wp\left(k^*\right) \geq 0$

$$\sigma_\wp\left(x^*\right) = \frac{\left|k^*\right| \sigma_\wp(0) - \left|k^*\right| \sigma_\wp\left(b^*\right) + b^* \sigma_\wp\left(k^*\right)}{k^{*2}}$$

2) $k^{*2}\sigma_{\wp}\left(x^{*}\right)-b^{*}\sigma_{\wp}\left(k^{*}\right)<0$

$$\sigma_{\wp}\left(x^{*}\right)=\frac{b^{*}\sigma_{\wp}\left(k^{*}\right)-\left|k^{*}\right|\sigma_{\wp}(0)+\left|k^{*}\right|\sigma_{\wp}\left(b^{*}\right)}{k^{*2}}$$

(146)

Which coincides with expression received from (144) on the basis of rule (120).

Let's consider linear equation in $\wp_C$ presentation (but in the same $\wp$ calibration). We have:

$$\tilde{k}_C^{\ *}\otimes\tilde{x}_C^{\ *}\oplus\tilde{b}_C^{\ *}=\tilde{0}_C^{\ *}\ ,\quad\left(\tilde{0}_C^{\ *}\equiv 0\right)\qquad(143')$$

According to the rules (121') and (122') we have:

$$\tilde{x}_C^{\ *}=-\frac{b_C^{*}}{k_C^{*}}$$

and

$$\sigma_{\wp C}\left(k_C^{\ *}\otimes x_C^{\ *}\oplus b_C^{\ *}\right)=\frac{\sigma_{\wp C}\left(k_C^{\ *}\otimes x_C^{\ *}\right)\sigma_{\wp C}\left(b_C^{\ *}\right)}{\sigma_{\wp C}\left(k_C^{\ *}\otimes x_C^{\ *}\right)+\sigma_{\wp C}\left(b_C^{\ *}\right)}=$$

$$=\frac{\sigma_{\wp C}\left(k_C^{\ *}\right)\sigma_{\wp C}\left(x_C^{\ *}\right)\sigma_{\wp C}\left(b_C^{\ *}\right)}{\sigma_{\wp C}\left(k_C^{\ *}\right)\sigma_{\wp C}\left(x_C^{\ *}\right)+\sigma_{\wp C}\left(b_C^{\ *}\right)\cdot\left|k_C^{\ *}\sigma_{\wp C}\left(k_C^{\ *}\right)+x_C^{\ *}\sigma_{\wp C}\left(x_C^{\ *}\right)\right|}=$$

$$= \frac{\dfrac{\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)\sigma_{\wp c}\left(b_c^*\right)}{\left|k_c^*\sigma_{\wp c}\left(k_c^*\right)+x_c^*\sigma_{\wp c}\left(x_c^*\right)\right|}}{\dfrac{\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)}{\left|k_c^*\sigma_{\wp c}\left(k_c^*\right)+x_c^*\sigma_{\wp c}\left(x_c^*\right)\right|}+\sigma_{\wp c}\left(b_c^*\right)} =$$

$$= \frac{\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)\sigma_{\wp c}\left(b_c^*\right)}{\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)+\sigma_{\wp c}\left(b_c^*\right)\cdot\left|k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)\right|}$$

so:

$$\frac{\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)\sigma_{\wp c}\left(b_c^*\right)}{\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)+\sigma_{\wp c}\left(b_c^*\right)\cdot\left|k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)\right|} =$$

$$= \sigma_{\wp c}(0)$$

and

$$\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)\sigma_{\wp c}\left(b_c^*\right) = = \left|k_c^*\right|\sigma_{\wp c}(0)\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}\left(x_c^*\right)+$$

$$+\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)\cdot\left|k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)\right|,$$

$$\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\left(\sigma_{\wp c}\left(b_c^*\right)-\sigma_{\wp c}(0)\right)\sigma_{\wp c}\left(x_c^*\right)=$$

$$=\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)\left|k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)\right|$$

Solvability condition is:

$$\sigma_{\wp c}\left(b_c^*\right)\geq\sigma_{\wp c}(0)$$

this condition is complementary of (145)

Consider two cases:

1) $k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)\geq 0$

$$\sigma_{\wp c}\left(x_c^*\right)=$$

$$=\frac{k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)}{b_c^*\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)+\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\left(\sigma_{\wp c}\left(b_c^*\right)-\sigma_{\wp c}(0)\right)}$$

2) $k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)-b_c^*\sigma_{\wp c}\left(x_c^*\right)<0$

$$\sigma_{\wp c}\left(x_c^*\right)=$$

$$=\frac{k_c^{*2}\sigma_{\wp c}\left(k_c^*\right)\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)}{b_c^*\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^*\right)-\left|k_c^*\right|\sigma_{\wp c}\left(k_c^*\right)\left(\sigma_{\wp c}\left(b_c^*\right)-\sigma_{\wp c}(0)\right)}$$

Present them in united formula by:

$$\sigma_{\wp c}\left(x_c^{\ *}\right)=$$

$$=\frac{k_c^{\ *2}\sigma_{\wp c}\left(k_c^{\ *}\right)\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^{\ *}\right)}{\left|b_c^{\ *}\sigma_{\wp c}(0)\sigma_{\wp c}\left(b_c^{\ *}\right)+k_c^{\ *}\sigma_{\wp c}\left(k_c^{\ *}\right)\left(\sigma_{\wp c}\left(b_c^{\ *}\right)-\sigma_{\wp c}(0)\right)\right|} \qquad (146')$$

Let us show that founded solution (146') is appropriate of (143') equation solution in view:

$$\tilde{x}_c^{\ *}=\left(-\tilde{b}_c^{\ *}\right)\div\tilde{k}_c^{\ *} \qquad (144')$$

where F-number $-\tilde{b}_c^{\ *}$ is defined by relation : calculated value = $-\tilde{b}_c^{\ *}$ and according to (128') $\sigma_{\wp_c}\left(-\tilde{b}_c^{\ *}\right)=\dfrac{\sigma_{\wp_c}\left(\tilde{b}_c^{\ *}\right)\cdot\sigma_{\wp_c}(0)}{\sigma_{\wp_c}\left(\tilde{b}_c^{\ *}\right)-\sigma_{\wp_c}(0)}$.

By (123') we may write the following:

$$\sigma_{\wp_c}\left(-b_c^{\ *}\div k_c^{*}\right)=\frac{k_c^{*2}\sigma_{\wp_c}\left(-b_c^{*}\right)\sigma_{\wp_c}\left(k_c^{*}\right)}{\left|b_c^{*}\sigma_{\wp_c}\left(-b_c^{*}\right)+k_c^{*}\sigma_{\wp_c}\left(k_c^{*}\right)\right|}=$$

$$=\frac{k_c^{*2}\cdot\dfrac{\sigma_{\wp_c}\left(b_c^{\ *}\right)\sigma_{\wp_c}(0)\sigma_{\wp_c}\left(k_c^{*}\right)}{\sigma_{\wp_c}\left(b_c^{\ *}\right)-\sigma_{\wp_c}(0)}}{\left|b_c^{*}\cdot\dfrac{\sigma_{\wp_c}\left(b_c^{\ *}\right)\sigma_{\wp_c}(0)}{\sigma_{\wp_c}\left(b_c^{\ *}\right)-\sigma_{\wp_c}(0)}+k_c^{*}\sigma_{\wp_c}\left(k_c^{\ *}\right)\right|}\ ,$$

which coincides with formula (146').

Fuzzy Quadratic Equations in Optimal Probabilistic Model

Suppose all calculations are made at $\wp$ calibration. Fuzzy quadratic equation has the following view:

$$\tilde{a}^{*}\tilde{x}^{*2}+\tilde{b}^{*}\tilde{x}^{*}+\tilde{c}^{*}=\tilde{0}^{*} \qquad \left(0^{*}\equiv 0\right) \qquad (147)$$

x^{*} is the solution of ordinary quadratic equation: $a^{*}x^{*2}+b^{*}x^{*}+c^{*}=0$. The solvability condition:

$$\sigma_{\wp}(0)\geq\sigma_{\wp}\left(c^{*}\right) \qquad (148)$$

Let us use rules (118)-(120) for $\sigma^{2}_{\wp}\left(x^{*}\right)$ dispersion calculation:

$$\sigma_{\wp}\left(a^{*}x^{*2}\right)+\sigma_{\wp}\left(b^{*}x^{*}\right)+\sigma_{\wp}\left(c^{*}\right)=\sigma_{\wp}(0) \qquad (149)$$

where

$$\sigma_{\wp}\left(a^{*}x^{*2}\right)=\left|a^{*}\sigma_{\wp}\left(x^{*2}\right)+x^{*2}\sigma_{\wp}\left(a^{*}\right)\right|=\left|2a^{*}x^{*}\sigma_{\wp}\left(x^{*}\right)+x^{*2}\sigma_{\wp}\left(a^{*}\right)\right|$$

(150)

$$\sigma_{\wp}\left(b^{*}x^{*}\right)=\left|b^{*}\sigma_{\wp}\left(x^{*}\right)+x^{*}\sigma_{\wp}\left(b^{*}\right)\right| \qquad (151)$$

Via putting (150) and (151) in (149), received:

$$\left|2a^* x^* \sigma_{\wp}\left(x^*\right) + x^{*2} \sigma_{\wp}\left(a^*\right)\right| + \left|b^* \sigma_{\wp}\left(x^*\right) + x^* \sigma_{\wp}\left(b^*\right)\right| + \sigma_{\wp}\left(c^*\right) = \sigma_{\wp}(0)$$

(152)

It is clear that solvability conditions must be satisfied:

$$\left|2a^* x^* \sigma_{\wp}\left(x^*\right) + x^{*2} \sigma_{\wp}\left(a^*\right)\right| \le \sigma_{\wp}(0) \ ,$$

$$\left|b^* \sigma_{\wp}\left(x^*\right) + x^* \sigma_{\wp}\left(b^*\right)\right| \le \sigma_{\wp}(0), \ \sigma_{\wp}\left(c^*\right) \le \sigma_{\wp}(0) \qquad (153)$$

Let us consider four cases according of sings of quantities, the modules of which figures in (152) and (153):

1) $2a^* x^* \sigma_{\wp}\left(x^*\right) + x^{*2} \sigma_{\wp}\left(a^*\right) \ge 0$ and $b^* \sigma_{\wp}\left(x^*\right) + x^* \sigma_{\wp}\left(b^*\right) \ge 0$

Inequality (152) would be rewrite in the following view:

$$2a^* x^* \sigma_{\wp}\left(x^*\right) + x^{*2} \sigma_{\wp}\left(a^*\right) + b^* \sigma_{\wp}\left(x^*\right) + x^* \sigma_{\wp}\left(b^*\right) + \sigma_{\wp}\left(c^*\right) = \sigma_{\wp}(0)$$

(152')

thus,

$$\sigma_{\wp}\left(x^*\right) = \frac{\sigma_{\wp}(0) - \left(\sigma_{\wp}\left(a^*\right) x^{*2} + \sigma_{\wp}\left(b^*\right) x^* + \sigma_{\wp}\left(c^*\right)\right)}{2a^* x^* + b^*} \qquad (154')$$

For the considered case generalized Vietta theorem is valid:

$$\sigma_{\wp}\left(x_1^* + x_2^*\right) = \sigma_{\wp}\left(x_1^*\right) + \sigma_{\wp}\left(x_2^*\right) =$$

$$= \frac{\sigma_{\wp}(0) - \left(\sigma_{\wp}\left(a^*\right) x_1^{*2} + \sigma_{\wp}\left(b^*\right) x_1^* + \sigma_{\wp}\left(c^*\right) \right)}{2a^* x_1^* + b^*} +$$

$$+ \frac{\sigma_{\wp}(0) - \left(\sigma_{\wp}\left(a^*\right) x_2^{*2} + \sigma_{\wp}\left(b^*\right) x_2^* + \sigma_{\wp}\left(c^*\right) \right)}{2a^* x_2^* + b^*} =$$

$$= \frac{\sigma_{\wp}\left(a^*\right)\left(x_2^{*2} - x_1^{*2} \right) + \sigma_{\wp}\left(b^*\right)\left(x_2^* - x_1^* \right)}{\sqrt{b^{*2} - 4a^* c^*}} =$$

$$= \frac{b^* \sigma_{\wp}\left(a^*\right) - a^* \sigma_{\wp}\left(b^*\right)}{a^{*2}}$$

On received relation note the following: as it is supposed that $\sigma\left(x_1^* \right) \geq 0$ and $\sigma\left(x_2^* \right) \geq 0$, so $\sigma\left(x_1^* + x_2^* \right) \geq 0$, further, appropriate dispersions of numbers $\frac{\tilde{b}^*}{\tilde{a}^*}$ and $(-1)\frac{\tilde{b}^*}{\tilde{a}^*}$ are equal, thus:

$$\sigma\left(x_1^* + x_2^* \right) = \sigma\left((-1)\frac{b^*}{a^*} \right) \quad ,$$

i.e

$$\tilde{x}_1^* + \tilde{x}_2^* = (-1)\frac{\tilde{b}^*}{\tilde{a}^*}$$

Let us consider $\tilde{x}_1^* \tilde{x}_2^*$ roots product. The appropriate dispersion is calculating via formula (119), we have:

$$\sigma_{\wp}\left(x_1^* x_2^*\right) = \left| x_1^* \sigma_{\wp}\left(x_2^*\right) + x_2^* \sigma_{\wp}\left(x_1^*\right) \right| =$$

$$= \left| \frac{x_1^* \sigma_{\wp}(0) - x_1^* \left(\sigma_{\wp}\left(a^*\right) x_2^{*2} + \sigma_{\wp}\left(b^*\right) x_2^* + \sigma_{\wp}\left(c^*\right) \right)}{2a^* x_2^* + b^*} + \right.$$

$$\left. + \frac{x_2^* \sigma_{\wp}(0) - x_2^* \left(\sigma_{\wp}\left(a^*\right) x_1^{*2} + \sigma_{\wp}\left(b^*\right) x_1^* + \sigma_{\wp}\left(c^*\right) \right)}{2a^* x_1^* + b^*} \right| =$$

$$= \left| \frac{a^* \left(\sigma_{\wp}(0) - \sigma_{\wp}\left(c^*\right) \right) + c^* \sigma_{\wp}\left(a^*\right)}{a^{*2}} \right| =$$

$$= \left| \frac{a^* \sigma_{\wp}\left(0 - c^*\right) - \left(0 - c^*\right) \sigma_{\wp}\left(a^*\right)}{a^{*2}} \right|$$

By (132) dispersion $\sigma_{\wp}\left(0-c^{*}\right)$ is appropriate of opposite F-number $-c^{*}$, so it is possible to write:

$$\sigma_{\wp}\left(x_1^{*}x_2^{*}\right)=\sigma_{\wp}\left((-1)x_1^{*}x_2^{*}\right)=\sigma_{\wp}\left(\frac{0-c^{*}}{a^{*}}\right)$$

i.e.

$$\frac{\tilde{0}-\tilde{c}^{*}}{\tilde{\alpha}^{*}}=(-1)\tilde{x}_1^{*}\,\tilde{x}_2^{*} \qquad (156')$$

2)

In this case equation (152) will have the view:

$$-2a^{*}x^{*}\sigma_{\wp}\left(x^{*}\right)-x^{*2}\sigma_{\wp}\left(a^{*}\right)-b^{*}\sigma_{\wp}\left(x^{*}\right)-$$

$$-x^{*}\sigma_{\wp}\left(b^{*}\right)+\sigma_{\wp}\left(c^{*}\right)=\sigma_{\wp}(0) \qquad (146'')$$

further,

$$\sigma_{\wp}\left(x^{*}\right)=\frac{\sigma_{\wp}(0)-\left(-\sigma_{\wp}\left(a^{*}\right)x^{*2}-x^{*}\sigma_{\wp}\left(b^{*}\right)+\sigma_{\wp}\left(c^{*}\right)\right)}{-2a^{*}x^{*}-b^{*}} \qquad (154'')$$

As in the previous case, it is easy to show that:

$$\tilde{x}_1^{*}+\tilde{x}_2^{*}=(-1)\frac{\tilde{b}^{*}}{\tilde{a}^{*}}$$

for roots product we will receive:

$$\sigma_{\wp}\left(x_1^* x_2^*\right) = \left| \frac{\left(\sigma_{\wp}(0) - \sigma_{\wp}\left(c^*\right)\right)a^* - c^* \sigma_{\wp}\left(a^*\right)}{a^{*2}} \right| \qquad (157)$$

It is easy to see that this standard deflection is appropriate of the following:

$$\frac{(-1)\left(\tilde{0} - \tilde{c}^*\right)}{\tilde{\alpha}^*} = \tilde{x}_1^* \tilde{x}_2^* \qquad (156'')$$

Individual cases consideration shows, that they are not correspond to Vieta's fuzzy theorem and so were not considered as solutions of equation (147).

It is interesting to consider solution of (147) in $\wp_C$ representation.

Apparently it is possible to receive analogous solutions for equations of high level and also for linear equations systems.

Solution of Fuzzy differential equations

At modeling of the real systems (which do not contain possibilistic or stochastic uncertainty) frequently we come to differential equations. Theory of differential equations is deeply and widely advanced. If at each concrete problem we use powerful numerical method then the model development does not represent with principle complexity itself.

Let's consider the example:

$$\dot{\tilde{x}} + \tilde{a}\tilde{x} = \tilde{0}$$

The equation for the calculated values appropriate to this one is:

$$\dot{x}^* + a^* x^* = 0 \qquad x^*(t) = Ae^{-a^* t}$$

Using the ”energetic function” we’ll have:

$$\sigma\left(\dot{\tilde{x}} + \tilde{a}\tilde{x}\right) = \sigma\left(\tilde{0}\right)$$

$$\sigma\left(\dot{\tilde{x}}\right) + \left|a^*\sigma(\tilde{x}) + x^*\sigma(\tilde{a})\right| = \sigma\left(\tilde{0}\right)$$

$$2\beta\sigma(\tilde{x}_c) + \begin{cases} a^*\sigma(\tilde{x}) + x^*\sigma(\tilde{a}), a^*\sigma(\tilde{x}) + x^*\sigma(\tilde{a}) \geq 0 \\ -a^*\sigma(\tilde{x}) - x^*\sigma(\tilde{a}), a^*\sigma(\tilde{x}) + x^*\sigma(\tilde{a}) < 0 \end{cases} = \quad \sigma\left(\tilde{0}\right)$$

$$\frac{\beta c}{\sigma(\tilde{x})} + \begin{cases} a^*\sigma(\tilde{x}) + x^*\sigma(\tilde{a}) = \sigma\left(\tilde{0}\right) \\ -a^*\sigma(\tilde{x}) - x^*\sigma(\tilde{a}) = \sigma\left(\tilde{0}\right) \end{cases}$$

$$\begin{cases} \beta c + a^*\sigma^2(\tilde{x}) + x^*\sigma(\tilde{a})\sigma(\tilde{x}) = \sigma\left(\tilde{0}\right)\sigma(\tilde{x}) \\ \beta c - a^*\sigma^2(\tilde{x}) - x^*\sigma(\tilde{a})\sigma(\tilde{x}) = \sigma\left(\tilde{0}\right)\sigma(\tilde{x}) \end{cases}$$

$$\begin{cases} a^*\sigma^2(\tilde{x}) + \left(x^*\sigma(\tilde{a}) - \sigma\left(\tilde{0}\right)\right)\sigma(\tilde{x}) + \beta c = 0 \\ a^*\sigma^2(\tilde{x}) + \left(x^*\sigma(\tilde{a}) + \sigma\left(\tilde{0}\right)\right)\sigma(\tilde{x}) - \beta c = 0 \end{cases}$$

$$\begin{cases} D_1 = \left(x^*\sigma(\tilde{a}) + \sigma\left(\tilde{0}\right)\right)^2 - 4a^*\beta c \\ D_2 = \left(x^*\sigma(\tilde{a}) + \sigma\left(\tilde{0}\right)\right)^2 + 4a^*\beta c \end{cases}$$

$$\sigma(\tilde{x}) = \frac{-\left(x^*\sigma(\tilde{a}) - \sigma\left(\tilde{0}\right)\right) \pm \sqrt{D_1}}{2a^*} \qquad \sigma(\tilde{x}) = \frac{-\left(x^*\sigma(\tilde{a}) + \sigma\left(\tilde{0}\right)\right) \pm \sqrt{D_2}}{2a^*}$$

As the determination of the uncertainty nature is the main task of the initial information processing, here is an opportunity to expand an area of use for differential equations even in those cases when parameters determining differential equations contain new types uncertainty via modeling of these situations with the help of new concept - canonically conjugated fuzzy subsets.

1*) The given formulas (118-120) allow representation of mean standard errors of sum, product and division by mean standard errors of composing components linearly.

Let's consider arithmetic operations separately:

We have: $\Delta(x_1 + x_2) = \left|\left(x_1 - x_1^{*}\right) + \left(x_2 - x_2^{*}\right)\right| \le \left|\left(x_1 - x_1^{*}\right)\right| + \left|\left(x_2 - x_2^{*}\right)\right|$ (*1)

It's possible to rewrite (*) in the following way:

$$\Delta(x_1 + x_2) = a\left(x_1, x_2; x_1^{*}, x_2^{*}\right)\left(\Delta x_1 + \Delta x_2\right) \quad (*2)$$

Where the function:

$$a\left(x_1, x_2; x_1^{*}, x_2^{*}\right) = \frac{\left|\left(x_1 - x_1^{*}\right) + \left(x_2 - x_2^{*}\right)\right|}{\left|x_1 - x_1^{*}\right| + \left|x_2 - x_2^{*}\right|} \quad (*3)$$

Thus if $f\left(x = x_1 + x_2\right)$ is the function of $x = x_1 + x_2$ random quantity distribution, then

$$\left(\Delta\left(x = x_1 + x_2\right)\right)^{*} = \int_{-\infty}^{+\infty}\int dx dx_2\, f\left(x_1 x_2\right)\Delta x = \int_{-\infty}^{+\infty}\int dx dx_2\, a\left(x_1, x - x_1; x_1^{*}, x_2^{*}\right) f\left(\Delta x, \Delta x_1\right)\left(\Delta x_1 + \Delta x_2\right) =$$

$$= \int_{-\infty}^{+\infty}\int dx dx_2\, a\left(x, x_2; x_1^{*}, x_2^{*}\right) f\left(x, x_2\right)\left(\Delta x_1 + \Delta x_2\right) = a\left(\bar{x}_1, \bar{x}_2; x_1^{*}, x_2^{*}\right)\Delta x_1 + a\left(\bar{\bar{x}}_1, \bar{\bar{x}}_2; x_1^{*}, x_2^{*}\right)\Delta x_2$$

(*4)

Upon receiving the last formula, we used mean value generalized theorem [36]. Analogically we can act in case of product and division operations. Thus to be more accurate, the rules should be written in the following way:

$$(\Delta(x_1 + x_2))^* = a_1\sigma_1 + a_2\sigma_2 \quad ; \quad \sigma_{x_1+x_2} = \alpha_1\sigma_1 + \alpha_2\sigma_2 = \sigma_1' + \sigma_2'$$

$$(\Delta(x_1 \bullet x_2))^* = b_1\sigma_1 + b_2\sigma_2 \quad ; \quad \sigma_{x_1 x_2} = \beta_1\sigma_1 + \beta_2\sigma_2 = \sigma_1'' + \sigma_2''$$

$$\left(\Delta\left(\frac{x_1}{x_2}\right)\right)^* = c_1\sigma_1 + c_2\sigma_2 \quad ; \quad \sigma_{\frac{x_1}{x_2}} = \gamma_1\sigma_1 + \gamma_2\sigma_2 = \sigma_1''' + \sigma_2'''$$

In case of canonically conjugated subsets, we will have

$$(\Delta(x_{c1} + x_{c2}))^* = \left(\frac{\alpha_1}{\sigma_{c1}} + \frac{\alpha_2}{\sigma_{c2}}\right)^{-1} = \frac{\alpha_1\alpha_2\sigma_{c1}\sigma_{c2}}{\alpha_1\sigma_{c2} + \alpha_2\sigma_{c1}}$$

$$(\Delta(x_{c1} \otimes x_{c2}))^* = \frac{\beta_1\beta_2\sigma_{c1}\sigma_{c2}}{\beta_1\sigma_{c2} + \beta_2\sigma_{c1}}$$

$$\left(\Delta\left(\frac{x_{c1}}{x_{c2}}\right)\right)^* = \frac{\gamma_1\gamma_2\sigma_{c1}\sigma_{c2}}{\gamma_1 x_2^*\sigma_{c1} + \gamma_2 x_1^*\sigma_{c2}}$$

Let's note that $\alpha_1\alpha_2 \geq 0$ (appropriately $\beta_1\beta_2 \geq 0$, $\gamma_1\gamma_2 \geq 0$). If we will bring appropriate notations, we will receive the given formulas (118-120).

References:

1. **Didier Dubois, Henri Prade. Teoria Vozmojnostei (Traduction Russe de "Théorie des Possibilités", Masson, Paris, 1985/1987), Radio i Sviaz, Moscou, 1990.**
2. **Didier Dubois, Henri Prade. Fuzzy Sets & Systems: Theory and Applications, Academic Press (APNet), V. V.144, 393 p., 1980.**
3. **Lotfi A. Zadeh . Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1, 3-28, 1978.**
4. თ.გაჩეჩილაძე, "კანონიკურად შეუღლებული არამკაფიო ქვესიმრავლეები და ცოდნის ახალი წარმოდგენა", საქ. მათემატიკოსთა ყრილობა, 1-4, VI, (მასალები), თბილისი, 1994.
5. А.Мессия – Квантовая Механика, Москва 1982г.
6. **P.A.M. Dirac. Principe of Quantum Mechanics. GIFML (1960)**
7. **A. Deluca and S. Termini, A definition of a non-probabilistic entropy in the setting of fuzzy set theory, Int. Control20, 1972, 301-312.**
8. **Didier Dubois, Henri Prade. Fuzzy Sets and Systems (réédition en CDROM de l'ouvrage publié en 1980), Academic Press, New York, Fuzzy Logic CDROM Library, 1996.**
9. **P.A.M. Dirac, on Quantum Theory, "The Development Of Quantum Mechanics" Conferenza Tenuta il 14 Aprile 1972, in Roma, Accademia Nazionale dei Lincei, 1974**
10. **George J. Klir , Tina A. Folger, Fuzzy sets, uncertainty, and information, Prentice-Hall, Inc., Upper Saddle River, NJ, 1987**
11. **Didier Dubois, Henri Prade. Fundamentals of Fuzzy Sets, Kluwer, Boston, Mass., The Handbooks of Fuzzy Sets Series, 2000.**
12. **Т.Гачечиладзе, "Модифицированная вероятностная модель характеристической функции нечеткого подмножества", Applied Mathem. Informatics, 1, N1, 1996.**

13. M.Tsintsadze – Probabilistic Model of Optimal Membership Function. Proceedings of Iv.Javakhishvili .Tbilisi State University, Vol 354(24) (2004) Tbilisi

14. M.Tsintsadze - Optimal Membership Function's Probabilistic Model for Fuzzy Differential Equations. Bulletin of the Georgian Academy of Sciences. Vol 171(3)(2005). Tbilisi

15. Abraham Kandel, Fuzzy mathematical techniques with applications, Addison-Wesley Longman Publishing Co., Inc., Boston, MA, 1986

16. Didier Dubois, Henri Prade. Théorie des Possibilités. Applications à la Représentation des Connaissances en Informatique, Collection Méthode + Programmes, Masson, Paris, 2, 1985

17. Lotfi A. Zadeh. Fuzzy probabilities, Information Processing and Management 19, 148-153, 1984.

18. Lotfi A. Zadeh. Probability Theory and Fuzzy Logic are Complementary rather than Competitive, Technometrics, 37, 271-276,1995.

19. T.Gachechiladze, F.kriado . "Fuzzy Random Events and Corresponding Probability Measures", Real Academia de Cientis (Mathematicas), LXXXIX, N1,2,1995.

20. T.Gachechiladze, "Entropy of fuzzy Sets" FSS, 88, N1,1997.

21. L. Zadeh, The concept of linguistic variables and its application to approximate reasoning, Informat. Sci. 8 (1975) 199-249.

22. A. Kandel, On the control and evaluation of uncertain processes, IEEE Trans. Automatic Control AC-25 6 (1980) 1182-1187.

23. D. Dubois, H. Prade, Theory of possibility: an approach to computerize processing of uncertainty, Plenum Press, New York, 1988.

24. F.Criado Torralba, T.Gachechiladze, H.Meladze, G.Tsertsvadze. Fuzzy Models of Language Structures // IEEE Transactions on Fuzzy Systems – 2002 – vol.10, №4 – pp.421-435

25. Т.Гачечиладзе, "Нечеткие математические модели лингвистической статистики", Изд. РАН (теория и системы уравнения), N5, 1998.

26. Т.Гачечиладзе, G.Tsertsvadze. “Математическая модель многошагового процесса поддержки принятия решения в задаче прогноза землетрясений“.თსუ შრომები (სერია გამოყ. მათემატიკა), ტ.330 (19), 1998.

27. T.Gachechiladze, F.kKriado, H.Meladze,G.Tsertsvadze. “A combined Entropy for Evalution of Fuzzy Random Events Probability“, (Rep. Enlarged Session of Inst. Appl. Mathemat. Vol. 13 N4, 1998.

28. Т.Гачечиладзе, F.Kriado “Нечеткие математические модели лингвистической статистики II“,Изд. РАН (теория и системы уравнения), N5, 1999.

29. Klir, G. J. and Yuan, B., Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, PTR, Upper Saddle River, NJ, 1995

30. L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975.

31. A. Kandel - Knowledge Discovery in Time Series Databases, IEEE Transactions on Systems, Man, and Cybernetics 31B(2001)

32. M.Tsintsadze - კანონიკურად შეუღლებულ არამკაფიო ქვესიმრავლეთა თეორია, საქართველოს მათემატიკოსთა IV ყრილობა, თეზისების კრებული, გვ. 154, 2005

33. M.Tsintsadze - On Theory of “Canonically Conjugated” Fuzzy Subsets. Electronic Scientific Journal: Computer Sciences and Telecommunications, 2006.
http://gesj.internet-academy.org.ge/en/list_artic_en.php?b_sec=comp

34. M.Tsintsadze - On Optimal Fuzzy Numbers- Bulletin of the Georgian Academy of Sciences, Vol 174(1), 2006

35. M.Tsintsadze - On theory of information functions presence. Universität Dortmund, Eldorado - Ressourcen aus und für Lehre, Studium und Forschung, 2006
https://eldorado.uni-dortmund.de/handle/2003/22659

36. Г.М Фихтеньольц – Дифференциальное и Интегральное Исчисление. Москва 1984г.